Course Title

Finite element method for structural dynamic
And stability analyses

Lecture – 03
Rayleigh Ritz method and method of weighted residues

By
Prof. CS Manohar
Professor
Department of Civil Engineering

Indian Institute of Science,
Bangalore-560 012
India

We have been discussing approximate methods for vibration analysis as a build-up to developing the finite element method, so in today's lecture we will be talking about what is known as Rayleigh's Ritz method and method of weighted residuals. Before we get into the

Finite element method for structural dynamic and stability analyses

Module-1 Approximate methods and FEM

Lecture-3
Rayleigh Ritz method and method of weighted residuals

Prof C S Manohar
Department of Civil Engineering
IISc, Bangalore 560 012 India

Recall

Exact solutions to PDE-s with inhomogeneous coefficients is in general not possible even for linear systems.

Approximate solutions
- Natural frequencies and mode shapes
- Forced response characteristics

A prelude to developing FEM

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details we will quickly recall what we have been doing exact solutions to partial differential equations with inhomogeneous coefficients is in general not possible even for linear systems, so approximate solutions are invariably needed, right now we are discussing approximate solution strategies for finding natural frequencies and mode shapes, and we will now move to forced response characteristics as well and this discussion is a prelude to the development of finite element method.

Rayleigh's quotient

Discrete MDOF system

$$R(u) = \frac{u^t K u}{u^t M u}$$

- $R(u)$ has units of $(\text{rad/s})^2$.

- If $u = \phi_n \Rightarrow R(u) = \frac{\phi_n^t K \phi_n}{\phi_n^t M \phi_n} = \omega_n^2$.

- $\omega_1^2 \leq R(u) \leq \omega_N^2$

- $u = \phi_n + \varepsilon y \Rightarrow R(u) = \omega_n^2 + O(\varepsilon^2)$

Rayleigh's principle

Axially vibrating bar

$$R[\phi(x)] = \frac{\int_0^L AE(x)\phi'^2(x)\,dx}{\int_0^L m(x)\phi^2(x)\,dx}$$

Euler Bernoulli beam

$$R[\phi(x)] = \frac{\int_0^L EI(x)\phi''^2(x)\,dx}{\int_0^L m(x)\phi^2(x)\,dx}$$

$$\omega_1^2 \leq R[\phi(x)]$$

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So we introduced during the previous lectures a quantity known as the Rayleigh's quotient and for a discrete multi-degree freedom system we showed that Rayleigh's quotient U, U is a N cross 1 vector, where N is a degree of freedom, is given by U transpose KU divided by U transpose MU, this quantity is known as Rayleigh's quotient. This has units of radian per second a whole square and if this vector U coincides with the nth eigenvector of the system then Rayleigh's quotient becomes exactly equal to the nth natural frequency of the system, or the nth eigenvalue which is square of the nth natural frequency, we can also show that the Rayleigh's quotient is bounded between the first eigenvalue and the last eigenvalue, moreover we have also shown that if the vector U is in the neighborhood of the true eigenvector Phi N, that is Phi N + epsilon Y then we can show that the Rayleigh's quotient will be in the neighborhood of the natural frequency with an error which is order epsilon square.

So we can also show that as you vary U, R(u) reaches stationary values in the neighborhood of true eigenvectors of the system and it reaches its minimum value when U coincide with the first eigenvector this is known as the Rayleigh's principle, the main application of Rayleigh's principle is to estimate with simple approximations to the mode shape, the first mode shape, the first natural frequency that is the main application of Rayleigh's quotient.

Now this is for discrete multi-degree freedom systems for a continuous system like actually vibrating bar or Euler Bernoulli beam we have shown that Rayleigh's quotient is now a function of the trial function Phi(x) defined over 0 to L and for a actually vibrating rod it is given by this and for Euler Bernoulli beam it is given by this, for distributed parameter systems the Rayleigh's quotient provides a bound as shown here, there is no upper bound because for

distributed parameter system the highest natural frequency tends to infinity, therefore we can place a bound on Rayleigh's quotient as well.

So again if as we vary Phi(x) whenever Phi(x) coincides with the true eigen function of the system then Rayleigh's quotient corresponds to the true eigenvalue of the system which is square of the natural frequency, and again this function reaches a stationary value whenever Phi(x) is in the neighbor, Phi(x) reaches is in the neighborhood of the true eigen function, okay and it takes a minimum value when Phi(x) is equal to the first eigen function.

Example

k = 10⁵ N/m

$k = 10^5 \, N/m$

$m = 1000 \, kg$

For the system shown in the above figure the first eigen vector is obtained as $\phi_1^t = (1.13 \quad 1.63 \quad 1.78)$ and the first natural frequency as $\omega_1 = 4.15$ rad/s. Due to a design modification the third mass is changed by 15%. Estimate the change in the natural frequency

Modified system

$$M = \begin{bmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 1.5 \times 1.15m \end{bmatrix} ; K = \begin{bmatrix} 2.5k & -1.5k & 0 \\ -1.5k & 4.5k & -3k \\ 0 & -3k & 3k \end{bmatrix}$$

Now some illustrative examples, suppose we consider a three degree freedom system with K as shown here, M as shown here, and let's assume that the kind of problem that we wish to solve is as follows for this system shown here the first eigenvector is obtained as shown here 1.13, 1.63, and 1.78, and the first natural frequency as 4.15 radian per second. Now due to a design modification of the modification the third mass is changed by about 15% that means this mass I increased by 15%. Now the question is how to estimate the change in the natural frequency, that analysis but that is not what is intended here we wish to find an approximation to the change in the natural frequency, so for this system for the modified system the mass matrix is shown here so this M3 is increased by 15% that means this will go to this and K remains the same as it was earlier. And what we'll do now is since the first eigenvector is already given to

Use $\phi' = (1.13 \quad 1.63 \quad 1.78)$ as the trial vector and estimate the Rayleigh's quotient as

$$R(\phi) = \frac{\phi' K \phi}{\phi' M \phi} = 16.10 \ (\text{rad/s})^2$$

$\Rightarrow \omega_{MOD} \approx 4.0131$

$\Rightarrow$ Estimate of the change in natural frequency=

$\big[(4.15\text{-}4.0\text{: natural frequency}$

Note: For the modified system the exact natural frequency

$s*0121$ rad/s

us we'll use this as the trial vector so, and compute the Rayleigh's quotient so for this value, for this eigenvector and with the modified mass matrix the Rayleigh's quotient turns out to be 16.10 from which I get the modified natural frequency as this.

Now so therefore the estimate of the change in natural frequency is about 3.3%, so a 15% change in mass results in about 3.3% change in natural frequency as per this formulation, for sake of reference we can note that if you were to do an exact analysis of the modified system you will get the first natural frequency as 4.0121 radian per second. So this is a one simple

Example
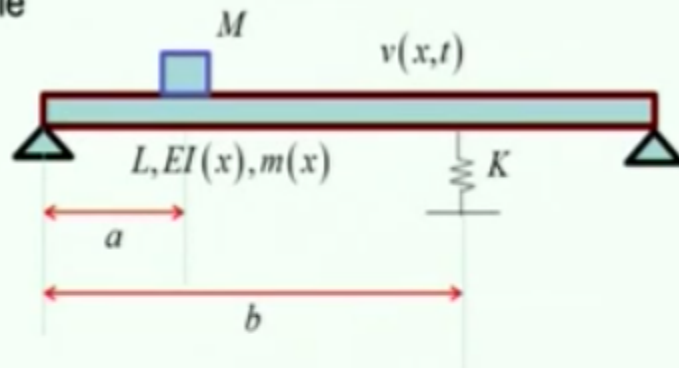
$$V(t) = \frac{1}{2}\int_0^L EI\left[v''(x,t)\right]^2 dx + \frac{1}{2}Kv^2(b,t)$$

$$T(t) = \frac{1}{2}\int_0^L m\dot{v}^2(x,t)dx + \frac{1}{2}M\dot{v}^2(a,t)$$

Normal mode oscillations: $v(x,t) = \phi(x)\cos(\omega t - \alpha)$

situation where Rayleigh's quotient can serve with some useful purpose. Now an example for a distributed parameter system so here I am considering a simply supported beam which carries a point mass M, and also it is mounted on a spring K, if this K is 0, and M is 0 we know the exact solution to the natural frequencies of the system, this is amenable for exact solution that is well known. Now the objective is to find characterize the first natural frequency of the system using Rayleigh's quotient, so we write the expression for the potential energy and the kinetic energy, this is the strain energy stored in the beam, and this energy stored in this spring, this is at X = B therefore this is 1/2 K V square B,T and kinetic energy will be, the kinetic energy stored in the beam which is this plus the kinetic energy stored in this point mass which is 1/2 MV dot square at A,T X = A now if the system undergoes normal mode oscillations it is understood that the form of the oscillation is as shown here that means all points on this structure vibrate harmonically at the same frequency, so the form of the solution will be Phi(x) into cos Omega T – alpha. So now I substitute that and compute the kinetic energy so we can get the kinetic energy as a term inside the brace and cos square Omega T – alpha, and similarly for, this is the potential energy, this is kinetic energy which is given here.

Now the maximum value of V(t) is reached when this cos square Omega T - alpha takes its maximum value and that is 1 because cosine function is, cosine function lies between plus minus 1. so cos square Omega T - alpha would lie between 0 and 1, so similarly T(t) the maximum value will be taken when this sin square Omega T - alpha is 1, so the maximum

$$V(t) = \left\{ \frac{1}{2} \int_0^L EI \left[\phi''(x) \right]^2 dx + \frac{1}{2} K \phi^2(b) \right\} \cos^2(\omega t - \alpha)$$

$$T(t) = \omega^2 \left\{ \frac{1}{2} \int_0^L m \phi^2(x) dx + \frac{1}{2} M \phi^2(a) \right\} \sin^2(\omega t - \alpha)$$

$$V_{max} = \left\{ \frac{1}{2} \int_0^L EI \left[\phi''(x) \right]^2 dx + \frac{1}{2} K \phi^2(b) \right\}$$

$$T_{max} = \omega^2 \left\{ \frac{1}{2} \int_0^L m \phi^2(x) dx + \frac{1}{2} M \phi^2(a) \right\}$$

$$R \left[\phi(x) \right] = \omega^2 = \frac{\int_0^L EI(x) \left[\phi''(x) \right]^2 dx + K \phi^2(b)}{\int_0^L m(x) \phi^2(x) dx + M \phi^2(a)}$$

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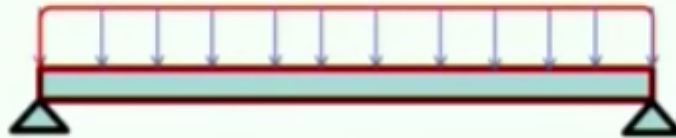potential energy is given by this, maximum kinetic energy is given by this, and since the system is conservative these two quantities must be equal to each other and we, this leads to the definition of the Rayleigh's quotient as shown here.

Now to proceed further we have to make a choice on Phi(x), this is the definition of Rayleigh's quotient for the problem, now we have to make the Rayleigh's quotient, what we can do is as I

- Mode shape of a simply supported beam with constant elastic and mass properties: $\phi(x) = \sin \dfrac{\pi x}{L}$
- $\phi(x) = x(L-x)$
- Deflected profile under an unit udl

- Choice of $\phi(x)$ determines how good $R[\phi(x)]$ is as an approximation to ω_1^2
- $\omega_1^2 \leq R[\phi(x)]$

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pointed out already if this mass is not there and if the spring is not there, we know that sin N
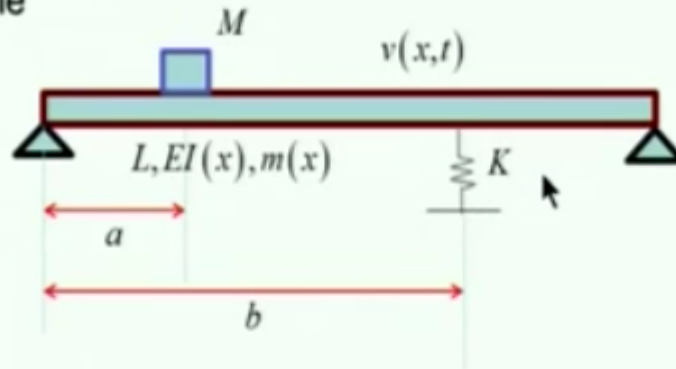
Example

$$V(t) = \frac{1}{2}\int_0^L EI\left[v''(x,t)\right]^2 dx + \frac{1}{2}Kv^2(b,t)$$

$$T(t) = \frac{1}{2}\int_0^L m\dot{v}^2(x,t)dx + \frac{1}{2}M\dot{v}^2(a,t)$$

Normal mode oscillations: $v(x,t) = \phi(x)\cos(\omega t - \alpha)$

Phi(x) by L, for N= 1, 2, 3, 4, etcetera are the set of exact eigen functions. Now for this case we can as well take the trial function to be sin Phi X by L, or I can take a polynomial like X into L – X, or I can apply UDL on this beam and find out the deflected profile and use that as the trial

•Mode shape of a simply supported beam with constant elastic and

mass properties: $\phi(x) = \sin\dfrac{\pi x}{L}$
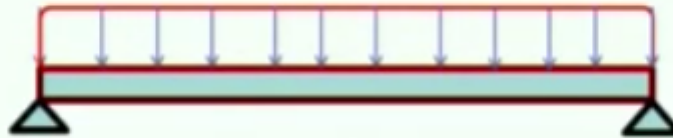
•$\phi(x) = x(L-x)$

•Deflected profile under an unit udl

•Choice of $\phi(x)$ determines how good $R\left[\phi(x)\right]$ is as

an approximation to ω_1^2

•$\omega_1^2 \leq R\left[\phi(x)\right]$

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function, so there are many options that we can follow. Now the choice of this Phi(x) determines how good this Rayleigh's quotient is as an approximation to the first natural frequency, no matter which choice you make this inequality will be honored that means Raleigh's quotient will be always greater than or equal to the square of the first natural frequency.

$$R[\phi(x)] = \omega^2 = \frac{\int_0^L EI(x)[\phi''(x)]^2 \, dx + K\phi^2(b)}{\int_0^L m(x)\phi^2(x) \, dx + M\phi^2(a)}$$

Let $EI(x) = EI; m(x) = m; \phi(x) = \sin\dfrac{\pi x}{L}$

$$\Rightarrow \omega^2 = \frac{EI\left(\dfrac{\pi^2}{L^2}\right)^2 \dfrac{L}{2} + K\sin^2\dfrac{\pi b}{L}}{\dfrac{mL}{2} + M\sin^2\dfrac{\pi a}{L}}$$

$$K = 0, M = 0 \Rightarrow \omega^2 = \frac{EI}{m}\left(\frac{\pi^2}{L^2}\right)^2 \Rightarrow \omega = 9.87\left(\frac{1}{L^2}\right)\sqrt{\frac{EI}{m}} \quad \text{(OK)}$$

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So now suppose if you take Phi(x) to be sin Phi X by L, now you get the natural frequency to be given by this you can carry out this integration, they are very simple you will get this. Now clearly when K = 0 and M = 0 the natural you know the frequency should revert back to the exact solution because the mode shape is exact in that situation, so that indeed happens we get for K = 0 and M = 0, the square of the natural frequency as this, which leads to the estimate of first natural frequency to be this, this is in fact Phi square 1 by L square by M, which is the exact solution.

$$R[\phi(x)] = \omega^2 = \frac{\int_0^L EI(x)[\phi''(x)]^2 \, dx + K\phi^2(b)}{\int_0^L m(x)\phi^2(x) \, dx + M\phi^2(a)}$$

Let $EI(x) = EI; m(x) = m; \phi(x) = x(L-x)$

$\Rightarrow \phi''(x) = -2$

$$\Rightarrow \omega^2 = \frac{4EIL + Kb^2(L-b)^2}{\dfrac{mL^5}{30} + Ma^2(L-a)^2}$$

$$\bullet M = 0, K = 0 \Rightarrow \omega = \frac{\sqrt{120}}{L^2}\sqrt{\frac{EI}{m}} = 10.95\frac{1}{L^2}\sqrt{\frac{EI}{m}}$$

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Now on the other hand if you now take the trial function to be X into L – X, so it satisfies the geometric boundary condition at X = 0, and at X = L, but if you compute the second derivative Phi double prime of X, you will see that this trial function doesn't satisfy the natural boundary condition, that is the bending moment that the supports is not 0 according to this trial function, but this doesn't matter in this application of the method because to compute Rayleigh's quotient and the highest derivative of Phi that I need is Phi double prime, so there is no problem here as far as application is concerned so if I do that this integration is straightforward and I get the Rayleigh's quotient in this case to be this, and if in this case if K = 0 and M = 0, if you compute you will get this as 10.95 into this quantity which, you know it is higher than 9.87 into the same quantity that we got when we use the exact solution.

Numerical illustration

L=3 m; E=210 GPa; ρ=7800 kg/m^3; $a = L/3$; $b = 3L/4$

Beam cross section (rectangular): width=0.1 m; depth=0.2 m

$$M = \frac{mL}{2};\ K = \frac{2EI}{L}$$

Results on the estimate of the first natural frequency (rad / s)

Beam parameters	Exact	$\phi(x) = \sin\left(\dfrac{\pi x}{L}\right)$	$\phi(x) = x(L-x)$
$M = 0, K = 0$	328.51	328.51	364.63
$M = \dfrac{mL}{2};\ K = \dfrac{2EI}{L}$	–	270.31	297.42

Now for a K not equal to 0 and M not equal to zero, to gain an understanding how this method works, we can assign some numerical values, so I have taken span to be three meters, X modulus to be 210 gigapascals or density A and B and beam cross section being rectangular with this width and depth, and mass or point mass is half of the beam mass, and stiffness of the spring is this, and if I now use 2 trial function Phi(x) = sin Phi X by L, and Phi(x) is X into L – X, for the case when M = 0 and K = 0, we get the exact solution is possible that also I have tabulated, and this trial function now as I've already said coincides with the exact solution therefore we should expect the exact solution which is as it should be. For X into L –X, I get 364.63 as an estimate for 328.51, so this is still a bound, nothing wrong but this is like as you can see this may not be an acceptable approximation.

Now for non-zero values of M and K this solution turns out to be 270.31 and this gives an answer 297.42. Now we don't know the exact solution, so now the question now arises if you are using Rayleigh's quotient how do you improve the result? Suppose if you had started with this you have got 297.42, and by some logic suppose you move to this shape function you get a much better answer, right but this is fortuitous, you use happen to know the exact mode shape therefore you are able to get this a much lesser value for Rayleigh's quotient than a simple function like this. But how do we systematically lower the Rayleigh's quotient, right so that takes us into the next question that is how to lower the value of Rayleigh's quotient, this leads to the method known as Rayleigh Ritz method.

$$R[\phi(x)] = \frac{\int_0^L EI(x)\phi''^2(x)\,dx}{\int_0^L m(x)\phi^2(x)\,dx}$$

$$\phi(x) = \sum_{n=1}^{N} a_n \psi_n(x)$$

- $\psi_n(x), n = 1, 2, \cdots, N$: a set of known linearly independent functions which satisfy all the boundary condtions.

- $a_n, n = 1, 2, \cdots, N$: a set of unknown constants which need to be determined

- Strategy: Select $a_n, n = 1, 2, \cdots, N$ such that $R[\phi(x)]$ is minimized.

Now the main objective of Rayleigh Ritz method is to lower the value of Rayleigh's quotient, so let's start with the Rayleigh quotient for the beam, I am using the beam as an example an inhomogeneous beam with flexural rigidity varying with in space, and mass per unit length varying in space, what we do is we expand the trial function in a set of linearly independent functions sin(x), okay this sin(x) are taken to be known functions, whereas these ANs are unknowns, so this sin are a set of known linearly independent function which satisfy all the boundary conditions. Let us start with this and let us revise this, revisit this question as we go along but for moment we'll assume that it satisfy all the boundary condition.

Now ANs are a set of unknown constant which need to be determined, now the strategy is now as follows now by introducing a series representation like this in my definition of Rayleigh's quotient I have introduced a set of undetermined parameters so I have a handle now to lower the Rayleigh's quotient, so the strategy that we will take is we'll minimize Rayleigh's quotient

$$R[\phi(x)] = R[a_1, a_2, \cdots, a_N] = \Omega^2 = \frac{\int_0^L EI(x)\left[\sum_{n=1}^{N} a_n \psi_n''(x)\right]^2 dx}{\int_0^L m(x)\left[\sum_{n=1}^{N} a_n \psi_n(x)\right]^2 dx} = \frac{A}{B}$$

Condtion for optimality: $\dfrac{\partial R}{\partial a_i} = 0$ for $i = 1, 2, \cdots, N$

$$\Rightarrow \frac{1}{B^2}\left(B\frac{\partial A}{\partial a_i} - A\frac{\partial B}{\partial a_i}\right) = 0 \Rightarrow \left(B\frac{\partial A}{\partial a_i} - A\frac{\partial B}{\partial a_i}\right) = 0$$

$$B\int_0^L 2EI(x)\left[\sum_{n=1}^{N} a_n \psi_n''(x)\right]\psi_i''(x)dx - A\int_0^L 2m(x)\left[\sum_{n=1}^{N} a_n \psi_n(x)\right]\psi_i(x)dx = 0$$

$$\Rightarrow \int_0^L EI(x)\left[\sum_{n=1}^{N} a_n \psi_n''(x)\right]\psi_i''(x)dx - \Omega^2 \int_0^L m(x)\left[\sum_{n=1}^{N} a_n \psi_n(x)\right]\psi_i(x)dx = 0$$

with respect to this unknown coefficients, so select A N such that this Rayleigh's quotient is minimized, so that leads to, suppose after I represent Phi(x) in terms of these N parameters, now this R now is a function of N parameters, is no longer a functional it's a set of N parameters, let us call it as capital Omega square and this is written like this.

Now the condition for optimality is Dou R by Dou AI must be equal to 0, for I = 1, 2, 3 and N, so this simple calculation can be done as follows, we can call this denominator as A, and numerator as B, so I am asked to find out Dou R by Dou AI, A is a function of A1, A2, A3, B is a function of A1, A2, A3 therefore I use the rule of differentiation, so I get 1 by B square where B is the denominator, into B into Dou A by Dou AI - A into Dou B by Dou AI, that must be equal to 0. Now since I am equating this to 0, this B square is inconsequential so I can forget that so I get the condition for optimality to be this.

Now if I divide in this case the both sides by B, I get A by B, A by B is nothing but Omega square, so this will be therefore Dou A by Dou AI into Omega square Dou B by Dou AI = 0, so that is what we are doing here, now if I want Dou A by Dou AI, I have to differentiate this with respect to AI, AI is one of the terms here so you differentiate this become 2 into this summation into sin I double prime of X, so that is what is written here. And this 2 is because of this 2, similarly I get the differentiation with respect to the denominator this is equal to 0, now I divide by B and I get omega square, and this I have to do for I = 1, 2, 3 and N, so it leads to capital N number of equations.

$$\int_0^L EI(x)\left[\sum_{n=1}^N a_n \psi_n''(x)\right]\psi_i''(x)\,dx - \Omega^2 \int_0^L m(x)\left[\sum_{n=1}^N a_n \psi_n(x)\right]\psi_i(x)\,dx = 0$$

Denote

$$K_{in} = \int_0^L EI(x)\psi_n''(x)\psi_i''(x)\,dx; \quad M_{in} = \int_0^L m(x)\psi_n(x)\psi_i(x)\,dx$$

$$\Rightarrow \sum_{n=1}^N K_{in}a_n - \Omega^2 \sum_{n=1}^N M_{in}a_n = 0, i = 1, 2, \cdots, N$$

$$\Rightarrow [K]\{a\} = \Omega^2 [M]\{a\}$$

$[K], [M]$ are $N \times N$ matrices

$(a_1 \quad a_2 \quad \cdots \quad a_N)'$ is a $N \times 1$ column

So we denote now, K IN as EI sin N double prime, sin I double prime DX, and M IN as a term here, as M(x) sin N, sin IX DX. Now the consequently this equation can be written as N = 1, K IN AN - Omega square, N = 1, M IN AN = 0, for I running from 1 to N, so this is a set of capital N number of equations in the unknowns A1, A2, A3, A capital N, so this can be put in the matrix form, I get KA = Omega square MA, so this K is a N cross N, capital N cross N square matrix, M is the square matrix of the same size and if you see here the expression for K IN and M IN, K IN is same as KNI and M IN is same as M NI that would mean K and M are symmetric, N by N matrices, so this A is N cross 1 column.

Remarks

- $Ka = \Omega^2 Ma$ represents an algebraic eigenvalue problem
- The formulation leads to a N-dof model for the system
- K = Generalized stiffness matrix; M = Generalized mass matrix
- $K = K^t; M = M^t$
- $\phi(x) = \sum\limits_{n=1}^{N} a_n \psi_n(x) \Rightarrow a_k = 0 \forall k > N \Rightarrow$ Increase in system stiffness

$\Rightarrow$ Natural frequencies are overestimated.

- If we select $\psi_n(x)$ such that $\int\limits_0^L m(x)\psi_n(x)\psi_k(x)dx = \delta_{nk} \Rightarrow M = I$

$\Rightarrow Ka = \Omega^2 a$

- The method also provides approximation to the kth eigenfunction as

$\phi_k(x) = \sum\limits_{n=1}^{N} a_{kn}\psi_n(x)$

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So this KA = Omega squared MA represents an algebraic eigenvalue problem, K we call it as generalized stiffness matrix, M as generalized mass matrix, K is symmetric, M is symmetric. Now actually to get a representation like this we should since Phi is a function of X, we should really be writing this as N = 1 to infinity. Now since we are terminating this expansion at lowercase N equal to capital N, this would mean AK = 0, for all K greater than N, so to achieve this we need to supply external forces to the system which increases the stiffness of the system, consequently the natural frequencies get overestimated.

Now we can achieve some computational simplicities if we select sin(x) to be orthogonal with respect to M(x) as shown here this is chronicle delta function, then what happens is this generalized mass matrix will be an identity matrix, so in which case as an eigenvalue problem will be the standard eigenvalue problem K = Omega squared A. Now once you find the eigenvalues you will get associated eigenvectors, and using those eigenvectors we can also construct the approximation to the eigen functions which is Phi K(x) A KNs summation or A KN sin(x), so this method, the basic aspiration was to lower the Rayleigh's quotient, but as a by-product we actually what we have done is we ended up with discretizing the system as a N degree of freedom system, and from that we are able to get approximations to not only the first natural frequency but up to the capital N number of natural frequency and associated mode shapes, so this is a byproduct of the method, so this is a way of, this can also be viewed as way of discretizing a continuous system as a multi degree freedom system, discrete multi-degree freedom system.

$$Ka = \Omega^2 Ma$$

$$\phi(x) = \sum_{n=1}^{1} a_n \psi_n(x) \Rightarrow \Omega = {}^1\Omega_1$$

$$\phi(x) = \sum_{n=1}^{2} a_n \psi_n(x) \Rightarrow \Omega = {}^2\Omega_1, {}^2\Omega_2$$

$$\phi(x) = \sum_{n=1}^{3} a_n \psi_n(x) \Rightarrow \Omega = {}^3\Omega_1, {}^3\Omega_2, {}^3\Omega_3$$

$$\vdots$$

$$\phi(x) = \sum_{n=1}^{N} a_n \psi_n(x) \Rightarrow \Omega = {}^N\Omega_1, {}^N\Omega_2, \cdots {}^N\Omega_N$$

Let $\omega_n, n = 1, 2, \cdots$ be the true (unknown) natural frequencies.

$${}^s\Omega_r \geq \omega_r, r = 1, 2, \cdots, s$$

$$\lim_{s \to \infty} {}^s\Omega_r \to \omega_r^2 \quad [\text{Eigenvalues converge from above}]$$

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Now we'll examine this solution bit more carefully, suppose if I retain only one term, so I get a single degree freedom approximation so from that model I can get only an estimate to the first natural frequency. On the other hand if I now retain two terms I will get a two degree freedom approximation therefore I will get approximation to the first natural frequency and a new information that I will get is an approximation to the second natural frequency, similarly a 3d, three term expansion will lead to estimate of first and second natural frequency and also the third natural frequency, so on and so forth so a capital N degree of freedom system will give approximation to the first N natural frequency. Now we can show that if omega N are the true but unknown natural frequencies, these omegas that is this omega that I'm showing here I have this bounding property, that means these eigenvalue converge from the above, so I will show it a numerical example what it means is as the degree of freedom increases the natural frequency approach the true natural frequency from above, how do I show that, maybe we can work through this example once again, suppose the same problem a beam with a point mass and a

Example

M

$v(x,t)$

$L, EI(x), m(x)$

K

a

b
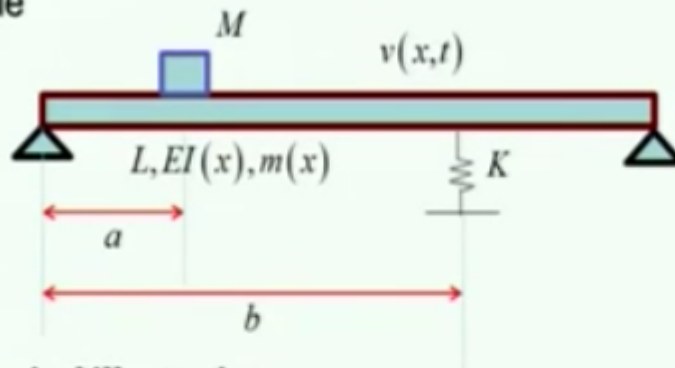
Numerical illustration

$L=3$ m; $E=210$ GPa; $\rho=7800$ kg/m^3; $a=L/3$; $b=3L/4$

Beam cross section (rectangular): width=0.1 m; depth=0.2 m

$$M = \frac{mL}{2}; \quad K = \frac{2EI}{L}$$

$$v(x) = \sum_{n=1}^{N} a_n \psi_n(x); \psi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

 spring and the numerical data as well as we had used in a previous example, so what I will do is now I will take the trial function to be a linear combination of sin N Phi X by L, okay now just again let us quickly recall when K = 0 and M = 0, sin N Phi X by L is the exact nth mode shape for the system, nth eigen function of the system, so with K and M not 0 this is no longer the exact solution but we can use that as a trial function, so that is what we are doing.

$$Ka = \Omega^2 Ma$$

$$\phi(x) = \sum_{n=1}^{N} a_n \psi_n(x); \psi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

$$K_{ij} = EI\left(\frac{i\pi}{L}\right)^4 \delta_{ij} + K \sin\left(\frac{i\pi b}{L}\right)\sin\left(\frac{j\pi b}{L}\right)$$

$$M_{ij} = \frac{mL}{2}\delta_{ij} + M \sin\left(\frac{i\pi a}{L}\right)\sin\left(\frac{j\pi a}{L}\right)$$

Estimates of the natural frequencies in rad/s

$N = 1$	270.309				
$N = 2$	266.024	1127.975			
$N = 3$	265.983	1127.917	2960.057		
$N = 4$	265.867	1124.862	2960.056	4624.599	
$N = 5$	265.846	1123.404	2960.0545	4549.745	7535.873

Now we will get now the eigenvalue problem as shown here where K IJ we can work through the steps and show that K IJ is given by this, M IJ is given by this, and delta IJ again is a chronicle delta that we can get. Now the fact that for the first two terms here we are getting chronicle delta is not surprising because sin(x) is the exact mode shape for the beam problem, so sin(x) is orthogonal to AI(x) that is sin N double Prime into sin K double prime, EI(x) integral 0 to L, DX is chronicle delta function so that is known, so that is why we are getting a chronicle deltas here, but of course K IJ and M IJ are not diagonal because of these terms involving the discrete spring and the point mass.

Now if we now take only one term I get 270.309 as approximation, if I now take two terms, I get now for the first natural frequency an estimate of 266.024 and an estimate for second natural frequency as well, so three terms I get a slightly different answer, for first natural frequency and a new information on third natural frequency so on and so forth if you continue I will listed here for first 5 modes, so for a 5 degree of freedom system these 5 numbers are respectively the estimates of the first 5 natural frequency. Now if you look at this column, you can see that it is converging from the top, similarly this is converging from the top so this happens for all the eigenvalue, so this is one of the properties of Rayleigh Ritz method provided you use linearly independent trial functions, these are very useful property.

Field equation: $\left[EI(x)v''(x,t)\right]'' + m(x)\ddot{v}(x,t) + c(x)\dot{v}(x,t) = f(x,t)$

ICS: $v(x,0) = v_0(x), \dot{v}(x,0) = \dot{v}_0(x)$

BCS: Appropriate geometric and natural BCS

We assume an approximate solution of the form

$$v(x,t) = \sum_{n=1}^{N} a_n(t)\phi_n(x)$$

$a_n(t), n = 1,2,\cdots,N$: Unknown functions of time to be determined

(generalized coordinates)

$\phi_n(x), n = 1,2,\cdots,N$: Known trial functions

Now we have been talking now till now about finding approximations to the natural frequencies and mode shapes, but we could as well ask the question now, how to get an approximation to the force response itself, so how do we approach that? So there is a class of methods known as method of weighted residuals, so for sake purpose of illustration I will consider again a beam problem, beam with inhomogeneous flexural rigidity mass per unit length and damping properties driven by a distributed loading F(x,t), now let us assume that initial conditions are initial displacement is this, initial velocity is this, and appropriate boundary conditions involving geometric and natural boundary conditions have been specified, it could be simply supported cantilever fixed, fixed so on and so forth.

Now in this class of methods the starting point is we assume the solution to be represented as a series, N = 1 to capital N, AN(t) Phi N(x), this AN(t) are the unknown functions of time to be determined we do not know that, these are called generalized coordinates, this Phi N(x) are the known trial functions. Now what conditions these should satisfy, we will see as we go along, so what I do now? I substitute this assumed form of the solution into the governing field equation and if I substitute here I will get, this is the field equation and this is assumed solution, now I

$$\left[EI(x)v''(x,t)\right]'' + m(x)\ddot{v}(x,t) + c(x)\dot{v}(x,t) = f(x,t)$$

$$v(x,t) = \sum_{n=1}^{N} a_n(t)\phi_n(x)$$

$$\Rightarrow \left[EI(x)\sum_{n=1}^{N} a_n(t)\phi_n''(x)\right]'' + m(x)\sum_{n=1}^{N} \ddot{a}_n(t)\phi_n(x) + c(x)\sum_{n=1}^{N} \dot{a}_n(t)\phi_n(x)$$

$$= f(x,t) + e(x,t)$$

$e(x,t) =$ Error or the Residue

For the exact solution $e(x,t) = 0 \, \forall t \geq 0 \, \& \, \forall x \in [0,L]$.

For an approximate solution this is not so.

Strategy:

Select the unknowns $a_n(t), n = 1, 2, \cdots, N$ such that a specified measure of total error is minimized in some sense. There is no unique sense of "minimization". Depending upon the sense of minimization, we have different methods.

have substituted here, what happens is this equation won't be exactly satisfied we don't expect that this equation will be exactly satisfied and we will be left with an error, if indeed this is an exact solution this error will be 0 for all X and T, but that is not going to, I mean that's not the situation that we are discussing, so there will be a error or this is also known as the residue.

For the exact solution this error is 0 for all times and for all X and 0 to L, for an approximate solution as I said this is not so, now what is the strategy now? We have to still select AN(t) which are not known, the strategy that we follow is we select these unknown functions such that a specified measure of the total error is minimized in some sense, we can't expect D(x,t) to be 0 for all X, suppose you fix a time T and if we can find AN so that E(x,t) = 0 for all X then there's the exact solution, but that is not what we are doing, we want to minimize a measure of this error, a measure of total error, but this measure itself has no unique interpretation, there is no unique sense of minimization because there is no unique sense of criterion of total, you know error and how to minimize it, depending upon the sense of minimization we have different methods, method known as method of least squares,

Method of least squares

$$e(x,t) =$$

$$\left[EI(x)\sum_{n=1}^{N} a_n(t)\phi_n''(x) \right]'' + m(x)\sum_{n=1}^{N} \ddot{a}_n(t)\phi_n(x) + c(x)\sum_{n=1}^{N} \dot{a}_n(t)\phi_n(x) - f(x,t)$$

Total meansquare error $\varepsilon(t) = \int_0^L e^2(x,t)dx$

Select $a_n(t), n = 1,2,\cdots,N$ such that $\dfrac{\partial \varepsilon(t)}{\partial a_n} = 0$ for $n = 1,2,\cdots,N$

$$\Rightarrow \int_0^L \frac{\partial e(x,t)}{\partial a_n} e(x,t)dx = 0 \text{ for } n = 1,2,\cdots,N$$

$$\int_0^L w_n(x,t)e(x,t)dx = 0 \text{ for } n = 1,2,\cdots,N; w_n(x,t) = \frac{\partial e(x,t)}{\partial a_n}$$

$$\Rightarrow M\ddot{a} + C\dot{a} + Ka = P(t)$$

what we do is, this is the residue which is given as shown here, and we define what is known as total meansquare error, which is epsilon of T, which is integral 0 to L, E square of x,t DX we are squaring because we don't want positive and negative errors to cancel, so we could have taken the integral of error or 0 to L, but if for certain parts of the beam error is positive and for certain other parts of the beam it is negative we get a false impression the total error is 0, so we square that and find the total error.

Now we select N(t) so that this epsilon of T is minimized with respect to A N for N = 1 to N, so leads to a set of capital N number of equations for the generalized coordinates AN(t), so if you do that so you will get 2E Dou E by Dou XT DX = 0 that 2 is irrelevant because on the right hand side is 0, so this is for 1 to M, so this can be written as in general WN x,t E (x,t) DX = 0, where WN x,t is this function, okay, for N equal to this. Now if you write all these equations you can show that the resulting equation will be of the form MA double dot + CA dot + KA = P(t). This P(t) will be of course integral of F(x,t) multiplied by this quantity and integrated 0 to L and that need not have unit of force, this function need not have units of force so the interpretation of these matrices also needs to be carefully done, and they need not have the traditional units and they may or may not be symmetric, okay, so that also we have to bear in mind.

$$e(x,t) =$$

$$\left[EI(x)\sum_{n=1}^{N} a_n(t)\phi_n''(x) \right]'' + m(x)\sum_{n=1}^{N} \ddot{a}_n(t)\phi_n(x) + c(x)\sum_{n=1}^{N} \dot{a}_n(t)\phi_n(x) - f(x,t)$$

$$\Rightarrow \frac{\partial e(x,t)}{\partial a_n} = \frac{\partial}{\partial a_n}\left[EI(x)\sum_{n=1}^{N} a_n(t)\phi_n''(x) \right]'' = \left[EI(x)\phi_n''(x) \right]''$$

$$\Rightarrow \int_0^L \left[EI(x)\phi_n''(x) \right]'' e(x,t)\,dx = 0 \text{ for } n = 1, 2, \cdots, N$$

$$\Rightarrow M\ddot{a} + C\dot{a} + Ka = P(t)$$

$$M_{ij} = \int_0^L \left[EI(x)\phi_i''(x) \right]'' m(x)\phi_j(x)\,dx \neq M_{ji}$$

$$C_{ij} = \int_0^L \left[EI(x)\phi_i''(x) \right]'' c(x)\phi_j(x)\,dx \neq C_{ji} \;;\; P_j(t) = \int_0^L \left[EI(x)\phi_j''(x) \right]'' f(x,t)\,dx$$

$$K_{ij} = \int_0^L \left[EI(x)\phi_i''(x) \right]'' \left[EI(x)\phi_j''(x) \right]'' dx = K_{ji}$$

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Now let us see if I can find the expression for that the weight, so what I am supposed to find is Dou E by Dou AN, right so Dou by Dou N of the residue you can see that it is given by EI Phi N double prime of X, entire thing differentiated twice, this is the weight function in this method, that is a WN, so the criterion for minimization of the error is given by this for N = 1, 2 N and this is independent of time, so W is although I wrote it as x,t it is actually W(x) only, so this is M IJ is given by this, C IJ is given by this, PJ is given by this, and K IJ is this, so you can see here M IJ is not same as M JI, and K IJ is of course symmetric because of the way these terms appear here, but it is not true for C IJ and P(t) is may not have units of force, okay so these are some issues that you may have to bear in mind when you are solving and interpreting this solution.

$$e(x,t) =$$

$$\left[EI(x)\sum_{n=1}^{N} a_n(t)\phi_n''(x)\right]'' + m(x)\sum_{n=1}^{N}\ddot{a}_n(t)\phi_n(x) + c(x)\sum_{n=1}^{N}\dot{a}_n(t)\phi_n(x) - f(x,t)$$

Select $a_n(t), n = 1, 2, \cdots, N$ such that $e(x,t) = 0$ for $x = x_n; n = 1, 2, \cdots, N$

$$\Rightarrow \int_0^L \delta(x - x_n)e(x,t)dx = 0 \text{ for } n = 1, 2, \cdots, N$$

$$\int_0^L w_n(x,t)e(x,t)dx = 0 \text{ for } n = 1, 2, \cdots, N; w_n(x,t) = \delta(x - x_n)$$

$$\Rightarrow M\ddot{a} + C\dot{a} + Ka = P(t)$$

$x_1 \quad x_2 \qquad \cdots \qquad x_N$

NPTEL

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In another strategy what is known as method of collocation, what we do is we again look at this residue which is as E (x,t) which is this, we select this AN(t) such that this error is 0 for a given T, this error is 0 at N chosen points along the beam that means I select X1, X2, XN and demand that at all these capital N number of points error is exactly equal to 0, so that requirement leads to capital N number of equations for the generalized coordinates and that is the governing equation.

Now we can see that this is equivalent to taking the weight function to be a direct delta function, so this is same as direct delta of X - XN E(x,t) DX = 0, that means the statement that E (x,t) is 0 for XN = N, XN N = 1 to N is equivalent to this. Now that is we can write this criteria as integral 0 to L, WN(x,t) it's actually WN(x) because there is no time here this time can be deleted from this E(x,t) DX = 0, where WN is this. Now this leads to again a set of ordinary differential equations which are coupled and you can verify that these matrices will be un-symmetric and non-diagonal and this will be the equation.

Now the problem with this method is although the error is exactly 0 here we don't have an idea how small or large the error is at an X which doesn't coincide with any of these X, so this you know we don't get any idea on that so I have to be careful in using the collocation method.

$e(x,t) =$

$$\left[EI(x) \sum_{n=1}^{N} a_n(t)\phi_n''(x) \right]'' + m(x) \sum_{n=1}^{N} \ddot{a}_n(t)\phi_n(x) + c(x) \sum_{n=1}^{N} \dot{a}_n(t)\phi_n(x) - f(x,t)$$

$$\int_0^L \delta(x - x_n) e(x,t) dx = 0 \text{ for } n = 1, 2, \cdots, N$$

$$\Rightarrow M\ddot{a} + C\dot{a} + Ka = P(t)$$

$$M_{ij} = \int_0^L \delta(x - x_i) m(x) \phi_j(x) dx = m(x_i)\phi_j(x_i) \neq M_{ji}$$

$$C_{ij} = \int_0^L \delta(x - x_i) C(x) \phi_j(x) dx = C(x_i)\phi_j(x_i) \neq C_{ji}$$

$$K_{ij} = \int_0^L \delta(x - x_i) \left[EI(x)\phi_j''(x) \right]'' dx = \left[EI(x_i)\phi_j''(x_i) \right]'' \neq K_{ji}$$

$$P_i(t) = f(x_j, t)$$

So the details of the matrices are obtained here, so this is the criterion for finding the generalized coordinates and M IJ turns out to be M XI Phi J XI and which is obviously not equal to M JI and similarly C IJ is obtained as this, and K IJ is obtained as this, this is again not symmetric, PJ(t) is F(xj,t) so this is how we get this equation. A modified version of this is what

Subdomain Collocation Method

$$e(x,t) =$$

$$\left[EI(x)\sum_{n=1}^{N}a_n(t)\phi_n''(x)\right]'' + m(x)\sum_{n=1}^{N}\ddot{a}_n(t)\phi_n(x) + c(x)\sum_{n=1}^{N}\dot{a}_n(t)\phi_n(x) - f(x,t)$$

Select $a_n(t), n = 1, 2, \cdots, N$ such that

$$\int_{x_{n-1}}^{x_n} e(x,t)\,dx = 0 \text{ for } n = 1, 2, \cdots, N; x_0 = 0; x_N = L$$

$$\int_{0}^{L} w_n(x,t)e(x,t)\,dx = 0 \text{ for } n = 1, 2, \cdots, N; w_n(x,t) = \{U(x-x_n)-U(x-x_{n+1})\}$$

$$\Rightarrow M\ddot{a} + C\dot{a} + Ka = P(t)$$

$$x_0 \quad x_1 \quad x_2 \quad \cdots \quad x_{N-1} \quad x_N$$

is known as subdomain collocation method, it is somewhat similar to collocation method but the criteria is that we again divide the beam into capital N number of segments, but instead of demanding at all these points the error is 0 we demand that over each of these segments, the average error is 0, that is the total error over this segment is 0, total error over this segment is 0, so I write this as integral XN - 1 to XN, E(x,t) DX = 0 for n from 1, 2 to capital N, where X naught is 0, XN is capital L. Now this can be rewritten as integral 0 to L, WN(x,t) it is WN(x), E(x,t) DX = 0, where WN is a box function written in terms of few side step function, so for this method subdomain collocation method, the weight function is a box function, okay, so this is a modified version of collocation method.

Galerkin's method

$$e(x,t) =$$

$$\left[EI(x) \sum_{n=1}^{N} a_n(t) \phi_n''(x) \right]'' + m(x) \sum_{n=1}^{N} \ddot{a}_n(t) \phi_n(x) + c(x) \sum_{n=1}^{N} \dot{a}_n(t) \phi_n(x) - f(x,t)$$

Select $a_n(t), n = 1, 2, \cdots, N$ such that

$$\Rightarrow \int_0^L \phi_n(x) e(x,t) dx = 0 \text{ for } n = 1, 2, \cdots, N$$

$$\left[\text{The error is orthogonal to the trial functions} \right]$$

$$\int_0^L w_n(x,t) e(x,t) dx = 0 \text{ for } n = 1, 2, \cdots, N; w_n(x,t) = \phi_n(x)$$

$$M\ddot{a} + C\dot{a} + Ka = P(t)$$

There is another method known as Galerkin's method, this is going to be used often in our course, here the idea is the weight function is selected to be the trial function itself, so we are expanding V as in terms of Phi N(x) and a weight function we take it as Phi N(x) itself, so this leads to that means the error is orthogonal to the trial functions, so this leads to capital N number of equations and we will see now what is the form of this equation, but from this statement we can see that we are going to get a set of N ordinary differential equations. So how what exactly happens here? Again let us start with the residue term, so the Galerkin's statement

$$e(x,t) =$$

$$\left[EI(x)\sum_{n=1}^{N}a_n(t)\phi_n''(x)\right]'' + m(x)\sum_{n=1}^{N}\ddot{a}_n(t)\phi_n(x) + c(x)\sum_{n=1}^{N}\dot{a}_n(t)\phi_n(x) - f(x,t)$$

$$\int_0^L \delta(x-x_n)e(x,t)dx = 0 \text{ for } n = 1,2,\cdots,N$$

$$\Rightarrow M\ddot{a} + C\dot{a} + Ka = P(t)$$

$$M_{ij} = \int_0^L \delta(x-x_i)m(x)\phi_j(x)dx = m(x_i)\phi_j(x_i) \neq M_{ji}$$

$$C_{ij} = \int_0^L \delta(x-x_i)C(x)\phi_j(x)dx = C(x_i)\phi_j(x_i) \neq C_{ji}$$

$$K_{ij} = \int_0^L \delta(x-x_i)\left[EI(x)\phi_j''(x)\right]'' dx = \left[EI(x_i)\phi_j''(x_i)\right]'' \neq K_{ji}$$

$$P_j(t) = f(x_j,t)$$

is this and you can see here if you implement this and M IJ will be M(x) Phi(x) Phi G (x) DX and this is M JI, CIJ is C(x) Phi (x) Phi J(x) this is C JI, and K IJ is actually this, we can see what happens to, this is the fourth derivative present here and there is no derivative on this, we will see what happens to this term, but in any case this is the K IJ. Now PJ(t) is the forcing function, now what happens to K IJ, so now this if you integrate by parts you do it twice so

$$K_{ij} = \int_0^L \left[EI(x)\phi_i''(x)\right]'' \phi_j(x)dx$$

$$= \left\{\left[EI(x)\phi_i''(x)\right]' \phi_j(x)\right\}\Big|_0^L - \int_0^L \left[EI(x)\phi_i''(x)\right]' \phi_j'(x)dx$$

$$= \left\{\left[EI(x)\phi_i''(x)\right]' \phi_j(x)\right\}\Big|_0^L + \left\{\left[EI(x)\phi_i''(x)\right]\phi_j'(x)\right\}\Big|_0^L + \int_0^L EI(x)\phi_i''(x)\phi_j''(x)dx$$

If we take trial functions to satisfy all the boundary condtions, then the first two terms in the above expression would be zero.

$$\Rightarrow K_{ij} = \int_0^L EI(x)\phi_i''(x)\phi_j''(x)dx = K_{ji}$$

upon each integration one of these derivative passes on to the weight function, so after you integrate by parts twice you get the integral 0 to L we will have Phi I of double prime of X, Phi J double prime of X, now if we take all trial functions to satisfy all the boundary conditions then these quantities inside the brace, these two terms will be 0, because this encapsulate or the class of admissible geometric and natural boundary conditions and this will be 0, and we get K IJ to be given by this, and K IJ in this case is symmetric.

$$e(x,t) =$$

$$\left[EI(x) \sum_{n=1}^{N} a_n(t) \phi_n''(x) \right]'' + m(x) \sum_{n=1}^{N} \ddot{a}_n(t) \phi_n(x) + c(x) \sum_{n=1}^{N} \dot{a}_n(t) \phi_n(x) - f(x,t)$$

$$\int_0^L w_n(x,t) e(x,t) dx = 0 \text{ for } n = 1, 2, \cdots, N; w_n(x,t) = \phi_n(x)$$

$$\Rightarrow M\ddot{a} + C\dot{a} + Ka = P(t)$$

$$M_{ij} = \int_0^L m(x) \phi_i(x) \phi_j(x) dx = M_{ji}$$

$$C_{ij} = \int_0^L c(x) \phi_i(x) \phi_j(x) dx = C_{ji}$$

$$K_{ij} = \int_0^L \left[EI(x) \phi_i''(x) \right]'' \phi_j(x) dx$$

So in this method I am getting again equation of the form MA double dot + CA dot + K = P(t) but the advantage is M C K are symmetric, and there are further advantages I will come to that as we go along it is something to do with the demands on differentiability of the trial functions.

Initial condtions

$$v(x,0) = \sum_{n=1}^{N} a_n(0)\phi_n(x) \ \& \ \dot{v}(x,0) = \sum_{n=1}^{N} \dot{a}_n(0)\phi_n(x)$$

Collocation method

$$v(x_i,0) = \sum_{n=1}^{N} a_n(0)\phi_n(x_i) \ \& \ \dot{v}(x_i,0) = \sum_{n=1}^{N} \dot{a}_n(0)\phi_n(x_i); i = 1,2,\cdots,N$$

$$\Rightarrow \{v(x_i,0)\} = [\phi_n(x_i)]\{a(0)\} \ \& \ \{\dot{v}(x_i,0)\} = [\phi_n(x_i)]\{\dot{a}(0)\}$$

$$\{a(0)\} = [\phi_n(x_i)]^{-1}\{v(x_i,0)\} \ \& \ \{\dot{a}(0)\} = [\phi_n(x_i)]^{-1}\{\dot{v}(x_i,0)\}$$

Least squares and Galerkin

$$\{a(0)\} = \left[\int_0^L \phi_k(x)\phi_n(x)dx\right]^{-1}\left\{\int_0^L v(x,0)\phi_k(x)dx\right\}$$

$$\{\dot{a}(0)\} = \left[\int_0^L \phi_k(x)\phi_n(x)dx\right]^{-1}\left\{\int_0^L \dot{v}(x,0)\phi_k(x)dx\right\}$$

Now how do we get initial conditions, we have got now the governing equation for the generalized coordinates, we need to find A(0) and A dot (0) so how do we get that? So this is a assumed solution V(x,t) is AN(t) Phi N(x) therefore at T = 0, I get this displacement at T = 0, the velocity is given by this, so the unknowns are AN dot (0) AN(0), so this in collocation method again we can demand that the error is 0 for N points and I get these equations for I = 1 to N, and you can solve this equation to get the required initial conditions. Similarly this is the equation for the initial displacement, initial velocity, in the least squares and Galerkin methods A(0) is given by the inverse of this matrix into the integral or the initial condition as shown here, okay, so this again requires some effort to evaluate the initial conditions.

Least squares	$\displaystyle\int_0^L \frac{\partial e(x,t)}{\partial a_n} e(x,t)\,dx = 0$ for $n = 1, 2, \cdots, N$
Collocation	$\displaystyle\int_0^L \delta(x - x_n) e(x,t)\,dx = 0$ for $n = 1, 2, \cdots, N$
Galerkin	$\displaystyle\int_0^L \phi_n(x) e(x,t)\,dx = 0$ for $n = 1, 2, \cdots, N$
Subdomain collocation	$\displaystyle\int_0^L \{U(x - x_{n-1}) - U(x - x_n)\} e(x,t)\,dx = 0$ for $n = 1, 2, \cdots, N$
Petrov-Galerkin	$\displaystyle\int_0^L \psi_n(x) e(x,t)\,dx = 0$ for $n = 1, 2, \cdots, N$

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So let me summarize what we have talked now, there is a least-squares method, Collocation method, Galerkin's method, Subdomain Colocation and Petrov-Galerkin, there is one more thing known as Petrov-Galerkin, I didn't talk about it, here it is similar to Galerkin technique except that the weight functions are taken to be another set of sin(x), the sin(x) is different from Phi N(x), that technique is known as Petrov-Galerkin. So in all these approaches you can see here one of the common term that is present in all these expressions is E(x,t), what distinguishes one method from the other is what multiplies this residual term in the integrand, for least squares it is this, for Collocation it is this, for Galerkin it is Phi N(x), for subdomain collocation it is this box function, for Petrov-Galerkin it is sin(x), so all these methods

$$\underset{0}{\overset{L}{\int}} \underbrace{w_n(x)}_{\text{Weight}} \underbrace{e(x,t)}_{\text{Residue}} dx = 0 \text{ for } n = 1, 2, \cdots, N$$

Method of weighted resuidues

Choices to be made

- A set of N trial functions
- A set of N weight functions

Issues $\begin{cases} \text{Order to which the functions need to be differentiable} \\ \text{Satisfaction of boundary condtions} \end{cases}$

- Value of N

nevertheless can be put as a weight function into a residue = 0, for N = 1, 2, N, so this class of methods is known as method of weighted residues.

Now to implement these methods there are certain choices that we have to make, first set of N trial functions then a set of N weight functions we have to make, choose. Now issues, when you are selecting trial function or the weight function order to which the functions need to be differentiable and how well they satisfy the boundary conditions, and also how many terms need to be included, so we'll revisit these questions but bear in mind these questions need to be addressed, so the general format is we have the field equation and we assume this solution and

Field equation: $\left[EI(x)v''(x,t)\right]'' + m(x)\ddot{v}(x,t) + c(x)\dot{v}(x,t) = f(x,t)$

ICS: $v(x,0) = v_0(x), \dot{v}(x,0) = \dot{v}_0(x)$

BCS: Appropriate geometric and natural BCS

$$v(x,t) = \sum_{n=1}^{N} a_n(t)\phi_n(x)$$

$$M\ddot{a} + C\dot{a} + Ka = P(t); a(0), \dot{a}(0)$$

What MWR achieves?

A PDE governing the behavior of a continuous system has been replaced by an equivalent set of ODE-s (IVP-s) with a view to obtain an approximate solution.

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after applying the method of weighted residual in any one of this form I get this set of equations, of course the meaning of MCK and P would be different depending on which method you have used, so what does method of weighted residuals achieves? A partial differential equation governing the behavior of a continuous system has been replaced by an equivalent set of ordinary differential equations which are initial value problems with a view to obtain an approximate solution, so that's what we are doing. So this is a way of discretizing an infinite dimensional system by a finite dimensional system, so in fact finite element method also achieves this but in a somewhat different way as we will see shortly.

$$\left[EI(x)v''(x,t)\right]'' + m(x)\ddot{v}(x,t) = 0$$

Boundary condtions $\begin{cases} \text{Geometric (I)} \\ \text{Natural (II)} \end{cases}$

Normal mode oscillations: $v(x,t) = \phi(x)\cos(\omega t - \theta)$

$$\Rightarrow \left[EI(x)\phi''(x)\right]'' = m(x)\omega^2\phi(x) \quad \text{(III)}$$

Classification of trial functions

	Geometric BCs (I)	Natural BCs (II)	Field Eqn. (III)
Admissible	Satisfied	Not satisfied	Not satisfied
Comparison	Satisfied	Satisfied	Not satisfied
Eigenfunction	Satisfied	Satisfied	Satisfied

Now the trial functions there is a scheme of classifying trial functions depending on the extent to which they satisfy the boundary conditions, see this is a field equation for example for a vibration problem this is a field equation and we have boundary conditions, one on geometric quantities, another one that is the force quantities the bending moment and shear force, call it as set one and set two. So in normal mode oscillation this is a solution that we assume and this eigenvalue problem that we get. Now we say that a trial function is admissible if geometric boundary conditions are satisfied we need not have to satisfy the natural boundary condition nor the field equation, my field equation I mean this, by comparison functions we mean those trial functions which satisfy both geometric and natural boundary conditions but the field equation is not satisfied. On the other hand an eigen function is an exact solution to the problem therefore it satisfies the geometric boundary conditions, natural boundary conditions as well as the field equation, so as you see here the least that we should expect from a trial function that it should be admissible, okay, so we will see what is the implication of this as we go along it is also tied up with the requirements on differentiability of the trial functions, so if a function is admissible the requirements is much less whereas if it is comparison function where you need to satisfy conditions on bending moment and shear force the requirements on differentiability increases and the class of functions that we can select shrinks, so ideally we would like to deal with admissible functions so that we have a huge class of selection, functions to select from, as we go in this direction that choice of trial functions you know narrows.

Rayleigh's quotient

$$R\left[\phi(x)\right] = \frac{\int_0^L EI(x)\phi''^2(x)dx}{\int_0^L m(x)\phi^2(x)dx}$$

Trial functions must be at least admissble

and possess nonzero derivatives at least up to the 2nd order.

Now let me just go back to the methods one by one and see what type of requirements each method imposes. In the Rayleigh's quotient method I need to evaluate only the second derivative of the trial function so they must be admissible, okay, and possess nonzero derivatives at least up to the second order, okay.

Rayleigh Ritz method

$$\int_0^L EI(x)\left[\sum_{n=1}^{N} a_n \psi_n''(x)\right]\psi_i''(x)\,dx - \Omega^2 \int_0^L m(x)\left[\sum_{n=1}^{N} a_n \psi_n(x)\right]\psi_i(x)\,dx = 0$$

$$K_{in} = \int_0^L EI(x)\psi_n''(x)\psi_i''(x)\,dx; \quad M_{in} = \int_0^L m(x)\psi_n(x)\psi_i(x)\,dx$$

$$[K]\{a\} = \Omega^2[M]\{a\}$$

Trial functions must be at least admissible and

possess nonzero derivatives at least up to 2nd order.

Least squares, & Collocation

Trial functions need to be comparison functions

and possess nonzero derivatives up to 4th order.

Galerkin

Trial functions need to be comparison functions.

In Rayleigh-Ritz method again the trial function needs to be at least admissible and possess nonzero derivatives up to the second order, because if you see the implementation of Rayleigh's method I need to compute this K IJ and M IJ they don't require functions beyond second derivative, so trial functions need not have you know we are happy to use trial functions which have the only second order derivatives exist, okay, so this is requirement, whereas in least squares and collocation trial functions need to be comparison functions, okay you can verify that you need derivatives up to the fourth order. In Galerkin, the trial functions need to be comparison functions, so we will see some of this.

Assumed mode method and Lagrange's equation

Solving of $\left[EI(x) y'' \right]'' + m(x)\ddot{y} = 0 +$ appropriate BC-s and IC-s

is equivalent to minimizing $A = \int\limits_{t_1}^{t_2}\int\limits_{0}^{l} \frac{1}{2}\left\{ m\dot{y}^2(x,t) + EI\left[y''(x,t) \right]^2 \right\} dx\, dt$

Approximate solutions can be developed based on

- $\left[EI(x) y'' \right]'' + m(x)\ddot{y} = 0 +$ appropriate BC-s and IC-s $\left[\mathbf{MWR} \right]$

or by optimizing $A = \int\limits_{t_1}^{t_2}\int\limits_{0}^{l} \frac{1}{2}\left\{ m\dot{y}^2(x,t) + EI\left[y''(x,t) \right]^2 \right\} dx\, dt$

Now there is yet another approach which is slightly different from what we have been discussing that is known as Assumed mode method that employ Lagrange's equation, this is slightly different, so again let us consider for purpose of illustration the free vibration problem of an inhomogeneous beam so we have the field equation and appropriate boundary conditions and initial conditions, so we have seen that solution of this differential equation is equivalent to minimizing this functional, okay, this we have seen this Hamilton's principle, so the idea of this Assumed Mode Method is to approach the problem of minimizing the action integral at the stage of developing the approximation, that means I won't develop this partial differential equation but I will develop an approximate method based on the minimization of action integral itself, so what does that mean? That means the approximate solution can be developed either by addressing the field equation or by optimizing the action integral so either we can adopt this or
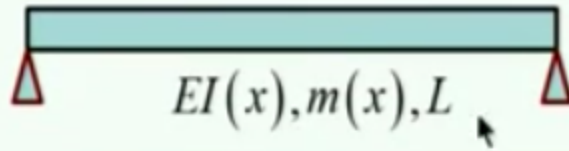
Approximate solution: $v(x,t) = \sum_{n=1}^{N} a_n(t)\phi_n(x); 0 \le x \le l$

$a_n(t); n = 1, 2, \cdots, N = $ unknown generalized coordinates

$\phi_n(x); n = 1, 2, \cdots, N = $ known trial functions; taken to satisfy BCS

For example, $\phi_n(x) = \sin\dfrac{n\pi x}{l}$

$$T(t) = \frac{1}{2}\int_0^l m(x)\left\{\sum_{n=1}^{N} \dot{a}_n(t)\phi_n(x)\right\}^2 dx$$

$$V(t) = \frac{1}{2}\int_0^l EI(x)\left\{\sum_{n=1}^{N} a_n(t)\phi_n''(x)\right\}^2 dx$$

this, so for purpose of illustration suppose if I take a beam line simply supported beam, the approximate solution is let us take it to be N = 1 to N(t) sin(x), where N(t) is set of unknown generalized coordinates and Phi N(x) are known trial function taken to satisfy all the boundary conditions, say for example Phi N(x) is sin N PI X by 1, now this is the expression for kinetic energy, this is the expression for potential energy.

$$A = \int_{t_1}^{t_2} \left(\frac{1}{2} \int_0^l m(x) \left\{ \sum_{n=1}^{N} \dot{a}_n(t)\phi_n(x) \right\}^2 dx - \frac{1}{2} \int_0^l EI(x) \left\{ \sum_{n=1}^{N} a_n(t)\phi_n''(x) \right\}^2 dx \right) dt$$

$$= \int_{t_1}^{t_2} L\left[a_1(t), a_2(t), \cdots, a_N(t), \dot{a}_1(t), \dot{a}_2(t), \cdots, \dot{a}_N(t) \right] dt$$

Governing equations: $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{a}_k} \right) - \dfrac{\partial L}{\partial a_k} = 0; k = 1, 2, \cdots, N$

$$\frac{\partial L}{\partial \dot{a}_k} = \int_0^l m(x) \sum_{n=1}^{N} \dot{a}_n(t)\phi_n(x)\phi_k(x) dx = \sum_{n=1}^{N} \dot{a}_n(t) M_{nk};$$

$$M_{nk} = \int_0^l m(x)\phi_n(x)\phi_k(x) dx$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{a}_k} \right) = \sum_{n=1}^{N} \ddot{a}_n(t) M_{nk}$$

So now I will form the Lagrange, Lagrangian for this, this is this, in terms of these generalized coordinates and their derivatives, and on this I will apply the Lagrange equation N number of times, okay, if I do this this is simple calculation we can get the mass and the stiffness, elements

$$\frac{\partial L}{\partial a_k} = -\int_0^l EI(x) \sum_{n=1}^{N} a_n(t)\phi_n''(x)\phi_k''(x) dx = -\sum_{n=1}^{N} a_n(t) K_{nk};$$

$$K_{nk} = \int_0^l EI(x)\phi_n''(x)\phi_k''(x) dx$$

$$\Rightarrow \sum_{n=1}^{N} \ddot{a}_n(t) M_{nk} + \sum_{n=1}^{N} a_n(t) K_{nk} = 0; k = 1, 2, \cdots, N$$

$$M\ddot{a} + Ka = 0; a(0) = a_0; \dot{a}(0) = \dot{a}_0$$

of mass and stiffness coefficients and we can get the equation MA double dot + KA = 0 with certain initial conditions. This K IJ that is K NK EI Phi and double prime, Phi K double prime which is symmetric in this case, similarly if you look at M NK it is symmetric, so this is similar to what we got in Galerkin's method.

$M\ddot{a} + Ka = 0; a(0) = a_0; \dot{a}(0) = \dot{a}_0$

Remarks

- M and K are $N \times N$ matrices
- $a = \begin{pmatrix} a_1(t) & a_2(t) & \cdots & a_N(t) \end{pmatrix}^t$: generalized dof-s; N=dof.
- $M = M^t; K = K^t$
- M is positive definite
- K is positive semidefinite
- $a = \begin{pmatrix} a_1(t) & a_2(t) & \cdots & a_N(t) \end{pmatrix}^t$ does not have direct physical meaning
- M and K depend upon choice of trial fucntions $\phi_n(x); n = 1, 2, \cdots, N$

Again

A PDE governing the behavior of a continuous system has been replaced by an equivalent set of ODE-s (IVP-s) with a view to obtain an approximate solution.

NPTEL

So we can see here in this case M and K are N by N matrices and these are the generalized coordinates with N degrees of freedom, both M and K are symmetric actually we can show that M is positive definite and K is positive semi-definite, one of the limitations of all the approaches including this one is that these generalized coordinates you know this vector of generalized coordinate does not have direct physical meaning, okay, they need to be substituted into a series expansion and evaluate that series then only it emerges as a quantity in which we can assign a physical meaning like displacement or velocity things like that. Now clearly this M and K depend on choice of these trial functions. So again in this approach also a partial differential equation governing the behavior of a continuous system has been replaced by an equivalent set of ODEs with a view to obtain an approximate solution.

Strong (operational) form, Weighted residual form, and weak (variational) form of governing equations

Objective : To find solutions in the form $v(x) = \sum_{n=1}^{N} a_n \phi_n(x)$

Strong (operational) form

Field equation: $\left[EI(x)v''(x) \right]'' + Pv''(x) + k(x)v(x) = f(x)$

BCS: Appropriate geometric and natural BCS

Weighted residual statement

$$\int_0^L w(x)\left[\left[EI(x)v''(x) \right]'' + Pv''(x) + k(x)v(x) - f(x) \right] dx = 0$$

Select a set of n weight functions and obtain governing equations for $a_n, n = 1, 2, \cdots, N$. Trial functions $\phi_n(x)$ $n = 1, 2, \cdots, N$ need to be differentiable up to fourth order. No such restrictions on weight functions.

Now in discussion of variational approaches to solution of partial differential equations, there is set of terminology that we should become now familiar, I will introduce them now. We talk about what is known as strong form, a weighted residual form and a weak form of the governing equations, so what they are? Now the discussion on these forms of equation is basically in the context of finding approximate solution in this form V(x) is N = 1, AN Phi(x), it is in this context that we talk about these different forms, in the strong form we use a field equation itself that means we write the governing partial differential equation and want to develop the approximate solution starting from this, okay so this field equation plus appropriate geometric and natural boundary conditions is used to develop approximations based on strong form.

In the weighted residual form what we do is, operationally what we do F(x) is taken to this side and we multiply by a weight function and demand that this integral is 0, so this is like virtual displacement instructional mechanics problems, okay, so I will come to that in due course, so the idea is we can select a set of N weight functions and obtain governing equations for the unknowns AN(t), AN in this case I am considering a static problem, it is ANs only, so the trial function Phi N(x) here you know if you are using this method you see in this method you would need V differentiated 4 times, that means you need D4 Phi N by DX 4, okay, whereas W(x) there is no such requirement, it can be, it should be integral but there is no requirements on its derivatives.

$$\Rightarrow \int_0^L w(x) \Big[\big[EI(x)v''(x) \big]'' + Pv'(x) + k(x)v(x) - f(x) \Big] dx = 0$$

$$\left\{ \left(\big[EI(x)v''(x) \big]' + Pv' \right) w(x) \right\}_0^L - \left\{ \big[EI(x)v''(x) \big] w'(x) \right\}_0^L +$$

$$\int_0^L \left\{ w''(x) \big[EI(x)v''(x) \big] + Pv'(x)w'(x) + k(x)v(x)w(x) - f(x)w(x) \right\} dx = 0$$

$$\Rightarrow F_2 w(L) - F_1 w(0) - M_2 w'(L) + M_1 w'(0) +$$

$$\int_0^L \left\{ w''(x) \big[EI(x)v''(x) \big] + Pv'(x)w'(x) + k(x)v(x)w(x) - f(x)w(x) \right\} dx = 0$$

$$M_1 = EI(0)v''(0); M_2 = EI(L)v''(L)$$

$$F_1 = \left(\big[EI(x)v''(x) \big]' + Pv' \right) @ x = 0; F_2 = \left(\big[EI(x)v''(x) \big]' + Pv' \right) @ x = L$$

Now in weak form what we do is, we carry out integration by parts of this weighted residual statement and see if I can pass on some demands on differentiability of these trial functions to the weight function, what does that mean? Suppose let us do that integration by parts I get, you do integration by parts twice and I get here if you carefully see, one of the term is W double prime X, EI(x), V double prime, this PV prime, W prime, KVW - F(x) into W(x) DX = 0 plus these boundary terms. Now we know that EI V double prime, prime is nothing but the shear force I can call it as F2, so it is W(L) this is W is here and this is shear force W(0) bending moment at the support into W1 prime of L and this, plus this, this equation is known as the weak form of the governing equation.

Now in this case if you carefully see the demand on differentiability of trial function is only up to the second order, the original equation was fourth order, now trial function need to be differentiable only up to the second order, but the weight function now need to possess derivatives up to second order, that means the demands on differentiability on V(x) the trial functions has been weakened and that has been passed on to the weight function, in that sense it is a weak form, that mean demand and continuity goes down on the trial functions, so for

Illustrations: Let the beam be simply supported at $x = 0$ and $x = L$ and be subjected to end moments M_1 and M_2.

$$F_2 w(L) - F_1 w(0) - M_2 w'(L) + M_1 w'(0) +$$

$$\int_0^L \left\{ w''(x)\left[EI(x)v''(x) \right] + Pv'(x)w'(x) + k(x)v(x)w(x) - f(x)w(x) \right\} dx = 0$$

$$M_1 = EI(0)v''(0); M_2 = EI(L)v''(L)$$

$$F_1 = \left(\left[EI(x)v''(x) \right]' + Pv' \right) @ x = 0; F_2 = \left(\left[EI(x)v''(x) \right]' + Pv' \right) @ x = L$$

$$\int_0^L \left\{ w''(x)\left[EI(x)v''(x) \right] + Pv'(x)w'(x) + k(x)v(x)w(x) - f(x)w(x) \right\} dx$$

$$- M_2 w'(L) + M_1 w'(0) = 0$$

Select $\phi(x)$ such that $v(0) = v(L) = 0 \ \& \ w(L) = w(0) = 0$

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purpose of illustration let us consider the beam be simply supported at X = 0 and X = L and it be subjected to bending moment M1 and M2, so the weak form of this equation is this, now M1 and M2 are the applied bending moments so they will substitute now for, that is M1 and M2 will appear there, but now the beam is simply supported therefore what I will do is, I will demand that this W(l) and W(0) which is a weight function must be 0 at X = 0 and X = L, so if I do that I will get the weak form in this, okay, so that means inhomogeneous boundary conditions now explicitly appear in the weak form, so now we have to select Phi(x) and weight functions so that V(0) is 0, V(l) is 0, because that is a geometric boundary conditions because a simply supported beam at X = 0 and X = L, and this weight function also need to, we take that this weight function also need to satisfy this, these need not be 0 therefore this condition will also be satisfied okay, so again let me emphasize that the trial functions are here in this approximation need to be differentiable only up to the order 2, whereas in the weighted residual form they need to be differentiable up to order 4, whereas there was no demand on weight function, but here the demand on differentiability of trial function and weight function are balanced okay, so this is the advantage of the method so if trial functions need to be only differentiable up to the second order, the pool of function from which we can select the trial function becomes very large as compared to those function which need to be differentiated up to fourth order, okay that is the main advantage of this matter.

Consider the term

$$\left\{\left(\left[EI(x)v''(x)\right]'+Pv'\right)w(x)\right\}_0^L=\left(\left[EIv''\right]'+Pv'\right)(L)\ \underbrace{w(L)}_{\substack{\text{Primary}\\ \text{Variable}=v(L)}}-$$
$$\underbrace{\left(\left[EIv''\right]'+Pv'\right)(0)}_{\text{Secondary variable}}\ \underbrace{w(0)}_{\substack{\text{Primary}\\ \text{Variable}=v(0)}}$$

Similarly

$$\left\{\left[EI(x)v''(x)\right]w'(x)\right\}_0^L=\underbrace{\left[EI(L)v''(L)\right]}_{\text{Secondary variable}}\ \underbrace{w'(L)}_{\substack{\text{Primary}\\ \text{Variable}=v'(L)}}-\underbrace{\left[EI(0)v''(0)\right]}_{\text{Secondary variable}}\ \underbrace{w'(0)}_{\substack{\text{Primary}\\ \text{Variable}=v'(0)}}$$

The dependent variable expressed in the same form as the weight function appearing in the boundary term= primary variables

Coefficients of weight functions: secondary variables

So now there are certain other terminologies that we should be aware of, so if we go back to the problem of beam on elastic foundation carrying axial load, I think this is a beam carrying axial loads we got these two terms in the variational form, so the coefficients of weight function we call them as secondary variables, okay. The coefficient of W(x) is this and this is declared as a secondary variable, the primary variable V is written in the same form as what is appearing here that is W(l) means V(l) this is a primary variable, okay, so in this case for example this will be the secondary variable, this is a primary variable, similarly at the other end the bending moment becomes a secondary variable, the primary variable is the slope, right because this is the function that multiplies the gradient of the weight function, so that coefficient of weight function or its derivative are called secondary variable. This is the primary variable is in the same form as what the weight function appears but W is replaced by V, so this is the classification, so that means the dependent variable expressed in the same form as a weight function appearing in the boundary term is known as primary variable.

Remarks

- In the weighted residual statement the trial functions $\phi_n(x), n = 1, 2, \cdots, N$ must be differentiable up to the fourth order. $w(x)$ can be any integrable function. Equations for the undetermined coefficients can be obtained by choosing a set of N weight functions.

- In the weak form we distribute the requirements on differentiabilty evenly between weight function and the trial function. Thus the continuity requirements on the trial functions is "weakened" and hence the name weak form. We have larger pool of functions to select from for the trial functions.

Now coefficients of weight functions are called secondary variable, so we'll make now a few remarks, the weighted residual statement, in the weighted residuals statement as I already said the trial functions Phi N(x) must be differentiable up to the fourth order W(x) on the other hand which is the weight function can be any integrable function, so equations for the undetermined coefficients can be obtained by choosing a set of N weight functions, so weight functions come from a very large pool, whereas trial functions come from very small, relatively smaller pool, in the weak form on the other hand we distribute the requirements on differentiability evenly between weight function and the trial function, thus the continuity requirements on the trial functions is weakened and hence the name weak form. We have a larger pool of function to select from further trial functions in this case.

Remarks (continued)

• The trial functions need to satisfy only the geometric boundary condtions as the natural boundary condtions are included in the weak form.

• Number of primary and secondary variables are the same. They appear in pairs (e.g., translation and shear force; slope and bending moment). Only one item in each pair can be specified at the boundary.

 The trial function therefore need to satisfy only the geometric boundary conditions as the natural boundary conditions are included in the weak form, for example I got a bending moment term that appeared explicitly in the weak form, right so and also we talked about primary and secondary variables, we will talk about this again in a later part of the course the number of primary and secondary variables are the same whenever we consider even ordered you know partial differential equations which often is what we do, they appear in pairs, for example translation and shear force, slope and bending moment and so on and so forth, only one item in each pair can be specified at the boundary, for example at the simply supported end we can specify displacement to be 0, we cannot say displacement and shear force to assume certain values that is not on. Similarly if you talk about slope, bending moment is 0 and you cannot say slope is also 0, okay so only 1 item in each pair can be specified at the bond.

Limitations

$$v(x,t) = \sum_{n=1}^{N} a_n(t) \phi_n(x)$$

• $\phi_n(x), n = 1, 2, \cdots, N$ are global in nature, that is, they are valid for all $x \in [0, L]$.

• Constructing these functions for simple geometries is relatively easy. Not so when geometries become more complicated.

• $a_n(x), n = 1, 2, \cdots, N$ are generalized coordinates and they do not have direct physical meaning.

Now in the methods that we have discussed so far one of the common features that we have seen is that the trial functions are valid over the entire domain of the structure being studied, that means these are globally well valid shape function, now trial functions, this is okay if you are talking about simple geometries like line elements that we are considering but for more complicated problems it is not easy to construct global trial functions, one more difficulty that we have is the generalized coordinates N(t) they do not have direct physical meaning, unless I substitute in this series and sum it up I would not know, only then I would arrive at a physically meaningful quantity, if I say A3(t) is at some time instant point naught 2, it doesn't specify much you know it is a, we cannot interpret directly in any useful way, so we would like to deal with not simply line elements we would like to deal with trusses like this,
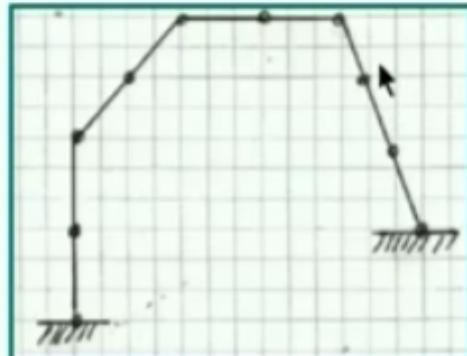
frames like this and you can imagine constructing global trial function for a trust like this would not be easy it will be impossible, similarly constructing a trial function for a much less, you know much more modest structure like this also is not easy.

So in the next class we will see how we can address these limitations and we will start talking about the development of finite element method, which aims to overcome and the difficulties that I just mentioned, so with this we'll conclude this lecture.