Indian Institute of Science
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NP-TEL
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Technology Enhanced Learning

Course Title
Finite element method for structural dynamic
And stability analyses
Lecture – 32
Dynamic analysis of stability and
Analysis of time varying systems
By

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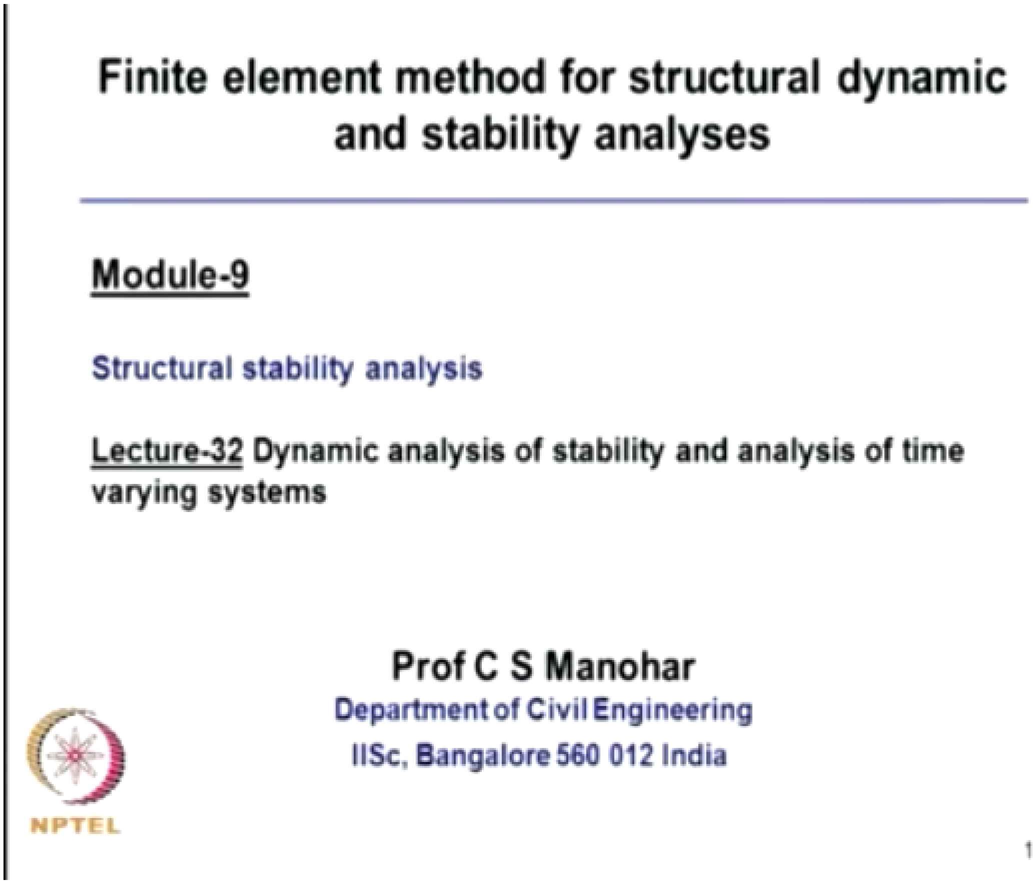


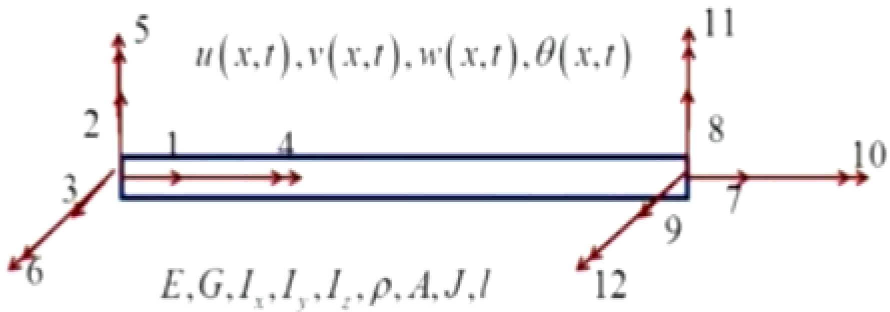
Previous lecture we considered questions about imperfection sensitive structures, we also derived the outline the derivation of geometric stiffness for a plate element, and considered the form of the function U for a Timoshenko beam element, 3D beam element.

3D beam element

Is it that computational effort increases and we need to handle larger sized matrices? OR

Are there any new phenomena that we need to be concerned about?

Refer to Lecture-12



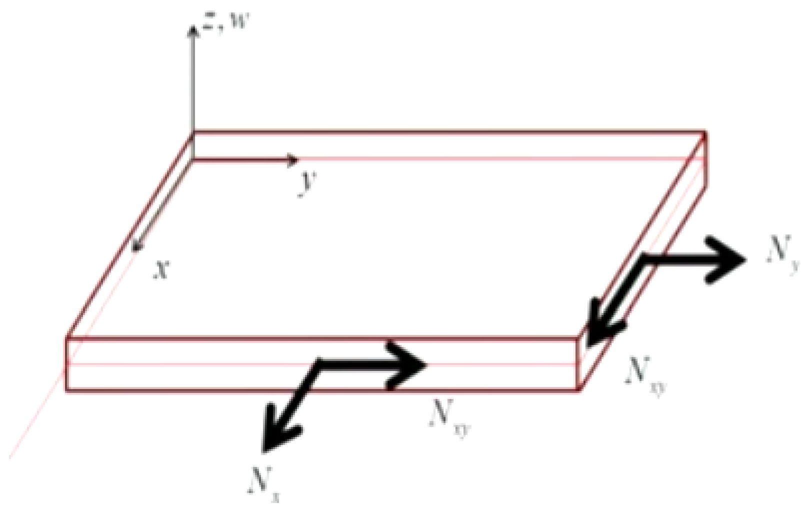
3D beam element : a simplified model

Effects included: coupling between axial forces with

- bending along two axes
- torsion about longitudinal axis

We will quickly recall some of the details, so for a 3D beam element the question that we asked is, is it simply that the computational effort increases with no new phenomena kicking in, or are there any new questions that we need to address. It turns out that there will be certain newer terms that we need to take care of, that takes into account coupling between axial forces with bending along two axis torsion about longitudinal axis, and also coupling between torque and bending moment. This model is a simplified model a more refined model would include other effects.

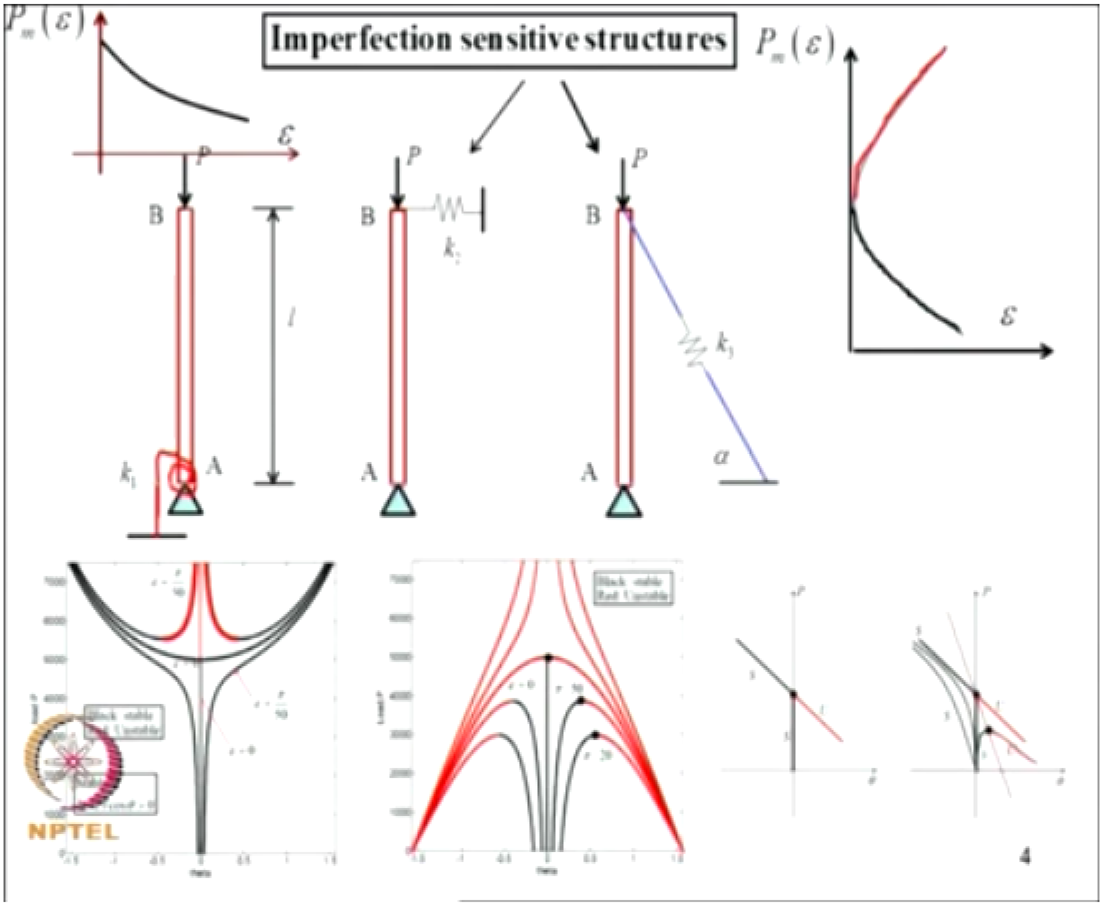
Geometric stiffness matrix for a thin plate bending element (Kirchoff-Loeve plate)



Membrane forces

$$N_x = \int_{-t/2}^{t/2} \sigma_{xx} dz; \;\; N_y = \int_{-t/2}^{t/2} \sigma_{yy} dz; \;\; N_{xy} = \int_{-t/2}^{t/2} \sigma_{xy} dz$$

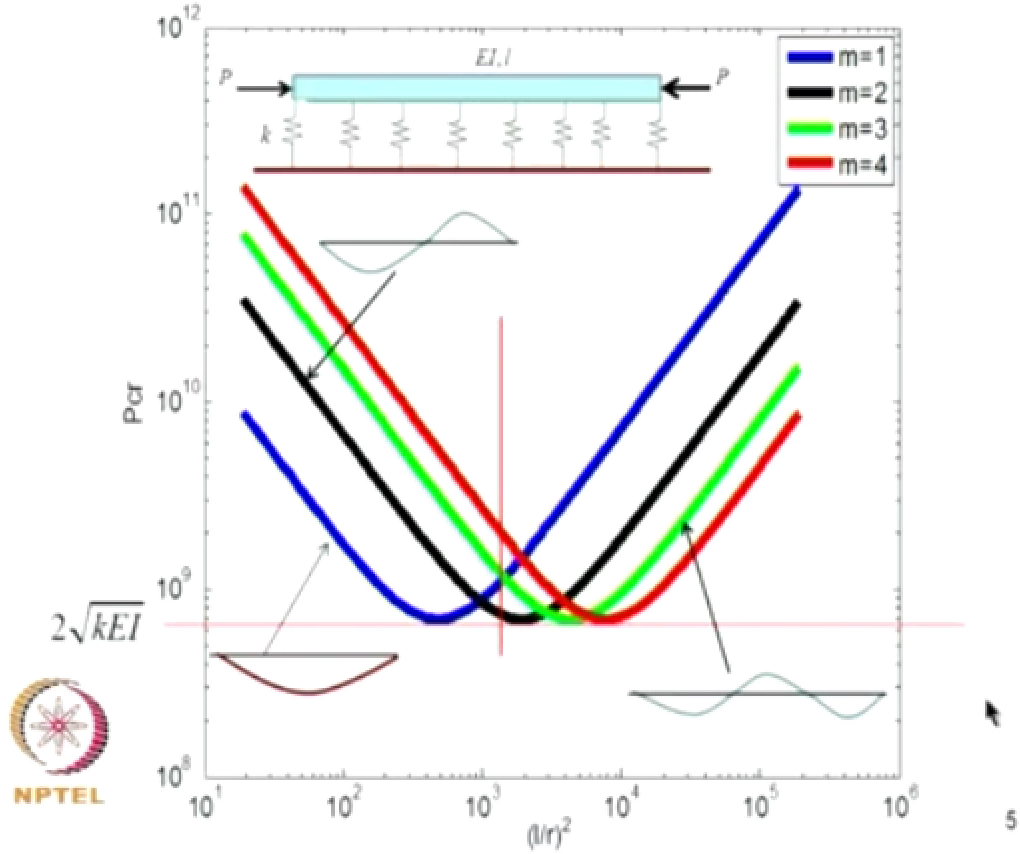
We also derived the geometric stiffness matrix for a thin plate element, we didn't complete the derivation, but the basic step involved in arriving at the geometric stiffness matrix was outlined.



We considered to discuss questions on imperfection sensitivity, we consider 3 archetypal mechanical models which basically consisted of rigid link A, B loaded in 3 different, supported in 3 different ways, these 3 systems were hypothetically designed to possess same critical load and the contention was that if one were to perform an experiment the critical load although theoretically, all the critical load for all the 3 systems should be exactly equal, it will turn out this structure will have A, there is a first system will have higher critical load than second one, and second one will have higher value then the third one, why that happens was the question that we were trying to answer and that we approached by considering the load deflection diagram by including large deformations and imperfection, so the bifurcation diagram for the first system had a rising load path whose rate of changed dramatically as we came close to the point of bifurcation, but this path did not encounter any unstable point, so the structure was able to carry the load even after you know large loads were applied.

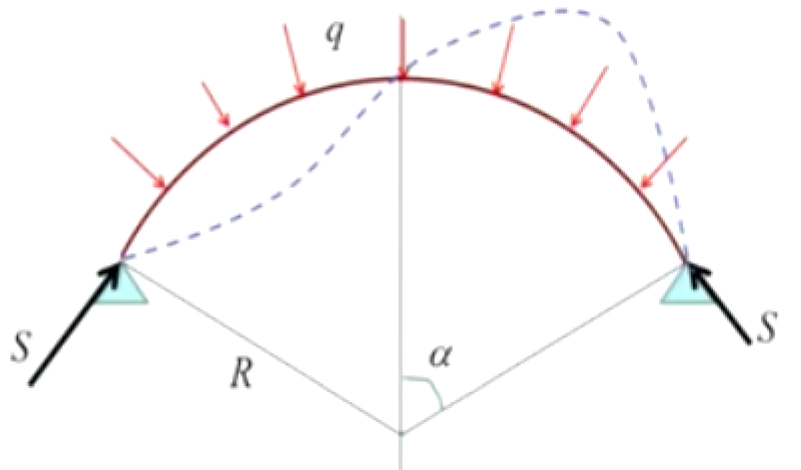
Whereas for the second system the load deflection diagram raised and it reached a point beyond which it was no longer able to carry the load and it bifurcated into 2 unstable parts which was symmetric about this. In presence of imperfection the load deflection path rises up to a point and encounters an unstable branch, and there by limiting the load carrying capacity to these values, so the maximum load that this system can carry is limited by the imperfection that is present, so greater the imperfection lesser the ability to carry that is depicted in this graph for this second system.

The third system in which the spring was placed like this, again the load deflection path raised up to a point and it bifurcated into a stable path and an unstable path, the unstable path was dropping and in presence of imperfection the load carrying capacity was dictated by the behavior of the unstable branch, and there was substantial lowering of load carrying capacity because of imperfections.



We also considered problems of buckling of beams, beams on Winkler's foundation specifically this graph is for a structure which is simply supported and on elastic foundation, we found that for certain L/R ratio the half sine wave shape of the mode shape leads to the lowest critical load, but this is not generally true for certain other choices of L/R ratio the full sine wave which corresponds to M = 2 becomes a critical load for the system, so this is one special feature associated with beeps on elastic foundation.

Circular arch



Two hinged circular arch under uniform pressure.
Would not have created BM in pre-buckled state.
What is critical value of q?
That is, what is the value of q, which induces bending in the arch?

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Timoshenko and Gere, p207

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We also studied the problem of a ring and derived the critical load that, external load that can be applied axis symmetrically on that ring we derived the value of that using simple formulation. Now we will continue that discussion as applied to a circular arch, and then we will take up the topic for today's class that is questions about stability analysis using structural dynamic concepts. So this is a circular arch which is loaded as shown here, it's a two hinged circular arch under uniform pressure, it would not have created bending moment in pre-buckled state, so but what is our critical value of Q, that is what is the value of Q which induces bending in the arch, so this is discussed in the book by Timoshenko and Gere, the simple, based on the formulation given there

$$\frac{d^2w}{d\theta^2}+w=-\frac{MR^2}{EI}=-\frac{SwR^2}{EI}=-\frac{(qR)wR^2}{EI}$$

$$\frac{d^2w}{d\theta^2}+w\left(1+\frac{qR^3}{EI}\right)=0 \text{ with BCS: } w(0)=0, w(2\alpha)=0$$

$$\frac{d^2w}{d\theta^2}+k^2w=0 \text{ with } k^2=\left(1+\frac{qR^3}{EI}\right)$$

$$w(\theta)=A\cos k\theta+B\sin k\theta$$

$$w(0)=0\Rightarrow A=0$$

$$w(2\alpha)=0\Rightarrow \sin 2k\alpha=0\Rightarrow 2k\alpha=n\pi, n=1,2,\Rightarrow k=\frac{\pi}{\alpha}$$

$$\Rightarrow\left(1+\frac{qR^3}{EI}\right)=\frac{\pi^2}{\alpha^2}\Rightarrow q_c=\frac{EI}{R^3}\left(\frac{\pi^2}{\alpha^2}-1\right)$$

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Exercise: Using Euler-Bernoulli beam elements, verify this result. 7

we can write the expression for bending moment and one gets this expression which we already saw for a ring in this form, so the governing equation here is as shown here with boundary condition W(0) and W(2) alpha = 0.

Now we define K square as this quantity and write the solution as A cos K theta + B sine K theta, now imposing the boundary condition W(0) = 0, I get A = 0, and W(2) alpha = 0 gives me sine 2K alpha = 0, from which I get K to be pi/alpha, so the critical load is given by this and QC turns out to be EI/R cube into this quantity, this is exact result assuming thin, cross-section is thin, so the exercise that I said is you can select an circular arch like this and use Euler-Bernoulli beam elements and discretize this and formulate the structure elastic and geometric stiffness matrix and evaluate the critical value of Q and compare it with this exact value.

Dynamic analysis of stability and analysis of time varying systems

Topics

- Dynamic analysis of structures in presence of static initial stresses
- Parametrically excited systems: stability analysis
- Statically loaded structures which require dynamic considerations to establish stability
- Response analysis of time varying systems

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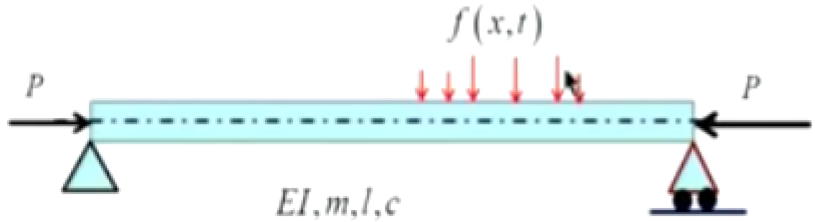
Okay, so with this now we will move on to a new topic, this is dynamic analysis of stability and analysis of time varying systems. Now we will consider a few topics, the first topic that we look at is how to perform dynamic analysis of structure in presence of static initial stresses, so this is generalization of beam column study to dynamic problems. Then we will discuss what are known as parametrically excited systems and consider questions on stability, I will explain what it means shortly. Then we will consider certain situations where we deal with statically loaded structures, but it turns out that to correctly infer their stability characteristics you need to perform a dynamic analysis, right, these are known as follower force problems, lot of discussion on that on this question in the existing literature, there are questions on whether this exists in the first place also have been raised, but it poses an interesting question, so we'll just take a look at that. And then what we'll do is we'll consider some time varying systems and see how we can use finite element method and discretize that type of problems, okay by time varying systems I mean some of the structural matrices will become functions of time. So I will explain all this in due course, this is just a glimpse of what the topics we are going to touch upon.

Dynamic analysis of a beam column



$$EI\frac{\partial^4 y}{\partial x^4}+P\frac{\partial^2 y}{\partial x^2}+m\frac{\partial^2 y}{\partial t^2}+c\frac{\partial y}{\partial t}=f(x,t)$$

$$\text{BCs: } y(0,t)=0,\ EI\frac{\partial^2 y}{\partial x^2}(0,t)=0,\ y(l,t)=0,\ EI\frac{\partial^2 y}{\partial x^2}(l,t)=0$$

$$\text{ICs: } y(x,0)=y_0(x),\ \frac{\partial y}{\partial t}(x,0)=v_0(x)$$

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So we’ll start with the simple problem of a beam which is actually loaded and carries a transverse load F(x,t), this is a beam equation and these are the boundary condition as appropriate for simple boundary conditions. Now the question I would like to ask is what is the influence of presence of axial load on response of this system, if P = 0 we already studied this how to carry out this analysis, when P is not equal to 0 what happens? So this is analogous to the beam column example where loading was F(x) we knew how to analyze the problem with P = 0, and by including P what was the effect was the question that we consider. So when it comes to dynamic analysis of this type of problems the question arises what is the role of P in determining the natural frequencies and mode shapes of the system that has bearing on force response analysis subsequently.

$$EI\frac{\partial^4 y}{\partial x^4} + P\frac{\partial^2 y}{\partial x^2} + m\frac{\partial^2 y}{\partial t^2} + c\frac{\partial y}{\partial t} = f(x,t) \equiv EIy^{iv} + Py'' + m\ddot{y} + c\dot{y} = f(x,t)$$

Undamped free vibration analysis

$$EI\frac{\partial^4 y}{\partial x^4} + P\frac{\partial^2 y}{\partial x^2} + m\frac{\partial^2 y}{\partial t^2} = 0$$

$$y(x,t) = \phi(x)\exp(i\omega t)$$

$$\Rightarrow \left\{EI\phi^{iv} - P\phi'' + m\omega^2\phi\right\}\exp(i\omega t) = 0 \Rightarrow EI\phi^{iv} + P\phi'' - m\omega^2\phi = 0$$

$$\phi^{iv} + \alpha^2\phi'' - \lambda^4\phi = 0 \text{ with } \alpha^2 = \frac{P}{EI} \;\&\; \lambda^4 = \frac{m\omega^2}{EI}$$

This constitutes an eigenvalue problem. Let us seek solution in the form

$$\phi(x) = \phi_0 \exp(sx)$$

$$\Rightarrow s^4 + \alpha^2 s^2 - \lambda^4 = 0$$

$$s^2 = \frac{-\alpha^2 \pm \sqrt{\alpha^4 + 4\lambda^4}}{2}$$

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So let's first start with to consider this equation and I will use prime for its derivative with respect to special variable, and we'll start with the free vibration, undamped free vibration analysis, we will use a method of separation of variables and look for a solution where all points on the structure vibrate harmonically at the same frequency omega such motions are possible if this phi and omega satisfy this equation. Now E raise to I omega T cannot be 0, therefore I'll get this equation here and I will rearrange these terms, I will write for P/EI alpha square and for M omega square/EI lambda to the power of 4, so this is an eigenvalue problem, the eigenvalue being the load P, okay, all other parameters are known of course a load P and omega are both, you know P is given, omega is the eigenvalue parameter, if you are doing stability analysis for a given omega of course that will be a different question, the question we are considering is P is given and omega is a eigenvalue. This constitutes an eigenvalue problem, let us seek solution in the form, this is a linear ordinary differential equation with constant coefficients therefore exponential function should satisfy this equation, so we seek the solution in the form of phi naught E raise to SX, substitute this we get the characteristic equation as shown here, and the

$$s^2 = \frac{-\alpha^2 \pm \sqrt{\alpha^4 + 4\lambda^4}}{2}$$

$$s_1^2 = \frac{-\alpha^2 + \sqrt{\alpha^4 + 4\lambda^4}}{2} > 0 \Rightarrow (s_1)_{1,2} = \pm\varepsilon$$

$$s_2^2 = \frac{-\alpha^2 - \sqrt{\alpha^4 + 4\lambda^4}}{2} < 0 \Rightarrow (s_2)_{1,2} = \pm i\delta$$

$$\phi(x) = A\cosh\varepsilon x + B\sinh\varepsilon x + C\sin\delta x + D\cos\delta x$$

$$\phi'(x) = A\varepsilon\sinh\varepsilon x + B\varepsilon\cosh\varepsilon x + C\delta\cos\delta x - D\delta\sin\delta x$$

$$\phi''(x) = A\varepsilon^2\cosh\varepsilon x + B\varepsilon^2\sinh\varepsilon x - C\delta^2\sin\delta x - D\delta^2\cos\delta x$$

$$\left.\begin{aligned}\phi(0) = 0 &\Rightarrow A + D = 0\\ \phi''(0) = 0 &\Rightarrow A\varepsilon^2 - D\delta^2 = 0\end{aligned}\right\} \Rightarrow A = D = 0$$

$$\left.\begin{aligned}\phi(l) = 0 &\Rightarrow B\sinh\varepsilon l + C\sin\delta l = 0\\ \phi''(l) = 0 &\Rightarrow B\varepsilon^2\sinh\varepsilon l - C\delta^2\sin\delta l = 0\end{aligned}\right\} \begin{bmatrix}\sinh\varepsilon l & \sin\delta l\\ \varepsilon^2\sinh\varepsilon l & -\delta^2\sin\delta l\end{bmatrix}\begin{Bmatrix}B\\ C\end{Bmatrix} = 0$$

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roots of this characteristic equation are given by this, this is a by quadratic equation so S square is this and consequently S itself is given by roots of this, S1 1,2 will be +- epsilon and this will be + - I Delta.

You can see here that S1 square will be positive, whereas S2 square will be negative, therefore roots leading from S2 will be pure imaginary complex conjugate whereas this will be real, so the solution will have sine and cosine hyperbolic, as well as sine and cosine functions, hyperbolic terms are associated with epsilon, and sine and cosine with delta. Now the A, B, C, D are the 4 constants, we have 4 boundary conditions, so we can run through that calculation and I get the characteristic equation, okay, so the characteristic equation is obtained by demanding that the

$$\begin{bmatrix} \sinh \varepsilon l & \sin \delta l \\ \varepsilon^2 \sinh \varepsilon l & -\delta^2 \sin \delta l \end{bmatrix} \begin{Bmatrix} B \\ C \end{Bmatrix} = 0$$

For nontrivial solutions, $\begin{vmatrix} \sinh \varepsilon l & \sin \delta l \\ \varepsilon^2 \sinh \varepsilon l & -\delta^2 \sin \delta l \end{vmatrix} = 0$

$$\Rightarrow -\left(\delta^2 + \varepsilon^2\right) \sinh \varepsilon l \sin \delta l = 0$$

$$\left(\delta^2 + \varepsilon^2\right) \neq 0$$

$\sinh \varepsilon l = 0 \Rightarrow \varepsilon = 0 \Rightarrow \alpha^2 = \sqrt{\alpha^4 + 4\lambda^4} \Rightarrow \lambda = 0$ Not ok.

$$\Rightarrow \sin \delta l = 0 \Rightarrow \delta_n l = n\pi, n = 1, 2, \cdots$$

$$\Rightarrow \phi_n(x) = \phi_{0n} \sin \frac{n\pi x}{l}$$

$$s^4 + \alpha^2 s^2 - \lambda^4 = 0 \;\&\; s_2^2 = -\delta^2 \Rightarrow$$

$$\delta^4 - \delta^2 \alpha^2 - \lambda^4 = 0$$

$$\left(\frac{n\pi}{l}\right)^4 - \left(\frac{n\pi}{l}\right)^2 \alpha^2 - \lambda^4 = 0$$

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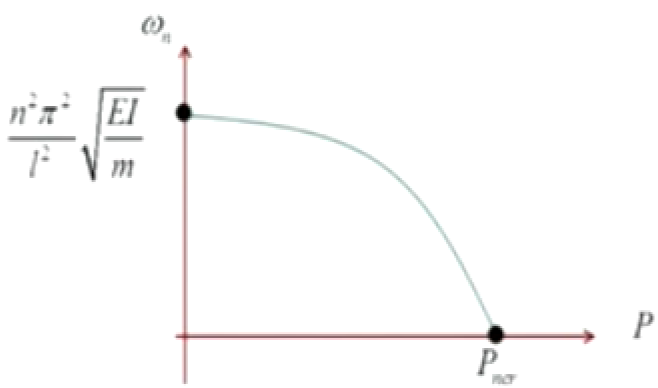
determinant of this coefficient matrix must be equal to 0 that leads to this characteristic equation. Now here delta square + epsilon square cannot be 0 so that is ruled out, sine H epsilon lambda if you assume that it is 0 then it becomes epsilon will be 0, and consequently lambda will be 0 and that is not acceptable, so this is not acceptable, so the only possibility sine delta L is 0 from which I get delta NL is N pi or delta N is N pi/L, so these are the mode shapes and I substitute into this and I get the expression for natural frequencies, so if I simplify that expression I get

$$\left(\frac{n\pi}{l}\right)^4 - \left(\frac{n\pi}{l}\right)^2\alpha^2 - \lambda^4 = 0 \Rightarrow \left(\frac{n\pi}{l}\right)^4 - \left(\frac{n\pi}{l}\right)^2\frac{P}{EI} = \frac{m\omega_n^2}{EI}$$

$$\omega_n^2 = \frac{EI}{m}\left(\frac{n\pi}{l}\right)^4\left[1-\frac{P}{EI}\frac{l^2}{n^2\pi^2}\right]$$

$$\omega_n = \frac{n^2\pi^2}{l^2}\sqrt{\frac{EI}{m}}\left[1-\frac{P}{P_{ncr}}\right]^{1/2} = \omega_{n0} f(P)$$



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natural frequency in terms of axial loads and other parameters of the problem as shown here, P critical is the Euler’s buckling load, so natural frequency is obtained as a natural frequency when P is 0 and that is omega N naught and multiplied by a nonlinear function F(p). Now if I plot this it turns out that natural frequency will become 0 when P is P critical, okay and when of course P = 0 you get back to the exact natural frequency of a beam without any axial load.

Remarks

- Due to presence of P, the natural frequencies get lowered. This has important implications in characterizing resonant characteristics of the system.
- In this case, it is observed that the mode shapes are not affected by P. This may not be true in general.
- At $P = P_{cr}$, $\omega_1 = 0$
- Orthogonality relations

$$\int_0^l m(x)\phi_n(x)\phi_k(x)\,dx = 0, \text{ for } n \neq k$$

$$\int_0^l \left[EI(x)\phi_n''(x)\phi_k''(x) + P\phi_n'(x)\phi_k'(x) \right] dx = 0, \text{ for } n \neq k$$

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So we can make some remarks, due to presence of P the natural frequencies get lowered. This has important implications in characterizing resonant characteristics of the system, because resonance is one of the things that we are primarily interested, and because of axial loads if natural frequency is going to become less then we have to take a call on that, in this case it is observed that the mode shapes are not affected by P, in this case means in the case of simply supported beam this may not be true in general, instead of simply supported beam if you consider an alternative other alternatives we may not get the same mode shapes, at P = P critical omega 1 it becomes 0. Now we already derived the orthogonality relation of vibration mode shape with respect to mass, this is this, but with respect to now stiffness and axial load there will be a modified version of orthogonality relations, I'll leave this as an exercise for you to verify.

Forced response analysis

$$EI\frac{\partial^4 y}{\partial x^4} + P\frac{\partial^2 y}{\partial x^2} + m\frac{\partial^2 y}{\partial t^2} + c\frac{\partial y}{\partial t} = f(x,t)$$

$$y(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x)$$

$$\Rightarrow \ddot{a}_n + 2\eta_n\omega_n\dot{a}_n + \omega_n^2 a_n = q_n(t)$$

Special case: $c = 0,\ f(x,t) = 0 \Rightarrow \ddot{a}_n + \omega_n^2 a_n = 0$

- $P < P_{cr} \Rightarrow a_n(t) = A_n \cos\omega_n t + B_n \sin\omega_n t$

Motion remains bounded for all times for all initial conditions

- $P = P_{cr} \Rightarrow a_n(t) = A_n + B_n t$

Motion grows linearly in time even for small initial conditions

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Now we can do a forced response analysis, we will expand this solution in terms of generalized coordinates use orthogonality of the mode shapes and we will get the uncoupled equations in this form, where N runs from 1 to infinity. Now as a special case we can consider undamped free vibration and I get AN double dot + omega N square N = 0, now if P is less than P critical AN(t) is given in terms of sine and cosine, that would mean motion would remain bounded for all times and for all initial conditions, because sine omega T and cos omega T are bounded function, but on the other end if P = P critical, AN(t) will be a linear function increasing in time, so motion grows linearly in time even for small initial conditions, okay.

- $P > P_{cr} \Rightarrow a_n(t) = A_n \exp(\gamma_n t) + B_n \exp(-\gamma_n t)$ with $\gamma_n^2 = -\omega_n^2$

Motion grows exponentially in time even for small initial conditions

Presence of damping

- $P < P_{cr} \Rightarrow$ motion decays exponentially in time
- $P \geq P_{cr} \Rightarrow$ no qualitiative change in the behavior

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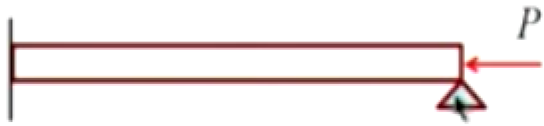
Now on the other hand if P is greater than P critical AN(t) becomes you know there are exponential functions with gamma and being this and motion grows exponentially in time even for small initial conditions, so presence of damping of course what happens is P less than P critical motion decays exponentially in time, but if P is greater than or equal to P critical there is won't be any qualitative change in the system behavior, okay, so this roughly captures the effect of axial load on dynamic behavior, this is reasonably straightforward analysis so one can



$$\frac{ml}{420}\begin{bmatrix}156 & 22l & 54 & -13l\\ 22l & 4l^2 & 13l & -3l^2\\ 54 & 13l & 156 & -22l\\ -13l & -3l^2 & -22l & 4l^2\end{bmatrix}\begin{Bmatrix}0\\0\\0\\ \ddot{u}_4\end{Bmatrix}+\frac{EI}{l^3}\begin{bmatrix}12 & 6l & -12 & 6l\\ 6l & 4l^2 & -6l & 2l^2\\ -12 & -6l & 12 & -6l\\ 6l & 2l^2 & -6l & 4l^2\end{bmatrix}\begin{Bmatrix}0\\0\\0\\u_4\end{Bmatrix}$$

$$-\frac{P}{30l}\begin{bmatrix}36 & 3l & -36 & 3l\\ -3l & 4l^2 & -3l & -l^2\\ -36 & -3l & 36 & -3l\\ 3l & -l^2 & -3l & 4l^2\end{bmatrix}\begin{Bmatrix}0\\0\\0\\u_4\end{Bmatrix}=0$$

$$\frac{ml}{420}4l^2\ddot{u}_4+\left(\frac{EI}{l^3}4l^2-\frac{P}{30l}4l^2\right)u_4=0\Rightarrow \ddot{u}_4+\underbrace{\frac{420}{ml}\left(\frac{EI}{l^3}-\frac{P}{30l}\right)}_{\omega_1^2}u_4=0$$

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consider some simple examples and illustrate using finite element concepts, suppose if I consider a propped cantilever beam and so the degree of freedom 1, 2, 3, 4, so for the boundary conditions 1, 2, 3, for the given boundary condition 1, 2, 3 are 0 and U4 is nonzero there's the only degree of freedom, so this is the mass matrix, this is the elastic stiffness matrix, and this is a geometric stiffness matrix, so you can assemble this equation and you get omega 1 square as given by this, so obviously when P = P critical this term inside the bracket will be parentheses will be equal to 0, okay, so this is for a propped cantilever, for a simply supported beam again using the same

$$\frac{ml}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ \ddot{u}_2 \\ 0 \\ \ddot{u}_4 \end{Bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ 0 \\ u_4 \end{Bmatrix}$$

$$-\frac{P}{30l}\begin{bmatrix} 36 & 3l & -36 & 3l \\ -3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ 0 \\ u_4 \end{Bmatrix} = 0$$

$$\cdots \begin{bmatrix} \cdots l^2 & -3l^2 \\ \cdots l^2 & 4l^2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_2 \\ \ddot{u}_4 \end{Bmatrix} + \left[\frac{EI}{l^3}\begin{bmatrix} 4l^2 & 2l^2 \\ 2l^2 & 4l^2 \end{bmatrix} - \frac{P}{30l}\begin{bmatrix} 4l^2 & -l^2 \\ -l^2 & 4l^2 \end{bmatrix}\right]\begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix} = 0$$

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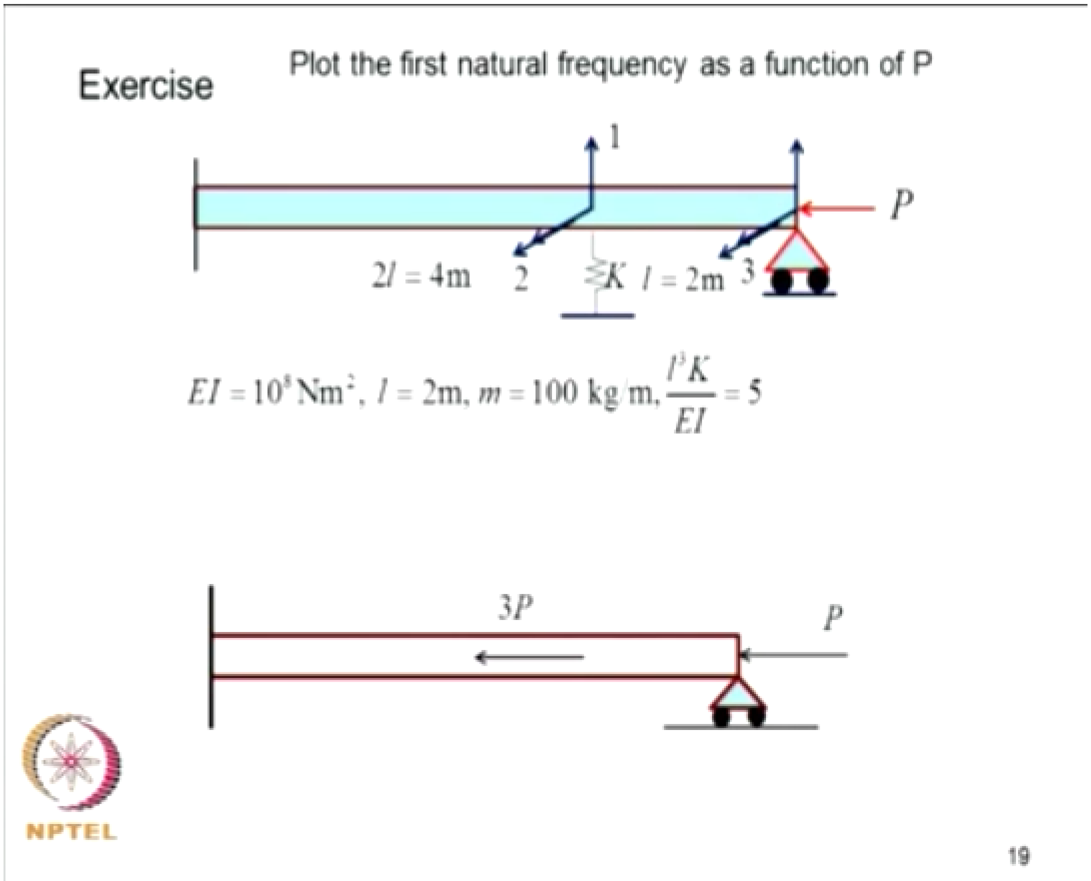
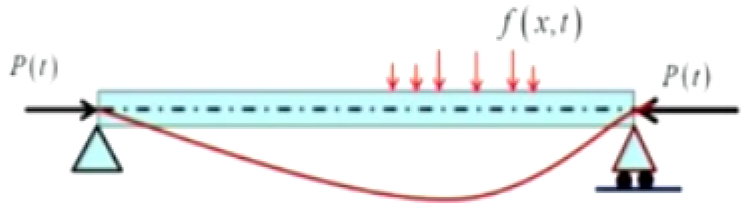
node numbering 1, 2, 3, 4, we see that 1 and 3 are 0, and 2 and 4 are the degrees of freedom, so we can assemble these matrices and obtain 2 x 2 set of equations governing the system from which you can do an eigenvalue analysis, and again see for what value of P the eigenvalues become 0 and so on and so forth, so this is I'll leave it as an exercise for you to pursue, but this gives you a flavor of how to proceed.



So an exercise is we'll consider a propped cantilever which is loaded on a spring, supported on a spring as shown here and we have considered this problem earlier for P = 0 we have got, solve this problem earlier in one of the earlier lectures, so if you go back you will get the elastic stiffness matrix and mass matrix for this problem. Now what I want you to do is to formulate the geometric stiffness matrix so you have to do first actual stress analysis, and then use formulate the geometric stiffness and then analyze the natural frequencies for this system, this is an exercise, repeat this exercise for this problem, okay you can use the same data the thing is here there is a 3P and a P load, so the initial axial thrust diagram will be different for this based on this you have to perform the analysis, these two are exercise problems.

Parametrically excited systems



$$EI\frac{\partial^4 y}{\partial x^4}+P(t)\frac{\partial^2 y}{\partial x^2}+m\frac{\partial^2 y}{\partial t^2}+c\frac{\partial y}{\partial t}=f(x,t)$$

BCs: $y(0,t)=0, EI\frac{\partial^2 y}{\partial x^2}(0,t)=0, y(l,t)=0, EI\frac{\partial^2 y}{\partial x^2}(l,t)=0$

ICs: $y(x,0)=y_0(x), \frac{\partial y}{\partial t}(x,0)=v_0(x)$

NPTEL

$P(t)$ = Parametric excitation

$f(x,t)$ = External excitation

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Now we will now consider what are known as parametrically excited systems, an archetypal system is a simply supported beam the axial load P now instead of being a constant it becomes a function of time, okay so if that happens the governing equation will be wherever there is P I will write P(t) and there is a lateral load F(x,t), so this F(x,t) we call it as an external excitation, and P(t) as parametric excitation, because P(t) multiplies dou square Y/dou X square and can be interpreted as a parameter of the problem, and an excitation appears as a parameter of the problem and therefore it is called parametric excitation. So the question we can ask is how do such systems behave? How do we analyze the response of the systems? What is resonance in such systems and so on and so forth, so let us see some of these questions.

Let $f(x,t)=0$, & $P(t)=P_0\cos\Omega t$

$\Omega=0, P_{cr}=\dfrac{\pi^2 EI}{l^2}$

$\Omega\neq 0, P_0<P_{cr}$, one would expect the beam to oscillate axially.

However, for certain values of Ω, the beam can get into large amplitude bending oscillations, even for small values of P_0 (Parametric resonance).

Note that when such oscillations occur, Ω need not coincide with any of the bending/axial natural frequencies of the beam.

Under this condition, the amplitudes can grow exponetially.

Conversely, if $P_0>P_{cr}$, the structure can remain stable for certain values of Ω.

NPTEL

$P(t)$ = External excitations for axial oscillations

= Parametric excitations for bending oscillations

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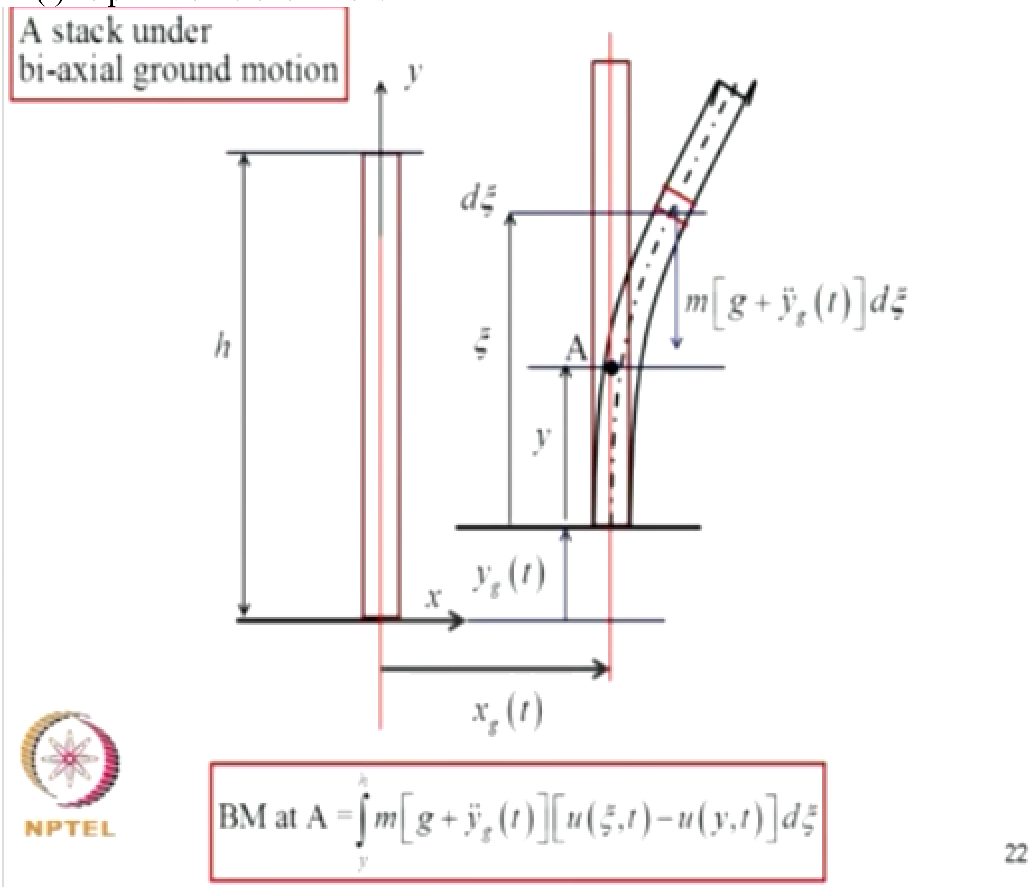
So let's assume for sake of discussion that the external load is 0, F(x,t) is 0, and P(t) is a harmonic load, P naught cos omega T, now if Omega is 0, critical load is given by pi square EI/L square it turns out that when omega is not 0, and if P naught is less than P critical that means the amplitude of the load is much less than or less than the critical buckling load, but depending on, if that happens we would expect that the beam to oscillate only axially, see we expect beam to oscillate predominantly axially, because P is much less than P critical, therefore the neighboring, the geometric stiffness effect will not be very much pronounced so it will basically oscillate the axial mode, but for certain values of the driving frequency omega, the driving frequency of parametric excitation, it turns out that the beam can get into large amplitude bending oscillations, even for small values of P naught this is known as parametric resonance.

We should note that when such oscillations occur omega need not coincide with any of the bending or actual natural frequencies of the beam, so what are those frequencies at which the beam can lose stability, not necessarily correspond to the natural frequencies of the beam, under this condition we will show shortly that the amplitudes can grow exponentially, conversely on the other hand if P naught is already greater than P critical so assume that this axial load P naught is more greater than P critical, then we would expect structure to lose its stability, but for certain values of omega capital Omega the structure remains stable for certain values of omega, that means the axial load will have a stabilizing effect on the beam which would otherwise be unstable, okay, so the term parametric excitation one has to carefully interpret if you consider axial vibration of this beam P(t) is an external excitation, but if you consider bending oscillations P(t) becomes parametric excitation, so P(t) is external excitation for axial oscillations whereas it is a parametric excitation for bending oscillation, so since our interest in this discussion is basically on bending oscillations and coupling between the influence of axial loads on bending we call P(t) as parametric excitation.



Now where do we encounter such type of systems? So we can consider a tall stack a chimney under say a bi-axial ground motion, suppose this system is subjected to earthquake ground motion which is XG(t) and YG(t), now I want to analyze this problem let's assume that it's Euler-Bernoulli beam and we will include the effect of gravity, and the vertical acceleration, in formulating the deflection of this beam, so due to XG(t) the beam would be here at some time T and YG(t) is this, so the coordinate of this point at time T will be XG(t), YG(t), this is a deflected profile, this one, so if you consider any element D sai, this is D sai, and the vertical load that acts on it is M D sai is a mass of that element into gravity + YG double dot(t) because in the event of an earthquake there will be vertical acceleration also, so this load will be acting like this.

Now if you consider the bending moment at cross-section A it will be this load into the arm, what is the arm? This displacement which is what we are calling U, U(sai,t) – U(y), this distance is Y, this is sai, so the bending moment will be due to this elementary strip is M D sai into G + YG double dot multiplied by the arm U(sai) – U(Y) and this you have to integrate from, if you want bending moment here you have to integrate from this to H, where H is the height of this thing, so I get Y to H, M into G + YG double dot U(sai,t) –U(y,t) D sai, so using this now I can

$$\text{BM at A} = \int_y^h m\left[g+\ddot{y}_g(t)\right]\left[u(\xi,t)-u(y,t)\right]d\xi$$

$$\frac{\partial^2}{\partial y^2}\left[EI\frac{\partial^2 u}{\partial y^2}+\int_y^h m\left[g+\ddot{y}_g(t)\right]\left[u(\xi,t)-u(y,t)\right]d\xi\right]+m\frac{\partial^2 u}{\partial t^2}$$

$$+c\left(\frac{\partial u}{\partial t}-\dot{x}_g(t)\right)=f(x,t)$$

$$u(0,t)=x_g(t);u'(0,t)=0$$

$$EIu''(h,t)=0;\left(EIu''\right)'(h,t)=0$$

$$\left(EIu''\right)''+m\left[g+\ddot{y}_g(t)\right]\left[u'-(h-y)u''\right]+m\ddot{u}+c\left(\dot{u}-\dot{x}_g\right)=f(x,t)$$

Parametrically excited system under time dependent boundary conditions.

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write the equation for the oscillation of the beam, this is dou square/dou Y square into the bending moment + M dou square U/dou T square inertia + C dou U/dou T - XT dot damping = F (x,t). Now what are the boundary conditions? At X = 0, U(0,t) is XG(t) because it's applied ground motion, so this ground motion is getting applied here so that is the boundary condition, and U prime(0,t) is 0 at the top bending moment and shear force are 0, okay.
Now you can carry out this integration and you can show that the governing equation will be EIU double prime double prime + these terms + MU double dot + this is equal to F(x,t). Now what you should observe is the field variable here is dependent variable U is multiplied by YG(t) here, so YG double dot(t) appears as a parametric excitation in this problem, and SG(t) is now in this formulation is appearing as a boundary condition, so it is a parametrically excited system under time dependent boundary conditions, okay, how do we use finite element method to solve this problem, okay, so that's one question that we can consider as we go along.

$$\left(EIu''\right)'' + m\left[g + \ddot{y}_g(t)\right]\left[u' - (h-y)u''\right] + m\ddot{u} + c\left(\dot{u} - \dot{x}_g\right) = f(x,t)$$

$$u(0,t) = x_g(t); u'(0,t) = 0$$

$$EIu''(h,t) = 0; \left(EIu''\right)'(h,t) = 0$$

$$w(y,t) = u(y,t) - x_g(t)$$

$$w(y,t) = \sum_{n=1}^{N} a_n(t)\phi_n(x)$$

$$\ddot{a}_n + 2\eta_n\omega_n\dot{a}_n + \omega_n^2 a_n + \frac{\left[g + \ddot{y}_g(t)\right]}{h}\sum_{\eta=1}^{N}\alpha_{nj}a_j(t) = -\gamma_n\ddot{x}_g(t)$$

$$M\ddot{X} + C\dot{X} + \left[K + K_0(t)\right]X = Q(t)$$

$$X(0) = X_0; \dot{X}(0) = \dot{X}_0$$

NPTEL

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Now we can do some simple substitutions, I can introduce another dependent variable W, I define as U - XG which is a relative displacement, and I substitute here and for moment I do this the time dependent boundary condition will become time independent, because at Y = 0, W(0,t) will be XG - XG which is 0, so the time dependent term present in the boundary conditions will be removed by this substitution, and in the resulting equation if you now do a kind of a series expansion this we get now for the modal equation this set of equations. Again you should notice, due to the presence of parametric excitation the generalized coordinate will not get uncoupled, so this is an approximation resulting from method of weighted residuals, it's not an exact you know a solution to the problem because the equations are not uncoupled here, because equation for AN has all the other generalized coordinates, so in the matrix form I can put it as MX double dot + CX dot + K + K naught(t) into X = Q(t), so this K naught(t) is a time varying stiffness matrix for the problem which contains here YG double dot(t), so this is a parametric excitation or you can interpret as a time varying system.

$$M\ddot{X} + C\dot{X} + \left[K + K_0(t)\right]X = Q(t)$$

$$X(0) = X_0 : \dot{X}(0) = \dot{X}_0$$

- Coupled mdof system with time dependent stiffness matrix
- Notion of natural frequencies and mode shapes not valid
- Normal modes (with ground motions=0) would not uncouple the equation of motion

Questions

How to solve these equations?

Are there any new phenomenological features associated with the response?

What happens to the notion of resonance?

What happens to the notion of instability?

NPTEL

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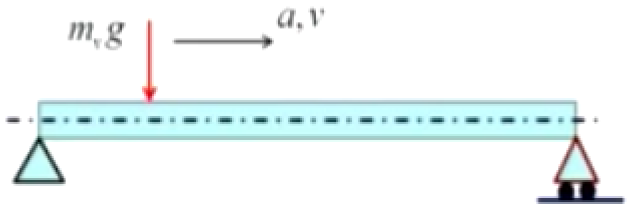
So this is a coupled multi degree freedom system with time-dependent stiffness matrix, consequently the notion of natural frequencies and mode shapes are not valid for these type of systems, if you ignore K naught(t) and if you were to find the natural frequency and mode shapes those set of Eigen solutions will not uncouple this equation, so they are no longer than natural coordinates for this problem, in fact for this type of problems natural coordinates do not exist. Normal modes with ground motions equal to 0 would not uncouple the equation of motion, now we can ask a few questions, how to solve these equations? Are there any new phenomenological features associated with the response? What happens to the notion of resonance? And what happens to the notion of instability? Some of these questions can be posed and we can address

Structures under moving loads

Vehicle as a moving force



$$EIy^{iv} + m\ddot{y} + c\dot{y} = m_v g\delta\left(x - vt - \frac{1}{2}at^2\right) \text{ for } 0<t<t_{exit}$$

$$= 0 \text{ for } 0<t<t_{exit}$$

$$y(x,t)=\sum_{n=1}^{N} a_n(t)\phi_n(x)$$

$$\Rightarrow \ddot{a}_n + 2\eta_n\omega_n\dot{a}_n + \omega_n^2 a_n = m_v g\phi_n\left(vt-\frac{1}{2}at^2\right) \text{ for } 0<t<t_{exit}$$

$$= 0 \text{ for } 0<t<t_{exit}$$

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these questions, but before we do that we will review one more type of problem that is of great interest to civil engineers, that is how those structures behave under moving loads, moving loads could be vehicular traffic for example on a bridge, so for sake of discussion we will assume that bridge has been idealized as a single span simply supported beam, we start with a simple model where the vehicle model is represented by a single concentrated force and that force is equal to the weight of the vehicle, so the vehicle is moving with velocity V and acceleration A, so assuming Euler-Bernoulli beam hypothesis we can set up the governing equation of motion, so at T = 0, the load enters the span and it will be on this span till TE, T exit, it exists there and after the load leaves the bridge there is no external force on the beam, during that time the load is on the bridge the point of application of the load keeps changing so the forcing function is expressed in terms of a direct delta function and delta(x-vt-1/2 AT square) at any time T the position will be VT + 1/2 AT square, so that is what we are writing.

Now a solution to this problem can be constructed using a modal expansion, this is an externally driven system, this is not a parametrically excited system, so I will get this phi N(x) if they are the exact mode shapes this will uncouple the equation and I will get this equation for T less than T exit this and afterwards it is 0.

$$\Rightarrow \ddot{a}_n + 2\eta_n\omega_n\dot{a}_n + \omega_n^2 a_n = m_v g\phi_n\left(vt - \frac{1}{2}at^2\right) \text{ for } 0 < t < t_{exit}$$

$$= 0 \text{ for } 0 < t < t_{exit}$$

$$a = 0, \phi_n(x) = \phi_{n0}\sin\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow \ddot{a}_n + 2\eta_n\omega_n\dot{a}_n + \omega_n^2 a_n = m_v g\phi_{n0}\sin\left(\frac{n\pi vt}{l}\right) \text{ for } 0 < t < t_{exit}$$

$$= 0 \text{ for } 0 < t < t_{exit}$$

Critical condition: $\omega_1 = \frac{\pi v}{l}$

$\Rightarrow$ Critical velocity $v_c = \frac{\omega_1 l}{\pi} = \frac{l}{\pi}\frac{\pi^2}{l^2}\sqrt{\frac{EI}{m}} = \frac{\pi}{l}\sqrt{\frac{EI}{m}}$

- Resonant conditions can prevail if a series of loads pass at velocity = v_c

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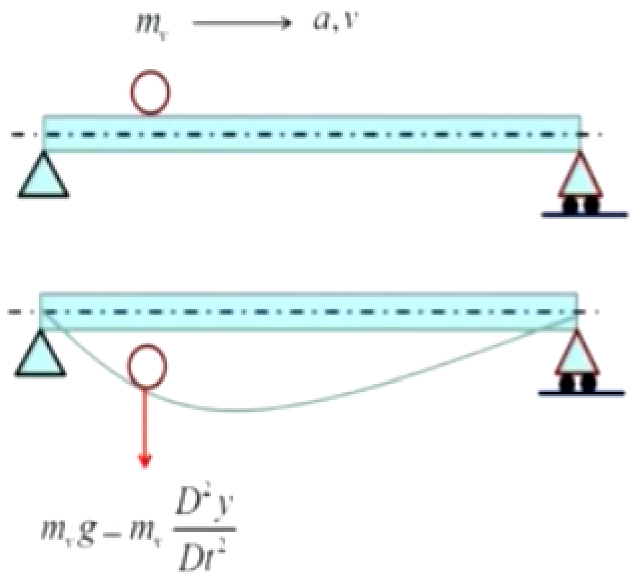
Now for simply supported beam we know that the mode shapes are given by sine N pi X/L, and if you assume that beam is moving with sorry the vehicle is moving with constant velocity, acceleration will be 0 and the simplified equation will be given by this. Now obviously now there is a critical condition that is possible if the natural frequency of the system coincides with the driving frequency here, because mode shapes are sinusoidal in space the driving on generalized coordinates will become harmonic in time, okay, so a harmonic excitation on a single degree freedom system can potentially cause resonance, but you should understand that this excitation is a transient excitation because this term exists only for the time duration T between 0 and T exit afterwards it becomes 0, therefore strictly speaking when a single concentrated load passes the beam we cannot talk about resonance, but still a critical condition can be taken to prevail if omega 1 is given by pi V/L, that pi V/L so this leads to the concept of a critical velocity which is given by pi/L square root EI/M.

Now although for a single concentrated load passing on the beam resonant condition is not possible but a series of concentrated loads can pass on the beam and that is typical of vehicular traffic on a bridge, and therefore if all the loads were to travel with this velocity the structure would indeed get into resonance and this velocity would indeed become critical velocity from the point of view of creating a resonant condition.

Vehicle as a moving mass



$$\frac{D}{Dt} y\left[x(t),t\right] = \frac{\partial y}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial y}{\partial t} = \left(v\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) y$$

$$\frac{D^2}{Dt^2} y\left[x(t),t\right] = \left(v\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)\left(v\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t}\right) = v^2\frac{\partial^2 y}{\partial x^2} + 2v\frac{\partial^2 y}{\partial x \partial t} + a\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial t^2}$$

Now we will improve upon this model, we will assume that vehicle is now represented as a moving mass, not as a moving force, so what is the implication? So as the vehicle travels on the bridge at some point suppose if it is here the mass of the vehicle into gravity is one force that acts on the beam, but since the beam itself is accelerating the mass of the vehicle into acceleration of the beam is another force, but now this mass is rolling on an inclined path, actually the beam is oscillating and this mass is rolling on that, so when I consider acceleration I should take total derivatives it's not the ordinary derivative, so if I write for example what is D/DT of Y(x,t) this, it is dou Y/dou X into dou X/dou T + dou Y/dou T into this, so the operator D/DT is given by this, so based on that D square Y/DT square the total derivative can be shown to be given by this, so these terms are known as Coriolis terms, because there is a rolling motion and a transverse oscillation which interact producing this Coriolis force.

$$EIy^{iv} + m\ddot{y} + c\dot{y} = \left\{m_v g - m_v \frac{D^2 y}{Dt^2}\right\}\delta\left(x - vt - \frac{1}{2}at^2\right) \text{ for } 0<t<t_{exit}$$

$$= 0 \text{ for } 0<t<t_{exit}$$

$$\frac{D^2}{Dt^2} y\left[x(t),t\right] = v^2 \frac{\partial^2 y}{\partial x^2} + 2v\frac{\partial^2 y}{\partial x \partial t} + a\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial t^2}$$

For $0<t<t_{exit}$

$$EIy^{iv} + m\ddot{y} + m_v \frac{D^2 y}{Dt^2}\delta\left(x - vt - \frac{1}{2}at^2\right) + c\dot{y} = m_v g\delta\left(x - vt - \frac{1}{2}at^2\right)$$

$$EIy^{iv} + m\ddot{y} + m_v\left\{v^2 \frac{\partial^2 y}{\partial x^2} + 2v\frac{\partial^2 y}{\partial x \partial t} + a\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial t^2}\right\}\delta\left(x - vt - \frac{1}{2}at^2\right) + c\dot{y}$$

$$= m_v g\delta\left(x - vt - \frac{1}{2}at^2\right)$$

$$M(t)\ddot{X} + C(t)\dot{X} + K(t)X = F(t); X(0) = X_0; \dot{X}(0) = \dot{X}_0$$

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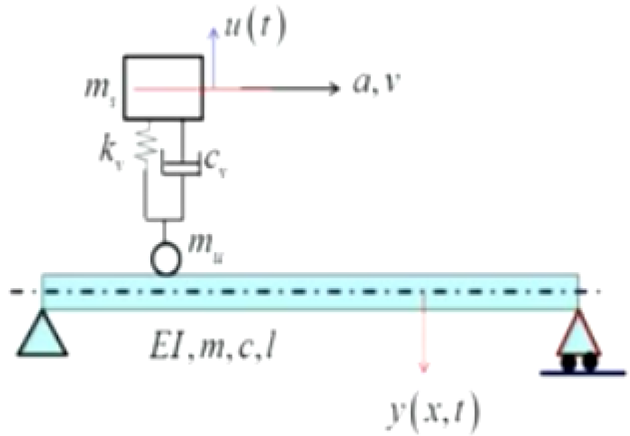
Now what is the consequence of all this in the equation of motion? So again when writing equation of motion the force should now include MVG + this term we get this, this is a point of application at time T the position of the V will be at distance VT + 1/2 AT square, so this is direct delta (X - VT – 1/2 AT square) and after the vehicle leaves the bridge of course this term will be 0, now this D square Y/DT square is in fact given by this. So now if you focus our discussion on T between 0 to T exit the equation will be given by this, so if we now expand this and reorganize the term we will see that the mass matrix, so the mass term has all these additional terms that is dou Y/dou X, dou Y/dou X dou T, dou Y/dou X etcetera, and this is a time varying term, the time varying term here is a capture to a direct delta function, but it is still a function of both X and T, so I get a partial differential equation now where the dependent variable is multiplied by a function which is function of both X and T, right, so therefore this term constitutes the parametric excitation term, so of course the load due to moving mass, the weight of the vehicle drives as an external excitation that continues to be present, so if I were to, we'll see how to discretize this, we will visit this problem, we will develop a finite element formulation for this, and we will see how the equations can be derived but as a prelude we can see that if one were to do that the governing equation will be of this form, the mass, damping, and stiffness matrices will now be functions of time, and of course there is a right hand side which is F(t), so again here we can ask several questions, how do we actually formulate this? How do we use finite element method and formulate this equation, again questions on possible phenomenological novelties in system behavior and actual quantification of the response, we want to find out beam displacement, strains, and stresses reaction transfer and so on and so forth, we have to analyze this problem, how do we do that? And how do we eventually compute stresses and reactions and so on and so forth, bending moment, shear force, etcetera.

Vehicle as a moving sdof oscillator



m_u = unsprung mass

m_s = sprung mass

A further refinement to this model now we'll include the stiffness characteristics of the vehicle, in the first case we simply considered vehicle as a force which is weight of the vehicle, in the second case we included the inertial property of the vehicle, now vehicle itself will have elastic properties there will be suspensions and so on and so forth, so we can model the vehicle now as a single degree freedom system with spring stiffness KV and a dissipation dampers CV, and we will divide the mass into what is known as an unsprung mass, and a sprung mass, so this system now enters the beam at T = 0, and exits at T exit, and Y is the displacement and properties of the beam are EI, M, C and L, and MU, MS, KV, CV are the properties of the vehicle and the degree of freedom associated with the vehicle I take it as U(t) and we will assume that the vehicle is moving with a velocity V and acceleration A.

So now the question is as the vehicle traverses the beam, the vibration of this system comprises of, vibration of the beam as well as the vibration of vehicle, and they mutually interact, that means the vehicle oscillations will modify the oscillation of the beam, and beam oscillations will modify the oscillation of the vehicle and this is a problem of vehicle structure interaction, so we for $0 < t < t_{exit}$

$$m_s\ddot{u} + c_v\left\{\dot{u} - \frac{D}{Dt}y\left[x(t),t\right]\right\} + k_v\left\{u - y\left[x(t),t\right]\right\} = 0$$

$$EIy^{iv} + m\ddot{y} + c\dot{y} = f(x,t)\,\delta\left(x - vt - \frac{1}{2}at^2\right)$$

$$f(x,t) = (m_u + m_s)g + k_v\left\{u - y\left[x(t),t\right]\right\} + c_v\left\{\dot{u} - \frac{D}{Dt}y\left[x(t),t\right]\right\} - m_u\frac{D^2}{Dt^2}y\left[x(t),t\right]$$

$f(x,t)$ = wheel force

$\Rightarrow$

$$M(t)\ddot{X} + C(t)\dot{X} + K(t)X = F(t);\; X(0) = X_0;\; \dot{X}(0) = \dot{X}_0$$

NPTEL

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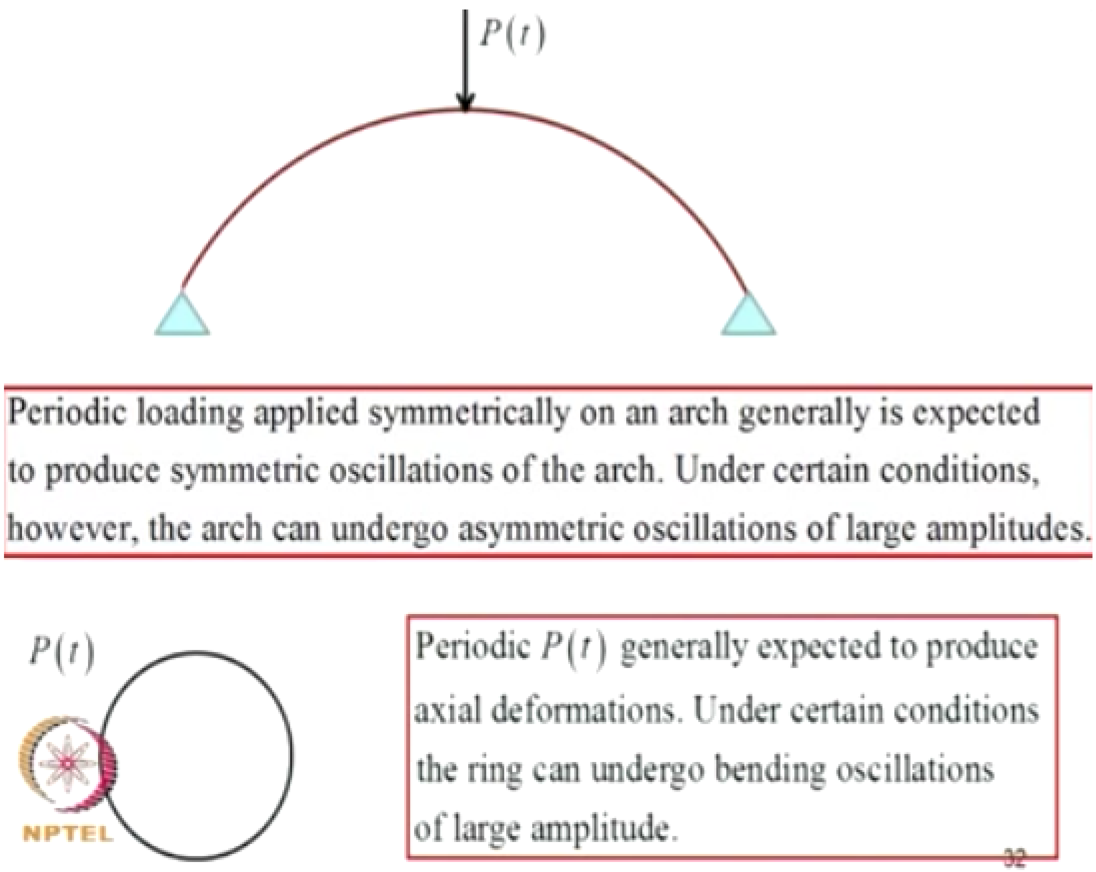
will do two things now we formulate this problem and see what are the governing equations, and then see how to use finite element method to discretize that, that will be taken up later.

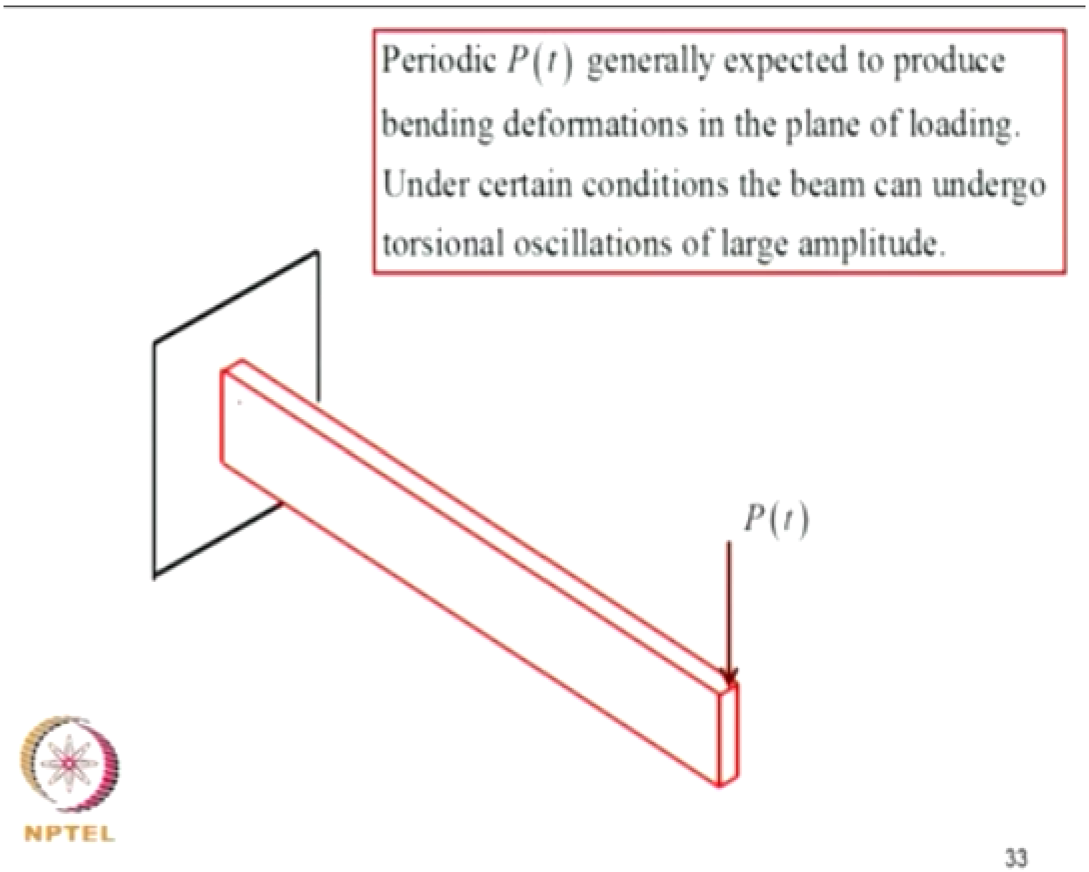
Now we will focus on T lying between 0 and T exit, so the equation for the vehicle that is MS U double dot + KV into U - Y here + CV into Y – here, right and the inertial term will be MS + MU into U double dot, and MU of course we will see how these terms come up, so MS U double dot + CV U dot into the total derivative, because this mass is rolling on a deflected profile, so again I have to take a total derivative, and this is the force in the spring, the beam itself carries the V load, I will call it as V load, let us call it as F(x,t), F(x,t) and a point of application keeps changing X - VT – 1/2 AT square, what this V load consists of? The gravity load due to the weight of the vehicle which is MU + MS into G + the reaction transform from the spring which is this, that is spring of the vehicle and this is a reaction transform of a damper of the vehicle which is this, minus the inertia due to the unsprung mass, this is this, so we will see, we will consider this problem in some detail in the following lectures, we will discretize this using finite element method, and we will see in that course of doing that, that the discretized form of equation will have this generic form M(t) X double dot + C(t) X dot + K(t) X = F(t) and certain initial conditions.

So and here again I get a system where mass, damping and stiffness matrices are functions of time, again the notion of natural frequencies in mode shapes and uncoupling the equation through natural coordinates etcetera are no longer applicable here, before embarking on



qualitative analysis of this type of system we will mention a few more examples where this type of questions become relevant, suppose if you consider an arch carrying a load it is symmetric circular arch, and P(t) if it is periodic, periodic loading applied on symmetrically loaded arch, generally is expected to produce symmetric oscillations of the arch okay under certain conditions however the arch can undergo asymmetric oscillations of large amplitude, there would be concept of modes are no longer applicable here, because there can be energy transfer from what we consider as generalized coordinates, suppose there is no geometric interactions then the first mode and second mode they're all uncoupled, but because of parametric, presence of parametric excitation effects those so-called modes will get coupled so a symmetric arch like this which is symmetrically supported and symmetrically loaded can undergo large oscillation asymmetric oscillations.

Similarly if you consider the ring and if it is subjected to uniform external pressure, external load, P(t) if P(t) is periodic one generally expect that the ring will undergo axial oscillations, it will be simply by oscillating in this direction, but however under certain condition the ring can undergo bending oscillations of large amplitude this again due to parametric effects. Another



case is suppose you consider a rectangular beam as shown here loaded symmetrically with a load P(t) again if you apply, if a P(t) is periodic one would expect that such a load will create bending deformation in the plane of the loading, but under certain condition the beam can undergo torsional oscillations of large amplitude, so this oscillations will be excited by this load, so how that happens is another you know issue that we need to you know consider.

Stability of steady state motions

$$m\ddot{x} + c\dot{x} + kx + \alpha x^3 = P\cos\Omega t; x(0) = x_0; \dot{x}(0) = \dot{x}_0$$

$$\lim_{t\to\infty} x(t) = X\cos(3\Omega t - \theta) = x^*(t)$$

$$x(t) \to x^*(t) + \Delta(t) \quad [\Delta(t) \text{ is "small"}]$$

$$\Rightarrow$$

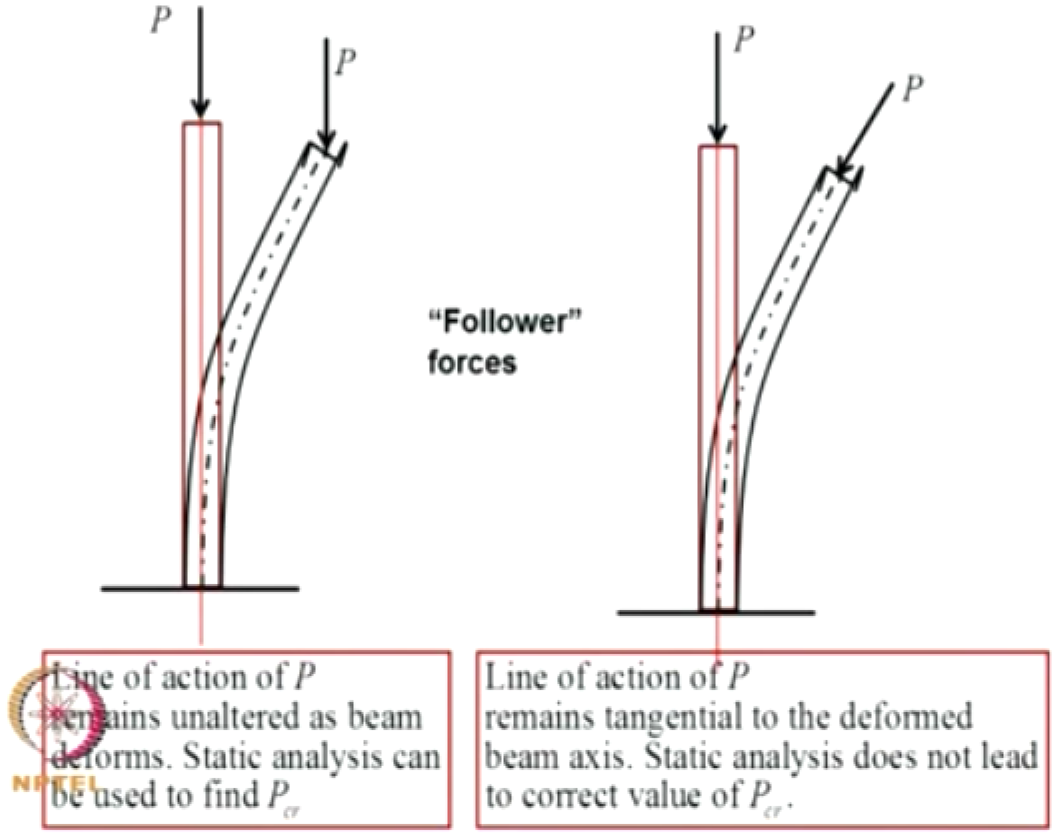
$$m\ddot{\Delta} + c\Delta + k\Delta + 3\alpha x^{*2}(t)\Delta = 0$$

$$\lim_{t\to\infty} |\Delta(t)| \to ?$$

$x^{*2}(t)$: acts as a parametric excitation

Now I briefly mentioned when we initiated discussion on stability of structures that we are considering stability of state of rest, and stability of steady state motions, the parametrically excited systems also arise in somewhat different context if one is studying stability of steady state motions this I had already shown once but let us quickly recall, if we have suppose a single degree freedom system with cubic nonlinear stiffness and it is driven harmonically by P cos omega T, suppose if we postulate that as T tends to infinity X(t) is periodic with this property, and call this as X*(t).

Now I want to know whether X*(t) is stable or not, just the periodic motion, so what I do I impress upon X*(t) a perturbation, delta(t) which is small, now upon substitution of this equation into this we can get an equation for delta and this equation will have X* square (t) which is the solution as a coefficient here, so X* square (t) acts as a parametric excitation for this problem, this is known as variational equation, in the variational equation the solution whose stability is being investigated appear as a parametric excitation term, in this type of questions we are not so much interested in the detailed time history of delta(t), we are basically interested in knowing what happens to amplitudes of delta T as T tends to infinity, so we are asking a relatively simpler question on whether delta(t) as T tends to infinity amplitude decay or not, so that question of course can be answered by complete solution of this equation, but since the question being asked has a simpler scope, there should be simpler ways of answering that question.



There is another class of problems that is of interest in stability analysis, suppose if we consider, this we have considered, suppose there is a cantilever beam and it is axially loaded and as it deforms, suppose if a neighboring equilibrium state is possible the load P continues to act you know what, the direction of load P doesn't change because of deformation of the beam, but suppose if you were to assume that load P, the point of application of load P also you know remains same but the direction remains tangential to the deformed geometry, okay, if this happens, now if we ask the question what would be the value of critical load for this system, okay, now so in this case line of action of P remains unaltered as beam deforms. Static analysis can be used to find P critical, that we have done.

Now here line of action of P remains tangential to the deformed beam axis, if you were to perform a static analysis as we have done for this case it turns out that for any value of P the structure would be stable, that is somewhat counterintuitive, structure cannot obviously carry large values of P, at some value of P there should be a problem. Now if we were to go back to the basics and ask how do we exactly study the stability for every increment in value of P we will give a perturbation and see how the oscillations caused by that perturbation decay or not, so the question of stability is being settled through actual analysis of a vibration problem, so if motivated by that if we were to pose this problem as a problem in dynamics, and analyze this problem it turns out that if we include inertial effects there will be a value of P, finite value of P at which the structure will lose the stability, that a pure static analysis will not be able to capture, so that is one novel element of this type of problems I will briefly touch upon this as we go along.

Problem 1

How to characterize resonances in systems governed by equations of the form

$$M(t)\ddot{X} + C(t)\dot{X} + K(t)X = 0; X(0) = X_0; \dot{X}(0) = \dot{X}_0$$

when the parametric excitations are periodic.

Problem 2

How to arrive at FE models for PDE-s with time varying coefficients?

Problem 3

Are there any situations in statically loaded systems, wherein one needs to use dynamic analysis to infer stability conditions?

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So to summarize, what we will do is will consider basically 3 types of problems, I am giving a overview of problems that we are going to consider. Now first question we will ask is how to characterize resonances in systems governed by equations of the form M(t) X double dot + C(t) X dot + K(t) X = 0, when the parametric excitations are periodic, okay so we talk about resonance in presence of periodic excitations, now the periodic excitation now appear as parameters, and then can we talk about resonance.

Second question we will consider is how to arrive at finite element models for partial differential equations with time varying coefficients, we saw that in the problem of say vehicle structure interaction the governing partial differential equation has a time varying term, and similarly the chimney under combined horizontal and vertical excitation also had a time varying term, so how do we use finite element method to discretize that. Last problem that we'll consider is are there any situations in statically loaded systems wherein one needs to use dynamic analysis to infer stability condition, so these 3 problems will consider in the following you know discussions.

Qualitative analysis of parametrically excited systems

$$\ddot{u}(t)+p_1(t)\dot{u}(t)+p_2(t)u(t)=0$$

$$u(0)=u_0;\dot{u}(0)=\dot{u}_0$$

$$p_i(t+T)=p_i(t), i=1,2$$

The governing equation is a linear second order ODE with time varying coefficients. It admits two fundamental solutions.

Digress

$$\ddot{x}+\omega^2 x=0$$

$$x(t)=a\cos\omega t+\bar{b}\sin\omega t=ax_1(t)+bx_2(t)$$

$$\text{with } x_1(t)=\cos\omega t, \& x_2(t)=\frac{\sin\omega t}{\omega}$$

Notice: $x_1(0)=1 \& \dot{x}_1(0)=0$

$x_2(0)=0 \& \dot{x}_2(0)=1$

Any solution can be expressed as linear combination of $x_1(t) \& x_2(t)$

So we will start by considering the first problem, qualitative analysis of parametrically excited systems, that is a topic suppose if I consider a single degree freedom system U double dot(t) + P1 (t) U dot(t) + P2(t), U(t) = 0, if mass is time varying I can divide that term and arrange the equation in this form, that amounts to multiplying this equation by M inverse.

Now let's assume that there are some nonzero initial condition and this P1 and P2 are periodic functions, now this governing equation is a linear second order ordinary differential equation with time varying coefficient, the equation is still linear therefore it admits two fundamental solutions, so what are these fundamental solutions? We can digress for a minute, suppose if you consider undamped free vibration of a single degree freedom system the governing equation is X double dot + omega square X = 0, and we know the solution is A cos omega T + B bar sine omega T, I can write this as AX1(t) + BX2(t) where X1 I will take as cos omega T, and X2 as sine omega T/omega, what is special about this X1 and X2? X1(0) is 1, X1 dot(0) is 0, similarly X2(0) is 0, X2 dot(0) is 1, so that means the initial conditions are 1, 0, 0, 1, if you start with, if you integrate this equation with those 2 sets of independent initial conditions 1, 0, and 0, 1, you get two solutions X1 and X2, and any solution to the problem can be constructed by superposing those 2 solution, these X1 and X2 are known as fundamental solutions. this type of fundamental solutions exist for all linear systems, so the point being made is this equation, this governing equation is a linear second order ordinary differential equation at time varying coefficient, it admits 2 fundamental solution.

$\ddot{u}(t)+p_1(t)\dot{u}(t)+p_2(t)u(t)=0$

Let $u_1(t)$ and $u_2(t)$ be the fundamental solutions of this equation.

$\Rightarrow u(t)=c_1u_1(t)+c_2u_2(t)$

Consider the governing equation at $t+T$

$\ddot{u}(t+T)+p_1(t+T)\dot{u}(t+T)+p_2(t+T)u(t+T)=0$

Since $p_i(t+T)=p_i(t), i=1,2,$ we get

$\ddot{u}(t+T)+p_1(t)\dot{u}(t+T)+p_2(t)u(t+T)=0$

$\Rightarrow$ If $u(t)$ is a solution $\Rightarrow u(t+T)$ is also a solution.

$\Rightarrow$

$u_1(t+T)=a_{11}u_1(t)+a_{12}u_2(t)$

$u_2(t+T)=a_{21}u_1(t)+a_{22}u_2(t)$

$\Rightarrow \{u(t+T)\}=[A]\{u(t)\}$

We are interested in nature of the solution as $t \to \infty$.

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Now let U1 and U2 be the two fundamental solutions, so I can write U(t) as C1, U1(t) + C2, U2(t), U(t) is solution to this problem. Now what we will do is we will consider this equation at time instant T + capital T, where capital T is a period of P1(t) and P2(t) if you do that I will have U double dot(t + T) + P1 (t + T) + U dot (t + T) + P2(t + T) into this equal to 0, but P1 and P2 are periodic, therefore P1(t + T) is simply P1(t), and P2(t + T) is P2(t), so the equation gets simplified I get this, so that would mean if U(t) is a solution, U(t + T) is also a solution, okay because U(t) satisfy this equation, and so also U(t + T), now therefore this U(t + T) can be expressed as linear combination of the two fundamental solutions A11, U1(t) + A12 U2(t). Now similarly U2(t) can be written in this form, or I can put this in a matrix form and write U(t + T) is A into U(t).

Now we are interested in nature of solutions as T tends to infinity, clearly you can easily see that this behavior will be controlled by this matrix A, okay and this matrix A is what we should focus on determining.

$$\{u(t+T)\} = [A]\{u(t)\}$$

$$\lim_{t\to\infty} u(t) \to ?$$

This is equivalent to asking $\lim_{n\to\infty} u(t+nT) \to ?$

$$u(t+T) = Au(t)$$

$$u(t+2T) = Au(t+T) = A^2u(t)$$

$$\vdots$$

$$u(t+nT) = Au(t+(n-1)T) = A^n u(t)$$

$\Rightarrow$ The behavior of $\lim_{n\to\infty} u(t+nT)$ is controlled by the behavior of $\lim_{n\to\infty} A^n$.

Intutively, one can see that this, in turn, depends upon the nature of eigenvalues of A.

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If we now consider the question what happens to U(t) as T tends to infinity, this equivalent to asking the question limit of N tending to infinity U(t + NT) what happens? Now I can write based on this equation U(t + T) is AU(t), U(t + 2T) is A into U(t + T) that itself is A square U(t), so it follows U(t + NT) is A to the power of N U(t), so the behavior of this quantity U(t + NT) as N tends to infinity is controlled by the behavior of A to the power of N as N tends to infinity, intuitively we can see at this stage that this would depend on eigenvalues of A.

Digress:

Consider the scalar case of

$x(t+T) = \alpha x(t)$

$\Rightarrow x(t+nT) = \alpha^n x(t)$

- $\lim_{n \to \infty} x(t+nT) \to 0$ if $|\alpha| < 1$
- $\lim_{n \to \infty} x(t+nT) \to \infty$ if $|\alpha| > 1$
- $x(t+T) = x(t) \Rightarrow x(t)$ is periodic with period T if $\alpha=1$
- $x(t+2T) = x(t) \Rightarrow x(t)$ is periodic with period $2T$ if $\alpha = -1$

Similar situation can be expected to prevail in analysing $u(t+T) = Au(t)$

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Again we will digress, we will consider first a scalar case suppose I have a map X(t + T) is alpha X(t), therefore X(t + NT) will be alpha N X(t), so as N tends to infinity this function will go to 0 if modulus of alpha is less than 1, and it will go to infinity if modulus alpha is greater than 1, if alpha equal to 1 what happens? The function remains the same so it is periodic with period capital T, if alpha = 1. On the other hand if alpha is – 1, the period will be 2T, okay because X2 will be - X1, X3 will be minus X2 which is again X1, so the period will be 2T okay, so similar situation can be expected to prevail in analyzing this vector equation.

$$\{u(t+T)\} = [A]\{u(t)\}$$

$$\lim_{t \to \infty} u(t) \to ?$$

This is equivalent to asking $\lim_{n \to \infty} u(t+nT) \to ?$

$$u(t+T) = Au(t)$$

$$u(t+2T) = Au(t+T) = A^2u(t)$$

$$\vdots$$

$$u(t+nT) = Au(t+(n-1)T) = A^n u(t)$$

$\Rightarrow$ The behavior of $\lim_{n \to \infty} u(t+nT)$ is controlled by the behavior of $\lim_{n \to \infty} A^n$.

Intutively, one can see that this, in turn, depends upon the nature of eigenvalues of A.

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So what we will do is in the next class we'll examine this issue how to investigate the behavior of U(t +NT) as N tends to infinity it will turn out that, we will be able to formulate this matrix A in terms of fundamental solutions of the governing equation and to answer this question we need to integrate the governing equations over one period of the parametric excitations for a set of independent initial conditions, using that information I will be able to answer this question whether the response becomes, how the response behaves, so there will be various situations that we will encounter, the response will be periodic with period capital T, with period capital 2T and it can decay to 0 and it can exponentially blow off, depending on the eigenvalues of matrix A, so this question we will take up in the next class, we will close this lecture at this point.

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