Indian Institute of Science
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NP-TEL
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Technology Enhanced Learning

Course Title

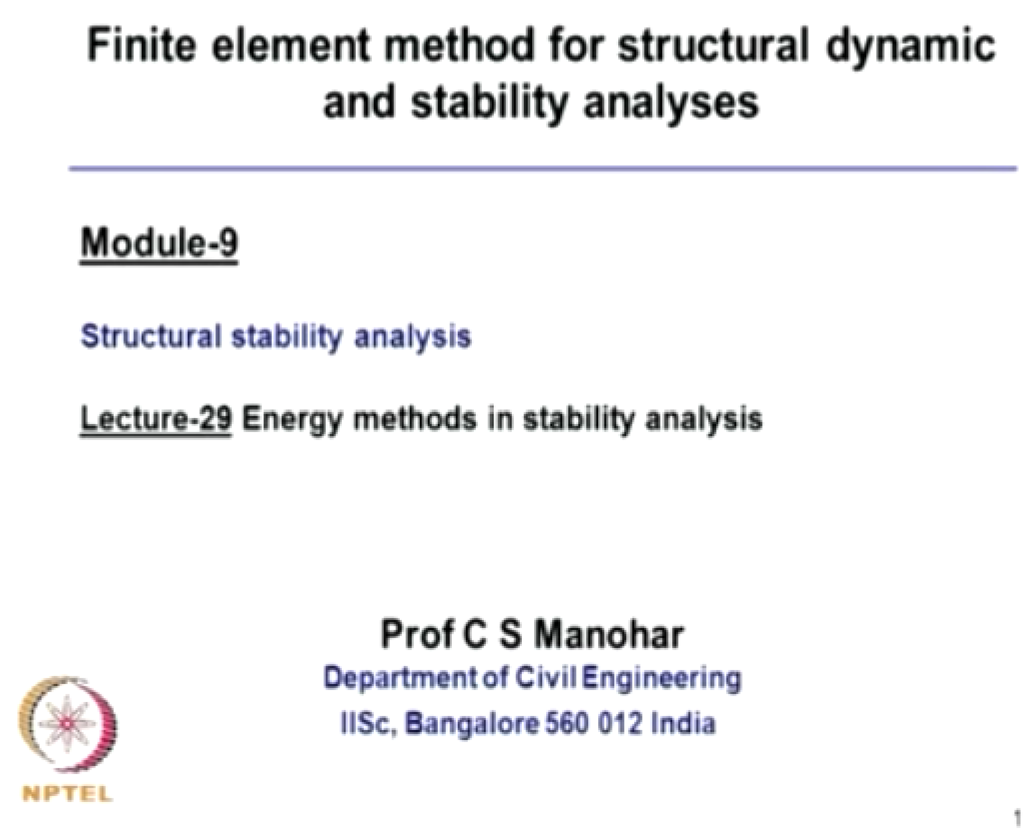
Finite element method for structural dynamic And stability analyses

Lecture – 29

Energy methods in stability analysis

By

Prof. CS Manohar
Professor
Department of Civil Engineering
Indian Institute of Science,
Bangalore-560 012
India



Towards the end of the last lecture we briefly touched upon questions related to energy methods in stability analysis, and we will be looking into those issues in detail in this lecture.

Equillibrium points or fixed points

$\dot{y}(t) = A(y); y(0) = y_0$

Points at which the system state is at rest. That is, $\dot{y}(t) = 0$.

These points are obtained as roots of the equation $A(y) = 0$

$$\ddot{x} + \omega^2 x = 0; x(0) = x_0, \dot{x}(0) = \dot{x}_0$$
$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = -\omega^2 y_1$$

Fixed point: $(0,0)$

$$\ddot{x} + 2\eta\omega\dot{x} + \omega^2 x = 0; x(0) = x_0, \dot{x}(0) = \dot{x}_0$$
$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = -2\eta\omega y_2 - \omega^2 y_1$$

Fixed point: $(0,0)$

$$\ddot{x} - \alpha x + \beta x^3 = 0; x(0) = x_0, \dot{x}(0) = \dot{x}_0$$
$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = \alpha y_1 - \beta y_1^3 = y_1(\alpha - \beta y_1^2)$$

Fixed points: $(0,0), \left(0, \pm\sqrt{\frac{\alpha}{\beta}}\right)$

Nonlinear systems have more than one fixed points.

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Before that we will quickly recall what we studied in the previous lecture, so we'll consider dynamical systems of the form Y dot = A(y) these are autonomous systems, and this is a vector of functions, so A is this is also vector of, Y is a vector N x 1 and A is a vector of functions, this is independent, time doesn't appear explicitly in this, then we say that it is an autonomous system.

We call points at which the system state is at rest, that is Y dot(t) = 0 as fixed points, they are the equilibrium points or the fixed points, and these points are obtained as roots of the equation A(y) = 0, so we saw that for an undamped single degree freedom system the origin is a fixed point, similarly for a damped single degree freedom system origin is a fixed point, for a system with negative linear elastic stiffness and a positive cubic nonlinear stiffness, we saw that the origin is a fixed point and also there are to a pair of fixed points located as shown here. So nonlinear systems have more than one fixed point the kind of examples that we studied we observed that nonlinear systems had more than one fixed point, where as linear systems of this kind exhibited only one fixed point.

Stability of equillibrium points

$\dot{x} = f(x,y); \dot{y} = g(x,y)$

Equillibrium points

$f(x^*, y^*) = 0$

$g(x^*, y^*) = 0$

$x(t) = x^* + \eta(t); y(t) = y^* + \xi(t)$

$$\begin{Bmatrix}\dot{\eta}(t)\\ \dot{\xi}(t)\end{Bmatrix} = \begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{bmatrix}\begin{Bmatrix}\eta(t)\\ \xi(t)\end{Bmatrix} \Rightarrow \begin{Bmatrix}\eta(t)\\ \xi(t)\end{Bmatrix} = \begin{Bmatrix}\alpha\\ \beta\end{Bmatrix}\exp(st) \Rightarrow \begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{bmatrix}\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix} = s\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix}$$

$s = a \pm ib$

If $a > 0$, $\lim_{t\to\infty}\begin{Bmatrix}\eta(t)\\ \xi(t)\end{Bmatrix} \to \infty \Rightarrow$ the fixed point (x^*, y^*) is unstable

If $a < 0$, $\lim_{t\to\infty}\begin{Bmatrix}\eta(t)\\ \xi(t)\end{Bmatrix} \to 0 \Rightarrow$ the fixed point (x^*, y^*) is stable

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Now we were interested in studying the stability of the motion in the neighborhood of fixed points so if X* star and Y* are the fixed points where F and G are 0, for example we consider a pair of first order equations X dot = F(x,y) and Y dot = G(x,y), the fixed points X*,Y* are obtained by solving this equation, and if we perturb the motion in the neighborhood of these fixed points, the perturbations evolve as per this equation, and we showed that the question on whether these perturbations grow in time can be answered by studying eigenvalues of this matrix and we showed that if A is greater than 0 where after doing this eigenvalue analysis for this Jacobian matrix we wrote the eigenvalues as A + - IB, and if the real part of the Eigen value is greater than 0 then motions grow in time and the fixed point X*, Y* star is unstable.
On the other hand if real part is negative the fixed point is stable, A = 0 is a case where the motion will be, perturbations will be periodic, so the question on whether fixed point is stable or unstable remains unresolved, so depending on the nature of these eigenvalues we classified

Classification of fixed points

Node

- both eigenvalues are real and are of the same sign
- the fixed point can be stable or unstable

Saddle

- both roots are real with one root positive and the other root negative
- the fixed point is unstable

Focus

- the roots are complex conjugates
- the fixed point could be stable or unstable

Center

- roots are pure imaginary
- linearized stability analysis is inadequate to answer the question on whether the fixed point is stable or unstable

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the fixed points as node, saddle, focus, and center, so in a node both eigenvalues are real and are of the same sign, so depending on the sign the fixed point can be stable or unstable. In a saddle both roots are real with one root positive and other root negative, and the fixed point is unstable because there is a positive real root. Focus, the roots are complex conjugates and the fixed point could be stable or unstable depending on the sign of the real part of the roots. Center, roots are pure imaginary and a linearized stability analysis is inadequate to answer the questions on whether the fixed point is stable or unstable, there is another way of looking at

Classification of fixed points

Robust cases :

(a) Repeller: both the eigenvalues have postive real part

(b) Attractor: both eigenvalues have negative real part

(c) Saddles: one eigenvalue is positive and the other negative

Marginal cases

(a) Centers: both eigenvalues are pure imaginary

(b) Higher order and non-isolated fixed points: at least one eigenvalue is zero.

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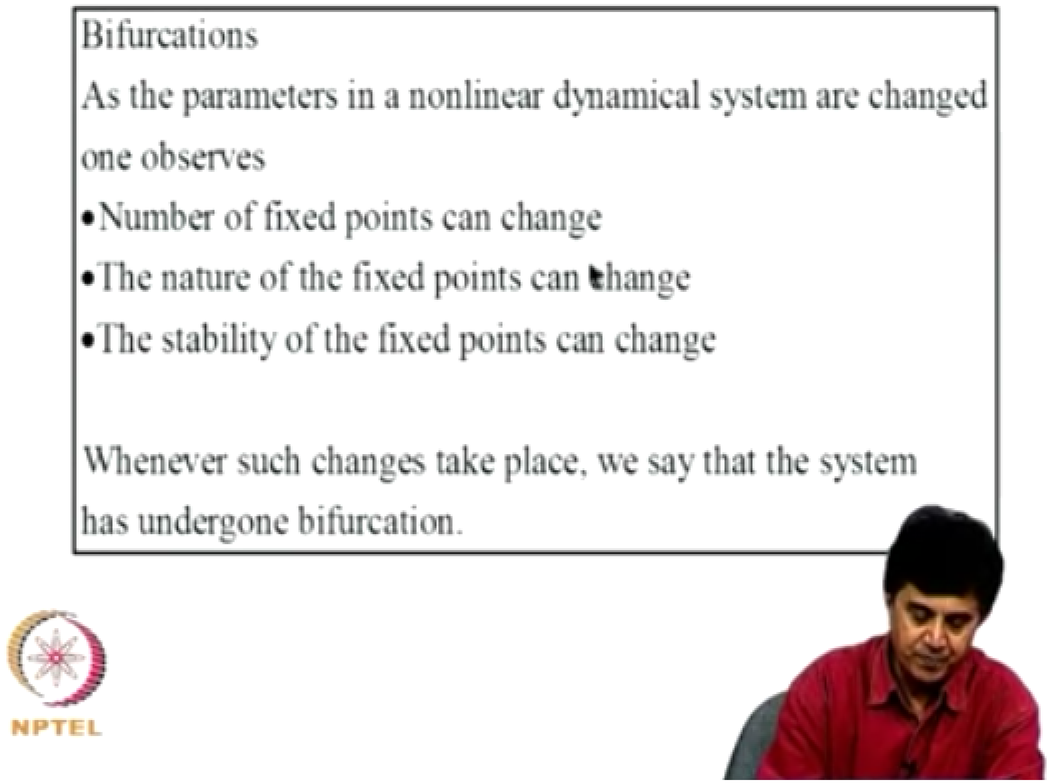
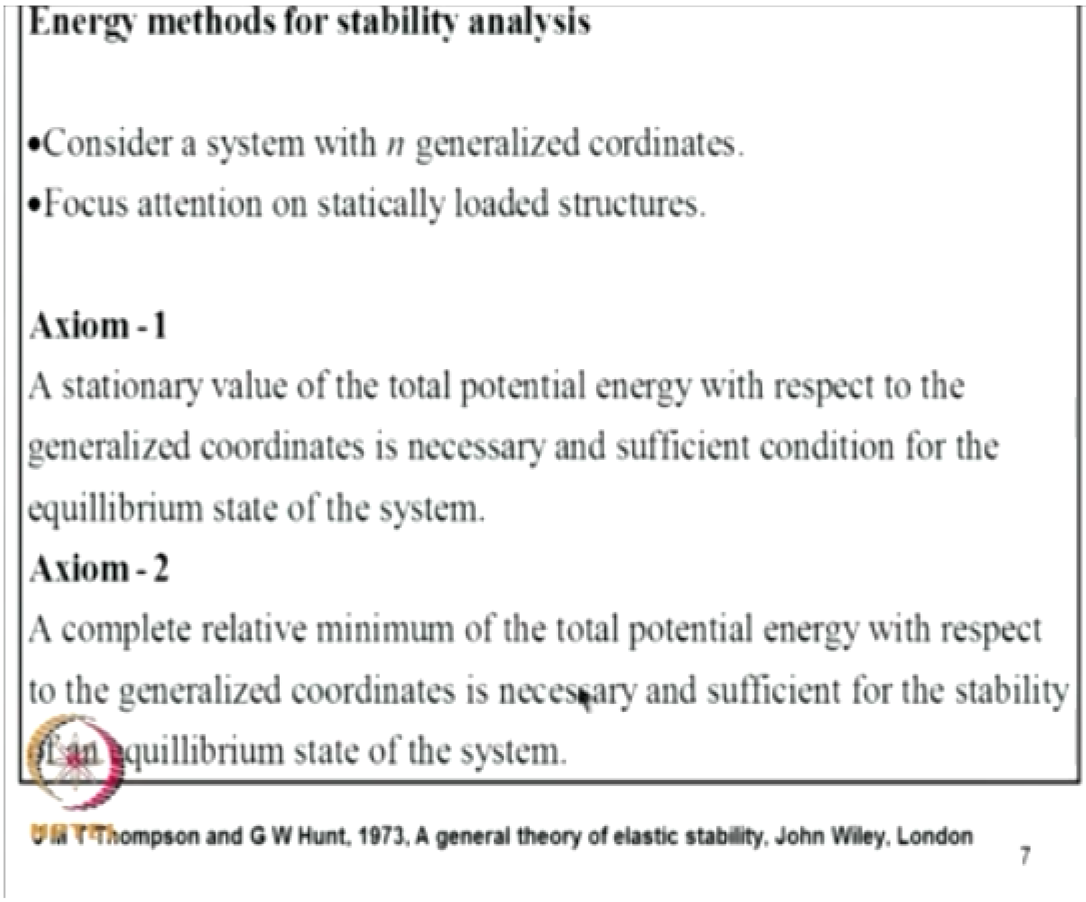
fixed points we can consider two possibilities, one is known as robust cases, here we talk about repellers when both eigenvalues have positive real part, and we talked about attractors when both eigenvalues have negative real part. Saddles are the ones where one eigenvalue is positive and the other eigenvalue is negative, marginal cases when both eigenvalues are pure imaginary, there are other situations like higher order and non-isolated fixed points where at least one of the eigenvalues will be 0, so we are not going to get into greater details on this, will later see how all these ideas are related to structural stability.



Now we use the word bifurcations, this is relation to the nature number, nature of the fixed points, as the parameters in a nonlinear dynamical system are change, one observes that number of fixed points can change, the nature of the fixed points can change, the stability of the fixed points can change, so whenever such changes take place we say that the system has undergone bifurcation, so there is various nomenclature associated with bifurcations depending on, before the bifurcation occurs what was the nature of the fixed points, and after the bifurcation occurs what is the nature of the fixed points, we saw that there is a node, focus, saddle, center, etcetera so there is half bifurcation where a periodic solution is born, the several you know classification of bifurcations, this again we are not going to get into greater details on these issues.



Now we are going to discuss in today's class energy methods for stability analysis, so what we'll do is we'll consider a system with N generalized coordinates, and focus attention on statically loaded structures, these energy methods are based on two axioms, the first axiom states that a stationary value of the total potential energy with respect to the generalized coordinates is necessary and sufficient condition for the equilibrium state of the system, the second axiom states that a complete relative minimum of the total potential energy with respect to the generalized coordinates is necessary and sufficient for the stability of an equilibrium state of the system, the first condition pertains to the equilibrium, the second condition pertains to the nature of the equilibrium whether it is stable or not, so this discussion is available in the monograph by Thompson and Hunt, so some of the material that I will be discussing are drawn from this resource.

Remarks

- Axioms are statements which are consistent with our physical experience of the world. These statements are deemed to be self-evident.
- Kinetic energy (and dissipation energy) do not enter these axioms

Theorem

Let $f(x)$ be a function with $a \le x \le b$ being the range of interest.

Let $f'(x^*) = f''(x^*) = \cdots = f^{n-1}(x^*) = 0$ & $f^n(x^*) \ne 0$.

n is even

$f(x^*)$ = minimum value of $f(x)$ at $x = x^*$ if $f^n(x^*) > 0$

$f(x^*)$ = maximum value of $f(x)$ at $x = x^*$ if $f^n(x^*) < 0$

n is odd

$f(x^*)$ is neither a maximum or a minimum

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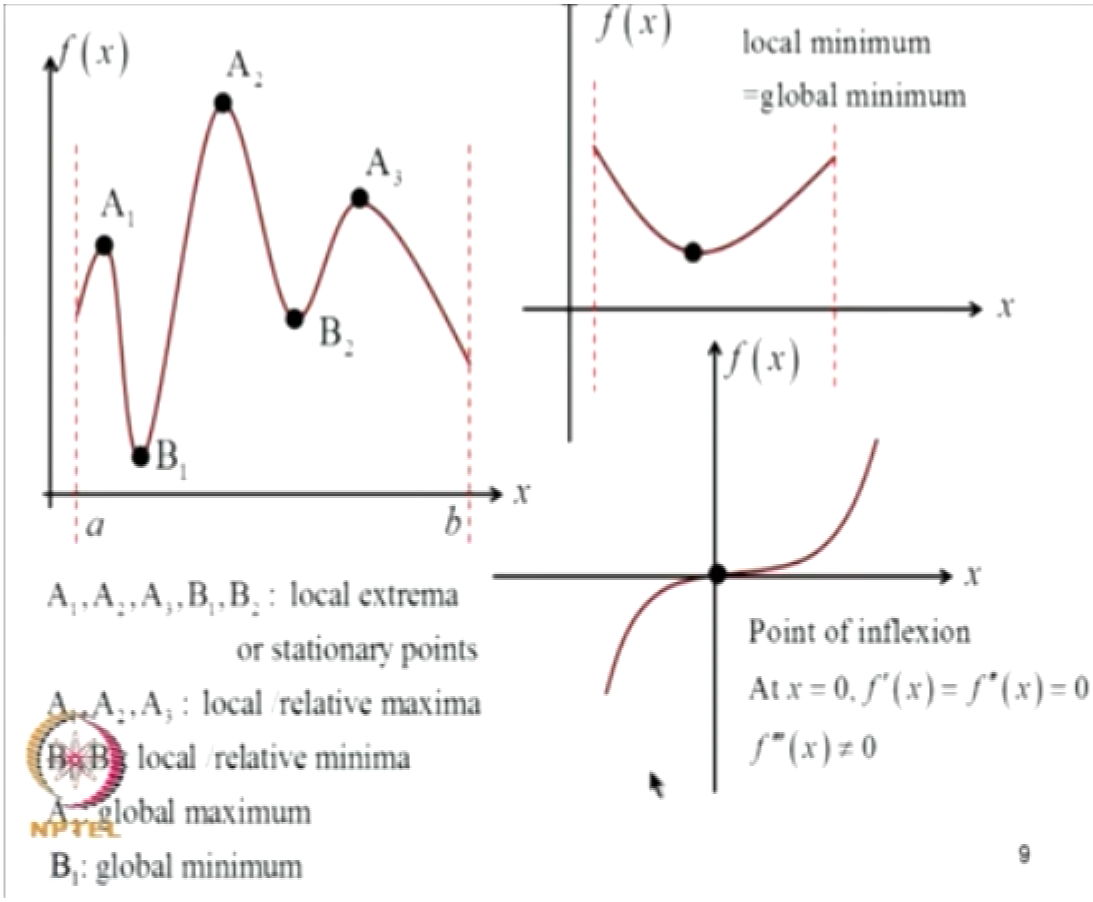
So we can make some remarks see axioms are statements which are consistent with our physical experience of the world, these statements are deemed to be self-evident, and also in these axioms the kinetic energy and dissipation energy have not entered the statement of these axioms, I am drawing your attention to that fact.

Now in the statement of these axioms there are issues like stationery value, relative minimum, and etcetera, etcetera, so we will quickly see a few results from optimization, so let F(x) be a function where X lies between A and B, and let F prime(x*), F double prime(x*) and FN -1 (x*) be 0, and FN(x*) is not 0, where this N is a, superscript N means I am considering nth order derivative, F is a function of a scalar variable here. If N is even F(x*) is minimum value of F(x) at X = X* if FN(x*) is greater than 0. On the other hand F(x*) is a maximum value of F(x) at X = X* if FN(x*) is less answer than 0, if N is odd F(x*) is neither a maximum nor a



minimum, so we can see a few schematics here this is a function F(x) versus X, and we have here A1, B1, A2, B2, A3, these are known as local extrema or stationary points, obviously F prime(x) is 0 in this depiction here. This A1, A2, A3 are known as local or relative maxima, B1 and B2 are known as local or relative minima, in the range of X between A and B, A2 is the global maximum, similarly B1 is a global minimum. In this function again lying between these two points A and B the local minimum and global minimum coincide. Here this point F prime(x) is 0, F double prime(x) is 0, but F triple prime(x) is not 0, this is a point of inflection, okay, so this also is embedded in the statement of this theorem.

Let $f(x)$ be a function of $x = (x_1, x_2, \cdots x_n)$.

Theorem

If $f(x)$ has an extreme point (minimum or maximum) at $x = x^*$, and if $f(x^*)$ exists then, $\frac{\partial f}{\partial x_i}(x^*) = 0 \; \forall i = 1, 2, \cdots, n$.

Theorem

A sufficient condition for a stationary point x^* to be an extreme point is that the Hessian matrix of $f(x)$ at $x = x^*$ is

positive definite when x^* is a minimum

positive negative definite when x^* is a maximum.

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Now let us consider a function of several variables let F(x) be a function of X1, X2, X3, XN, if F(x) is an extreme point minimum or maximum at X equal to X* and F(x*) exists then dou F/dou XI (x*) = 0 for I = 1 to N, okay, this is standard results from calculus. Then a sufficient condition for a stationary point X star to be an extreme point is that the Hessian matrix of F(x) at X = X* is positive definite when X* is a minimum. Negative definite when X* is a maximum, that means where you consider the Hessian matrix F(x), if it is positive definite when X* is minimum, it is negative definite when X* is a maximum.

H = Hessian matrix = $\left[\dfrac{\partial^2 f}{\partial x_i \partial x_j}\right] = H^t$

H is said to be positive definite if

(a) all the eigenvalues of H are >0, or

(b) $q^t H q > 0$ for any choice of $n \times 1$ vector q, or

(c) $|H_{11}| > 0, \begin{vmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{vmatrix} > 0, \cdots, |H| > 0$

G J Simitses and D J Hodges, 2006, Fundamentals of structural stability, Elsevier, Amsterdam.

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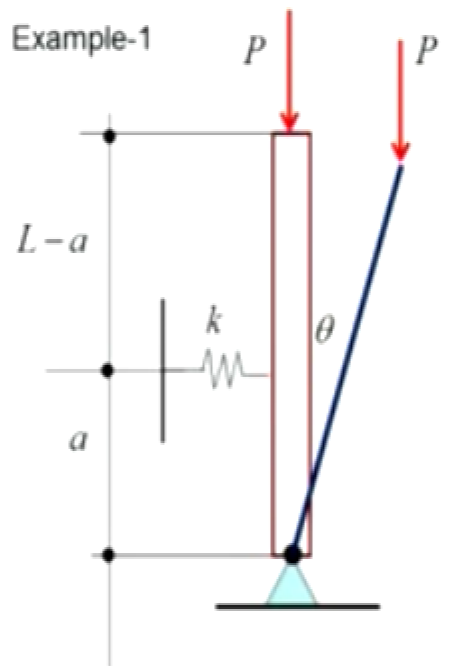
What is Hessian matrix? Hessian matrix is the matrix of second order derivative of H, so this is given by this elements of HIJ is dou square F/dou XI dou XJ, clearly since dou square F/ dou XI dou XJ = dou square F/dou XJ dou XI, this Hessian matrix is symmetric. Now we say that the Hessian matrix is positive definite if all the eigenvalues of H are greater than 0, or Q transpose HQ is greater than 0 for any choice of N x 1 vector Q, or we have few requirements H11, absolute value of H11 must be greater than 0, the 2 x 2 determinant H11, H12, H21, H22 must be greater than 0 so on and so forth, and determinant of H itself must be greater than 0. Now what we will do is we will try to use these axioms and try to gain understanding of how these axioms are applied by considering few examples, so these examples are drawn from this book by Simitses sand Hodges, the first example we consider is that of a rigid bar which is

Example-1



$$U = \frac{1}{2}ka^2\tan^2\theta - PL(1-\cos\theta)$$

Equillibrium

$$\frac{\partial U}{\partial \theta} = 0 \Rightarrow ka^2\tan\theta\sec^2\theta - PL\sin\theta = 0$$

$$\Rightarrow ka^2\frac{\sin\theta}{\cos^3\theta} - PL\sin\theta = 0$$

$$\Rightarrow \sin\theta\left(\frac{ka^2}{\cos^3\theta} - PL\right) = 0$$

$$\Rightarrow \sin\theta = 0 \text{ or } \frac{ka^2}{\cos^3\theta} - PL = 0$$

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loaded axially and which is supported by a spring K. Now as, if a neighboring equilibrium position exists then the potential is 1/2 KA square tan square theta - PL (1 - cos theta) this is KA square tan square theta is the strain energy stored here, now theta is a generalized coordinate, so there is only one generalized coordinate. For equilibrium dou U/dou theta must be equal to 0, so you differentiate this with respect to theta I get KA square tan theta secant square theta - PL sine theta must be equal to 0, so if you organize this terms we get the condition that sine theta into KA square by cos cube theta - PL must be equal to 0, so this quantity can be 0 by either sine theta being 0 or the term inside this bracket being parenthesis being 0, so we get the condition sine theta = 0 or KA square/cos cube theta - PL = 0, now this is the condition for equilibrium.

$$U = \frac{1}{2} ka^2 \tan^2\theta - PL(1-\cos\theta)$$

$$\frac{\partial U}{\partial \theta} = \sin\theta\left(\frac{ka^2}{\cos^3\theta} - PL\right)$$

$$\frac{\partial^2 U}{\partial \theta^2} = \cos\theta\left(\frac{ka^2}{\cos^3\theta} - PL\right) + \sin\theta\left(\frac{ka^2}{\cos^6\theta} 3\cos^2\theta\sin\theta\right)$$

$$= \cos\theta\left(\frac{ka^2}{\cos^3\theta} - PL\right) + \frac{3ka^2\sin^2\theta}{\cos^4\theta}$$

$$\theta = 0 \Rightarrow \frac{\partial^2 U}{\partial \theta^2} = \left(ka^2 - PL\right)$$

$$\theta = 0 \text{ is stable if } \left(ka^2 - PL\right) > 0 \Rightarrow P < \frac{ka^2}{L}$$

$$\theta \text{ such that } \left(\frac{ka^2}{\cos^3\theta} - PL\right) = 0 \Rightarrow \frac{\partial^2 U}{\partial \theta^2} = \frac{3ka^2\sin^2\theta}{\cos^4\theta}$$

$$\text{This is always} > 0 \Rightarrow \theta \text{ such that } \left(\frac{ka^2}{\cos^3\theta} - PL\right) = 0 \text{ is always stable}$$

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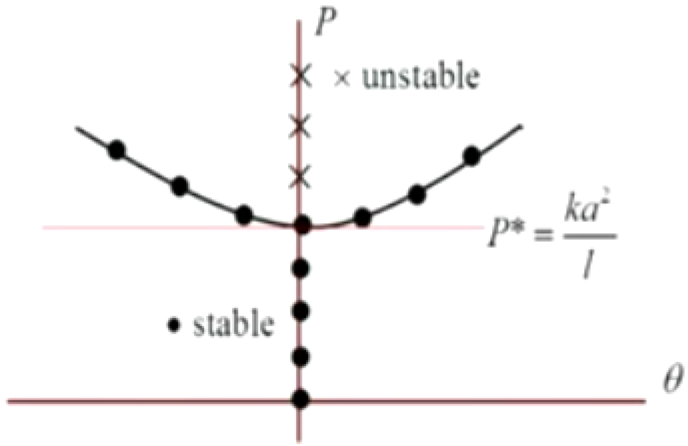
Now we want to examine the stability now, so we have to differentiate we have to find now the second derivative, dou square U/dou theta square, so dou U/dou theta we have obtained, this is given by this expression and upon differentiating this with respect to theta we get these terms and by reorganizing them we get the equation in this form. Now we will consider now the equilibrium position theta = 0, and see what happens to the sign of dou square U/dou theta square, so when theta = 0 we can see here cos theta is 1, cos cube theta is 1, this is 0 so I get dou square U/dou theta square is KA square – PL, theta = 0 is stable if this is greater than 0 or p is less than KA square/L, so this is a condition for stability of theta = 0, so P critical obviously is KA square/L. Now if theta is emanating from the second branch, okay, so in that case I have theta which satisfies this equation.

Now if you look at the expression for dou square U/dou theta square there is a term which is exactly equal to this and that is 0, and I am left with this term therefore dou square U/dou theta square for theta which satisfies this condition is given by this, now this quantity is always positive, you can see here K is positive, S square is positive, sine square theta is positive, cos to the power of 4 theta is positive, this would mean that any theta that satisfies this equation is always a stable equilibrium point.



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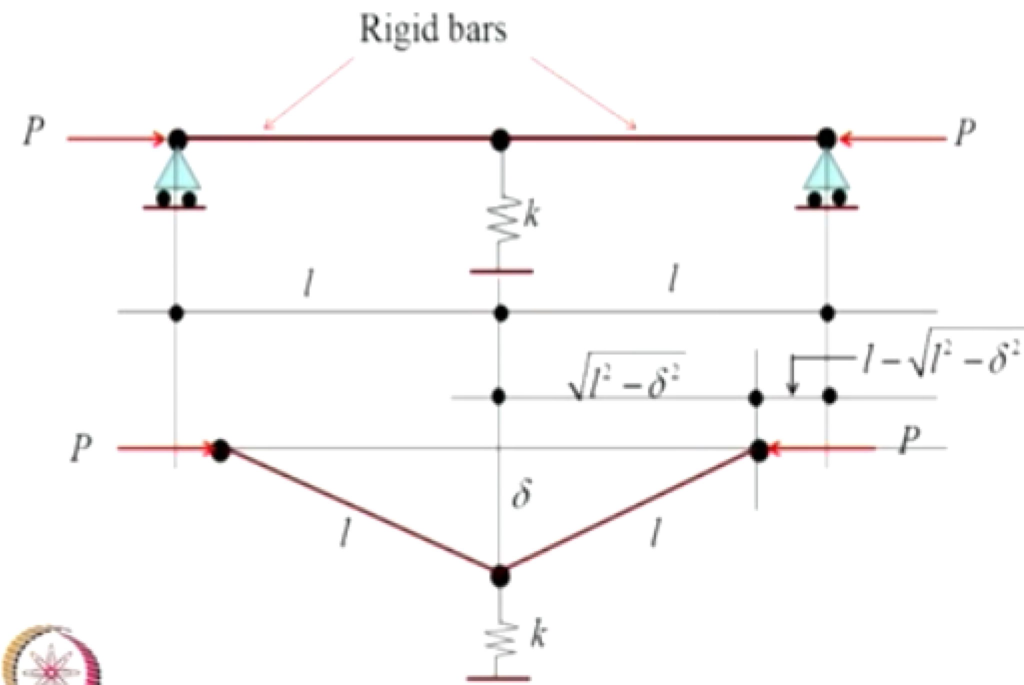
So now we can plot this initially till theta reaches this KA square, P reaches the value of KA square/L the only theta = 0 is the possible solution, and this is stable, this branch is actually the root of this equation KA square/cos cube theta - PL = 0 is this branch, this is always stable, and once KA square/L is crossed these points become unstable, so this is the load deflection diagram for the problem based on the two axioms that we have developed.

Example-2



$$U = \frac{1}{2}k\delta^2 - 2P\left[l-\sqrt{l^2-\delta^2}\right]$$

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Another example there are two rigid bars supported on spring here these dots represent hinges, and these bars are loaded axially through these loads P. Now as the load P increases these two bars are on rollers so they will move towards each other so at some point this roller would have moved by this amount, and this comes here and this roller would have moved by this amount it comes here, and consequently this link moves downwards right, so these loads are now doing work on, the work done by P on this system is P into this distance, and on the other hand because of this motion there will be energy stored in K, so what is the total strain energy, potential energy here? 1/2 K delta square - 2P, L square root L square + delta square, how do I get that? This is rigid link therefore this length is L, so this deflection is delta, so L square + delta square is sorry, this is a rectangle triangle therefore delta square + this square must be L, so based on that I get this distance as L square - delta square, so the distance by which the load has moved is L - L square by delta square and there are 2 such motions so this per factor, 2 comes here.

$$U = \frac{1}{2}k\delta^2 - 2P\left[l - \sqrt{l^2 - \delta^2}\right]$$

Equillibrium

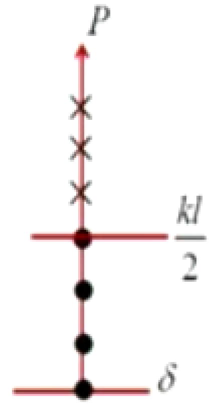
$$\frac{\partial U}{\partial \delta} = 0 \Rightarrow k\delta + 2P\frac{1}{2}\frac{1}{\sqrt{l^2 - \delta^2}}(-2\delta) = 0$$

$$\frac{\partial U}{\partial \delta} = k\delta - \frac{2P\delta}{\sqrt{l^2 - \delta^2}}$$

$$= k\delta - \frac{2P\delta}{l\sqrt{1 - \left(\frac{\delta}{l}\right)^2}} \approx k\delta - \frac{2P\delta}{l} = \delta\left(k - \frac{2P}{l}\right)$$

Equillibrium point: δ=0

$$\frac{\partial^2 U}{\partial \delta^2} = \left(k - \frac{2P}{l}\right) \Rightarrow \delta\text{=0 is stable if } k - \frac{2P}{l} > 0 \Rightarrow P < \frac{kl}{2}$$



Re-do the problem without linearizing the expression for U

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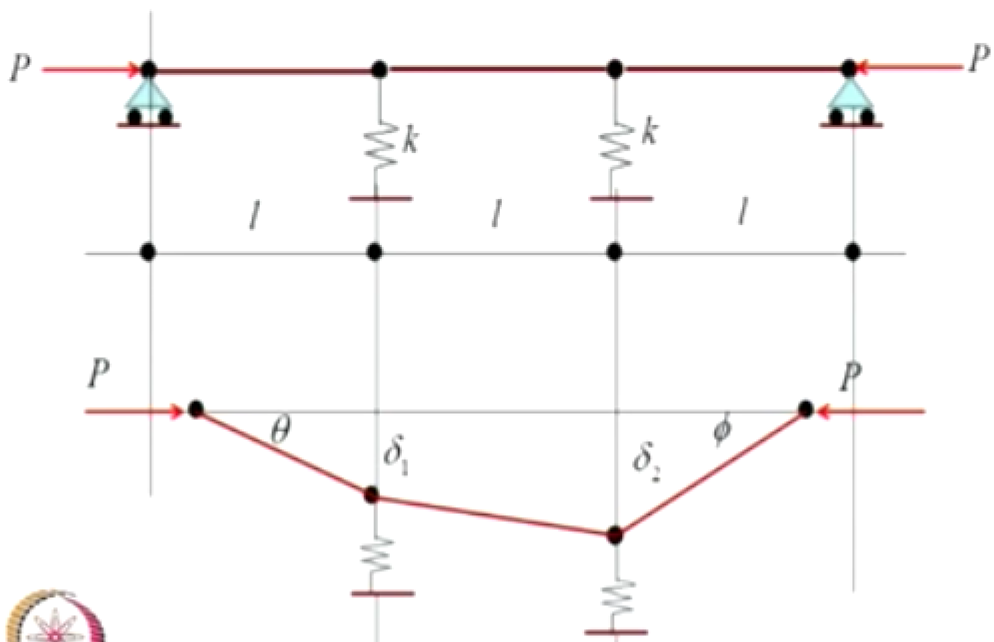
Now delta is a generalized coordinate, so I have got now U in terms of delta, for equilibrium I have dou U/dou delta = 0, so I differentiate I get this equal to 0 and this is the load deflection path which satisfies the equilibrium condition, this equal to 0.

Now for sake of simplification we can linearize this function so I take L outside and I have square root of 1 – delta/L whole square and if delta/L is small I can approximate this as K delta - 2P delta/L which is equal to delta K – 2P/L. Now the equilibrium point therefore is delta = 0, now what is the second order derivative? Del square U/dou delta square is you have to differentiate with respect to delta, I get K - 2P/L, for delta = 0 to be stable this quantity in the parenthesis must be greater than 0, so I get the condition K - 2P/L must be greater than 0 or P must be less than KL/2, so according to the linearized, after linearizing the problem we have got this result that delta = 0 is a stable branch till the time I reach, P reaches the value of KL/2 after that delta = 0 is unstable. We are unable to trace the other paths because we are linearized here so I have left this as an exercise, you don’t do this linearization, but plot this curve and evaluate the second-order gradient at each of these points and see whether the sign is positive or negative and then decide upon whether the point on this curve that is this equal to 0 curve is stable or not, so then you will be able to trace the complete solution, obviously this line will be part of this solution but there will be further information which is not encapsulated in this load deflection diagram.

Example-3



$$U = \frac{1}{2}k_1\delta_1^2 + \frac{1}{2}k_2\delta_2^2 - P\left\{3l - \left[\sqrt{l^2-\delta_1^2} + \sqrt{l^2-(\delta_2-\delta_1)^2} + \sqrt{l^2-\delta_2^2}\right]\right\}$$

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Now we have considered two examples with one generalized coordinate, now let us consider an example where we allow for 2 generalize coordinates, so this problem is conceptually similar to the one that we considered just now but it has now 3 rigid bars with the 2 hinges and again loaded axially, so upon the application of these loads, these roller moved to the right, this roller move to the left, and these springs gets compressed, and this is the snapshot of the deformed configuration from which I will be able to write the expression for the total potential energy, this is energy stored in K1, I am calling this as K1, K2 numerically both are equal, so this is K1 delta 1 square + 1/2 K2 delta 2 square, P 3L minus this distance, this distance so you can compute this distance by using simple geometrical arguments and we get U as a function of delta 1 and delta 2. So this is the total potential and now I have for equilibrium and to

$$U = \frac{1}{2}k_1\delta_1^2 + \frac{1}{2}k_2\delta_2^2 - P\left\{3l - \left[\sqrt{l^2-\delta_1^2} + \sqrt{l^2-(\delta_2-\delta_1)^2} + \sqrt{l^2-\delta_2^2}\right]\right\}$$

Equillibrium

$$\frac{\partial U}{\partial \delta_1} = 0 \;\&\; \frac{\partial U}{\partial \delta_2} = 0$$

$$\frac{\partial U}{\partial \delta_1} = k_1\delta_1 - P\left\{-\frac{1}{2}\frac{1}{\sqrt{l^2-\delta_1^2}}(-2\delta_1) - \frac{1}{2}\frac{2(\delta_2-\delta_1)}{\sqrt{l^2-(\delta_2-\delta_1)^2}}\right\}$$

$$= k_1\delta_1 - P\left\{\frac{\delta_1}{\sqrt{l^2-\delta_1^2}} - \frac{(\delta_2-\delta_1)}{\sqrt{l^2-(\delta_2-\delta_1)^2}}\right\}$$

$$\approx k_1\delta_1 - P\left\{\frac{\delta_1}{l} - \frac{(\delta_2-\delta_1)}{l}\right\}$$

For equillibrium, $k_1\delta_1 - P\left\{\frac{\delta_1}{l} - \frac{(\delta_2-\delta_1)}{l}\right\} = 0$

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differentiate this function with respect to delta 1 and delta 2 and equal to 0, so if I do that I get 2 equations, for example dou U/dou delta 1 I get this, and here again if I linearize if I assume that this delta 1 is small in relation to L I can linearize this function and I will get dou U/dou delta 1 is this, and for equilibrium this must be equal to 0.

$$U = \frac{1}{2}k_1\delta_1^2 + \frac{1}{2}k_2\delta_2^2 - P\left\{3l - \left[\sqrt{l^2-\delta_1^2} + \sqrt{l^2-(\delta_2-\delta_1)^2} + \sqrt{l^2-\delta_2^2}\,\right]\right\}$$

$$\frac{\partial U}{\partial \delta_2} = k_2\delta_2 - P\left\{-\frac{1}{2}\frac{1}{\sqrt{l^2-\delta_2^2}}(-2\delta_2) + \frac{1}{2}\frac{2(\delta_2-\delta_1)}{\sqrt{l^2-(\delta_2-\delta_1)^2}}\right\}$$

$$= k_2\delta_2 - P\left\{\frac{\delta_2}{\sqrt{l^2-\delta_2^2}} + \frac{(\delta_2-\delta_1)}{\sqrt{l^2-(\delta_2-\delta_1)^2}}\right\}$$

$$\approx k_2\delta_2 - P\left\{\frac{\delta_2}{l} + \frac{(\delta_2-\delta_1)}{l}\right\}$$

For equillibrium, $k_2\delta_2 - P\left\{\frac{\delta_2}{l} + \frac{(\delta_2-\delta_1)}{l}\right\} = 0$

A similar equation by differentiating this with respect to delta 2 can be done, you differentiate again assume that delta 2/L is a small quantity linearize this function and I get K2 delta 2 - P into this quantity equal to 0 as a condition for equilibrium, so there are now two conditions for

$$U=\frac{1}{2}k_1\delta_1^2+\frac{1}{2}k_2\delta_2^2-P\left\{3l-\left[\sqrt{l^2-\delta_1^2}+\sqrt{l^2-(\delta_2-\delta_1)^2}+\sqrt{l^2-\delta_2^2}\right]\right\}$$

Equillibrium

$$\frac{\partial U}{\partial \delta_1}=0\ \&\ \frac{\partial U}{\partial \delta_2}=0$$

$$\frac{\partial U}{\partial \delta_1}=k_1\delta_1-P\left\{\frac{\delta_1}{l}-\frac{(\delta_2-\delta_1)}{l}\right\}$$

$$\frac{\partial U}{\partial \delta_2}=k_2\delta_2-P\left\{\frac{\delta_2}{l}+\frac{(\delta_2-\delta_1)}{l}\right\}$$

For equillibrium,

$$\left.\begin{array}{l} k_1\delta_1-P\left\{\frac{\delta_1}{l}-\frac{(\delta_2-\delta_1)}{l}\right\}=0 \\ k_2\delta_2-P\left\{\frac{\delta_2}{l}+\frac{(\delta_2-\delta_1)}{l}\right\}=0 \end{array}\right\}\Rightarrow\begin{bmatrix} k-\frac{2P}{l} & \frac{P}{l} \\ \frac{P}{l} & k-\frac{2P}{l} \end{bmatrix}\begin{Bmatrix}\delta_1\\ \delta_2\end{Bmatrix}=0$$

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equilibrium I can write them together, before that we can also think of that is what we are doing here dou U/dou delta 1 = 0, dou U/dou delta 2 = 0 leads to this pair of equations in the matrix form this is the equation.

For nontrivial solutions

$$\begin{vmatrix} k - \frac{2P}{l} & \frac{P}{l} \\ \frac{P}{l} & k - \frac{2P}{l} \end{vmatrix} = 0 \Rightarrow \left(k - \frac{2P}{l}\right)^2 - \left(\frac{P}{l}\right)^2 = 0$$

$$\left(k - \frac{2P}{l} - \frac{P}{l}\right)\left(k - \frac{2P}{l} + \frac{P}{l}\right) = 0$$

$$\Rightarrow P = kl, P = \frac{kl}{3}$$

Stability

$$\frac{\partial U}{\partial \delta_1} = k_1\delta_1 - P\left\{\frac{\delta_1}{l} - \frac{(\delta_2 - \delta_1)}{l}\right\} \Rightarrow \frac{\partial^2 U}{\partial \delta_1^2} = k_1 - \frac{2P}{l}; \frac{\partial^2 U}{\partial \delta_1 \partial \delta_2} = \frac{2P}{l}$$

$$\frac{\partial U}{\partial \delta_2} = k_2\delta_2 - P\left\{\frac{\delta_2}{l} + \frac{(\delta_2 - \delta_1)}{l}\right\} \Rightarrow \frac{\partial^2 U}{\partial \delta_2^2} = k_2 - \frac{2P}{l}$$

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Now we can examine the stability by considering the second order gradients, so that is the Hessian matrix I have to find out and see whether Hessian matrix is positive definite or not, so I construct the second order derivatives and the Hessian matrix and for Hessian matrix to be

$$H = \begin{bmatrix} k - \frac{2P}{l} & \frac{2P}{l} \\ \frac{2P}{l} & k - \frac{2P}{l} \end{bmatrix}$$

For Hessian to be positive definite

$$k - \frac{2P}{l} > 0$$

$$\begin{vmatrix} k - \frac{2P}{l} & \frac{2P}{l} \\ \frac{2P}{l} & k - \frac{2P}{l} \end{vmatrix} > 0 \Rightarrow \left(k - \frac{3P}{l}\right)\left(k - \frac{P}{l}\right) > 0$$

$$\Rightarrow k - \frac{2P}{l} > 0 \Rightarrow \left(k - \frac{P}{l}\right) > 0$$

$$\Rightarrow \left(k - \frac{3P}{l}\right)\left(k - \frac{P}{l}\right) > 0 \Rightarrow \left(k - \frac{3P}{l}\right) > 0$$

$$\Rightarrow P_{cr} = \frac{kl}{3}$$

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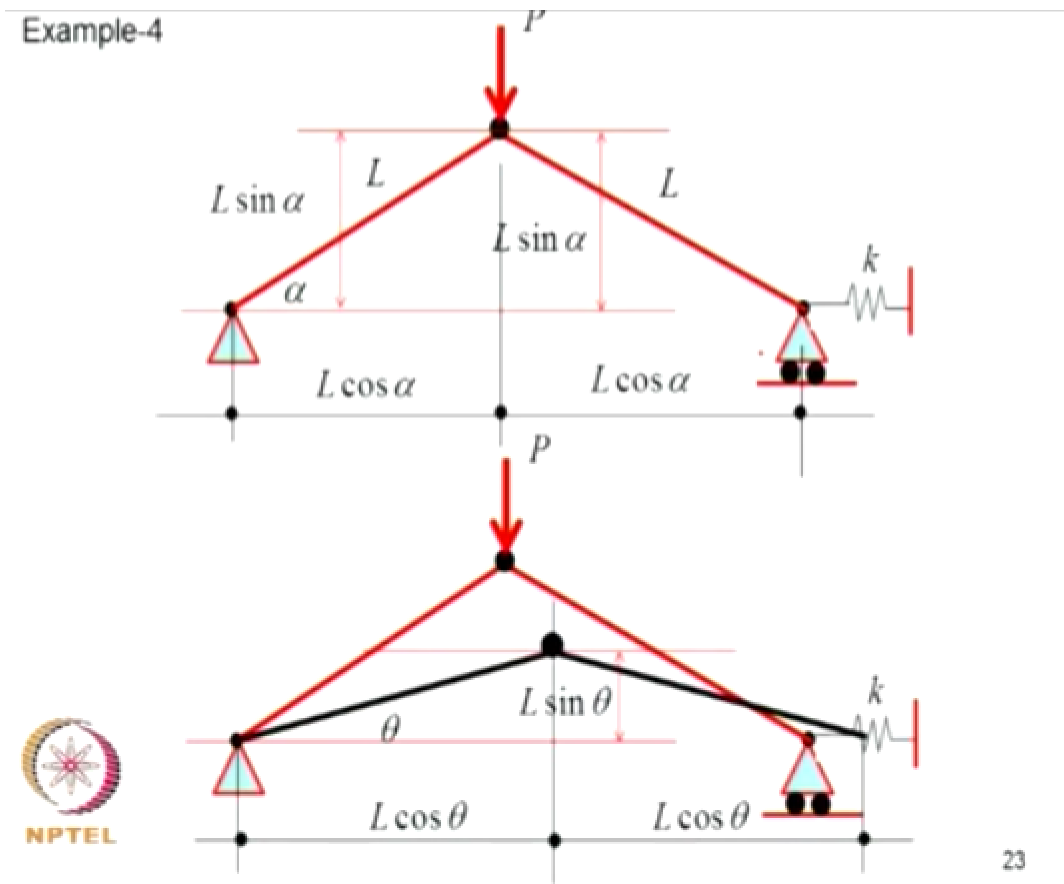
positive definite there are 2 criterion K - 2P/L must be greater than 0 and determinant of this must be greater than 0, so I get this condition, first condition is K - 2P/L must be greater than 0 second condition is this. Now if K - 2P/L is greater than 0 it automatically implies that K – P/L will be greater than 0 and hence this condition in conjunction with this really translates to this condition K - 3P/L must be greater than 0, so we get from this P critical to be KL/3.



Now in one of the early discussions on stability we consider the problem of snap through, we conceptually outlined how the solution might be, but now we will work out the solution in some detail, so for illustrating that we again consider 2 links, rigid links connected through a hinge here and this is supported on a spring as shown here, as shown here and when P = 0 this bar occupies this triangular position. Now as P is increased as shown here, as P is increased, this point moves down and this roller moves to the right there by compressing the spring, and this load will do the work through this distance and there will be strain energy stored here.

Now we will write the expression for the total potential in this case, you can note down the nomenclature here alpha is the angle that this link makes to the horizontal when P is 0, whereas theta is the angle made by the link to the horizontal when P is not 0, so theta is the generalized coordinate whereas alpha is a parameter of the problem, so we get the total potential to be given

$$U = \frac{1}{2}k\left(2L\cos\theta - 2l\cos\alpha\right)^2 - P\left(L\sin\alpha - L\sin\theta\right)$$

$$= 2kL^2\left(\cos\theta - \cos\alpha\right)^2 - PL\left(\sin\alpha - \sin\theta\right)$$

$$\frac{\partial U}{\partial\theta} = 4kL^2\left(\cos\theta - \cos\alpha\right)\left(-\sin\theta\right) + PL\cos\theta$$

$$\frac{\partial^2 U}{\partial\theta^2} = 4kL^2\left(\cos\theta - \cos\alpha\right)\left(-\cos\theta\right) + 4kL^2\left(-\sin\theta\right)\left(-\sin\theta\right) - PL\sin\theta$$

$$= -4kL^2\cos\theta\left(\cos\theta - \cos\alpha\right) + 4kL\sin^2\theta - PL\sin\theta$$

Equillibrium

$$\frac{\partial U}{\partial\theta} = 0 \Rightarrow 4kL^2\left(\cos\theta - \cos\alpha\right)\left(-\sin\theta\right) + PL\cos\theta = 0$$

$$\Rightarrow \frac{P}{4kL} = \sin\theta - \cos\alpha\tan\theta \quad \text{[Load deflection curve]}$$

Stability: $\frac{\partial^2 U}{\partial\theta^2} > 0 \Rightarrow -4kL^2\cos\theta\left(\cos\theta - \cos\alpha\right) + 4kL\sin^2\theta - PL\sin\theta > 0$

$$\Rightarrow \frac{\cos\alpha}{\cos\theta} - \cos^2\theta > 0$$

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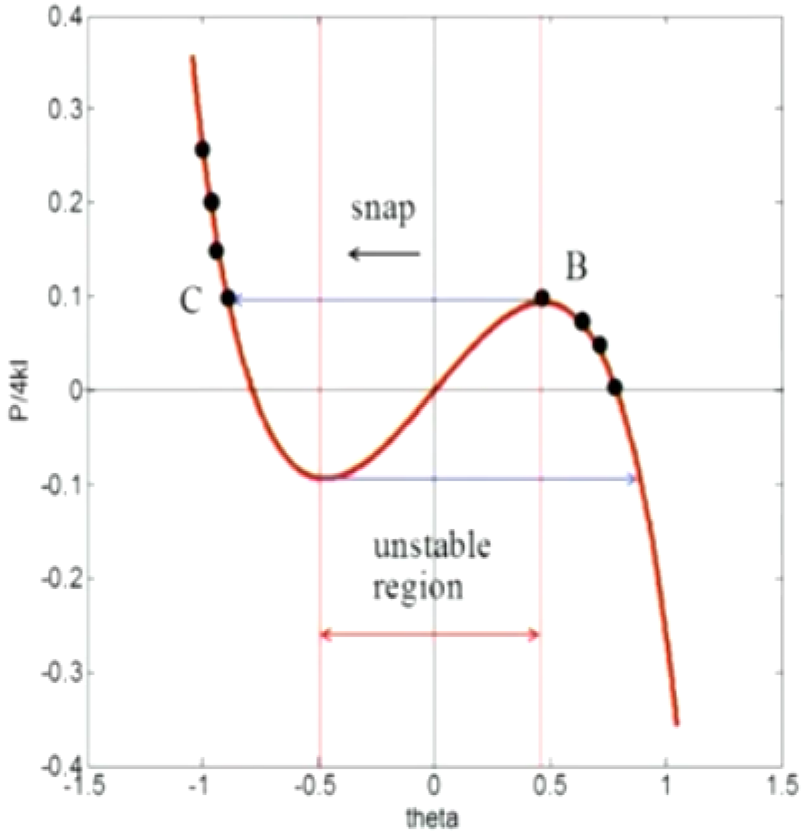
by the strain energy stored in the spring which is, what is the distance through which this point has moved this distance is 2L cos alpha, whereas this distance is 2L cos theta, so the difference between the 2 is the distance through which this roller has moved and that is here, 2L cos theta - 2L cos alpha whole square 1/2 K. Similarly the work done by P is P into this vertical distance and this vertical distance here is L sine alpha, whereas here it is L sine theta so the distance through which this unmoved is L sine alpha - L sine theta, okay, so that is this, so we can take out 2L outside and rewrite this expression, and now we are ready to investigate the stability.

For equilibrium dou U/dou theta, dou U/dou theta upon differentiating this with respect to theta I get this, second derivative I differentiate this with respect to theta and I get this expression, we can do a bit of simplification and we get dou U/dou theta and dou square U/dou theta square as outlined here. Now we examine the condition for equilibrium, the condition for equilibrium is dou U/dou theta = 0, so that means this quantity must be equal to 0, so we can rearrange the terms and if we do that I will get the equation as P/4KL is sine theta - cos alpha tan theta, so this is actually a load deflection curve which is non-linear, theta is the displacement and P is applied load, so a point P, theta from the points on load deflection curve, and for equilibrium they should obey this relation, okay.

Now every point on this load deflection curve we have to evaluate dou square U/dou theta square and look at the sign of that quantity, if it is positive that corresponding point on this load deflection path is stable, otherwise it is unstable, so I will put this dou square U/dou theta square I will evaluate this, and for this to be greater than 0 I get this condition that cos alpha/ cos theta - cos square theta must be greater than or equal to 0 when this condition is also satisfied.



Now graphically this curve looks as follows, this line that you are seeing here is the load deflection path given by this curve, this curve is this red line, when load is 0, Y axis is the load, when load is 0 the deflection will be this, this is the theta which is alpha, this is actually alpha this value will be alpha, as load is increased the theta goes on reducing and when it reaches this point B, so what I am doing is for every value of theta here I will evaluate dou square U/dou theta square and see what is the sign of this quantity, if it is positive it is stable otherwise unstable, so these points are stable, and moment I reach this point B what happens is this branch will be completely unstable, so the loading path unstable positions cannot be occupied so the structure actually snaps it comes here, this path is unstable.

Now upon increasing the load further it traces this path, now similarly if you start unloading the structure would be able to realize this path, but at this point it snaps to this, snaps upwards and comes to this point and again these points will be stable, okay, so this region is never occupied either in the forward path or in the reverse path so this region is unstable region. These values of theta cannot be realized by the structure so that is what this theory tab, and this as I said is a snap through phenomena where if you carry an umbrella during a rainy day, windy day, the curvature of the umbrellas, the sign of that suddenly changes under certain velocities and so that is the kind of thing that we are talking about but under static loads.

Remarks

- We need to use large deflection theory.
- No adjacent equillibrium states exist at B.
- In the neighbourhood of B, as load increases, the system loses its stability and moves to C (to a far away state).
- This behavior is called limit load buckling.
- Buckling of spherical caps and arches display this behavior.

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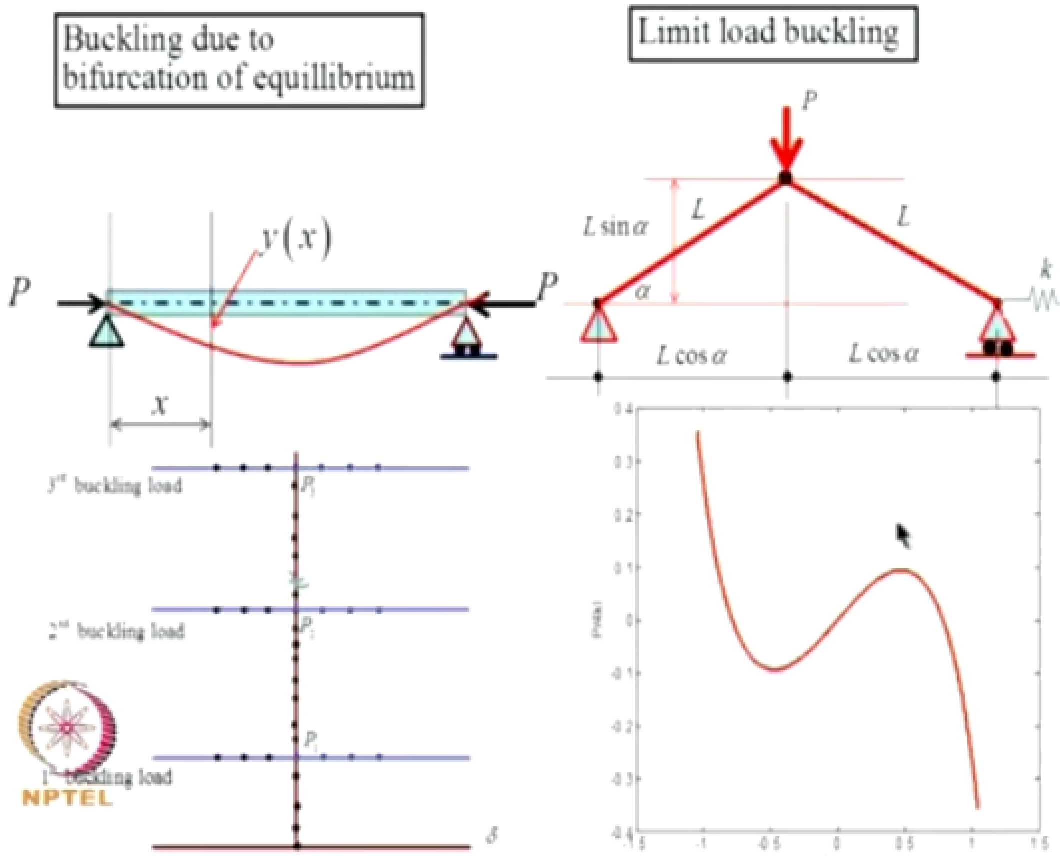
Now we can make few observations here to study this problem we need to use large deflection theory, if you linearize you won't get this details of this theory correctly, here at B there are no adjacent equilibrium positions possible, so in the neighborhood of B as load increases the system will loses its stability and moves to C to a faraway state this is C, it doesn’t come in the neighborhood, this behavior is called limit load buckling, okay the buckling of spherical caps and arches display this behavior that is curved elements.



Now we can compare this with buckling due to bifurcation equilibrium, so if you recall when we studied this problem of an ideal beam carrying a truly axial load Y = 0 was the equilibrium solution and we are interested to see if there is any other equilibrium position in the neighborhood of Y = 0, so by assuming that there exists in neighboring equilibrium position we wrote the equation, presuming that such an neighboring equilibrium exists and try to find out for what value of P that assumption is valid, so that led to this load deflection diagram as shown here, so when load is 0, P is 0, this delta is a mid-span deflection is 0, and as we gently increase value of P and at every increase suppose you pluck the beam and allow it to oscillate and see whether it returns to its original state or not assuming that system is damped it will return to its original path till P reaches a value P1, after that depending on the perturbations that you give it occupies a neighboring equilibrium position, okay, so that is the phenomena of buckling due to bifurcation of equilibrium.

Whereas in limit load buckling what happens is as you go on increasing the load the structure deforms, here the deflection is 0 as a load increases, but here as the load increases the deformation also increases, okay, so we started here, as the load increases the deflection is changing this is unlike this situation where with increase in P origin still continues to be stable, okay, and when it reaches this critical value it occupies a faraway position, in buckling due to

Buckling due to bifurcation of equillibrium

As load increases deflection remains at zero. Non-zero deflection becomes possible only when load reaches the criticial value.

Limit load buckling

The structure deforms with every increase in applied load. Both load and deformation evolve as load increases and at the critical stage when structure becomes unstable and the system gets into an equillibrium far from the original equillibrium position. As the load approaches the critical value, any small increase in the load results in large changes in deformation

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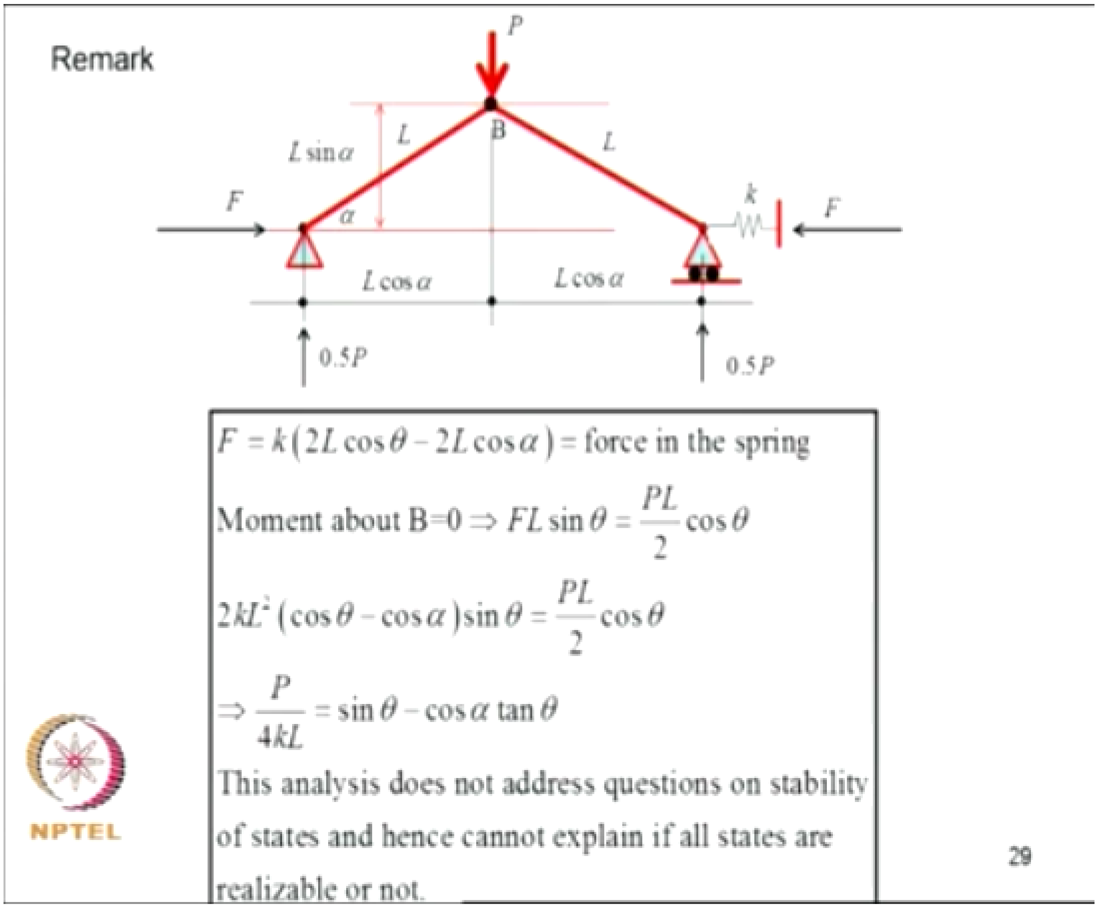
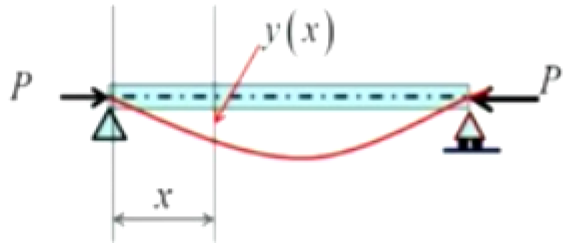
bifurcation of equilibrium as load increases deformation deflection remains at 0, nonzero deflection becomes possible only when load reaches the critical value till that time there is no deflection, whereas in limit road buckling the structure deforms with every increase in applied load, both load and deformation evolve as load increases, and the critical stage when structure becomes unstable and the system gets into an equilibrium far away from the original equilibrium position, so when that critical condition is reached the structure occupies a faraway equilibrium position, as the load approaches the critical value any small increase in the load results in large changes in deformation, even away from this critical condition with increases in load the deformation changes, whereas that doesn't happen here, so to analyze limit load buckling we need to perform a nonlinear analysis.



Now suppose if we consider the same problem purely based on equilibrium considerations, suppose I find out the reactions and take moment about B and write the equilibrium equation, so that can be done for example this reaction, vertical reaction by symmetry is P/2 and the force in the spring we already derived it to be K(12 cos theta - 2L cos alpha) the moment about B if you take here F into this distance, and this reaction into that will give me the required relation and I get the equilibrium path, that is the relation between P and theta, but this analysis does not address the questions on stability of states and hence cannot explain if all states are realizable or not, you will get this graph and if you plot it you will get a curve like this, but you will not know whether this path is realizable or not, okay, and also there will be a problem here if you consider this value of P, there are 3 possible solutions, so which solution will be realized? You see that question has to be answer, so that depends on how you are approached that point and this intermediate point cannot be realized at all, okay, so these details will not be delineated if I do simply an equilibrium analysis, so you need to perform a stability analysis as well.

Approximate methods for stability analysis

Rayleigh-Ritz method



$$V = \frac{1}{2}\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx - \frac{1}{2}\int_0^L P\left(\frac{dy}{dx}\right)^2 dx$$

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Now let's now return to problems of continuum, so far we considered some simple examples involving rigid links and some simple mechanical models, so let us return to problems of say stability of beams, so let us consider this problem of ideal simply supported beam carrying truly axial loads and we are considering whether there is an equilibrium position in the neighborhood possible or not, so I can write the expression for strain energy in the system as shown here. How do I now apply the axioms to investigate at what value of P such positions are possible?

Assume a deflection shape (trial function)

$y(x) = ax(l-x)$

a: generalized coordinate

$y'(x) = a(l-2x)$

$y''(x) = -2a$

$$\Rightarrow V = \frac{1}{2}\int_0^l EI(-2a)^2\,dx - \frac{1}{2}\int_0^L P\left(a(l-2x)\right)^2\,dx$$

$$V(a) = a^2\left(2EI - \frac{Pl^3}{6}\right)$$

$$\frac{dV}{da} = 2a\left(2EI - \frac{Pl^3}{6}\right)$$

$$\frac{d^2V}{da^2} = 2\left(2EI - \frac{Pl^3}{6}\right)$$

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So what we do is we use this Rayleigh-Ritz type of analysis, so this we have discussed earlier when we, in the very first few classes when we discussed approximate methods for determination of natural frequency, so we are used to the language so I will not repeat all the details, so we start with a trial function, for example if I take Y(x) to be AX(l-x), so as you can see here at X = 0, Y is 0 and X = L, Y is 0, so at least the geometric boundary condition is satisfied, this A is the generalized coordinate.

Now if you look at the expression for total potential it has D square Y/DX square and DY/DX so you find out Y prime(x) I get A(1-2x), Y double prime(x) – 2A, substitute this into V and perform this integration I get V(a) to be A square 2EI – PL cube/6, DV/DA simple you differentiate this with respect to A I get this, D square V/DA square I get this. Now I will need these two quantities to infer equilibrium and stability, for equilibrium DV/DA must be equal to

Equillibrium: $2a\left(2EI - \frac{Pl^3}{6}\right) = 0 \Rightarrow a = 0$

Stability: $\left(2EI - \frac{Pl^3}{6}\right) > 0 \Rightarrow a = 0$ is stable if $P < \frac{12EI}{l^3}$

$\Rightarrow P_{cr} = \frac{12EI}{l^3}$

Recall: $P_{cr}^{\text{Exact}} = \frac{\pi^2 EI}{l^3} = \frac{9.869EI}{l^3}$

Error: ~20%

Recall discussion on choice of trial functions when we discussed Rayleigh-Ritz method in the context of determination of natural frequencies.

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0, that means A would be 0, and this equilibrium position is stable if the second order derivative of V evaluated at this point is positive, so what is DV/DA at A = 0? It is a constant everywhere so I get this, and if it is greater than 0, A is stable, so for this to be greater than 0, P should be less than 12EI/L cube, so P critical according to this theory is 12EI/L cube. Now if you recall we have computed the critical load for this case and first critical load is pi square EI/L cube. Now this is 9.869EI/L cube and the when this approximate analysis gets an approximation of 12EI/L cube to this number, so there is about 20% error in this analysis.

Now if you recall the discussion on choice of trial functions when we discuss Rayleigh-Ritz method in the context of determination of natural frequencies, we classified the trial functions

Classification of trial functions

	Geometric BCs (I)	Natural BCs (II)	Field Eqn. (III)
Admissible	Satisfied	Not satisfied	Not satisfied
Comparison	Satisfied	Satisfied	Not satisfied
Eigenfunction	Satisfied	Satisfied	Satisfied

$y(x) = ax(l-x)$: admissible

$y(x)$: deflected profile of the beam under its own weight: comparison

$y(x)$: eigenfunction

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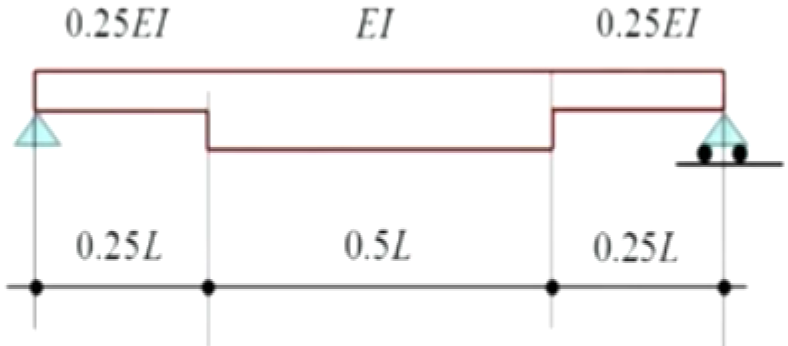
as being admissible, comparison, and Eigen functions, there are geometric boundary condition and natural boundary conditions, and there is a field equation, the trial function which satisfy the geometric boundary conditions and which did not satisfy the natural boundary conditions and field equations are called admissible function. If the trial function satisfies geometric boundary condition as well as natural boundary conditions, but not the field equation they are called comparison functions. Eigen functions are those which satisfy the geometric boundary condition, natural boundary condition, as well as the field equation, that means they are the exact solution.

Now if you look at the trial function that we have used Y = AX(l-x) is an admissible function, because the second derivative is a constant and bending moment would not be 0 at the supports. Now we can try other shape functions for example Y(x) could be a deflected profile of the beam under its own weight, suppose if I take the beam and apply a concentrated load, distributed load which makes its own self weight, this deflected profile will satisfy geometric boundary condition and obviously if the analysis is right the bending moment computed from this curve will be 0 at the two supports so it will be a comparison function, so you can use that A into this shape can be taken as a trial function, that's one way of constructing comparison functions. We can also consider for example if it is sine N pi X/L, there are 2 Eigen functions, okay, so if I use in the Rayleigh Ritz method sine pi X/L as my trial function I will get the exact solution, but you would not know in real problems what would be the Eigen function.

Example



$$V = \frac{1}{2}\int_0^{0.25l} \frac{EI}{4}\left(\frac{d^2y}{dx^2}\right)^2 dx + \frac{1}{2}\int_{0.25l}^{0.75l} EI\left(\frac{d^2y}{dx^2}\right)^2 dx + \frac{1}{2}\int_{0.75l}^{l} \frac{EI}{4}\left(\frac{d^2y}{dx^2}\right)^2 dx - \frac{1}{2}\int_0^{l} P\left(\frac{dy}{dx}\right)^2 dx$$

So to illustrate this let us consider a single span beam which is made up of stepped cross-sections, the middle half has a flexural rigidity of EI and the other quarter have EI/4, so I want to find out the, this structure is subjected to axial loads and I am interested in knowing the critical value of this load P where the equilibrium position Y = 0 loses its stability, so I will write the expression for V taking into account that this EI is different, and different cross sections so first I will integrate from 0 to 0.25L and then 0.25L to 0.75L, and 0.75L to 0.25L and I use EI/4, EI and EI/4 in these 3 cross-sections respectively. The axial load of course is constant throughout so I will take that, there is no need to split this, so this is the expression for the total potential.

$$V=\frac{1}{2}\int_0^{0.25l}\frac{EI}{4}\left(\frac{d^2y}{dx^2}\right)^2dx+\frac{1}{2}\int_{0.25}^{0.75l}EI\left(\frac{d^2y}{dx^2}\right)^2dx+\frac{1}{2}\int_{0.75l}^{l}\frac{EI}{4}\left(\frac{d^2y}{dx^2}\right)^2dx-\frac{1}{2}\int_0^l P\left(\frac{dy}{dx}\right)^2dx$$

$$y(x)=a_1\phi_1(x)+a_2\phi_2(x)=a_1\sin\left(\frac{\pi x}{l}\right)+a_2\sin\left(\frac{3\pi x}{l}\right)$$

$\sin\left(\frac{\pi x}{l}\right),\sin\left(\frac{3\pi x}{l}\right)$: comparison functions

(a_1,a_2): generalized coordinates

$$V(a_1,a_2)=\frac{EI\pi^4}{l^3}\left(0.216a_1^2-1.080a_1a_2+11.01a_2^2\right)-\frac{P\pi^2}{4l}\left(a_1^2+9a_2^2\right)$$

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Now I can try 2 term solution, I will take A1 sine pi X/L + A2 sine 3 pi X/L, now if the beam where to be uniform, okay sine pi X/L will be an Eigen function because it is a true mode shape of the, buckling mode shape of the beam. On the other hand if the since the beam is inhomogeneous sine pi X/L will be a comparison function, so I am considering these two comparison functions, and A1 and A2 are the generalized coordinates, so I can substitute this and perform this integration if I do that I get this as my total potential in terms of generalized coordinates A1 and A2.

$$V(a_1,a_2)=\frac{EI\pi^4}{l^3}\left(0.216a_1^2-1.080a_1a_2+11.01a_2^2\right)-\frac{P\pi^2}{4l}\left(a_1^2+9a_2^2\right)$$

Equillibrium: $\dfrac{\partial V}{\partial a_1}=0, \dfrac{\partial V}{\partial a_2}=0$

$$\begin{bmatrix} 0.432-\dfrac{Pl^2}{2EI} & -1.080 \\ -1.080 & 22.032-\dfrac{9Pl^2}{2EI} \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}=0$$

Stability: Hessian matrix is positive definite

$\Rightarrow$ Critical condition: $|H|=0$

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Now a condition for minimum, sorry equilibrium is dou V/dou N, dou A1 = 0, dou V/dou A2 = 0, so if I do that I get this matrix equations and the A1 = 0, and A2 = 0 is the equilibrium position is it stable, so I have to construct the Hessian matrix and evaluated the solution A1 = 0 and A2 = 0 and see whether it is positive, so the critical condition of course will be the determinant of H is 0, so if I do that I get the critical value for P to be given by this number.

$|H| = 0 \Rightarrow$

$$\begin{vmatrix} 0.432 - \frac{Pl^2}{2EI} & -1.080 \\ -1.080 & 22.032 - \frac{9Pl^2}{2EI} \end{vmatrix} = 0 \Rightarrow P^C = \frac{0.735\pi^2 EI}{l^2}$$

Note: $P^C = \frac{0.864\pi^2 EI}{l^2}$ (If only one term is retained, that is , a_2=0)

$$P_c^{\text{Exact}} = \frac{0.65\pi^2 EI}{l^2}$$

(retaining the second term reduces the error from 33% to 13%)

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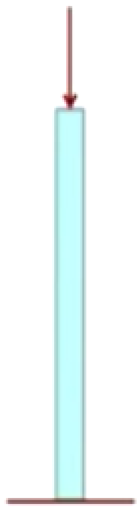
Now here of course if I retain only if the first term I can put A2 = 0, and the critical load corresponding to that will be this term equal to 0 which will be 0.864 pi square EI/L square, that is I am forcing A2 to be 0, so if only one term is retained I get the answer is 0.864 into this multiplier, if two terms are retained I get an answer 0.735 into this, it so happens that the exact solution for this problem is 0.65 pi square EI/L square, this would mean that retaining the second term reduces the error from 33% to about 13%, so it offers advantage.



$$V = \frac{1}{2}\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx - \frac{1}{2}\int_0^L P\left(\frac{dy}{dx}\right)^2 dx$$

$$y(x) = ax^2 \Rightarrow y'(x) = 2ax,\ y''(x) = 2a$$

Note: $y(x)$ = admissble

$$V = \frac{1}{2}\int_0^l EI(2a)^2 dx - \frac{1}{2}\int_0^L P(2ax)^2 dx = a^2\left(2EIl - \frac{2Pl^3}{3}\right)$$

Equillibrium: $\frac{\partial V}{\partial a} = 0 \Rightarrow 2a\left(2EIl - \frac{2Pl^3}{3}\right) = 0 \Rightarrow a = 0$

Stability: $\frac{\partial^2 V}{\partial a^2} = 2\left(2EIl - \frac{2Pl^3}{3}\right) > 0 \Rightarrow P < \frac{3EI}{l^2}$

$P_{cr} = \frac{3EI}{l^2}$ Note: $P_{cr}^{\text{Exact}} = 2.467\frac{EI}{l^2}$

Exercise:

Select $y(x) = a\phi(x)$,

where $\phi(x)$ = deflected profile of the beam under its own weight.

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Now we can consider a few more examples so that the ideas get fixed in our mind, so we consider now a cantilever beam loaded axially through load P as shown here, so this is the expression for the total potential I will try and take a trial function AX square, so Y prime X is 2X, Y double prime is 2A and YFX is admissible because at the free end we need conditions on bending moment and shear force that is will not be satisfied by this, so substitute I will get V to be this, so dou V/dou A = 0 is the condition for equilibrium and that means A = 0 is the equilibrium position. Now differentiate this with respect to A, again dou V/dou A and look at the sign of this quantity at A = 0, if I do this for A = 0 to be stable I get the condition that P must be less than 3I/L square, so according to this single term approximation the critical load is 3I/L square whereas the exact solution is 2.467EI/L square, so we can improve upon this solution by taking Y(x) to be A into phi X, where phi X is a deflected profile of the beam under its own weight, by that in this case what is meant is you have to consider the deflected profile of the beam this is phi(x), I don’t mean the weight in this direction of the applied load, but in this direction, okay, so that might offer some improvement.

Geometric BCs: $y(0)=0, y'(0)=0, y(l)=0$

$$y(x)=a_0+a_1x+a_2x^2+a_3x^3$$

$$y'(x)=a_1+2a_2x+3a_3x^2$$

$$y(0)=0 \Rightarrow a_0=0$$

$$y'(0)=0 \Rightarrow a_1=0$$

$$y(l)=a_2l^2+a_3l^3=0 \Rightarrow a_2=-a_3l$$

$$\Rightarrow y(x)=a\left(x^3-lx^2\right)$$

$$y'(x)=a\left(3x^2-2lx\right)$$

$$y''(x)=a(6x-2l)$$

$$\Rightarrow V(a)=2EIa^2l^3-\frac{1}{15}Pa^2l^5 \Rightarrow P_{cr}=30\frac{EI}{l^2}$$

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Now we will consider one more example, we'll consider 2 alternative trial functions, one in which the geometric boundary conditions are satisfied, and another one in which the force boundary condition is also satisfied, so let us consider this propped cantilever carrying this axial load P, the geometric boundary conditions are Y(0) is 0, Y prime(0) is 0 and Y(l) is 0, so I will now select a trial function which satisfy these 3 conditions, so I will start with Y(x) is A naught + A1X + A2X square + A3X cube, and I will differentiate this and by using these 3 conditions, I arrive at trial function which is Y(x) is A(X cube - LX square) okay, this satisfies all the boundary conditions, geometric boundary conditions.

You know substitute into the expression for total potential and if you can perform the requisite calculation I get P critical as 30EI/L square, so I have skipped those steps you can fill it up.

Geometric BCs: $y(0)=0, y'(0)=0, y(l)=0$

Consider the force BC while constructing the trial function.

$$y(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4$$

$$y'(x)=a_1+2a_2x+3a_3x^2+4a_4x^3$$

$$y(0)=0\Rightarrow a_0=0$$

$$y'(0)=0\Rightarrow a_1=0$$

$$y(x)=a_2x^2+a_3x^3+a_4x^4$$

$$y'(x)=2a_2x+3a_3x^2+4a_4x^3$$

$$y''(x)=2a_2+6a_3x+12a_4x^2$$

$$y(l)=0\ \&\ y''(l)=0$$

$$a_2=\frac{3a_4l^2}{2}, a_3=\frac{5a_4l}{2}\Rightarrow y(x)=al^4\left[\left(\frac{x}{l}\right)^4-\frac{5}{2}\left(\frac{x}{l}\right)^3+\frac{3}{2}\left(\frac{x}{l}\right)^2\right]$$

$$\Rightarrow V(a)=\frac{1}{2}EIa^2l^4\frac{9l}{5}-\frac{P}{2}a^2l^6\frac{3}{35}l\Rightarrow P_{cr}=21\frac{EI}{l^2}$$

Now on the other hand if I now consider the force boundary condition also while constructing the trial functions, so I will now assume a fourth order polynomial for Y(x), so 4 of these constants I will evaluate by imposing the 4 boundary conditions and the fifth one will be treated as the generalized coordinate, if I do that the polynomial that I get eventually will be of this form Y(x) is AL to the power of 4, X/L to the power of 4, etcetera as shown here, so this trial function is a comparison function, it satisfies the geometric boundary conditions as well as the force boundary condition. The force boundary condition is Y double prime L = 0, that EIY double prime L must be equal to 0 which is the bending moment.

Now with this I get the potential V(a) as shown here, and I use the two axioms and check the sign of the Hessian, check the condition for Hessian to be positive definite, and I get P critical to be 21EI/L square, so summary is in first choice I had a trial function which satisfies

Summary

Choice 1 (Geometric bcs satisfied; force bc not satisfied) $P_{cr} = 30\dfrac{EI}{l^2}$

Choice 2 (Geometric & force bcs) $P_{cr} = 21\dfrac{EI}{l^2}$

$P_{cr}^{\text{Exact}} = 20.14\dfrac{EI}{l^2}$

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geometric boundary condition, but force boundary condition was not satisfied, and I got a P critical of 30EI/L square.

In the second choice both geometric and force boundary conditions were satisfied and I got this answer, the true answer is exact answer is 20.14EI/L square, so you can see that the extra effort that we made in achieving, in satisfying the boundary condition is dividends here.

Galerkin's method

Starting point: governing field equation

$$\frac{d^2}{dx^2}\left[EI\frac{d^2 y}{dx^2}\right]+P\frac{d^2 y}{dx^2}=0$$

$$y(x)=\sum_{n=1}^{N}a_n\phi_n(x) \qquad \left[\ \phi_n(x):\ \text{admissible}\right]$$

$$R(x)=\frac{d^2}{dx^2}\left[EI\sum_{n=1}^{N}a_n\phi_n''(x)\right]+P\sum_{n=1}^{N}a_n\phi_n''(x)$$

Method of weighted residuals

$$\int_0^l R(x)\phi_k(x)\,dx=0,\ k=1,2,\cdots,N$$

$$\sum_{n=1}^{N}a_n\int_0^l\left\{EI\phi_n''(x)\right\}''\phi_k(x)\,dx+P\sum_{n=1}^{N}a_n\int_0^l\phi_n''(x)\phi_k(x)\,dx=0,\ k=1,2,\cdots,N$$

$$\sum_{n=1}^{N}a_nK_{nk}+P\sum_{n=1}^{N}a_nJ_{nk}=0,\ k=1,2,\cdots,N$$

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Now we talked about Rayleigh-Ritz method, we could also use the Galerkin's method, here the starting point will be the governing differential equation itself, so let us consider inhomogeneous beam where EI is a function of X, and the governing equilibrium equation is given by this, this can be obtained based on application of Hamilton's principle and we will be able to get this with along with all the appropriate boundary conditions, so that would be the starting point for applying Galerkin's method.

Now I assume Y(x) to be in given by this series AN phi N(x), where phi N(x) need to be at least admissible, so if I substitute this into this equation, the equation won't be satisfied we will be left with a residue. Now as we have seen in method of weighted residuals we weighted sum of this residue is taken to be 0 and the weighting function in Galerkin's method is taken to be the trial function itself, and I get a set of N equations, so if we multiply by phi K and integrate from 0 to L, I will get these terms and I can evaluate this term by parts and use the boundary conditions, and I will be able to simplify this I have not shown the simplification we have done that when we discussed the Galerkin's method in the context of vibration analysis, so I will get equations in this form, so the new thing is there is one more integral JNK given by this, here also we can simplify this by integrating once, then this is for K = 1 to N, and in a matrix form I can write it as KA + P(JA) = 0, and this matrix J is known as stability matrix.

$$\sum_{n=1}^{N} a_n K_{nk} + P\sum_{n=1}^{N} a_n J_{nk} = 0, k = 1, 2, \cdots, N$$

Eigenvalue problem

$$[K]\{a\} + P[J]\{a\} = 0$$

For nontrivial solutions $|K + PJ| = 0$

Leads to estimates of the N values of P at which the system admits a neighbouring equillibrium state.

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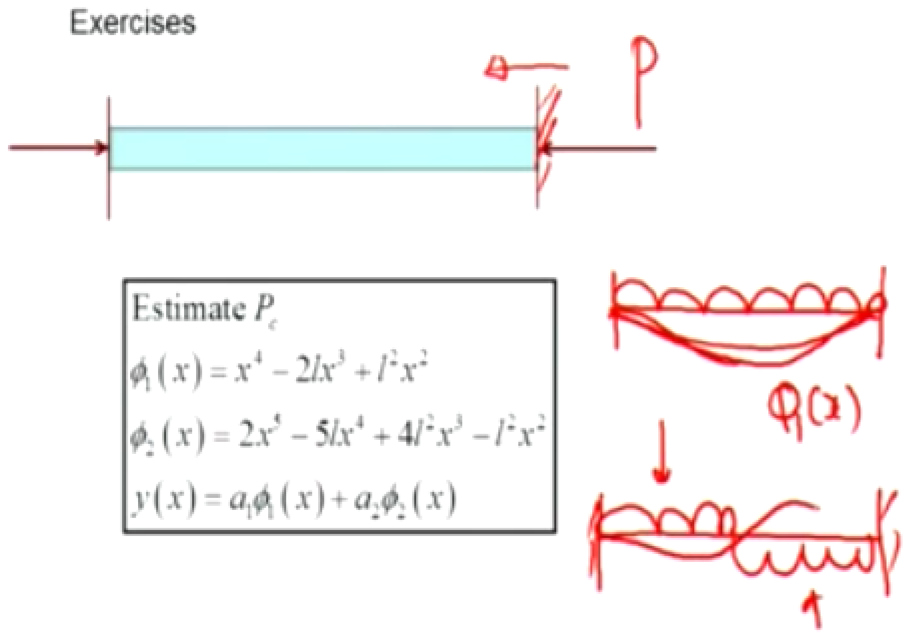
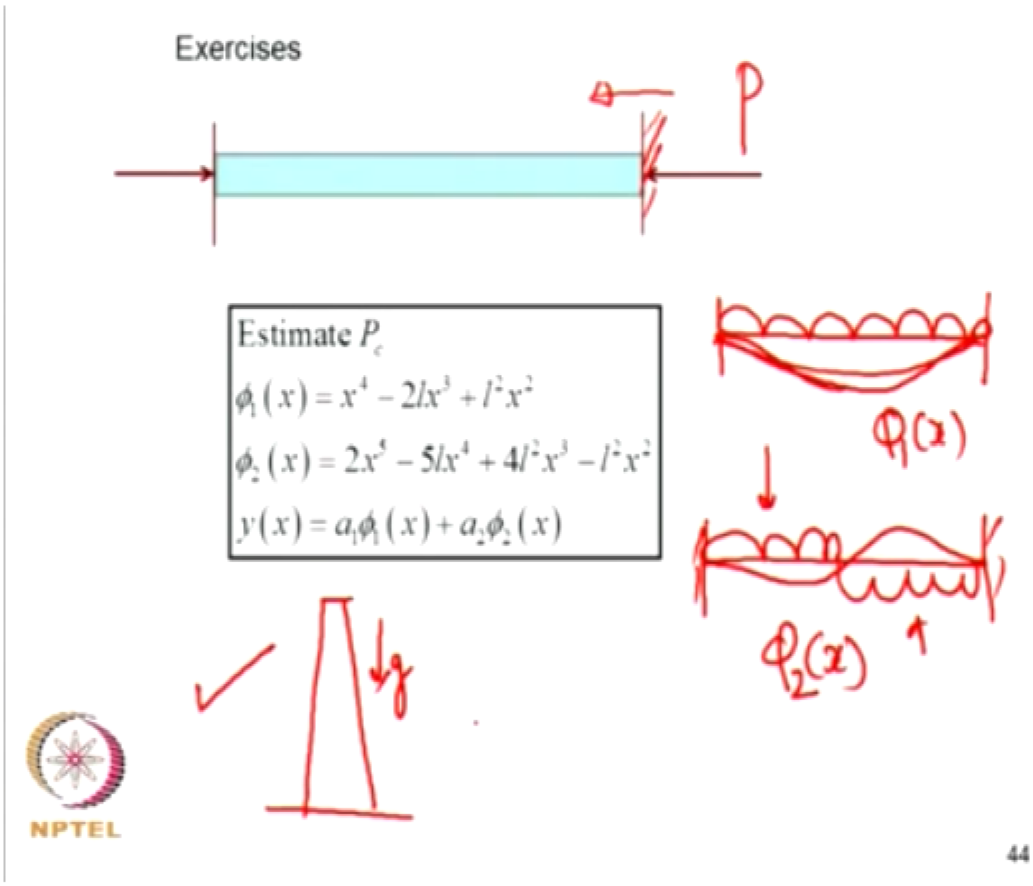
We want now, if you want a neighboring equilibrium position and non-trivial solution is needed for A, so the condition for non-trivial solution is the determinant of K + PJ must be equal to 0, so this and this is an eigenvalue problem with P as the eigenvalue, and A the deflected profile is the eigenvector, so this leads to estimates of N values of P at which the system admits a neighboring equilibrium state, okay, so this is Galerkin's method. So the Galerkin method you



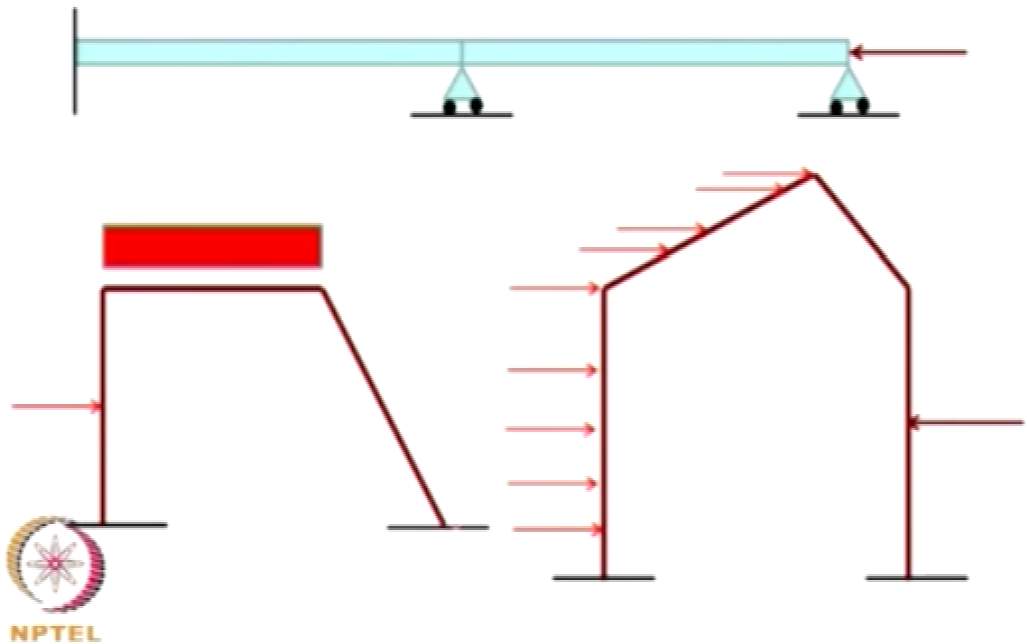
know we can use it for few examples, for example if you consider a beam fixed at the two ends carrying axial loads as shown here, again I want to recall that this fixity convention that we are using here should be taken to mean that the translation and rotation here are 0, but the axial deformations are permitted, otherwise this load will not do any work on the beam and that is not what is meant by writing this fixity condition.

Now how do we estimate, the problem here is to estimate PC, that is the critical value of this P by using 2 trial functions Galerkin's method, and these 2 trial functions the way they are derived is in one case I apply a UDL and find the solution, this is phi 1(x), in the other case I apply half the beam load this way, and other half this way that means this load acts upwards, this load acts downwards, so the deflected profile will be something like this, so this is phi 2(x) and they are explicitly given here and I leave it as an exercise that you use Galerkin's method and evaluate the P critical value.



Now there can be other examples that can be used, for example we consider the cantilever beam deflecting under its own weight, so we found out the possibility of the structure buckling under its own weight, so this problem can again be tackled using Galerkin's method, this is also left as an exercise.

How about these problems?



Constructing globally valid trial functions become increasingly difficult 45

Now so far we have considered simple structural elements, but when we come across built up structures like a 2 span beam or a portal frame or industrial shed carrying various types of loads, how do we proceed? How do we determine the equilibrium position and their stabilities? So here again if you can construct the global trial functions, the trial functions which are valid all across the structure you can still use Rayleigh-Ritz or Galerkin's type of approach, but they become increasingly unwieldy, so this situation we have encountered when we started discussing application of finite element method for vibration problems, so what was our remedy at that time? What we did was a structure like this is discretized into elements and the nodal displacement degrees of freedom, the nodal displacement values were taken as degrees of freedom and the field variables within an element were interpolated using the nodal values of the degrees of freedom at the nodes, and then we formulated the mass and stiffness matrix, we can do the same thing here instead of using global trial functions we can construct trial functions which are polynomials over pieces of these structural elements, and then demand continuity of deformation etcetera across these elements, so that is basically the idea of using finite element method for this type of problems.

Now how these ideas are developed is what we will consider in the next lecture, so we will close this lecture at this point.

Programme Assistance

Guruprakash P

Dipali K Salokhe

Technical Supervision

B K A N Singh

Gururaj Kadloor
Indian Institute of Science
Bangalore