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Course Title

Finite element method for structural dynamic And stability analyses

Lecture – 30

FEM for stability analysis. Euler-Bernoulli beam And general formulations

By

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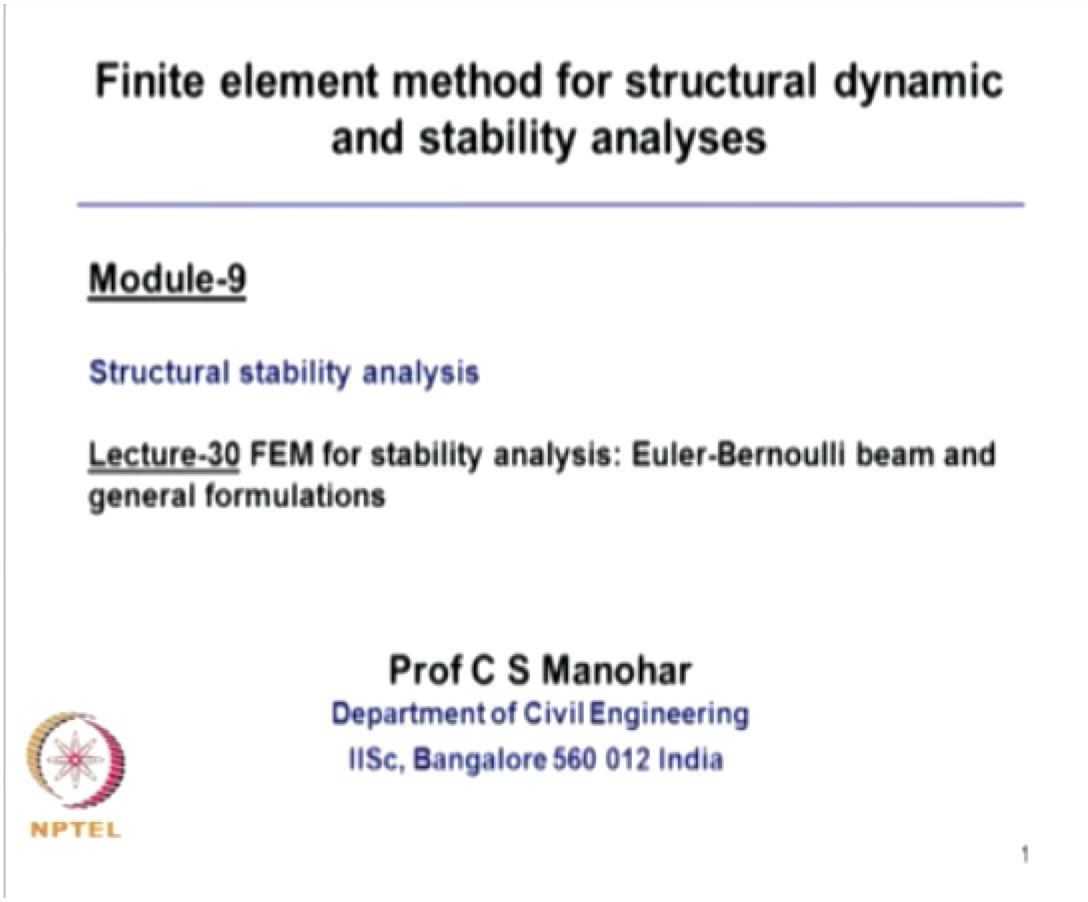


We have been discussing energy methods for structural stability analysis, so in the previous

Energy methods for stability analysis

- Consider a system with n generalized coordinates.
- Focus attention on statically loaded structures.

Axiom - 1

A stationary value of the total potential energy with respect to the generalized coordinates is necessary and sufficient condition for the equillibrium state of the system.

Axiom - 2

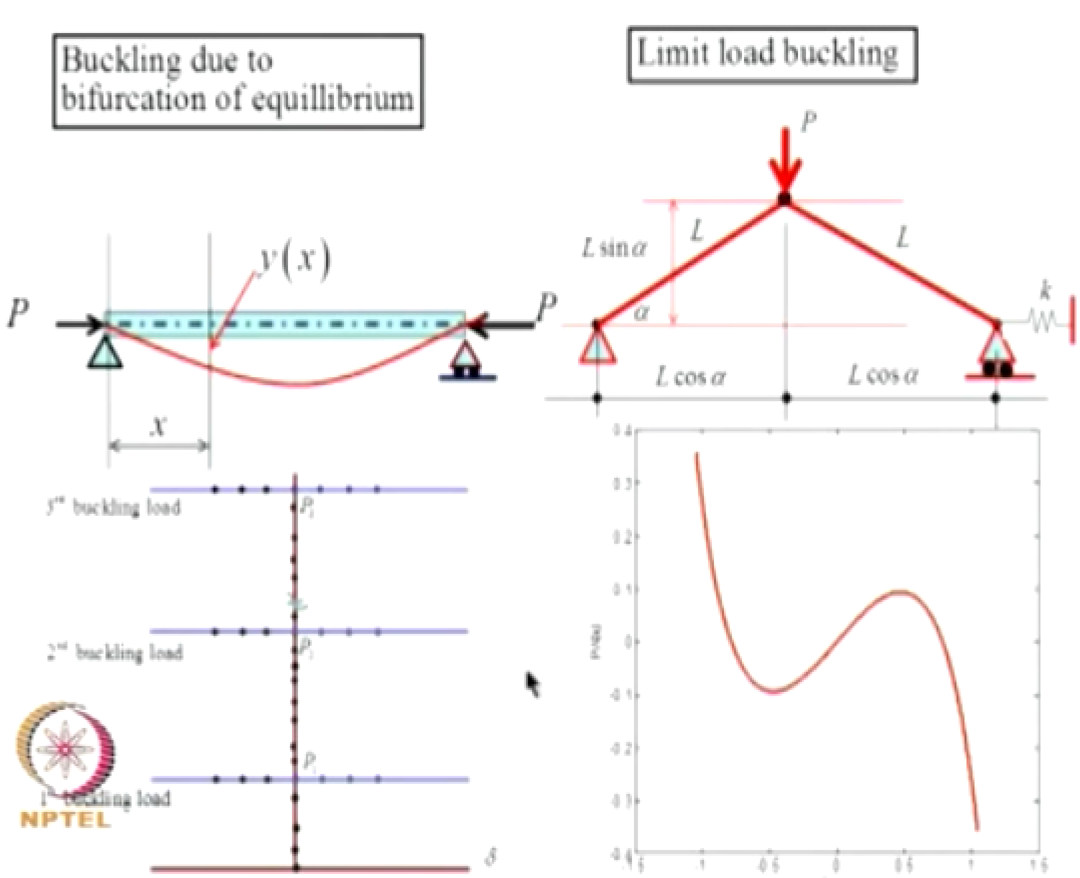
A complete relative minimum of the total potential energy with respect to the generalized coordinates is necessary and sufficient for the stability of an equillibrium state of the system.

J M T Thompson and G W Hunt, 1973, A general theory of elastic stability, John Wiley, London

2

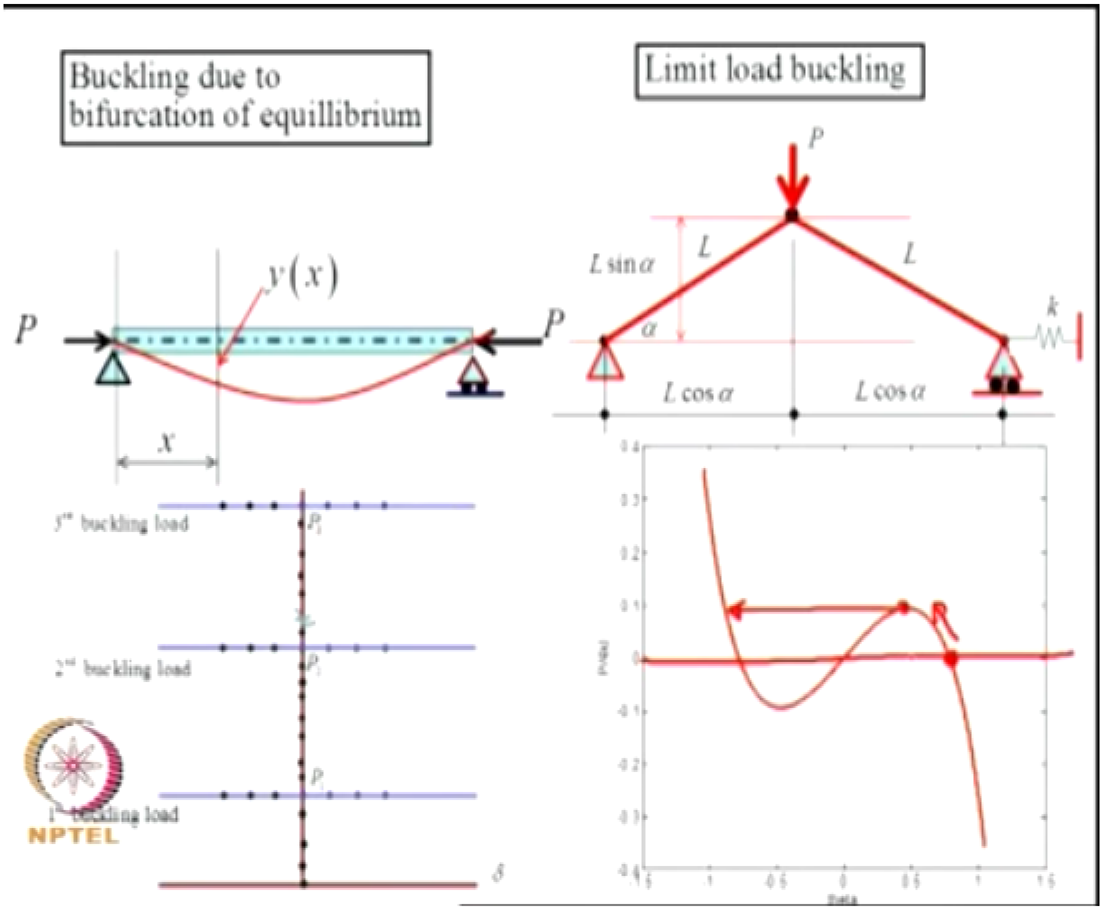
lecture we introduced the axioms, if we consider a system with N generalized coordinates, and if we focus attention on statically loaded structures, there are two axioms, first axioms says a stationary value of the total potential energy with respect to the generalized coordinates is necessary and sufficient condition for the equilibrium state of the system, this part is a condition for establishing the equilibrium.

The axiom 2 helps us to examine the stability of that equilibrium, so a complete relative minimum of the total potential energy with respect to the generalized coordinates is necessary and sufficient for the stability of an equilibrium state of the system, so based on this we

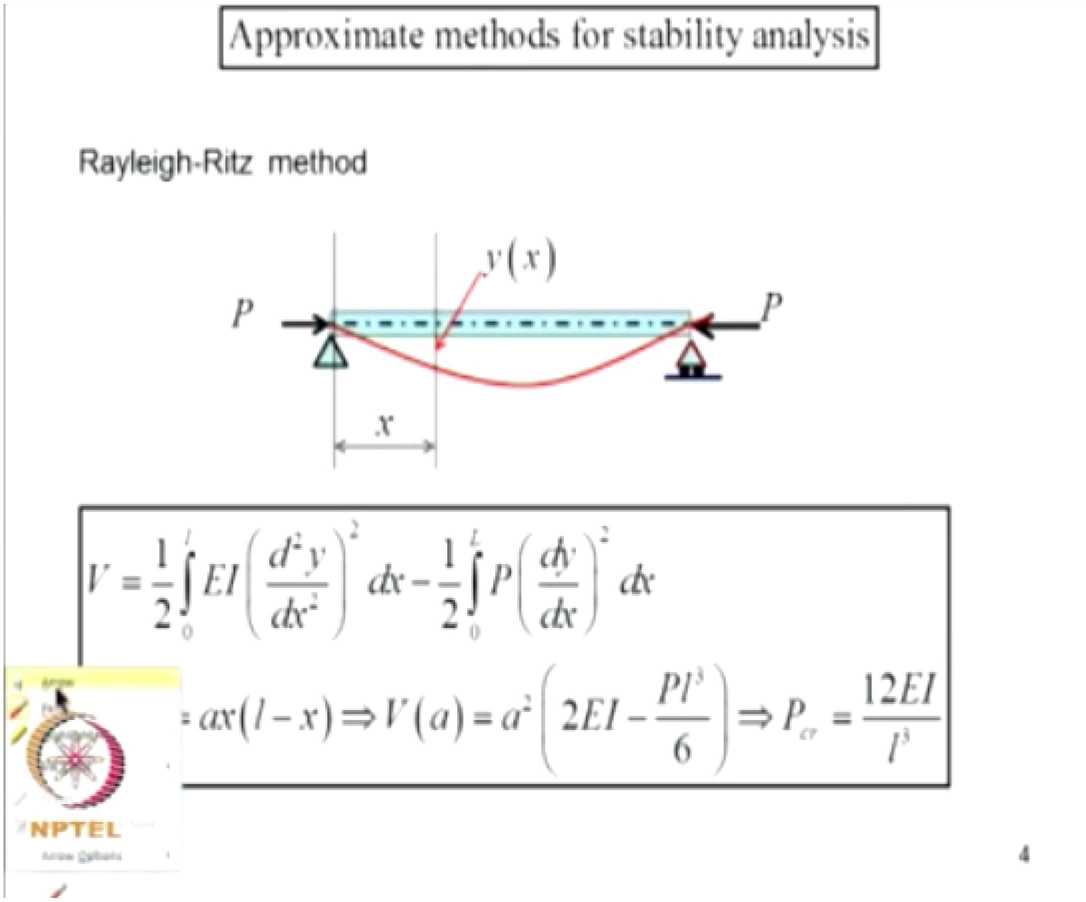


analyzed few problems and basically we considered simple systems to bring out the central ideas of the analysis, so we consider the beam column problem and in this case when P equal to, for small values of P, Y = 0 is the equilibrium position, and as P is increased we find that a neighboring equilibrium position becomes possible, and that is called, we discussed as buckling due to bifurcation of equilibrium, so the load deflection path has this feature there are infinite number of buckling loads, and the lowest one obviously is of the fundamental interest, so as the load is increased at some point when P coincides with the first critical or the first buckling load the zero position becomes unstable, and 2 new branches emerge.

In another type of problem what is known as limit load buckling example for which is shown here, there are 2 rigid links, load is supported through springs as shown here, and it is loaded by a force P and as the force P is increased the deflection of the system and the load increased together, and at some point both of them reach a critical stage when a faraway equilibrium position becomes possible, and this is known as limit load buckling, we start with here at P = 0, that this will be the deflection and as load is increased it will go, the load deflection path rises



and at this point it snaps and reaches this position, and this is a load deflection path, and for certain values of P the load deflection path is multi-valued, but although it is multi-valued there are certain portion of this load deflection path is not physically realizable, because those positions are unstable.



We also started discussing about approximate methods for stability analysis wherein we consider the expression for the V, that is total strain energy as shown here and we postulated the displacement field in terms of trial functions where E is a generalized coordinate, and we evaluated V as a function of A and then apply the 2 axioms to get the critical load.

Galerkin's method

Starting point: governing field equation

$$\frac{d^2}{dx^2}\left[EI\frac{d^2y}{dx^2}\right]+P\frac{d^2y}{dx^2}=0$$

$$y(x)=\sum_{n=1}^{N}a_n\phi_n(x) \qquad \left[\ \phi_n(x):\ \text{admissible}\right]$$

$$R(x)=\frac{d^2}{dx^2}\left[EI\sum_{n=1}^{N}a_n\phi_n''(x)\right]+P\sum_{n=1}^{N}a_n\phi_n''(x)$$

Method of weighted residuals

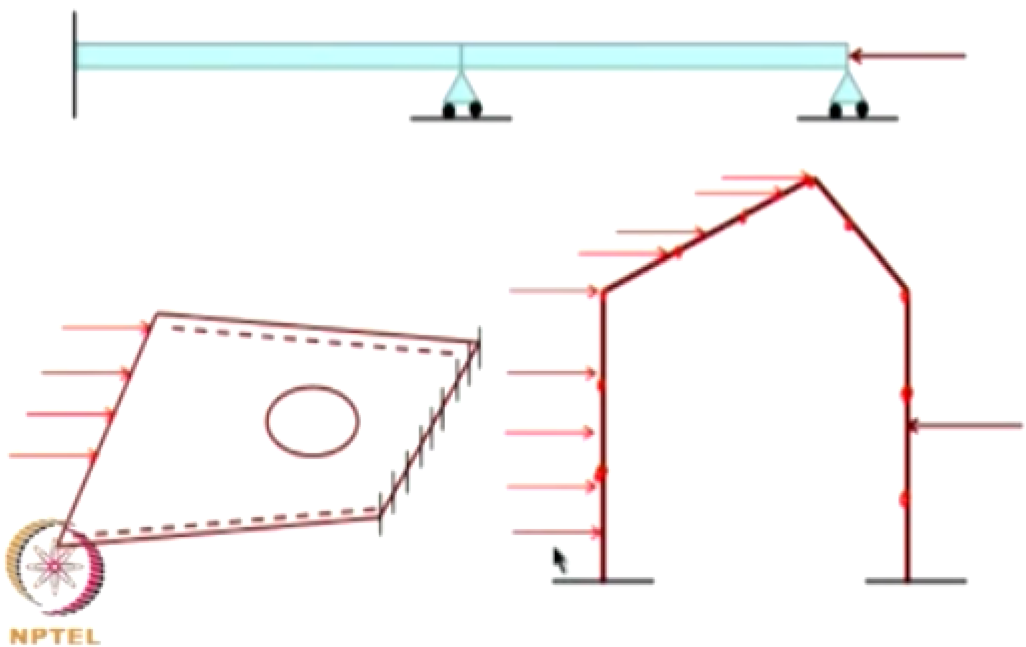
$$\int_0^l R(x)\phi_k(x)dx=0, k=1,2,\cdots,N$$

$$\Rightarrow[K]\{a\}+P[J]\{a\}=0$$

5

Here we started with the total potential energy whereas in Galerkin's method we could as well start with the governing differential equation and assume the solution in this form where phi N(x) are need to be admissible and we substitute this into the governing equation and we get a residue and we demand that this residue is orthogonal to the trial functions, and we get a set of capital N number of equations, and this leads to the equilibrium equation as shown here. And for non-trivial solutions we need to do an eigenvalue analysis to determine the critical load.

How about these problems?



Constructing globally valid trial functions become increasingly difficult 6

Now the question now that we should consider is how to generalize this to more a difficult class of problem, for example if beam is multi-span or we consider a plate with a hole with complex geometry and a hole and things like that, or a building frame which carries lateral loads and as well maybe there could be vertical loads as well, how do we perform the stability analysis of this structure? For example if we consider this type of problems, suppose this is the loading pattern that is acting on the structure we would like to know if keeping the entire loading pattern the same we increase the magnitude of these loads, so at some point we ask the question can the structure lose stability, okay, so that is the type of questions we wish to answer, so we use to tackle this problem the finite element method in the Rayleigh-Ritz and Galerkin's method we construct a trial functions which are globally valid, and as we have seen in the discussion, as in the discussion of vibration problems constructing trial functions which are globally valid for this type of built up structures is not possible, and what we do is we divide the domain of the structure into subdomains and within a sub domain we interpolate the field variables in terms of the nodal values of the field variables and that is that finite element method and that provides a powerful alternative to tackle problems of stability analysis as well.

FEM: powerful alternative

$$[K]\{a\}-P[J]\{a\}=F$$

$[J]$: Stability matrix

$P[J]$: geometric stiffness matrix

$[K]-P[J]$: Total stiffness matrix

$$[K]\{a\}-P[J]\{a\}=0$$

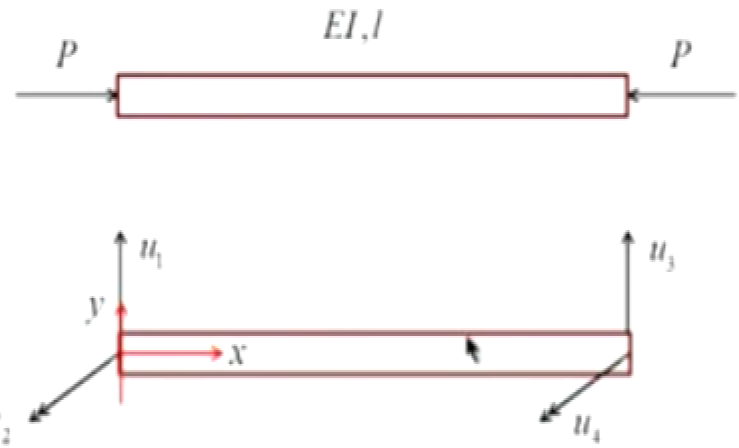
$$M\ddot{X}+C\dot{X}+[K-PJ]X=f(t)$$

$\vdots$

So as you will see shortly this leads to equilibrium equation of this form where J is the stability matrix and P(j) is the geometric stiffness matrix, and K - PJ is a total stiffness matrix, and we can either solve the response of the structure by allowing for effect of geometric stiffness or find out the critical conditions where the structure loses stability, we can do a static analysis, or we can do a dynamic analysis, so we will see some of this as we go along.

Euler-Bernoulli beam element

$$U = \frac{1}{2}\int_0^l EI\left(\frac{d^2v}{dx^2}\right)^2 dx - \frac{P}{2}\int_0^l \left(\frac{dv}{dx}\right)^2 dx$$



$$v(x) = \sum_{i=1}^{4} u_i \phi_i(x)$$

8

So let’s start with, as we did when we discuss vibration problems Euler-Bernoulli beam problems and then we will generalize to more complicated elements, so the total potential energy for Euler-Bernoulli beam we have derived it is in this form, so we considered a Euler-Bernoulli beam element carrying an axial load P, it has two nodes and it has 2 degrees of freedom, two displacement and rotation at each degree of freedom, and we assume that field variable displacement field in this form where UI are the nodal values and phi(x) are the trial functions, so as we have seen in the context of vibration problems we you know the cubic

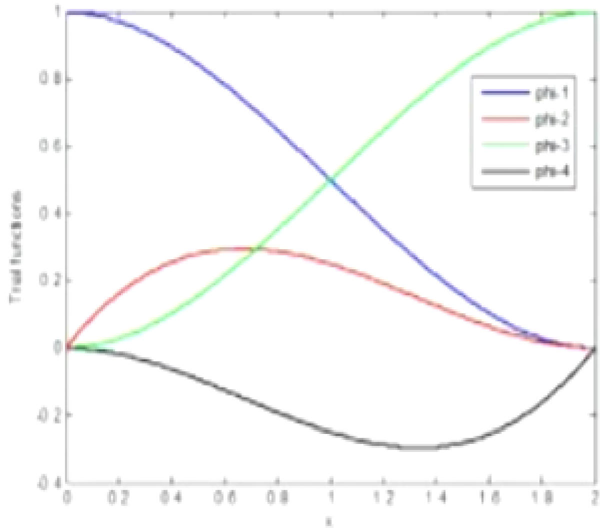
$$v(x)=\sum_{i=1}^{4} u_i\phi_i(x)$$

$$\phi_1(x)=1-3\frac{x^2}{l^2}+2\frac{x^3}{l^3}$$

$$\phi_2(x)=x-2\frac{x^2}{l}+\frac{x^3}{l^2};$$

$$\phi_3(x)=3\frac{x^2}{l^2}-2\frac{x^3}{l^3};$$

$$\phi_4(x)=-\frac{x^2}{l}+\frac{x^3}{l^2}$$



$$U=\frac{1}{2}\int_0^l EI\left(\frac{d^2v}{dx^2}\right)^2 dx-\frac{P}{2}\int_0^l\left(\frac{dv}{dx}\right)^2 dx$$

$$=\frac{1}{2}\int_0^l EI\left(\sum_{i=1}^{4} u_i\phi_i''(x)\right)^2 dx-\frac{P}{2}\int_0^l\left(\sum_{i=1}^{4} u_i\phi_i'(x)\right)^2 dx$$

polynomials are appropriate to handle this problem and these are the polynomials that we have used in the dealing with vibration problems, so see it is very clear there are 2 nodes and there are 2 degrees of freedom, because the functional here, the highest derivative is 2, therefore the nodal degrees of freedom are V and DV/DX, there are 2 nodes so we need a polynomial with 4 coefficient that would mean we need to is a cubic polynomial.

So we substitute now that into the expression for the total potential energy, and we get now the strain energy or total potential energy in terms of the nodal coordinates, so these are once we

$$U=\frac{1}{2}\int_0^l EI\left(\sum_{i=1}^{4}u_i\phi_i''(x)\right)^2 dx-\frac{P}{2}\int_0^l\left(\sum_{i=1}^{4}u_i\phi_i'(x)\right)^2 dx=U\left(u_1,u_2,u_3,u_4\right)$$

Equillibrium: $\dfrac{\partial U}{\partial u_i}=0, i=1,2,3,4$

$$\frac{\partial U}{\partial u_j}=\frac{1}{2}\int_0^l 2EI\left(\sum_{i=1}^{4}u_i\phi_i''(x)\right)\phi_j''(x)\,dx-\frac{P}{2}\int_0^l 2\left(\sum_{i=1}^{4}u_i\phi_i'(x)\right)\phi_j'(x)\,dx$$

$$=\sum_{i=1}^{4}K_{ij}u_i-P\sum_{i=1}^{4}J_{ij}u_j$$

with

$$K_{ij}=\int_0^l EI\phi_i''(x)\phi_j''(x)\,dx \;\&\; J_{ij}=\int_0^l\phi_i'(x)\phi_j'(x)\,dx$$

Equillibrium: $\sum_{i=1}^{4}K_{ij}u_i-P\sum_{i=1}^{4}J_{ij}u_j=0, i=1,2,3,4\Rightarrow[K]\{u\}-P[J]\{u\}=0$

10

substitute for this, these being polynomials they can easily be evaluated and for equilibrium we have dou U/dou UJ = 0 and with that we get this equation, and we define KIJ as EI phi double prime, phi J double prime X, DX, and JIJ as this. For equilibrium this must be equal to 0 for I= to 1, 2, 3, 4, and this I can arrange in a matrix form and this K is the elastic stiffness matrix, that we have seen earlier and this is a new thing J is the stability matrix, and P is the axial load, JIJ if you notice is independent of elastic constants and loads on the system, so that is why it called, that is why the name geometric stiffness. So we can evaluate those integrals and I get the

$$K_{ij} = \int_0^l EI\phi_i''(x)\phi_j''(x)\,dx \;\&\; J_{ij} = \int_0^l \phi_i'(x)\phi_j'(x)\,dx$$

$$K = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \;\&\; K_G = PJ = \frac{P}{30l}\begin{bmatrix} 36 & 3l & -36 & 3l \\ -3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix}$$

Hessian matrix

$$\frac{\partial U}{\partial u_j} = \sum_{i=1}^{4} K_{ij}u_i - P\sum_{i=1}^{4} J_{ij}u_j \Rightarrow \frac{\partial^2 U}{\partial u_j \partial u_k} = K_{jk} - PJ_{jk},\; j,k = 1,2,3,4$$

$$[H] = [K_{jk} - PJ_{jk}]$$

Equillibrium: $[K]\{u\} - P[J]\{u\} = 0$

Critical state: $|K_{jk} - PJ_{jk}| = 0$

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elastic stiffness matrix this we have seen earlier, and this is a new matrix that we have derived. Now to examine the critical condition we construct the Hessian, and this is the Hessian is a matrix of these elements which is KJK – PJK, and this is a hessian matrix, for equilibrium I need this equation to be satisfied and for a critical state the determinant of the Hessian must be

Remarks

- K is the elastic stiffness matrix which has been derived earlier
- $K_G = PJ$ is the geometric stiffness matrix
- It is called as the consistent geometric stiffness matrix since the shape functions which have been used in evaluating Khave also been used in evaluating this matrix.
- J is called the element stability matrix
- $K_T = K - PJ$ is called the total stiffness matrix
- K_G is independent of the EI of the beam and depends upon only the length of the beam and hence the name.
- Structure of U

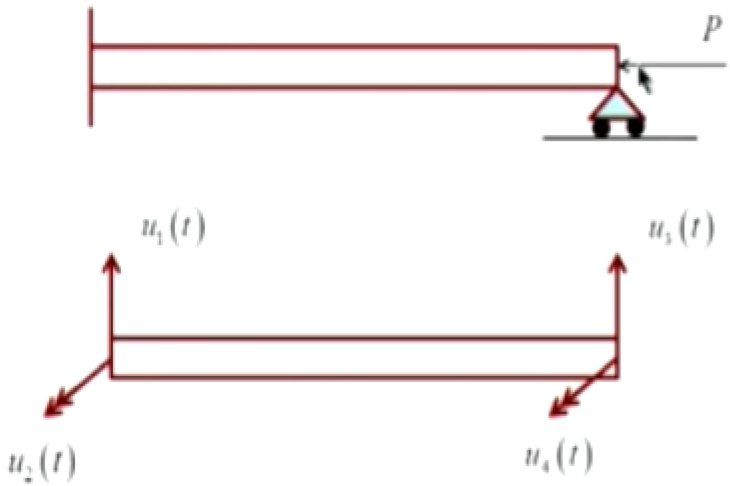
$$U = \frac{1}{2}\int_0^l EI\left(\sum_{i=1}^{4} u_i \phi_i''(x)\right)^2 dx - \frac{P}{2}\int_0^l \left(\sum_{i=1}^{4} u_i \phi_i'(x)\right)^2 dx$$

$$= \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4} u_i u_j K_{ij} - \frac{P}{2}\sum_{i=1}^{4}\sum_{j=1}^{4} u_i u_j J_{ij} = \frac{1}{2} u^t \left[K - PJ\right] u$$

12

0, so we can make few observations K is the elastic stiffness matrix which has been derived earlier, KG = PJ is the geometric stiffness matrix, it is called as a consistent geometric stiffness matrix, since the shape functions which have been used in evaluating K have also been used in evaluating this matrix, the same shape functions have been used, J is called the stability matrix. Now KT which is K - PJ is called a total stiffness matrix, so it modifies the elastic stiffness matrix by contribution from the geometric stiffness term. Now KG which is P into J is independent of EI of the beam and depends upon only the length of the beam and hence the name geometric stiffness, now we can also see the structure of U, so if we write this and expand we can see that U using KIJ notations KIJ and JIJ this can be written as 1/2 U transpose K - PJ into U.

Example 1



$$\boxed{u_1(t)=0; u_2(t)=0; u_3(t)=0}$$

Now based on this we can consider a few examples so that we get an idea on how to use this, so we'll start with the propped cantilever it has one degree of freedom, namely U4(t) U1, U2, and U3 are 0, so this is one of the simplest problem that we can think of, so for equilibrium I have 1, 2, 3, as 0 and U4 is to be determined, and this is again 0, 0, 0, U4 and I get the equilibrium

Equillibrium

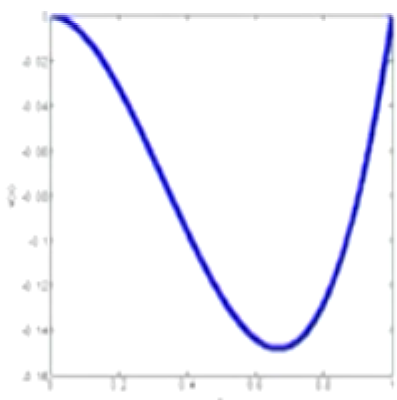
$$\frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ u_4 \end{Bmatrix} - \frac{P}{30l}\begin{bmatrix} 36 & 3l & -36 & 3l \\ -3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ u_4 \end{Bmatrix} = 0$$

$$\left(\frac{EI}{l^3}4l^2 - \frac{P}{30l}4l^2\right)u_4 = 0 \Rightarrow u_4 = 0$$

Stable if $\frac{EI}{l^3}4l^2 - \frac{P}{30l}4l^2 > 0 \Rightarrow P_{cr} = 30\frac{EI}{l^2}$

$$P_{cr}^{\text{Exact}} = 20.1997\frac{EI}{l^2}$$

Mode shape: $v(x) = \phi_4(x) = -\frac{x^2}{l} + \frac{x^3}{l^2}$



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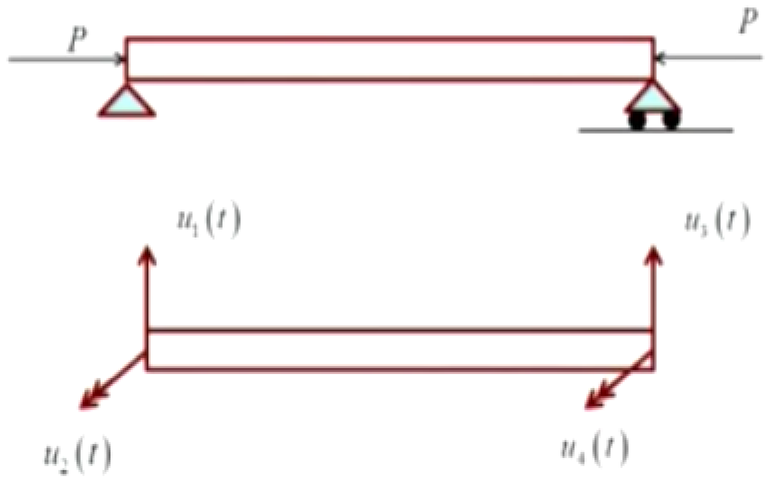
14

equation to be given by this, that would mean U4 = 0 is the equilibrium state, is this stable? Now you differentiate this with respect to U4 and demand that that derivative is greater than 0, so for that to happen P critical, P should be less than some number and the critical value is 30EI/L square, exact Euler buckling load for this case is, around 20.1997EI/ L square and the buckling mode shape according to this theory is given by phi 4(x) which is shown here, so this is a very simple application of the idea of elastic stiffness and geometric stiffness matrix.

Example 2



$$\boxed{u_1(t) = 0; u_3(t) = 0}$$

15

We can now consider an single span simply supported beam discretized as a single element, so there are 4 nodal degrees of freedom, 2 of which will be 0, U1 and U3 will be 0, the equilibrium

Equillibrium

$$\frac{EI}{l^3}\begin{bmatrix}12 & 6l & -12 & 6l\\ 6l & 4l^2 & -6l & 2l^2\\ -12 & -6l & 12 & -6l\\ 6l & 2l^2 & -6l & 4l^2\end{bmatrix}\begin{Bmatrix}0\\ u_2\\ 0\\ u_4\end{Bmatrix}-\frac{P}{30l}\begin{bmatrix}36 & 3l & -36 & 3l\\ -3l & 4l^2 & -3l & -l^2\\ -36 & -3l & 36 & -3l\\ 3l & -l^2 & -3l & 4l^2\end{bmatrix}\begin{Bmatrix}0\\ u_2\\ 0\\ u_4\end{Bmatrix}=0$$

$$\frac{EI}{l^3}\begin{bmatrix}4l^2 & 2l^2\\ 2l^2 & 4l^2\end{bmatrix}\begin{Bmatrix}u_2\\ u_4\end{Bmatrix}-\frac{P}{30l}\begin{bmatrix}4l^2 & -l^2\\ -l^2 & 4l^2\end{bmatrix}\begin{Bmatrix}u_2\\ u_4\end{Bmatrix}=0$$

Critical condition $|H|=0$

$$\left|\frac{EI}{l^3}\begin{bmatrix}4l^2 & 2l^2\\ 2l^2 & 4l^2\end{bmatrix}-\frac{P}{30l}\begin{bmatrix}4l^2 & -l^2\\ -l^2 & 4l^2\end{bmatrix}\right|=0$$

This is equivalent to finding the solutions of the eigenvalue problem

$$\frac{EI}{l}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}\phi-\frac{Pl}{30}\begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix}\phi=0\Rightarrow\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}\phi=\lambda\begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix}\phi;\lambda=\frac{Pl^2}{30EI}$$

equation now in the, this is K into U, and in U I have 0 is here, and this is a geometric stiffness matrix, now extracting the equations for U2 and U4 I get the equilibrium equation as this. Now the critical condition is given by determinant of Hessian matrix to be 0 and that leads to this condition, so this finding this condition in fact is equivalent to performing an eigenvalue analysis of K and KG that means I have, we need to perform this eigenvalue analysis, so we can do that and we get the equation to be this and from this I get lambda to be PI square/30, I mean by substituting for lambda this number and carrying, lambda will be the eigenvalue parameter I

$$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\phi = \lambda\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}\phi$$

$$\Phi = \begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}$$

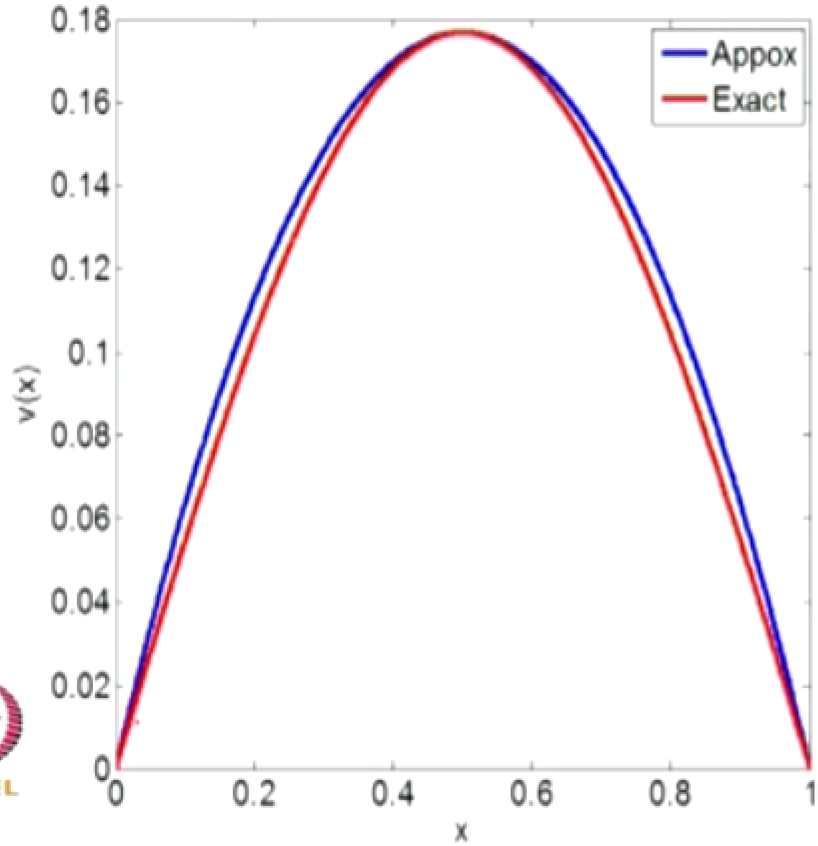
$$\lambda = \begin{Bmatrix} 0.4 \\ 2.0 \end{Bmatrix}$$

$$\lambda_{\min} = 0.4 \Rightarrow 0.4 = \frac{P_{cr}l^2}{30EI} \Rightarrow P_{cr} = \frac{12EI}{l^2}$$

$$P_{cr}^{\text{Exact}} = 9.869\frac{EI}{l^2}$$

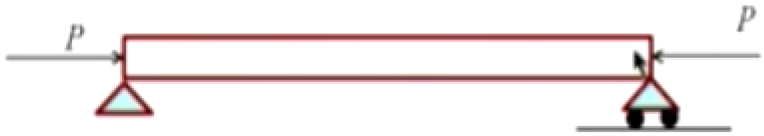
Mode shape: $v(x) = 0.7071\left[\phi_1(x) - \phi_4(x)\right]$

get the modal matrix to be this, and the 2 eigenvalues to be this, so the smallest of these eigenvalues is helpful in determining the lowest critical load, and using that we arrive at the solution that critical load is 12EI/L square, whereas the exact one is 9.869EI/L square and this is the mode shape, okay, so we can see how the mode shape look like, exact one is a sign half



sine wave, and the blue one is the approximate using these 2 cubic polynomial, so this is a model for, a simple model for critical condition establish simple model for establishing critical condition for a single spam.

Example 2



$$\boxed{u_1(t) = 0; u_3(t) = 0}$$

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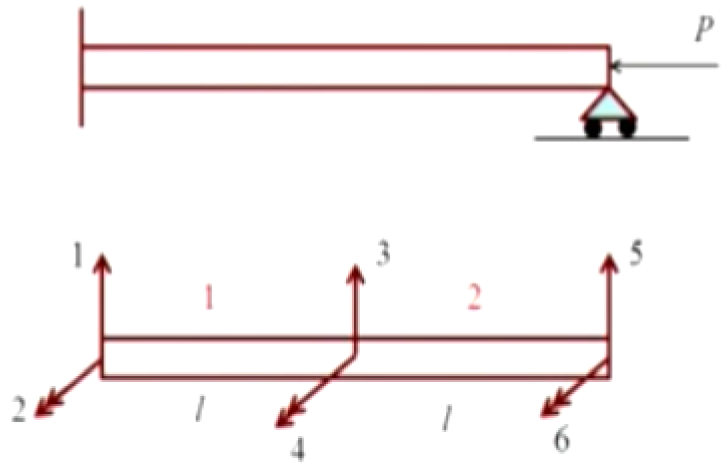
Example 3



$u_1(t)=0; u_2(t)=0; u_5(t)=0$

DOFs: 3,4,6

How to improve upon the results? So in the first example suppose if we introduce an additional node, okay, so now what happens we have 2 elements and the procedure for assembling matrices and transforming into global coordinates etcetera is quite similar to what we have done in dealing with vibration problem so that story need not be repeated now, so we have 2 elements and the boundary conditions will be U1 and U2 are 0 here, and U5 is 0, so degrees of freedom are 3, 4, and 6, so for the first element I formulate the elastic stiffness and geometric stiffness matrix, for second element I form this, and now using the fact that 3, 4, 6 are the

$$K_1 = \frac{EI}{l^3}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} ; K_{G1} = \frac{P}{30l}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 36 & 3l & -36 & 3l \\ -3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$K_2 = \frac{EI}{l^3}\begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \end{array}\begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array} ; K_{G1} = \frac{P}{30l}\begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 36 & 3l & -36 & 3l \\ -3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix} \end{array}\begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$u_1(t) = 0; u_2(t) = 0; u_5(t) = 0$
DOFs: 3,4,6

degrees of freedom we can assemble them either you can form the A matrices for the 2 elements and assemble them through matrix operations, or if you are working with pen and paper you can assemble them by you know observations, so we get the 3 x 3 structure, reduced structure elastic stiffness matter and reduced structure geometric stiffness matrix, for equilibrium it is KU = PJU and a critical condition is this, so this is the equilibrium equation and again by introducing lambda is PL square/30EI, now I get a 3 x 3 eigenvalue problem,

$$
\begin{array}{c} \\ K = \dfrac{EI}{l^3} \end{array}
\begin{array}{c} \begin{array}{ccc} 3 & 4 & 6 \end{array} \\ \begin{bmatrix} 24 & 0 & 6l \\ 0 & 8l^2 & 2l^2 \\ 6l & 2l^2 & 4l^2 \end{bmatrix} \end{array}
\begin{array}{c} \\ 3 \\ 4 \\ 6 \end{array}
; K_G = \frac{P}{30l}
\begin{array}{c} \begin{array}{ccc} 3 & 4 & 6 \end{array} \\ \begin{bmatrix} 72 & 0 & 3l \\ 0 & 8l^2 & -l^2 \\ 3l & -l^2 & 4l^2 \end{bmatrix} \end{array}
\begin{array}{c} \\ 3 \\ 4 \\ 6 \end{array}
$$

Equliibrium: $Ku = PJu$; Critical condition $|H| = 0 \Rightarrow |K - PJ| = 0$

$$
\frac{EI}{l^3}\begin{bmatrix} 24 & 0 & 6l \\ 0 & 8l^2 & 2l^2 \\ 6l & 2l^2 & 4l^2 \end{bmatrix}\begin{Bmatrix} u_3 \\ u_4 \\ u_6 \end{Bmatrix} = \frac{P}{30l}\begin{bmatrix} 72 & 0 & 3l \\ 0 & 8l^2 & -l^2 \\ 3l & -l^2 & 4l^2 \end{bmatrix}\begin{Bmatrix} u_3 \\ u_4 \\ u_6 \end{Bmatrix}
$$

$$
\Rightarrow \begin{bmatrix} 24 & 0 & 6l \\ 0 & 8l^2 & 2l^2 \\ 6l & 2l^2 & 4l^2 \end{bmatrix}\begin{Bmatrix} u_3 \\ u_4 \\ u_6 \end{Bmatrix} = \lambda\begin{bmatrix} 72 & 0 & 3l \\ 0 & 8l^2 & -l^2 \\ 3l & -l^2 & 4l^2 \end{bmatrix}\begin{Bmatrix} u_3 \\ u_4 \\ u_6 \end{Bmatrix} \text{ with } \lambda = \frac{Pl^2}{30EI}
$$

$$
\lambda_{\min} = 0.1726 \Rightarrow P_{cr} = 5.178\frac{EI}{l^2}; P_{cr}^{\text{Exact}} = 5.1999\frac{EI}{l^2}
$$

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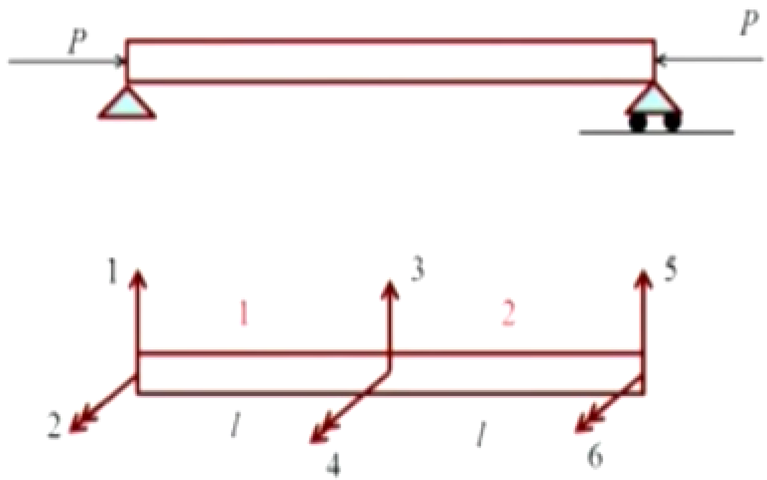
$$
\left(P_{cr} \text{ with sdof model} = 7.5\frac{EI}{l^2}; \text{note } 2l \rightarrow l\right)
$$

21

eigenvalue problem associated with a 3 x 3 matrix and the minimum eigenvalue if you calculate will turn out to be 0.1726 from which I get P critical as this, exact we have seen to be this, and now we have improved on the earlier solution, note that in this case I have taken element length to be L whereas in the first example the span of the beam was taken as L, therefore we need to do a slight adjustment in comparing the results.

Example 4



$$\boxed{u_1(t)=0; u_5(t)=0}$$

Similarly for a simply supported beam if we now improve upon accuracy we have now degrees of freedom will be U2, U3, U4, and U6, so 2, 3, 4, 6 are the degrees of freedom so once we

$$
K = \frac{EI}{l^3}\begin{array}{c c}
\begin{array}{cccc} 2 & 3 & 4 & 6 \end{array} & \\
\begin{bmatrix} 4l^2 & -6l & 2l^2 & 0 \\ -6l & 24 & 0 & 6l \\ 2l^2 & 0 & 8l^2 & 2l^2 \\ 0 & 6l & 2l^2 & 4l^2 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \\ 4 \\ 6 \end{array}
\end{array}
; K_G = \frac{P}{30l}\begin{array}{c c}
\begin{array}{cccc} 2 & 3 & 4 & 6 \end{array} & \\
\begin{bmatrix} 4l^2 & -3l & -l^2 & 0 \\ -3l & 72 & 0 & 3l \\ -l^2 & 0 & 8l^2 & -l^2 \\ 0 & 3l & -l^2 & 4l^2 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \\ 4 \\ 6 \end{array}
\end{array}
$$

Eigenvalue problem to be solved

$$A\phi = \lambda B\phi$$

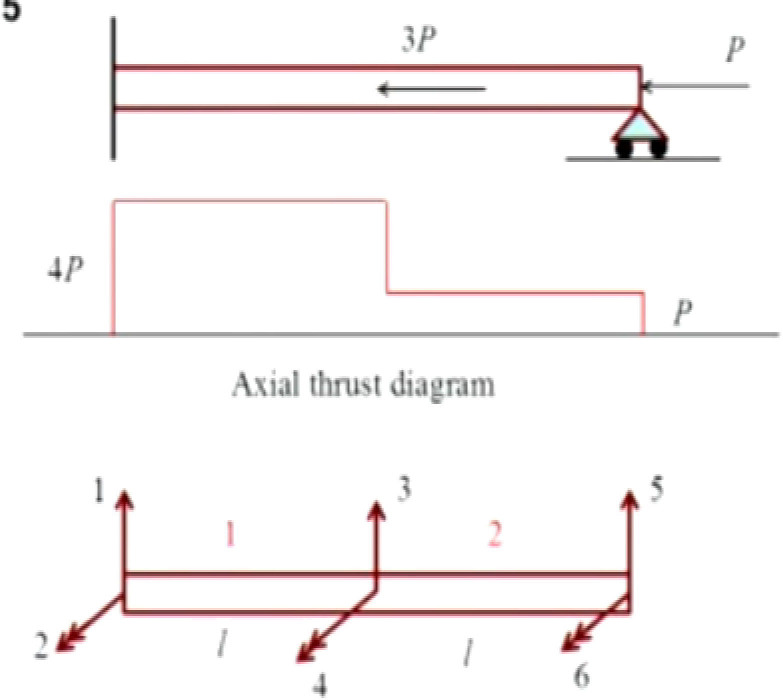
$$
A = \begin{bmatrix} 4 & -6 & 2 & 0 \\ -6 & 24 & 0 & 6 \\ 2 & 0 & 8 & 2 \\ 0 & 6 & 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 4 & -3 & -1 & 0 \\ -3 & 72 & 0 & 3 \\ -1 & 0 & 8 & -1 \\ 0 & 3l & -1 & 4 \end{bmatrix}; \lambda = \frac{P}{3EI}
$$

$$\lambda = \{0.0829 \quad 0.4000 \quad 1.0727 \quad 2.0000\}^T$$

$$\lambda_{\min} = 0.0829 \Rightarrow P_{cr} = 0.2487EI = 9.438\frac{EI}{(2l)^2}; P_{cr}^{\text{Exact}} = 9.8696\frac{EI}{(2l)^2}$$

assemble the structure stiffness matrix, elastic stiffness matrix, and geometric stiffness matrix we get these 2 matrices and hence the Eigenvalue problem to be solved is now 4 x 4 because there are 4 degrees of freedom and lambda which is now this parameter P/3I, if I solve this problem I get these are the 4 eigenvalues and similarly by using the smallest of these eigenvalues the P critical using this method turns out to be 9.438EI/L square, L is 2L here and this compares much better with the exact solution because we have increased the degree of freedom.

Example 5



$$u_1(t)=0; u_2(t)=0; u_5(t)=0$$

24

Now how do we tackle problems such as this? For example there is an axial load P here and an axial load 3P here, these are externally applied, so what we do is first we analyze the structure for finding the membrane forces which are the axial thrust here, so by simple logic we can see that the reaction at this end will be 4P, and the axial thrust diagram will be this, so again I will divide this beam into 2 elements and while forming the geometric stiffness for element 1, I will use for 4P as axial force, whereas when I form the geometric stiffness for the second element, P

$$K_{G_1} = \frac{4P}{30l}\begin{bmatrix} 4l^2 & -3l & -l^2 & 0 \\ -3l & 72 & 0 & 3l \\ -l^2 & 0 & 8l^2 & -l^2 \\ 0 & 3l & -l^2 & 4l^2 \end{bmatrix} \;\&\; K_{G_2} = \frac{P}{30l}\begin{bmatrix} 4l^2 & -3l & -l^2 & 0 \\ -3l & 72 & 0 & 3l \\ -l^2 & 0 & 8l^2 & -l^2 \\ 0 & 3l & -l^2 & 4l^2 \end{bmatrix}$$

Eigenvalue problem to be solved

$$A\phi = \lambda B\phi$$

$$A = \begin{bmatrix} 24 & 0 & 12 \\ 0 & 32 & 0 \\ 12 & 0 & 16 \end{bmatrix}; B = \begin{bmatrix} 180 & -18 & 6 \\ -18 & 80 & -4 \\ 6 & -4 & 16 \end{bmatrix} \lambda = \frac{P}{3EI}$$

$$\lambda = \{0.0764 \quad 0.3067 \quad 1.1899\}^t$$

$$\lambda_{min} = 0.0764 \Rightarrow P_{cr} = 9.1696\frac{EI}{(2l)^2}$$

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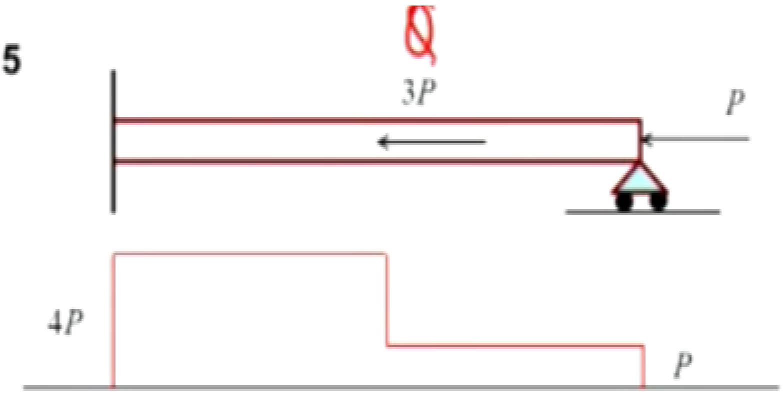
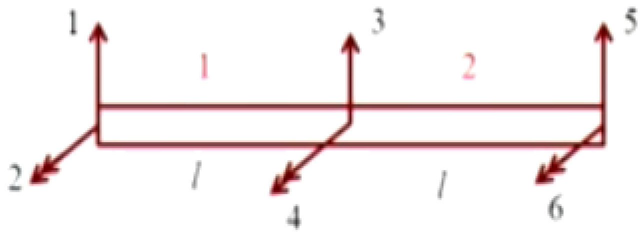
will be the axial load, so that is the only difference so for a first element the geometric stiffness is 4P/30L, for second one it is P/30L, all other details remain the same as far as elastic stiffness matrix is concerned, so we get the eigenvalue problem like this and lambda is again obtained as this, and P critical in this case turns out to be this.

Example 5

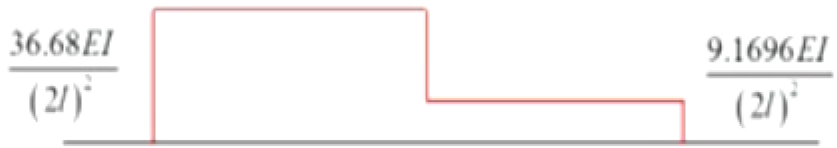


Axial thrust diagram



$$\boxed{u_1(t)=0; u_2(t)=0; u_5(t)=0}$$

Now what we are doing here is if we see when I increase p the two loads are increased in same proportion, for example if this load was Q and this was P then what I will do is I will increase P by alpha P, and Q also by alpha Q that means as I increase the load all the loads increase by the same factor there could be transverse loads and other loads as well, so I will find out the value of alpha at which critical condition is reached and that value of alpha is known as load factor, so in problems of elastic stability analysis we aim to find the load factor, by that what we mean is for a given structural configuration carrying certain specially distributed load keeping the loading pattern the same I will increase the entire loading pattern by a constant amount that is alpha, so the question we ask is how far alpha should be increased before the structure reaches a critical state, so this example is illustrative of that procedure that we will follow.

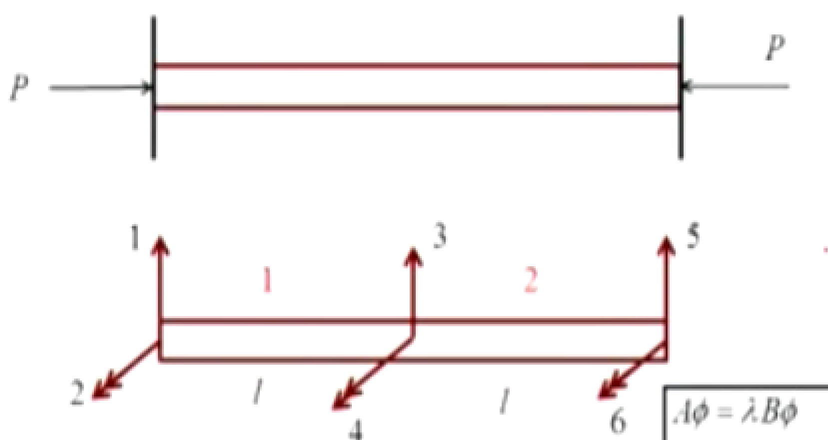


Axial thrust diagram on the verge of buckling

26

Now this is axial thrust diagram of the structure when it is on the verge of buckling, so this is the critical loading condition as far as axial loads are concerned. So some more examples

Example 6



$u_1(t)=0; u_2(t)=0; u_5(t)=0; u_6(t)=0$

DoF: 3,4

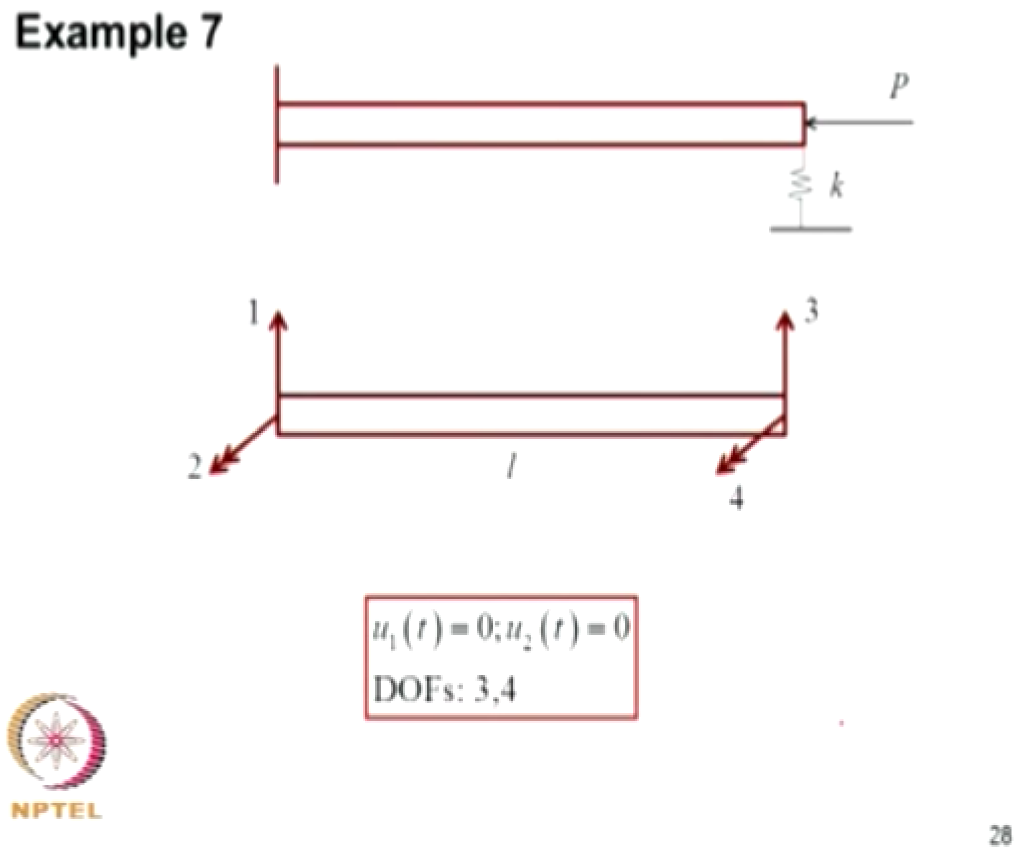
$A\phi = \lambda B\phi$

$A = \begin{bmatrix} 24 & 0 \\ 0 & 24 \end{bmatrix}; B = \begin{bmatrix} 72 & 0 \\ 0 & 72 \end{bmatrix}$

$\lambda = 1/3, 1 \Rightarrow P_{cr} = 40\frac{EI}{(2l)^2}$

$P_{cr}^{\text{Exact}} = 39.4784\frac{EI}{(2l)^2}$

suppose we have N casted beams I take 2 elements and introduce a node at the mid-span and here again we can perform the analysis and I get 40EI/2L square as against 39.4784 so these are some simple examples that you can, you know run through and to convince yourself that these steps are correct.



Now one more example suppose there is a cantilever beam which is mounted on a spring and a load P is applied, so we discretize there are 2 elements now, the first element is a beam and the second element is a spring, and the first element is discretized using single element with 2 nodes, and U1 and U2 are 0 and I have basically degrees of freedom 3 and 4, 3 is also the degree of freedom shared by K, so I construct the elastic stiffness matrix and geometric

$$K_1 = \frac{EI}{l^3} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 4l^2 & -6l & 2l^2 & 0 \\ -6l & 24 & 0 & 6l \\ 2l^2 & 0 & 8l^2 & 2l^2 \\ 0 & 6l & 2l^2 & 4l^2 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad K_2 = k \begin{array}{c} \begin{array}{cc} 3 & 4 \end{array} \\ \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ 3 \\ 4 \end{array}$$

$$\alpha = \frac{kl^3}{EI}$$

$$\frac{EI}{l^3}\begin{bmatrix} 12+\alpha & -6l \\ -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \frac{P}{30l}\begin{bmatrix} 36 & -3l \\ -3l & 4l^2 \end{bmatrix}$$

α	0	1	10	100	1000	10^4
$\frac{l^2 P_{cr}}{EI}$	2.486	3.3077	10.4815	29.0696	29.0696	29.9993

$$\alpha = 0, P_{cr}^{Exact} = 2.4674\frac{EI}{l^2}; \alpha \to \infty, P_{cr}^{Exact} = 20.1997\frac{EI}{l^2}$$

stiffness matrix and when we assemble I use a notation alpha as KL cube/EI and I get this equation, so clearly the critical condition, critical loads will be function of K, so that parametric study has been done when alpha equal to 0 it becomes a cantilever beam, and when alpha tends to infinity it becomes a propped cantilever, so for K = 0, and K equal to infinity we know the exact solutions, so here when alpha = 0 this is the exact solution, and when alpha tends to infinity this exact solution. So now as alpha is varied for alpha = 0, I get this factor as 2.486 that has to be compared with 2.4674 and as this number is increased this model converges to 29.993 this factor whereas it should really converge to 20.1997, but this difference is basically due to the simplicity of this model, this can be improved by introducing more number of elements and discretizing the structure with finer detail, okay.

Remarks

Natural frequencies: $K\phi = \omega^2 M\phi$

Buckling loads: $K\phi = \lambda K_G \phi$

P_{cr} is a function of applied load. The same structure can have different P_{cr} for different applied loads. In this aspect the buckling load is different from natural frequencies of a structure.

Load factor: a constant factor by which a given loading pattern needs to be increased so that the structure would be on the verge of buckling.

If axial stresses are tensile, the stiffness of the structure increases. This effect is called stress stiffening.

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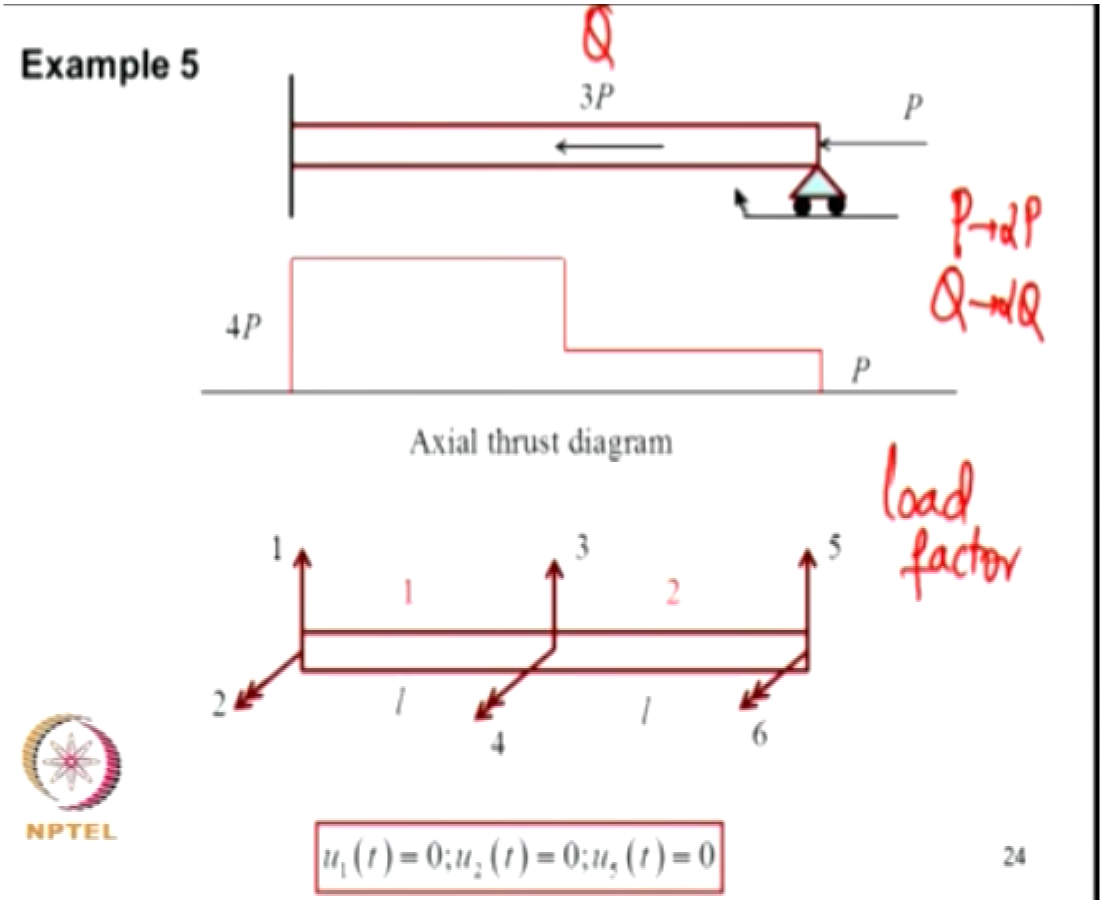
30

Let us make some remarks now based on what we have seen if you recall when we found natural frequencies we found solve this eigenvalue problem, this was associated with the structure elastic stiffness matrix and the mass matrix whereas to determine buckling loads we are again doing an eigenvalue analysis, and one of the matrix is the same that is elastic stiffness matrix, there is a new matrix KG which is a geometric stiffness matrix, the eigenvalue here were the natural frequencies, whereas here they correspond to the load factors or the critical loading conditions there will be N such situation in the nth degree of freedom model and lambda N eigenvalues provide that.

Now one important difference that we have to notice between these 2 problems is P critical is a function of the applied load the same structure can have different P critical for different applied loads, see for example we considered this problem because there is a load 3P the same propped



cantilever will have a different natural frequency then when suppose this 3P was not there, only P was there you will get another load factor. In this aspect the buckling load is different

Remarks

Natural frequencies: $K\phi = \omega^2 M\phi$

Buckling loads: $K\phi = \lambda K_G \phi$

P_{cr} is a function of applied load. The same structure can have different P_{cr} for different applied loads. In this aspect the buckling load is different from natural frequencies of a structure.

Load factor: a constant factor by which a given loading pattern needs to be increased so that the structure would be on the verge of buckling.

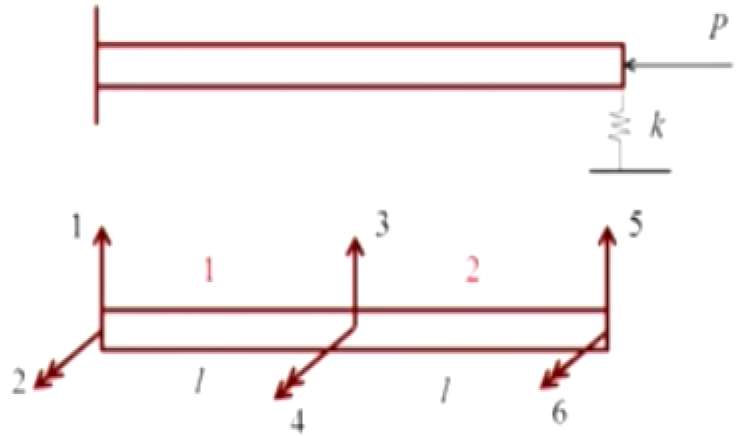
If axial stresses are tensile, the stiffness of the structure increases. This effect is called stress stiffening.

NPTEL

30

from natural frequency of a structure, whereas natural frequency of the structure unless you include geometric stiffness effects also into that is not dependent on applied loads, so load factor is a constant factor by which a given loading pattern needs to be increased so that the structure would be on the verge of bucking. If axial stresses are tensile the stiffness of the structure increases, this effect is called stress stiffening, so in a more general problem this will automatically get accounted for based on the sign of the axial thrust, so that automatically will be included in the analysis.

Exercise

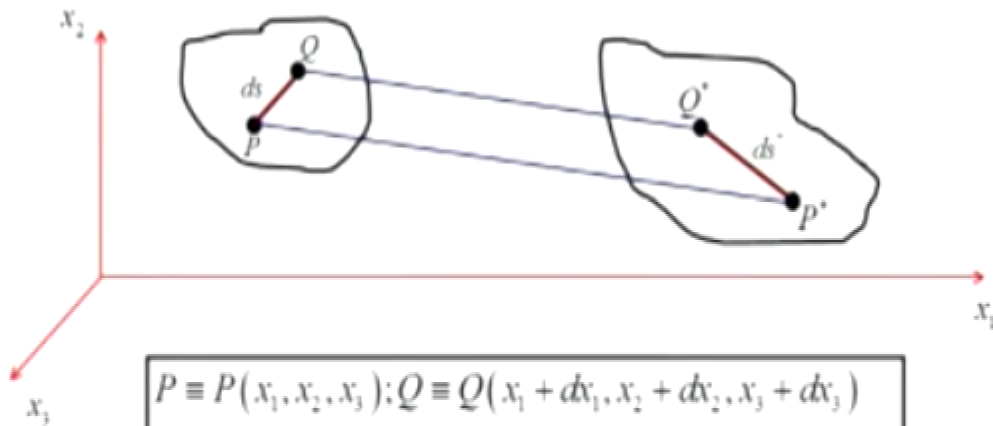


So a small exercise would be to improve upon the result that we obtain for this cantilever beam supported on the spring by introducing one more element, and you can redo this exercise for different values of K and see whether we get improved solutions. Now after this point we can

General formulation

Preliminaries: nonlinear strain-displacement relations:

Green-Lagrange strain tensor



$$P \equiv P(x_1, x_2, x_3); Q \equiv Q(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$$

$$P^* \equiv P(x_1^*, x_2^*, x_3^*); Q^* \equiv Q(x_1^* + dx_1^*, x_2^* + dx_2^*, x_3^* + dx_3^*)$$

$$PQ = ds; P^*Q^* = ds^*$$

$$x_1^* = x_1 + u_1; x_2^* = x_2 + u_2; x_3^* = x_3 + u_3$$

32

take a traditional route, and for example compute geometric stiffness matrix for a 3D beam, so 3D beam has bending along 2 planes twisting as we have seen earlier, that we can derive a geometric stiffness for that, then we can move on to a plate and other elements, but instead of taking each of these problems one by one we can, since given that we already have experience in formulating this element, we can take a look at the general formulation as applied for a 3-dimensional solid and then we can specialize it to specific structures of interest, so with that in mind we will first start with the preliminaries, as you have seen it is important to include nonlinear strain displacement relations in our formulations for stability analysis, so to be able to do that in a systematic manner we will quickly review the notion of Green-Lagrange strain tensor.

So we'll consider a Cartesian coordinate system X1, X2, X3 in which there is an object which is supported and loaded by surface tractions and body forces in certain manner, and our interest is this is the structural configuration before deformation, and this is the structural configuration after deformation, we are focusing our attention on a line element PQ which is in the un-deformed configuration and it will occupy a position P*, Q* in the deformed configuration, so this length is DS and this length is DS*, so we want to examine properties of these two line elements which will help us to define the strain, so we will consider P, coordinates is X1, X2, X3, Q which is X1 + DX1, X2 + DX2 and X3 + DX3, similarly P* and Q* have these coordinates. PQ length is DS, P* Q* is DS*, now the displacement field is clearly given by X1 star is X1 + U1, X2* is X2 + U2, and X3* is X3 + U3, so these relations define the displacements U1, U2, and U3, so this is a standard definition of displacement field.

Components of $ds = (dx_1, dx_2, dx_3)$

Components of $ds^* = (dx_1^*, dx_2^*, dx_3^*)$

Components of $PP^* = (u_1, u_2, u_3)$

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2$$

$$ds^{*2} = dx_1^{*2} + dx_2^{*2} + dx_3^{*2}$$

$$ds^{*2} - ds^2 = \left(dx_1^{*2} - dx_1^2\right) + \left(dx_2^{*2} - dx_2^2\right) + \left(dx_3^{*2} - dx_3^2\right)$$

$$x_i^* \equiv x_i^*(x_1, x_2, x_3)$$

$$\left.\begin{aligned} dx_1^* &= \frac{\partial x_1^*}{\partial x_1}dx_1 + \frac{\partial x_1^*}{\partial x_2}dx_2 + \frac{\partial x_1^*}{\partial x_3}dx_3 \\ dx_2^* &= \frac{\partial x_2^*}{\partial x_1}dx_1 + \frac{\partial x_2^*}{\partial x_2}dx_2 + \frac{\partial x_2^*}{\partial x_3}dx_3 \\ dx_3^* &= \frac{\partial x_3^*}{\partial x_1}dx_1 + \frac{\partial x_3^*}{\partial x_2}dx_2 + \frac{\partial x_3^*}{\partial x_3}dx_3 \end{aligned}\right\} \Rightarrow \{dx^*\} = [J]\{dx\}$$

NPTEL

33

Now components of DS are DX1, DX2, DX3, components of DS* is DX1*, DX2*, DX3*, similarly components of PP* this is a displacement field, P moves to P* and that is U1, U2, U3, so if you now find a length square of the length DX square this is given by DX1 square + DX2 square + DX3 square, similarly the length DS* square is given by this. Now you find the difference I get this equation, now each point in the deformed configuration is a function of the initial coordinates so I have XI* as XI star(x1, x2, x3), so we are selecting X1, X2, X3 as a coordinate system, so the coordinate system is with respect to the un-deformed geometry and that is what we are doing, so we can differentiate this with respect to, we can find the DX1*, DX2*, DX3* using this relation and I get DX* as J into DX.

$$\left\{dx^*\right\} = \left[J\right]\left\{dx\right\}$$

$$x_1^* = x_1 + u_1 ; x_2^* = x_2 + u_2 ; x_2^* = x_3 + u_3 \Rightarrow$$

$$\left\{dx^*\right\} = \begin{bmatrix} 1+\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & 1+\frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_1} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_1} & 1+\frac{\partial u_3}{\partial x_3} \end{bmatrix} \left\{dx\right\}$$

$$ds^{*2} - ds^2 = \left(dx_1^{*2} - dx_1^2\right) + \left(dx_2^{*2} - dx_2^2\right) + \left(dx_3^{*2} - dx_3^2\right)$$

$$\Rightarrow \frac{ds^{*2} - ds^2}{2} = \sum_{i=1}^{3}\sum_{j=1}^{3} \varepsilon_{ij}\, dx_i dx_j$$

$$\Rightarrow \frac{ds^{*2} - ds^2}{2ds^2} = \sum_{i=1}^{3}\sum_{j=1}^{3} \varepsilon_{ij} \frac{dx_i}{ds}\frac{dx_j}{ds} = \sum_{i=1}^{3}\sum_{j=1}^{3} \varepsilon_{ij} N_i N_j$$

34

And if I now write equation for X1* which is X1 + U1 X2* is X2 + U2 so on and so forth using that if I evaluate elements of J, I get them to be in terms of displacement, gradients of displacement field to be this. So now we will look at the change in this quantity DA* square - DS square, so I can write in this form DS* square – DS/2 I can put it in this form, so what I have to do is, I have for DX* in this expression should be written in terms of U1, U2, U3 using this relation, so it involves quite a bit of an algebra at the end of which I will be able to write this quantity DA* square - DS square/2 in terms of quantity is known as epsilon IJ and DXI DXJ, and if I divide by DS I get epsilon IJ in the summoned we have DXI /DS, DXJ/DS and this is nothing but the direction cosines of the line segments, so I call them as NI, NJ, so this epsilon IJ, and NJ, what are epsilon IJ? That has to be synthesized by actually performing these

$$\frac{ds^{*2}-ds^2}{2}=\sum_{i=1}^{3}\sum_{j=1}^{3}\varepsilon_{ij}\,dx_i\,dx_j$$

$$\varepsilon_{11}=\frac{\partial u_1}{\partial x_1}+\frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2+\left(\frac{\partial u_2}{\partial x_1}\right)^2+\left(\frac{\partial u_3}{\partial x_1}\right)^2\right]$$

$$\varepsilon_{22}=\frac{\partial u_2}{\partial x_2}+\frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_2}\right)^2+\left(\frac{\partial u_2}{\partial x_2}\right)^2+\left(\frac{\partial u_3}{\partial x_2}\right)^2\right]$$

$$\varepsilon_{33}=\frac{\partial u_3}{\partial x_3}+\frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_3}\right)^2+\left(\frac{\partial u_2}{\partial x_3}\right)^2+\left(\frac{\partial u_3}{\partial x_3}\right)^2\right]$$

$$2\varepsilon_{12}=\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_2}$$

$$2\varepsilon_{13}=\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3}+\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_3}+\frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_3}$$

$$2\varepsilon_{23}=\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}+\frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_3}+\frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_3}$$

35

calculations and it turns out that these epsilons are given by these relations, so you immediately recognize that the quantity is shown in the black are the equations represent the linear strain displacement relations, whereas the quantities that are shown in the red are the contributions due to nonlinear effects and this is a complete set of nonlinear strain displacement relation, so

Magnification factor

$$\lambda_{PQ} = \frac{ds^{*2} - ds^2}{2ds^2} = \sum_{i=1}^{3}\sum_{j=1}^{3} \varepsilon_{ij} \frac{dx_i}{ds}\frac{dx_j}{ds} = \sum_{i=1}^{3}\sum_{j=1}^{3} \varepsilon_{ij} N_i N_j$$

= strain of the line segment PQ

- $\varepsilon = \left[\varepsilon_{ij}\right] = \left[\varepsilon_{ji}\right]$ is called the Green Lagrange strain tensor at P.
- Rule for coordinate transformation: $\varepsilon' = C\varepsilon C^t$
- If $ds = ds^* \Rightarrow MF = 0 \Rightarrow \varepsilon_{ij} = 0, i, j = 1,2,3$
- $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_\alpha}{\partial x_i}\frac{\partial u_\alpha}{\partial x_i}\right)$
- Interpretation

N_1	N_2	N_3	λ_{PQ}
1	0	0	ε_{11}
0	1	0	ε_{22}
0	0	1	ε_{33}

36

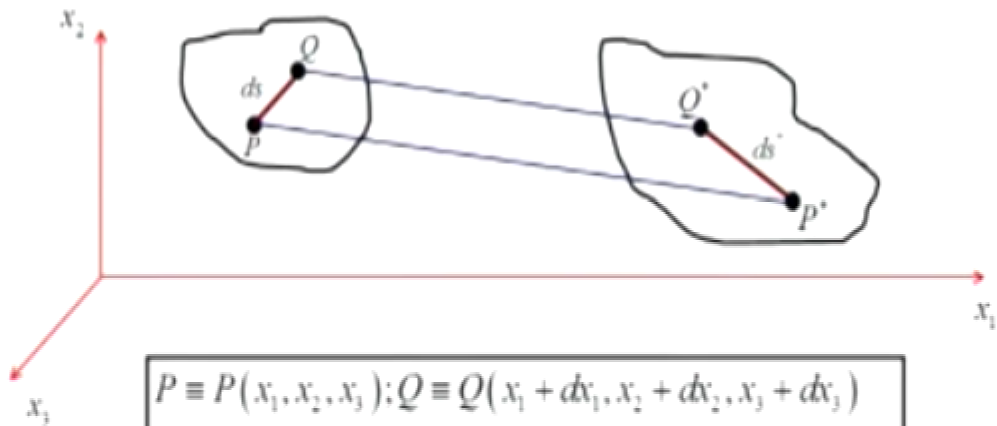
we call this quantity epsilon IJ as strain components, and the quantity DS* square - DS square/2 DS square is lambda PQ is called the strain of the line segment PQ, and that is given by epsilon IJ and NI NJ, so this quantity epsilon IJ that is a matrix, epsilon IJ is called the Green-Lagrange strain tensor at P, it is a tensor therefore the rules of coordinate transformation epsilon prime in the transform coordinate system is C epsilon C transpose.

Now if DS = D* the magnification factor lambda PQ is 0 and all epsilon IJ's will be 0, so notation we can write this in this form where repeated index imply summations and indices run from 1 to 3.

General formulation

Preliminaries: nonlinear strain-displacement relations:

Green-Lagrange strain tensor



$$P \equiv P(x_1, x_2, x_3); Q \equiv Q(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$$

$$P^* \equiv P(x_1^*, x_2^*, x_3^*); Q^* \equiv Q(x_1^* + dx_1^*, x_2^* + dx_2^*, x_3^* + dx_3^*)$$

$$PQ = ds; P^*Q^* = ds^*$$

$$x_1^* = x_1 + u_1; x_2^* = x_2 + u_2; x_3^* = x_3 + u_3$$

NPTEL

32

Now if you now take, you let us focus on this expression lambda PQ, NJ, now if I take N1 to be 1, and N2 to be 0, and N3 to be 0, on the right hand side I will be left with only one term which is epsilon 1 1, so what it means is if you are taking a line segment which is parallel to X axis here, and epsilon 1 1 represents the magnification of that line element due to deformation, that is the definition of strain that we are interested in. Similarly 0 1 0 if you take it is epsilon 2 2, Magnification factor

$$\lambda_{PQ} = \frac{ds^{*2} - ds^2}{2ds^2} = \sum_{i=1}^{3}\sum_{j=1}^{3} \varepsilon_{ij} \frac{dx_i}{ds}\frac{dx_j}{ds} = \sum_{i=1}^{3}\sum_{j=1}^{3} \varepsilon_{ij} N_i N_j$$

= strain of the line segment PQ

- $\varepsilon = \left[\varepsilon_{ij}\right] = \left[\varepsilon_{ji}\right]$ is called the Green Lagrange strain tensor at P.
- Rule for coordinate transformation: $\varepsilon' = C\varepsilon C^t$
- If $ds = ds^* \Rightarrow MF = 0 \Rightarrow \varepsilon_{ij} = 0, i, j = 1,2,3$
- $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_\alpha}{\partial x_i}\frac{\partial u_\alpha}{\partial x_j}\right)$

$$\lambda_{PQ} = \sum\sum \epsilon_{ij} N_i N_j$$

- Interpretation

N_1	N_2	N_3	λ_{PQ}
1	0	0	ε_{11}
0	1	0	ε_{22}
0	0	1	ε_{33}

36

that means say a line segment along Y axis will get amplified by epsilon 2 2, similarly the line segment along Z axis gets amplified by this.

Final direction of the line segment

Direction cosines of PQ$=N_1, N_2, N_3 = \frac{dx_1}{ds}, \frac{dx_2}{ds}, \frac{dx_3}{ds}$

Direction cosines of P*Q* $=N_1^*, N_2^*, N_3^* = \frac{dx_1^*}{ds^*}, \frac{dx_2^*}{ds^*}, \frac{dx_3^*}{ds^*}$

$$x_i^* \equiv x_i^*\left(x_1, x_2, x_3\right)$$

$$\frac{dx_i^*}{ds^*} = \frac{dx_i^*}{ds}\frac{ds}{ds^*} = N_i^*$$

$$\lambda_{PQ} = \frac{ds^{*2} - ds^2}{2ds^2} = \frac{1}{2}\left[\left(\frac{ds^*}{ds}\right)^2 - 1\right] \Rightarrow \frac{ds}{ds^*} = \frac{1}{\sqrt{2\lambda_{PQ}+1}}$$

$$\frac{dx_i^*}{ds} = \frac{d}{ds}x_i^*\left(x_1, x_2, x_3\right) = \frac{\partial x_i^*}{\partial x_1}\frac{dx_1}{ds} + \frac{\partial x_i^*}{\partial x_2}\frac{dx_2}{ds} + \frac{\partial x_i^*}{\partial x_3}\frac{dx_3}{ds}$$

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37

Now we would also like to evaluate the final direction of the line segment, so direction cosine of PQ is N1, N2, N3 which is given by this, and direction cosine of P*, Q* is N1*, N2*, N3* so now we wish to evaluate N1*, N2*, N3* in terms of strains and displacements so again we use this relation XI* star is this, and if I differentiate with respect to S* I get this and this is nothing but NI*. Now lambda PQ is given by this and from this I can manipulate and obtained DS/DS* to be given by this, so DXI*/DS therefore is obtained in this form, by writing for displacements

$$u_i = x_i^* - x_i \Rightarrow= x_i^* = u_i + x_i$$

$$\frac{dx_1^*}{ds} = \left(1 + \frac{\partial u_1}{\partial x_1}\right) N_1 + \frac{\partial u_1}{\partial x_2} N_2 + \frac{\partial u_1}{\partial x_3} N_3$$

$$\frac{dx_2^*}{ds} = \frac{\partial u_2}{\partial x_1} N_1 + \left(1 + \frac{\partial u_2}{\partial x_2}\right) N_2 + \frac{\partial u_2}{\partial x_3} N_3$$

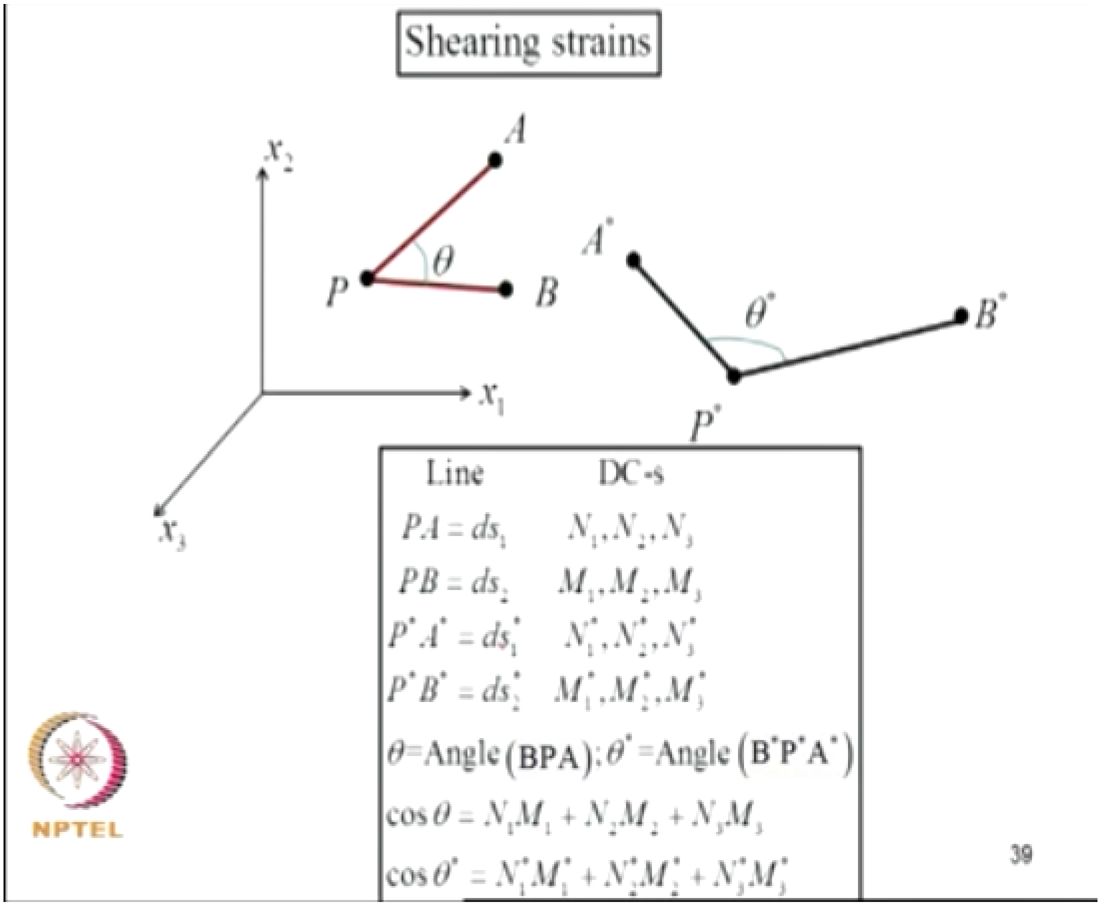
$$\frac{dx_3^*}{ds} = \frac{\partial u_3}{\partial x_1} N_1 + \frac{\partial u_2}{\partial x_3} N_2 + \left(1 + \frac{\partial u_3}{\partial x_3}\right) N_3$$

$$\begin{Bmatrix} N_1^* \\ N_2^* \\ N_3^* \end{Bmatrix} = \frac{1}{\sqrt{2\lambda_{PQ} + 1}} \begin{bmatrix} 1 + \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x3} \\ \frac{\partial u_2}{\partial x_1} & 1 + \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & 1 + \frac{\partial u_3}{\partial x_3} \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix}$$

$$N_i^* \sqrt{2\lambda_{PQ} + 1} = \left(\delta_{ij} + \frac{\partial u_i}{\partial x_j}\right) N_j$$

38

we can write the 3 equation DX1*/DS, DX2*/DS etcetera in this form, and consequently by rearranging these terms I get the direction cosines of the line segments in the deformed configuration to be given by the direction cosines in the un-deformed configuration through this matrix. This is important in the definition of shearing strain, so to define shearing strain what



we do, we consider a point P and erect 2 line segments at an angle theta and after deformation P will move to P*, A will move to A*, and B will move to B*, and the subtended angle will be theta*, so some notations PA is DS1, PB is DS2, P* A* is DS1*, P* B* is DS2*, so direction cosines of PA is N1, N2, N3, PB is M1, M2, M3 and the star quantities are for P*, B* and P* A*, theta is angle BAP, and theta * is angle B* A* P*.

$$\sqrt{2\lambda_{PA}+1}\begin{Bmatrix}N_1^*\\N_2^*\\N_3^*\end{Bmatrix} = \begin{bmatrix}1+\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x3}\\ \frac{\partial u_2}{\partial x_1} & 1+\frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3}\\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & 1+\frac{\partial u_3}{\partial x_3}\end{bmatrix}\begin{Bmatrix}N_1\\N_2\\N_3\end{Bmatrix}$$

$$\sqrt{2\lambda_{PB}+1}\begin{Bmatrix}M_1^*\\M_2^*\\M_3^*\end{Bmatrix} = \begin{bmatrix}1+\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x3}\\ \frac{\partial u_2}{\partial x_1} & 1+\frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3}\\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & 1+\frac{\partial u_3}{\partial x_3}\end{bmatrix}\begin{Bmatrix}M_1\\M_2\\M_3\end{Bmatrix}$$

$$\cos\theta^* = N_1^*M_1^* + N_2^*M_2^* + N_3^*M_3^*$$

$$\Rightarrow \sqrt{2\lambda_{PA}+1}\sqrt{2\lambda_{PB}+1}\cos\theta^* = \cos\theta + 2\varepsilon_{ij}N_iM_j$$

40

Now cos theta using this I can write it as N1 M1 + N2 M2 + N3 M3, and cos theta * is given by this, so now using the relation between the direction cosines of the line segment in the deformed configuration with those in the un-deformed configuration which we derived a while before for line, the 2 line segments I write this and if we now evaluate a cos theta* in terms of use and N1, N2, N3, it can be shown that this factor, there will be these factors so that if we absorb and take it to the cos theta * on the left hand side I get this quantity to be cos theta + 2 epsilon IJ, NI, MJ.

$$\sqrt{2\lambda_{PA}+1}\sqrt{2\lambda_{PB}+1}\cos\theta^{*}=\cos\theta+2\varepsilon_{ij}N_iM_j$$

Define $\Gamma_{AB}=\sqrt{2\lambda_{PA}+1}\sqrt{2\lambda_{PB}+1}\cos\theta^{*}$

Remarks

- Let $\theta=90^{\circ}\Rightarrow\cos\theta=1$

N_1	N_2	N_3	M_1	M_2	M_3	Γ_{AB}
1	0	0	0	1	0	$2\varepsilon_{12}=\Gamma_{12}$
1	0	0	0	0	1	$2\varepsilon_{13}=\Gamma_{13}$
0	1	0	0	0	1	$2\varepsilon_{23}=\Gamma_{23}$

- Let $\theta=90^{\circ}$ and deformation be small $\Rightarrow\theta^{*}\approx\frac{\pi}{2}$ & $\cos\theta^{*}\approx\frac{\pi}{2}-\theta^{*}$

$$\Rightarrow\sqrt{2\lambda_{PA}+1}\sqrt{2\lambda_{PB}+1}\cos\theta^{*}=\cos\theta+2\varepsilon_{ij}N_iM_j\approx\frac{\pi}{2}-\theta^{*}$$

= Traditional definition of strain

- $\sqrt{2\lambda_{PA}+1}\sqrt{2\lambda_{PB}+1}\cos\theta^{*}=\Gamma_{AB}=\cos\theta+2\varepsilon_{ij}N_iM_j$

is accepted as measure of shear strain.

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41

Now I define this quantity gamma AB as the shearing strain, so why that is sharing strain? For example if theta = 90 degrees that means the two line segments are at 90 degrees, one of the line segment is aligned with X-axis, other one is Y axis, if you compute gamma AB it turns out to be 2 epsilon 1 2, which is in the engineering, this is gamma 1 2, or gamma XY if you use XYZ notation, so similarly if you take a line segment along X axis and a line segment along Z axis, gamma AB turns out to be 2 epsilon 1 3. Now if theta is small we can approximate theta* as pi/2 and cos theta* as pi/2 – theta* and in which case gamma AB turns out to be pi/2 – theta* which is the definition of the shearing strain for small deformation. So the traditional definition of strain is recovered, so based on this we accept that gamma AB given by this quantity is a measure of shear strain, so this is some basics of Green-Lagrange strain tensor.

•All displacements are measured wrt the stationary reference coordinate system which is the original coordinate system for the structure in its undeformed configuration.

•This is true no matter how large the rotations of the stucture are.

•Green-Lagrange strain components are equal to zero for all rigid body rotations (small or large).

•The stationary coordinates are also called the material coordinates.

•This approach is called the total Lagrangian approach.

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Now a few observations, all displacements are measured with respect to the stationary reference coordinate system which is the original coordinate system for the structure in its un-deformed configuration, this is true no matter how large the rotation of the structure are. Now the Green-Lagrange strain components are equal to 0 for all rigid body motions, the rigid body motions can be small or large. The stationary coordinates are also called material coordinates. This approach to solving nonlinear problems is called total Lagrangian approach. So there are

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) + \frac{1}{2}\delta_{km}u_{k,i}u_{m,j}$$

Alternative notations for the coordinate system

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]$$

$$2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

$$2\varepsilon_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}$$

$$2\varepsilon_{23} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}$$

alternative notations in tensorial notation with repeated indices, all these relations can be compactly written in this form where repeated indices implies summation and comma implies differentiation, and in terms of XYZ coordinates we can write this expression instead of X1, X2, X3, I will write XYZ and I get this.

General formulation

Consider a 3D solid. Let an initial stress σ_0 exist in the body.

Assume that as the body strains, the stress σ_0 remains constant.

The work done as strain ε occurs is given by

$$U = \int_V \varepsilon^t \sigma_0 dV$$

$$\sigma_0 = \left\{\sigma_{xx0}, \sigma_{yy0}, \cdots, \sigma_{zx0}\right\}^t$$

$$\varepsilon = \left\{\varepsilon_{xx}, \varepsilon_{yy}, \cdots, \varepsilon_{zx}\right\}^t$$

We now use the nonlinear strain-dsiplacement relations and write

$$U = \int_V [\sigma_{xx0}\left\{\frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]\right\} + \cdots +$$

$$\sigma_{zx0}\left\{\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right\}]dV$$

Now with this brief introduction to nonlinear strain displacement relations, we will now return to the problem of a general formulation of stability in non-linear solid. So let us consider 3D solid, let an initial stress sigma naught exist in the body, okay, now assume that the body, as the body strains the stress sigma naught remains constant, the work done as strain epsilon occurs is given by epsilon transpose into sigma naught, the sigma naught is a multi-axial state of stress, in the beam problem if you recall what this assumption means? So when we analyze beam we found out the state of axial stress under this load P, and P/A was the sigma naught, sigma XX of P/A all other stress components were 0, so as a body strained this load did work on the deformation, that is what we were using in computing total potential.

Now in a 3-dimensional solid the generalization of that is there exist a multi-axial state of stress in the body and this remains constant as a body strain, thus here the axial load in the beam problem was not changing as a structure deformed, that was held constant, so the same assumption if you make it means sigma naught is constant as the body strains, so the work done is given by this. Now stress components there are 6 stress components, initial stress components, 3 normal stresses and 3 shear stresses and the strain components are 6. So now we compute the work done, this is epsilon transpose into sigma naught by including the nonlinear strain displacement relations, okay, suppose if you do that I will have to put all this now into this equation sigma XX naught will get multiplied by epsilon XX which is entirely all this term not just the first term, all these terms, so it is the interaction between sigma XX naught and these terms that create the issues that are related to buckling or loss of stability, similarly I get other terms.

$$U = \int_V \left(\sigma_{xx0} \left\{ \frac{\partial u}{\partial x} \right\} + \cdots + \sigma_{zy0} \frac{1}{2} \left\{ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right\} \right) dV +$$

$$+ \int_V \left[\sigma_{xx0} \left\{ \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial x} \right)^2 \right\} + \cdots + \sigma_{zy0} \frac{1}{2} \left\{ \frac{\partial u}{\partial y} \frac{\partial u}{\partial z} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \frac{\partial w}{\partial z} \right\} \right] dV$$

$$= U_1 + U_\sigma$$

$$U_1 = \int_V \left(\sigma_{xx0} \left\{ \frac{\partial u}{\partial x} \right\} + \cdots + \sigma_{zy0} \frac{1}{2} \left\{ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right\} \right) dV$$

leads to equivalent nodal forces due to σ_0.

$$U_\sigma = \int_V \left[\sigma_{xx0} \left\{ \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial x} \right)^2 \right\} + \cdots + \sigma_{zy0} \frac{1}{2} \left\{ \frac{\partial u}{\partial y} \frac{\partial u}{\partial z} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \frac{\partial w}{\partial z} \right\} \right] dV$$

contributes to the analysis of stability of the structure.

NPTEL

45

Now I will split this work done into a component U1 and U sigma, the U1 component is on the linear part of the strain displacement relations, and the other part we call it as U sigma is the contribution that is crucial for the analysis of stability of the structure, now this U1 if you

$$U_1 = \int_V \left(\sigma_{xx0} \left\{ \frac{\partial u}{\partial x} \right\} + \cdots + \sigma_{zx0} \frac{1}{2} \left\{ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right\} \right) dV = \int_V \varepsilon_l^t \sigma_0 \, dV$$

$$u = \sum_{i=1}^{s} N_i u_i \, ; \; v = \sum_{i=1}^{s} N_i v_i \, ; \; w = \sum_{i=1}^{s} N_i w_i$$

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [N]\{u\}_e \Rightarrow \varepsilon_l = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} N u_e = B u_e$$

$$\int_V \{u\}_e^t B^t \sigma_0 \, dV = \{u\}_e^t \int_V B^t \sigma_0 \, dV = \{u\}_e^t \int_V B^t D \varepsilon_0 \, dV \Rightarrow \text{Leads to equivalent nodal forces}$$

consider what it does there is an initial state of stress in a finite element analysis how will you take that into account, it will become simply set of equivalent nodal forces and this is what leads to that, so how to see that let us consider you want to be this, here I am retaining only the linear part of the strain displacement relations, next I will approximate UVW using finite element approximation so there are say S degrees of freedom N1, these Ns are the interpolation functions I represent like this, and by assembling UVW in a matrix N and in terms of nodal degrees of freedom I write in this form and the epsilon linear is given by this acting on Nu there is a BUE, so if you now consider U1 this will be UV transpose B transpose sigma naught DV, so this is nothing but UE transpose, B transpose, sigma naught DV, this is nothing but now for sigma naught I will write D into epsilon naught, where D is a matrix of elastic constants, this is essentially the equivalent nodal forces due to presence of an initial stress, so this becomes a load vector that has to be handled.

$$\delta = \left\{ \frac{\partial u}{\partial x} \quad \frac{\partial u}{\partial y} \quad \frac{\partial u}{\partial z} \quad \frac{\partial v}{\partial x} \quad \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial z} \quad \frac{\partial w}{\partial x} \quad \frac{\partial w}{\partial y} \quad \frac{\partial w}{\partial z} \right\}^T$$

$$\delta = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ \frac{\partial}{\partial y} & 0 & 0 \\ \frac{\partial}{\partial z} & 0 & 0 \\ 0 & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial y} \\ 0 & 0 & \frac{\partial}{\partial z} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \Xi \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

Now how about the remaining terms that is U sigma terms, so all the terms that are present here

$$U = \int_V \left(\sigma_{xx0} \left\{ \frac{\partial u}{\partial x} \right\} + \cdots + \sigma_{zx0} \frac{1}{2} \left\{ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right\} \right) dV +$$

$$+ \int_V \left[\sigma_{xx0} \left\{ \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial x} \right)^2 \right\} + \cdots + \sigma_{zx0} \frac{1}{2} \left\{ \frac{\partial u}{\partial y} \frac{\partial u}{\partial z} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \frac{\partial w}{\partial z} \right\} \right] dV$$

$$= U_1 + U_\sigma$$

$$U_1 = \int_V \left(\sigma_{xx0} \left\{ \frac{\partial u}{\partial x} \right\} + \cdots + \sigma_{zx0} \frac{1}{2} \left\{ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right\} \right) dV$$

leads to equivalent nodal forces due to σ_0.

$$U_\sigma = \int_V \left[\sigma_{xx0} \left\{ \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial x} \right)^2 \right\} + \cdots + \sigma_{zx0} \frac{1}{2} \left\{ \frac{\partial u}{\partial y} \frac{\partial u}{\partial z} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \frac{\partial w}{\partial z} \right\} \right] dV$$

contributes to the analysis of stability of the structure.

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are gradients dou U/dou X, dou V by you know dou W/dou X etcetera, etcetera, so I declare

$$\delta = \left\{ \frac{\partial u}{\partial x} \quad \frac{\partial u}{\partial y} \quad \frac{\partial u}{\partial z} \quad \frac{\partial v}{\partial x} \quad \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial z} \quad \frac{\partial w}{\partial x} \quad \frac{\partial w}{\partial y} \quad \frac{\partial w}{\partial z} \right\}^T$$

$$\delta = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ \frac{\partial}{\partial y} & 0 & 0 \\ \frac{\partial}{\partial z} & 0 & 0 \\ 0 & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial y} \\ 0 & 0 & \frac{\partial}{\partial z} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \Xi \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

NPTEL

47

this gradients of UVW with respect to XYZ, the 9 coordinates at delta, so this delta is related to UVW through this matrix, okay, so this I call it as capital XI UVW, where capital XI is this a matrix, this is similar to the matrix that we formulate when we write the elastic you know

$$U_\sigma = \int_V \left[\sigma_{xx0}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right\} + \cdots + \sigma_{zx0}\frac{1}{2}\left\{\frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right\}\right] dV$$

$$\delta = \Xi u$$

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [N]\{u\}_e \Rightarrow \delta = \Xi[N]\{u\}_e = [G]\{u\}_e$$

$$U_\sigma = \frac{1}{2}\int_V \delta^t \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} \delta\, dV \text{ with } s = \begin{bmatrix} \sigma_{xx0} & \sigma_{xy0} & \sigma_{xz0} \\ \sigma_{yx0} & \sigma_{yy0} & \sigma_{yz0} \\ \sigma_{xz0} & \sigma_{yz0} & \sigma_{zz0} \end{bmatrix}$$

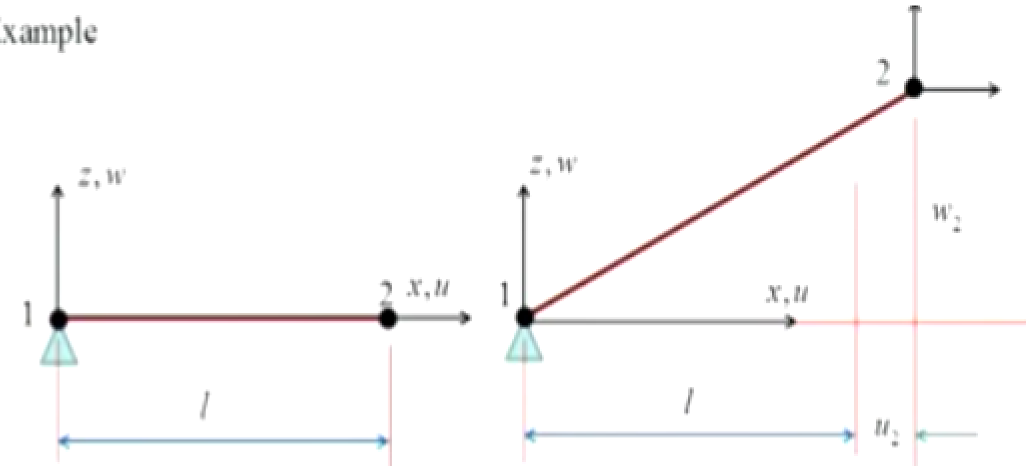
$$U_\sigma = \frac{1}{2}\int_V \{u\}_e^t [G]^t \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} [G]\{u\}_e\, dV = \{u\}_e^t K_\sigma \{u\}_e$$

$$\text{with } K_\sigma = \frac{1}{2}\int_V [G]^t \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} [G]\, dV = \text{Geometric stiffness matrix}$$

48

stiffness or in this case the effect of initial stress on linear part of strain displacement relation, so now I have U sigma to be this, and I am now making the substitution delta is capital XI into U, and again UVW I will interpolate in terms of set of interpolation functions, and nodal coordinates and delta consequently becomes capital XI into N into UV, this matrix capital XI into N I call it as capital G, so this is G into UE, this is exactly similar to B into UE, whereas B was this matrix of these operators, delta I have here capital XI to be this, so consequently or this complicated expression can be put neatly in this form, U sigma is 1/2 delta transpose into this matrix of initial stresses delta DV, S is a stress matrix due to initial stresses, so U sigma is this. Now for delta I will now make this substitution G into UV and I will get U sigma as you UE transpose K sigma UE, and K sigma is this matrix, this G transpose into this stress matrix into GDV, this is the geometric stiffness matrix in this general form, so when you have to apply this theory to specific structural elements you have to simply identify this G matrix, so for example if you are dealing with Euler-Bernoulli beam or Timoshenko beam etcetera you will know assume displacement form from which you will be able to derive the strain displacement relations and you will have to evaluate K sigma with that, so let us consider a simple example

Example



Consider an axially deforming rod in the xz plane.

Initial stresses: all components zero, except, σ_{xx0}.

$$u(x) = N_1 u_1 + N_2 u_2$$

$$w(x) = N_1 w_1 + N_2 w_2$$

$$N_1 = \frac{l-x}{x}; N_2 = \frac{x}{l}$$

NPTEL

49

of a axially deforming bar, suppose I have axially deforming bar in XZ plane so U and W are the displacement fields, so this is the un-deformed configuration and this is a deformed configuration, so this point now we will have U2, W2, as the degrees of freedom, it is not necessary that this should deform in its own axis, along its own axis.

Initial stresses we will assume that in this case all components will be 0 except sigma XX naught, now U and W I approximate using, there are 2 nodes and linear interpolation function will use N1 is L – X/L and N2 is X/L, so this is the representation for the field variables, there are 2 field variables U and W. Now G matrix in this case turns out to be this, and you can do

$$[G] = \frac{1}{l}\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

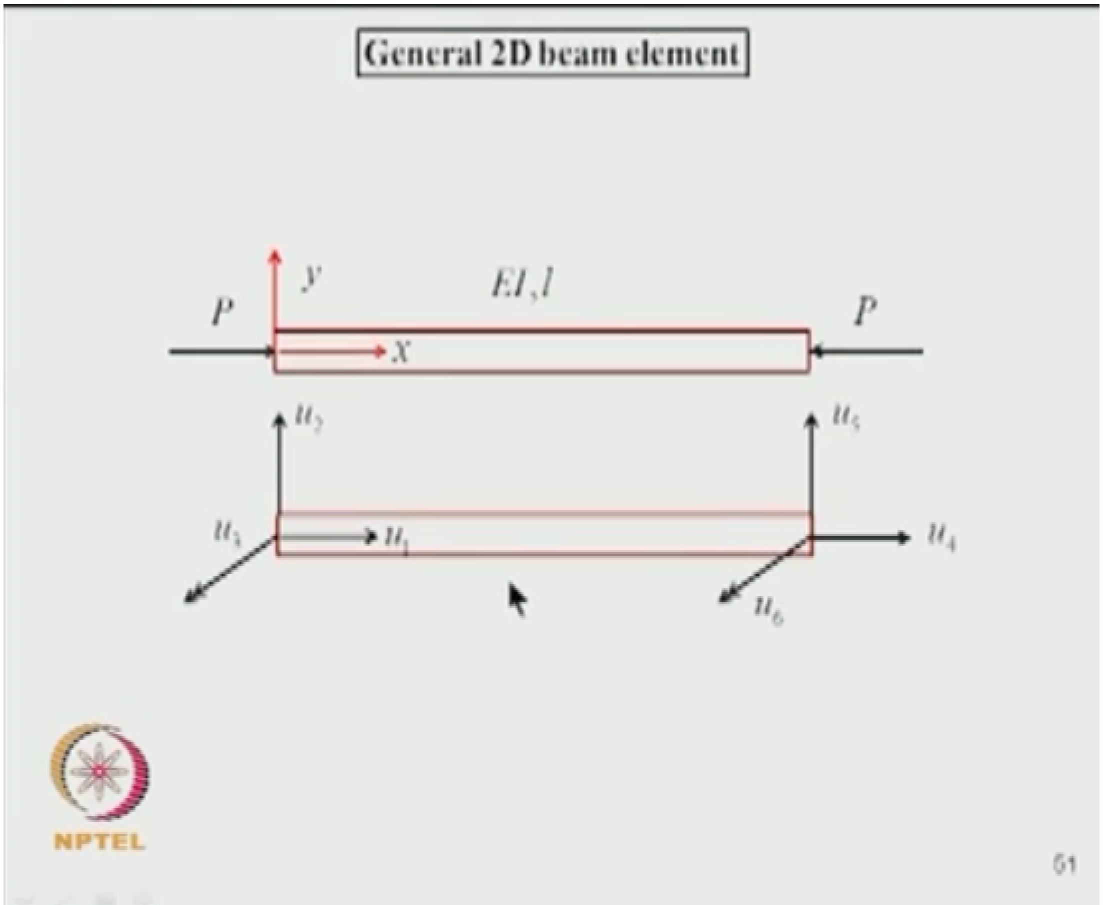
$$K_\sigma = \int_0^l [G]^t \begin{bmatrix} \sigma_{xx0} & 0 \\ 0 & \sigma_{xx0} \end{bmatrix} [G] A dx = \frac{P}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \text{ with } P = A\sigma_{xx0}$$

If $AE >> P$ it can be shown that

$$K_\sigma = \frac{P}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

this simple calculation and find out this integral of G transpose S into G ADX and it turns out that A is, and the GK sigma matrix is given by this, so this is the geometric stiffness associated with axial deformation where P is A into sigma XX naught.

Now if AE is taken to be much greater than P we can show that this matrix reduces to this, so if you see textbooks this K sigma for axial deforming bar sometimes will be given in this form, sometimes in this form, both are valid if you accept this assumption, otherwise this is a more general form which we can use.



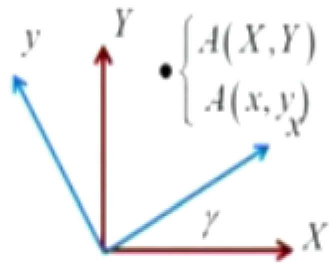
Now we will consider a general 2D beam element and this is a beam element carrying an axial load, and this is a 2 noded beam element with 3 degrees of freedom at each node U1 and U4 are the actual degrees of freedom, and U2, U3, U5, U6 are the flexural degrees of freedom, following the procedure that we have developed it can be shown that the geometric stiffness

$$K_G = \frac{P}{l}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & \frac{6}{5} & \frac{l}{10} & 0 & -\frac{6}{5} & \frac{l}{10} \\ 0 & \frac{l}{10} & \frac{2l^2}{15} & 0 & -\frac{l}{10} & -\frac{l^2}{30} \\ -1 & 0 & & 1 & 0 & 0 \\ 0 & -\frac{6}{5} & -\frac{l}{10} & 0 & \frac{6}{5} & -\frac{l}{10} \\ 0 & \frac{l}{10} & -\frac{l^2}{30} & 0 & -\frac{l}{10} & \frac{2l^2}{15} \end{bmatrix} \end{array}$$

62

matrix for this type of element can be shown to be given by this matrix, I leave this as an exercise, the kind of context in which this element becomes useful is depicted here, for example if you consider a member like this it is subjected to a combined action of axial deformation as well as bending, and when we formulate the geometric stiffness matrix for this type of situation we use this matrix, the other steps involving coordinate transformation assembly etcetera remains similar to what we have already done in previous occasions.

Transformation from local to global coordinate system



$$x = X\cos\gamma + Y\sin\gamma$$

$$y = -X\sin\gamma + Y\cos\gamma$$

$$\underbrace{\begin{Bmatrix} x \\ y \end{Bmatrix}}_{\text{Local}} = \begin{bmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix} \underbrace{\begin{Bmatrix} X \\ Y \end{Bmatrix}}_{\text{Global}}$$

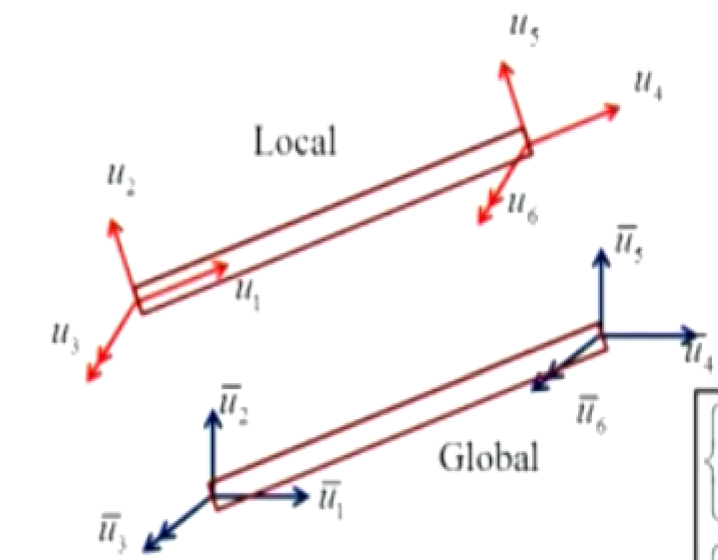
$$m = \cos\gamma; n = \sin\gamma$$

$$\underbrace{\begin{Bmatrix} x \\ y \end{Bmatrix}}_{\text{Local}} = \begin{bmatrix} m & n \\ -n & m \end{bmatrix} \underbrace{\begin{Bmatrix} X \\ Y \end{Bmatrix}}_{\text{Global}} = C \underbrace{\begin{Bmatrix} X \\ Y \end{Bmatrix}}_{\text{Global}}$$

$$C = \begin{bmatrix} m & n \\ -n & m \end{bmatrix} \quad \text{Note: } C^t = C^{-1}$$

53

The rules for coordinate transformation and assembly all that is similar so this we have seen already I will flash these slides you can go back to the earlier lectures for details, so this is how



$$\begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix} = \begin{bmatrix} m & -n \\ n & m \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix};$$

$$\begin{Bmatrix} \bar{u}_4 \\ \bar{u}_5 \end{Bmatrix} = \begin{bmatrix} m & -n \\ n & m \end{bmatrix} \begin{Bmatrix} u_4 \\ u_5 \end{Bmatrix};$$

$$\begin{Bmatrix} \bar{u}_3 \\ \bar{u}_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_6 \end{Bmatrix}$$

54

we transform the nodal degrees of freedom in global coordinates, relate the global degrees of freedom with local degrees of freedom through this relation, and this is the transformation

$$\begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \bar{u}_4 \\ \bar{u}_5 \\ \bar{u}_6 \end{Bmatrix} = \begin{bmatrix} m & -n & 0 & 0 & 0 & 0 \\ n & m & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

$$\bar{u} = T_0^t u; T_0^t = \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix}; L = \begin{bmatrix} m & -n & 0 \\ n & m & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow u = T_0 \bar{u}$$

NPTEL

55

matrix and I write U bar as T naught transpose into U where T naught is this, etcetera, so all this we have seen, now I can substitute that in the expression for total potential and the elastic

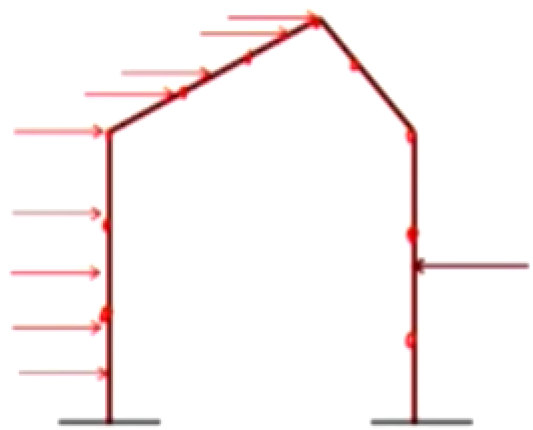
$$V(t) = \frac{1}{2}u^t(t)Ku(t) - \frac{P}{2}u^tJu$$
$$= \frac{1}{2}\bar{u}^tT_0^tKT_0\bar{u} - \frac{P}{2}\bar{u}^tT_0^tJT_0\bar{u}$$
$$= \frac{1}{2}\bar{u}^t\bar{K}\bar{u} - \frac{P}{2}\bar{u}^t\bar{J}\bar{u}$$
$$\bar{K} = T_0^tKT_0$$
$$\bar{J} = T_0^tJT_0$$

$\bar{K}$ = Element stiffness matrix in the transformed coordinate system
$\bar{J}$ = Element stability matrix in the transformed coordinate system

NPTEL

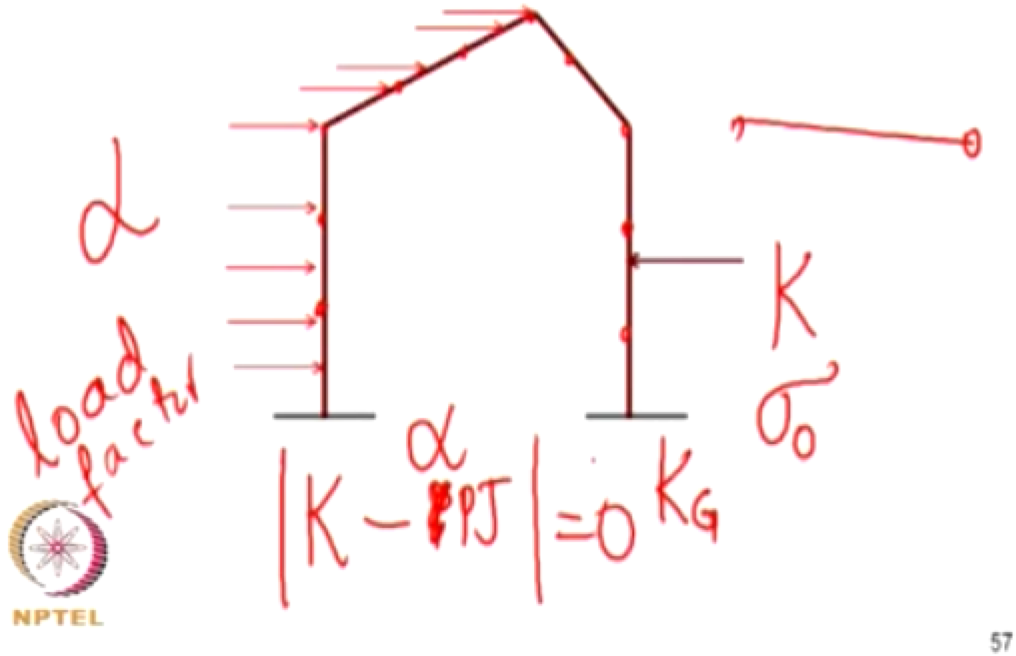
56

stiffness matrix in the global coordinate system we've already derived, similarly the stability matrix can also be derived using this formulation. So K bar and J bar are the element stiffness and stability matrices in the transform coordinate system, so this we need not again get into all the details.



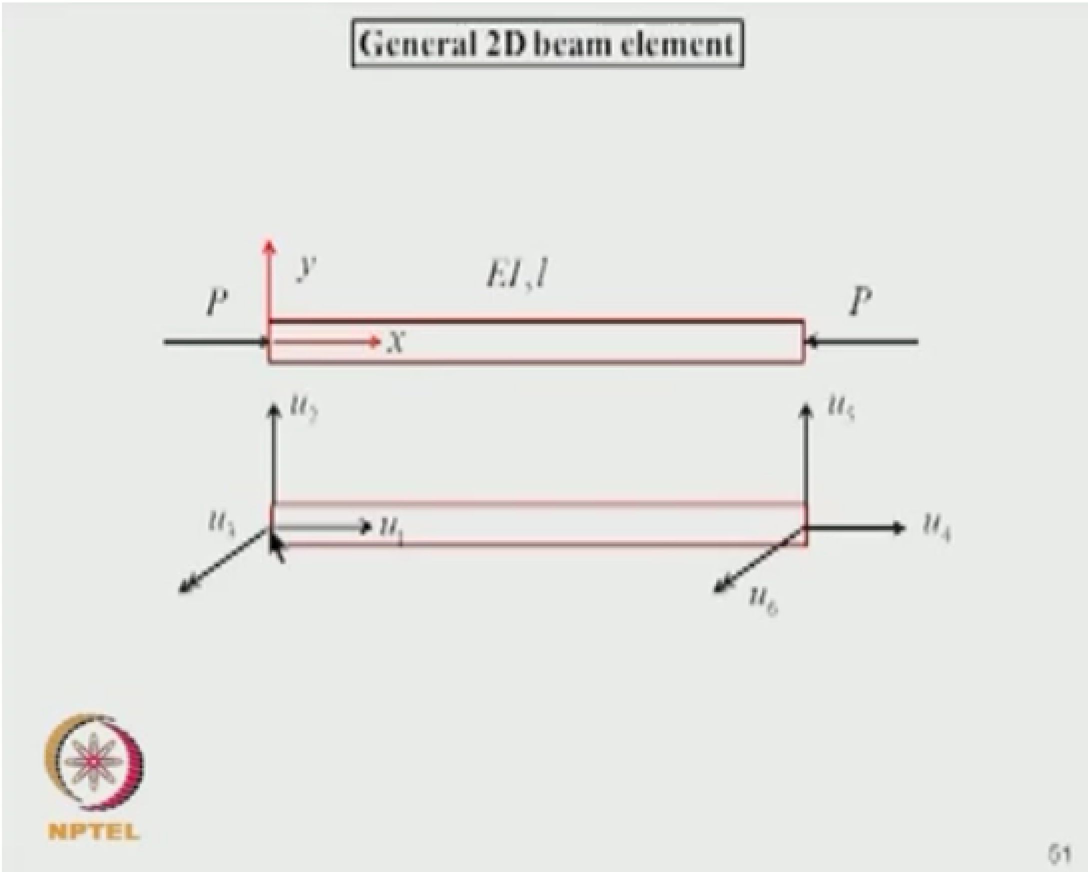
Now the next question is how do we solve this type of problems? So now we are equipped ourselves with analysis of a 2D beam element, so we can discretize this frame into different elements as shown here, and for each element I can use the 2D beam element that we have 2 noded beam element with 3 degrees of freedom per node for which I have the elastic stiffness matrix, and the geometric stiffness matrix, so first thing what we have to do is we have to perform a elastic stress analysis and find out the axial thrust in the structure, so the first step would be don't include geometric stiffness and analyze this problem, you will find the initial stress, that is what we have to first determine. Once the initial stress is determined for every element I know the state of initial stress, then the question that we are asking is this loading pattern that exists will be now multiplied by a factor alpha, and as this alpha is increased the initial state of stress is not, I mean assumed to be constant, so the question that we are asking is as the structure it strains the initial stress won't change, that is the assumption, we want to find out the value of alpha at which the structure as a whole loses its stability, so this alpha is known as load factor.

So the procedure for this would be to first of course we have to assemble the K matrix and perform the elastic stress analysis, and find out the initial state of stress then you formulate KG matrix then we have the eigenvalue problem to be solved, and we evaluate this, if those values, actually this is alpha, so I will write this in this form this number alpha that needs to be



increased so that this quantity becomes 0, the critical value of alpha at which structure loses its stability, so what we will do is we will close the lecture, this lecture at this stage, in the next class we will consider some details about this.

Now following this we need to consider how this formulation can be extended to 3 dimensional



situations, now one way to assume that is when we developed a general 3D beam element assuming that cross section is symmetric, what we did was we analyzed each of the mode of deformation separately, axial deformation separately, twisting along the longitudinal axis differently, bending in one plane and bending in other plane these four entities were analyzed separately and they were all subsequently assembled, so there is no qualitatively new feature associated with the built-up stiffness matrix, it was only that the amount of computational effort increased, there is no new concept that entered our formulation, but whereas when we come to

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) + \frac{1}{2}\delta_{km}u_{k,i}u_{m,j}$$

Alternative notations for the coordinate system

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]$$

$$2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

$$2\varepsilon_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}$$

$$2\varepsilon_{23} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}$$

NPTEL

doing the same problem for geometric stiffness analysis that is not so, it is not just that the axial loads interact with bending the other interaction can also take place, the initial state of stress is multi-axial, so in the initial state of stress there are torques, there are axial stresses, there are two bending moments, so there are 4 forms of initial you know stress resultants in the system, they interact, okay, so we need to include that I mean that is fairly obvious that is not going to be simple by looking at these expressions, there is complex interaction between sigma XX naught and these slopes it is not just that the axial load interacts with bending, so depending on to what extent we are able to capture these interactions, there are various forms of non-linear analysis possible, and to be able to perform a reasonable you know stability analysis we should capture at least a few important interactions and we will see some of these details in the next lecture, and that discussion on beams will be followed up by discussion on plates, and let us see how it goes, we will conclude this lecture at this stage.

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