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<u>Course Title</u>

Finite element method for structural dynamic And stability analyses

Lecture – 27

Nonlinear dynamical systems, Fixed points and bifurcations

By

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Finite element method for structural dynamic and stability analyses

Module-9

Structural stability analysis

Lecture-28 Nonlinear dynamical systems, fixed points, and bifurcations

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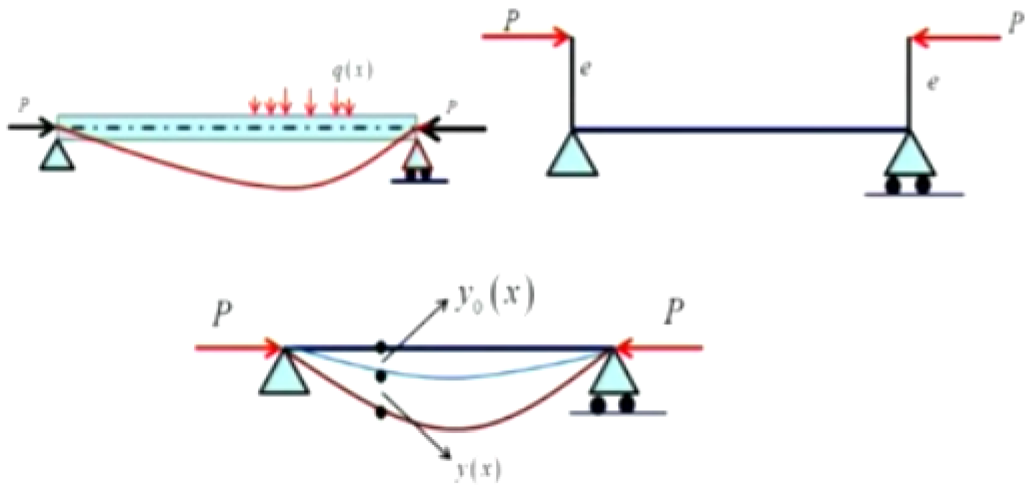
We will continue with the discussion on structural stability analysis, I would like to introduce in this lecture some elementary notions about fixed points and bifurcations in nonlinear dynamical systems.



These three problems are mathematically equivalent.
The
- transverse load
- eccentrically applied axial load
- initial imperfections

are manifestations of departures from an ideal situation.
How about the study of the ideal situation itself?

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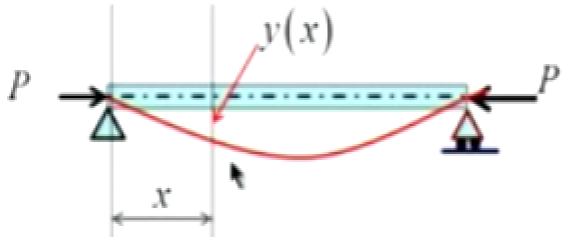
See we will quickly recall what we have been doing, we considered a beam columns, and we considered three situations which were departure from ideal situation like beam carrying axial load plus transverse loads, and this axial load being applied eccentrically or the axial load is applied on a beam which is not initially perfect, there is an initial imperfection Y naught (x), we showed that all these three problems are mathematically equivalent, and for certain values of P the response of all these three systems become large the structure loses its stability, then we ask this question all these three cases represented departure from an ideal situation, why such things happen, if to answer that we need to consider the study of ideal situation itself, then we showed



$$EI\frac{d^2y}{dx^2} + Py = 0; y(0) = 0; y(L) = 0$$

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} = 0; y(0) = 0, y(l) = 0; y'(0) = 0, y'(l) = 0$$

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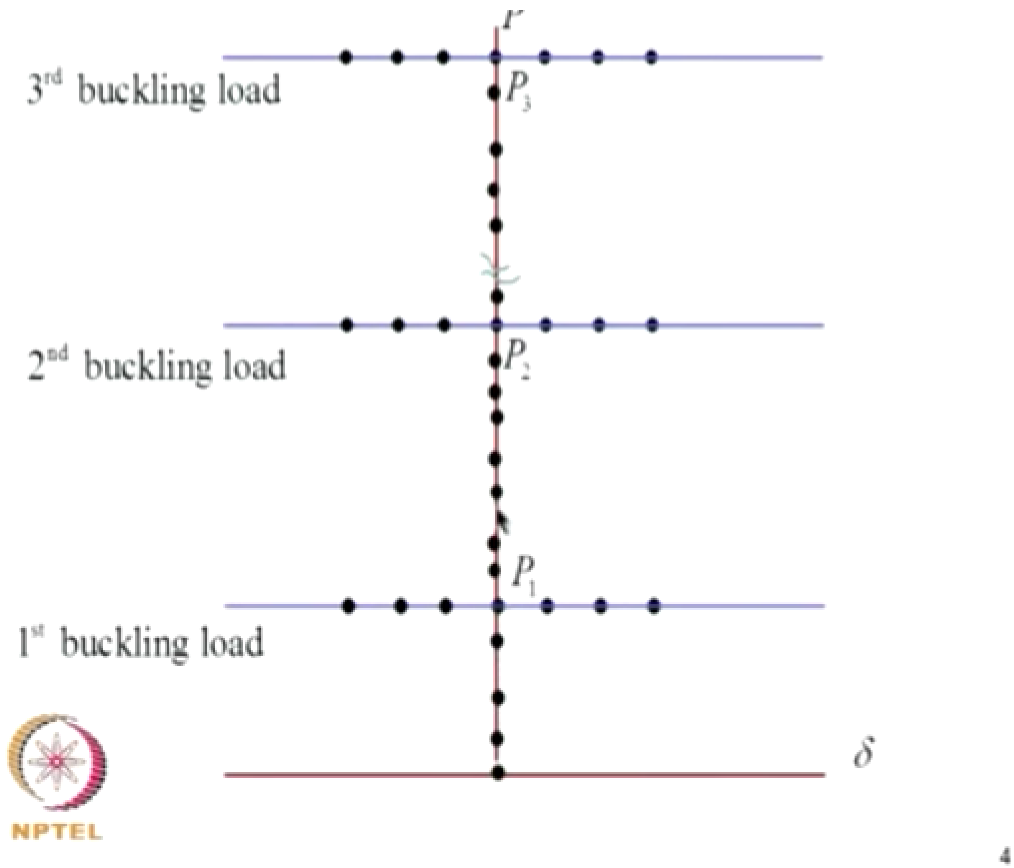
that if we consider a perfect straight beam which is loaded truly axially, and we investigated the condition under which an equilibrium position in the neighborhood of Y = 0 is possible, so assuming that such a solution is possible we searched for the value of P which can lead to that, so under the assumption that a neighborhood, an equilibrium position in the neighborhood is possible we were able to write the equilibrium equation for determinate systems we could write in the terms of bending moments, whereas for statically indeterminate systems we need to use the fourth order equation.

So I pointed out a couple of times that this term PY, that is P into this distance Y, this force is computed, this bending moment is computed with respect to the deformed configuration of the structure, whereas while finding reactions and other terms associated with the equations of motion we consider the un-deformed geometry so that difference we have to notice and we need to see later on how does that manifest in a more generic formulation of such type of problems.



Then we plotted the mid-span transverse deflection versus the applied loads and we showed that the load deflection path has infinitely many branches and as the load is increased at this point, at newer equilibrium states become possible and this is the overall profile of the equilibrium load deflection path.

Nonlinear dynamical systems, fixed points, and stability

Consider free vibration of sdof, nonlinear dynamical system governed by

$$\ddot{x} + 2\eta\omega\dot{x} + \omega^2 x + \varepsilon g(x,\dot{x}) = 0; x(0) = x_0, \dot{x}(0) = x_0$$

Define

$$y_1(t) = x(t)$$

$$y_2(t) = \dot{x}(t)$$

$\Rightarrow$

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = \ddot{x}(t) = -2\eta\omega\dot{x} - \omega^2 x - \varepsilon g(x,\dot{x}) = -2\eta\omega y_2 - \omega^2 y_1 - \varepsilon g(y_1, y_2)$$

$\Rightarrow$

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = -2\eta\omega y_2 - \omega^2 y_1 - \varepsilon g(y_1, y_2)$$

Now what we will do now is we will start considering certain issues related to dynamical systems and then we will relate all these to problems of stability analysis of structures as we go on. So for purpose of discussion we'll start with a free vibration analysis of a single degree freedom nonlinear dynamical system which is governed by this equation X double dot + 2 eta omega X dot + omega square X + epsilon into G(x, x dot) this is a nonlinear term which is equal to 0, we assume that this nonlinear term is a function of instantaneous values of X and X dot, it could as well be that this X could be function of tau, where tau varies from 0 to T which is the current time in which case the system will have hereditary non-linearity, but right now we need not consider that we are not going to consider that, so what we will do is we will introduce a new set of dependent variables I will call X(t) as Y1(t) and X dot(t) as Y2(t), now clearly Y1 dot(t) is X dot(t), but X dot(t) is Y2(t), therefore Y1 dot is Y2, Y2 dot(t) is X double dot(t), X double dot(t) we derived from the governing equation, so for X double dot we write - 2 eta omega X dot - omega square X - epsilon G (x, x dot), so that is what we are writing and in terms of Y1, Y2 this becomes, the equation for Y1 and Y2 thus can be written in a compact manner like this, this is a pair of first order equations which are mutually coupled, so Y1 has terms containing Y2, and Y2 has terms containing Y1.

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = -2\eta\omega y_2 - \omega^2 y_1 - \varepsilon g(y_1, y_2)$$

In general we consider equations of the form

$$\dot{x} = f(x, y)$$

$$\dot{y} = h(x, y)$$

$$\text{system state} = X = \begin{Bmatrix} x \\ y \end{Bmatrix}$$

Fixed points: system states become time invariant $\Rightarrow$

$$\dot{x} = f(x, y) = 0$$

$$\dot{y} = h(x, y) = 0$$

Fixed points are thus roots of the equations

$$f(x, y) = 0$$

$$h(x, y) = 0$$

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Now in general we consider equations of the form, suppose X and Y are the 2 dependent variables, and I consider a pair of first order equations X dot is some F(x,y), and Y dot is some H(x,y), we define the system state as a vector as X which is consist of X and Y. Now we define what are known as fixed points for this system, the fixed points are the system states at which the system state becomes time invariant, by that I mean those values of X and Y for which X dot and Y dot are 0, so that would mean at the fixed points we have F(x,y) = 0, H(x,y) = 0, so fixed points are those roots of this pair of equations.

$$\ddot{x}+\omega^2 x=0; x(0)=x_0, \dot{x}(0)=x_0$$
$$\dot{y}_1=y_2$$
$$\dot{y}_2=-\omega^2 y_1$$
Fixed point: $(0,0)$

$$\ddot{x}+2\eta\omega\dot{x}+\omega^2 x=0; x(0)=x_0, \dot{x}(0)=x_0$$
$$\dot{y}_1=y_2$$
$$\dot{y}_2=-2\eta\omega y_2-\omega^2 y_1$$
Fixed point: $(0,0)$

$$\ddot{x}-\alpha x+\beta x^3=0; x(0)=x_0, \dot{x}(0)=x_0$$
$$\dot{y}_1=y_2$$
$$\dot{y}_2=\alpha y_1-\beta y_1^3=y_1\left(\alpha-\beta y_1^2\right)$$
Fixed points: $(0,0),\left(0,\pm\sqrt{\frac{\alpha}{\beta}}\right)$

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What do they mean? So we will consider some simple examples a linear undamped single degree freedom system, Y1 dot is Y2, Y2 dot is - omega square Y1, so Y2 = 0, and Y1 = 0 are the fixed points, you consider a damped system X double dot + 2 eta omega X dot + omega square X = 0, I have Y1 dot = Y2, Y2 dot = - 2 eta omega Y2 - omega square Y1, again fixed points are Y2 = 0, and Y1 = 0, so origin is a fixed point.

Now we will consider another system X double dot - alpha X + beta X cube = 0, so Y1 dot is Y2, and Y2 dot is alpha Y1 - beta Y1 cube, so here Y1 can be pulled out I write it as alpha - beta Y1 square, so the fixed points are Y2 = 0, and this term equal to 0 means there are 2 possibilities Y1 = 0 or Y1 = plus or minus square root of alpha by beta, so this system has 3 fixed points, right, 0 0, 0, plus this and 0, minus this, now these are linear differential equations, this are nonlinear differential equation, so one thing that we can observe immediately is for systems governed by linear systems there is only one fixed point, whereas for a non-linear system there can be more than one fixed point.

Questions

- What are the fixed points of a dynamical system?
- How many fixed points can a system have?
- What happens if motion in the neighbourhood of a fixed point is perturbed?
- What happens if values of system parameters are varied?
    - Do the number of fixed points remain unaltered?
    - Do the nature of the motion in the neighbourhood of fixed points change because of changes in values of the system parameters?

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Now what are these fixed points of a dynamical system? What do they mean? What is their significance? And how many fixed points can a system have? What happens if motion in the neighborhood of a fixed point is perturbed? Okay, you have X dot = 0, Y dot = 0, now I slightly perturb the motion, will the motion also die out or it will grow or it will remain unbounded I mean neither decays to 0 nor becomes unbounded, what happens? Next, what is the role of system parameters in deciding upon the number of fixed points, for example what happens if values of system parameters are varied, do the number of fixed points remain unaltered that you can quickly see here beta equal to 0 the system has becomes linear, and there will be only one fixed point, but beta naught = 0 there will be 3 fixed point, so by changing beta I am changing the number of fixed points that is quite possible. Do the nature of the motion in the neighborhood of fixed points change because of changes in values of system parameters, okay, so these questions we need to understand and of course relate them to the larger issue of stability of systems. So let’s consider a linear undamped single degree freedom system so as we

$$\ddot{x}+\omega^2 x=0; x(0)=x_0, \dot{x}(0)=\dot{x}_0$$

$$\dot{y}_1 = y_2$$

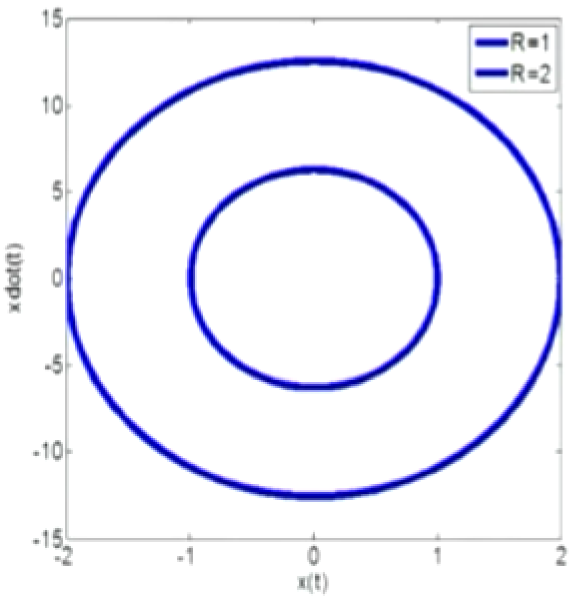
$$\dot{y}_2 = -\omega^2 y_1$$

Fixed point: $(0,0)$

$$x(t) = R\cos(\omega t-\theta)$$

$$\dot{x}(t) = -R\omega\sin(\omega t-\theta)$$

$$x^2+\frac{\dot{x}^2}{\omega^2}=R^2$$



Phase portrait

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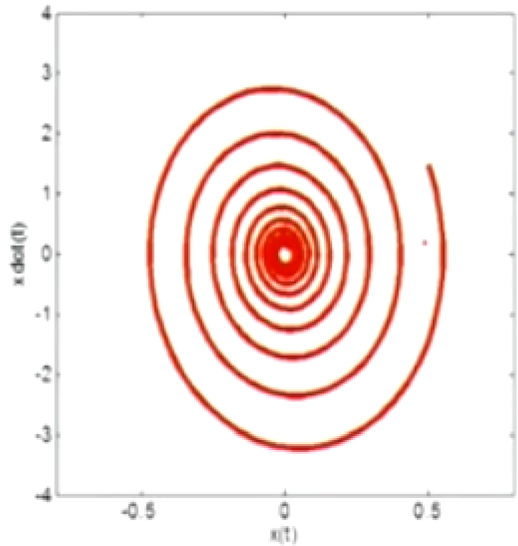
saw the governing equation is X double dot + omega square X = 0, and we will assume that the system starts from some initial condition X naught and X naught dot.

Now again I will write Y1 is Y2, Y2 dot is - omega square Y1, fixed point is 0,0, now we can solve this equation so X(t) we know it is given by R cos omega T – theta, and X dot(t) is given by this, where R and theta depend on these 2 initial condition, if initial conditions are changed R and theta would change. Now what I can do now is I can plot X(t) and X dot(t) as a pair of time histories, that is what we have been doing, but instead I can also plot X dot on one axis and X(t) on another axis and eliminate time from these plots, so if we do that what happens if we can quickly see here if you write X square + X dot square/omega square you can quickly see that this is R square, so that means if I plot X dot and X, or X dot/omega and X this will be a closed curve and it will be a circle if this scale is on Y axis I plot X dot/omega instead of just X dot, okay, so that means at T = 0, we start the motion somewhere here and as time advances this point moves in the face, this plane is known as phase plane, and this point moves in the phase plane and traces a closed curve.

For another set of initial conditions it raises another closed curve, okay, so they can exist by slight change of initial conditions in another closed curve in the vicinity of this, if I change the initial conditions slightly, okay this has some significance we will see in due course what it is.

$$\ddot{x}+2\eta\omega\dot{x}+\omega^2x=0;$$
$$x(0)=x_0,\ \dot{x}(0)=\dot{x}_0$$
$$\dot{y}_1=y_2$$
$$\dot{y}_2=-2\eta\omega y_2-\omega^2 y_1$$
Fixed point: $(0,0)$



$$x(t)=\exp(-\eta\omega t)\left(A\cos\omega_d t+B\sin\omega_d t\right)$$
$$\dot{x}(t)=-\eta\omega x(t)+\left(-A\omega_d\sin\omega_d t+B\omega_d\cos\omega_d t\right)$$
$$A=x_0$$
$$B=\frac{\dot{x}_0+\eta\omega x_0}{\omega_d}$$

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Now let’s consider damped system, so what is happening here? There is no dissipation in the system and both displacement and velocity are sinusoidal and they never decay to 0, and in the phase plane it raises a closed curve, so in the phase plane when I plot different trajectories of the system with different initial conditional, different system parameters and things like that what we get here is known as a phase portrait of the system.

Now if we consider now a damped single degree freedom system, again in this case we know the exact solution, so 0,0 is the fixed point and if I now again find X(t) and X dot(t), X(t) is given by E raised to – eta omega T into A cos omega DT + B sine omega DT, I am assuming under damped system, and X dot(t) will be given by this, where by using initial conditions I can find A and B, so now if the motion is initiated somewhere here in the phase plane X dot versus X(t) plane, it spirals and decays to 0, so how does the time history look like? This is typical plot of X(t), so X dot(t) will be something similar and when plotted and the phase plane it traces a spiral which eventually lands at the origin, and because at T tends to infinity X(t) goes to 0, X dot(t) goes to 0 and therefore this trajectory goes to 0, if you start from some another initial condition, similar another spiral will emanate and it will again as T tends to infinity goes to 0.

We can consider systems which are externally driven for example if I take a damped single

$$\ddot{x} + 2\eta\omega\dot{x} + \omega^2 x = f(t); x(0) = x_0, \dot{x}(0) = \dot{x}_0$$

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = -2\eta\omega y_2 - \omega^2 y_1 + f(t)$$

$$y = \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}$$

$$\dot{y} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\eta\omega \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ f(t) \end{Bmatrix} = Ay + F$$

$$\ddot{x} + 2\eta\omega\dot{x} + \omega^2 x + g(x, \dot{x}) = f(t); x(0) = x_0, \dot{x}(0) = \dot{x}_0$$

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = -2\eta\omega y_2 - \omega^2 y_1 - g(y_1, y_2) + f(t)$$

$$= \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} y_2 \\ -2\eta\omega y_2 - \omega^2 y_1 - g(y_1, y_2) + f(t) \end{Bmatrix} = A(y, t)$$

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degree freedom system with an external excitation F(t) I can write it as Y1 dot is Y2, Y2 dot is now - 2 eta omega Y2 - omega square Y1 + F(t), so if I now consider system states as Y1, Y2, I can write the governing equation in the so-called state space form where Y dot is given by, I will write this in the matrix form call this matrix as A, and this is A into Y + some capital F, so the governing equations here are of the form this.

On the other hand if the system has non-linear terms this type of matrix representation will not be possible, we will simply get a generic equation of the form this A is a function of Y, T, whereas here A is a matrix, whereas here it is a 2 x 1 vector of functions whereas it is 2 x 2 matrix here, so that is the difference. Now we say that system is autonomous if A is

Autonomous system

$\dot{y} = A(y); y(0) = y_0$

The system has autonomy to choose its frequency and amplitude of oscillations.

Non-autonomous system

$\dot{y} = A(y,t); y(0) = y_0$

External forcing influences the frequency and amplitude of oscillations.

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independent of T so that would happen clearly when there are no external excitations, so autonomous systems have the autonomy to choose their own frequency and amplitude of oscillations that is why I call it is autonomous. On the other hand if the system is driven externally, for example a linear system is driven externally by harmonic force at say a frequency capital Lambda, if even if the system natural frequency is omega, in the steady state the system will be obliged to oscillated lambda so it does not have the autonomy to decide its own frequency, so this in such systems A, in the representation that we have used A will be a function of time, so external forcing for example influences the frequency and amplitude of oscillation.

Equillibrium points or fixed points

$\dot{y}(t) = A(y); y(0) = y_0$

Points at which the system state is at rest. That is, $\dot{y}(t) = 0$.

These points are obtained as roots of the equation $A(y) = 0$

$\ddot{x} + \omega^2 x = 0; x(0) = x_0, \dot{x}(0) = x_0$

$\dot{y}_1 = y_2$

$\dot{y}_2 = -\omega^2 y_1$

Fixed point: $(0,0)$

$\ddot{x} + 2\eta\omega\dot{x} + \omega^2 x = 0; x(0) = x_0, \dot{x}(0) = x_0$

$\dot{y}_1 = y_2$

$\dot{y}_2 = -2\eta\omega y_2 - \omega^2 y_1$

Fixed point: $(0,0)$

$\ddot{x} - \alpha x + \beta x^3 = 0; x(0) = x_0, \dot{x}(0) = x_0$

$\dot{y}_1 = y_2$

$\dot{y}_2 = \alpha y_1 - \beta y_1^3 = y_1\left(\alpha - \beta y_1^2\right)$

Fixed points: $(0,0), \left(0, \pm\sqrt{\frac{\alpha}{\beta}}\right)$

Nonlinear systems have more than one fixed points.

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Now we will again return to the question of equilibrium points or the fixed point, thought they are used anonymously, so I consider now an autonomous system Y dot(t) = AY with Y(0) is Y naught, now we define fixed points of this system as points at which the system state is at rest that is Y dot = 0, so these points are obtained as roots of the equation A(y) = 0, so these are set of equations so as many equations as the size of Y and you will be able to solve for the roots of this equation. Again we have seen all this for a linear system we got 0 0 and undamped system, for damped system again 0 0 and here as we saw there are 3 points, so as I already pointed out nonlinear systems have more than 1 fixed points.

Stability of equillibrium points

Consider

$\dot{x} = f(x, y)$

$\dot{y} = g(x, y)$

The equillibrium points are given by the conditions

$$\left.\begin{array}{l} f(x^*, y^*) = 0 \\ g(x^*, y^*) = 0 \end{array}\right\} (x^*, y^*) \text{ : a set of equillibrium points.}$$

Let us examine the nature of motion around each of the equillibrium points by perturbing the equillibrium states by small amounts as:

$x(t) = x^* + \eta(t)$

$y(t) = y^* + \xi(t)$

$\dot{x} = \dot{\eta}(t) = f\left[x^* + \eta(t), y^* + \xi(t)\right]$

$\dot{y} = \dot{\xi}(t) = g\left[x^* + \eta(t), y^* + \xi(t)\right]$

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We now consider the question, what happens if we perturb the motion or the system states, equilibrium system states by small motion, if you apply a small force to the system when the system is at one of its equilibrium states, what happens? So let's consider X dot = F(x,y), Y dot is G(x,y), I am considering a 2 x 1 system of equation this can be generalized for more larger size problems. The equilibrium points are given by the conditions F(x*, y*) = 0, G(x*,y*) = 0, so this X*, Y* is a set of equilibrium points.

Now let us examine the nature of motion around each of the equilibrium points by perturbing the equilibrium states by small amounts, so what I will do, I will take X(t) as X* + eta(t), and X dot(t), and this is Y(t) as Y* + some XI(t). Now let us substitute this into this equation, X dot is X* dot + eta dot but X* dot is 0, because X* is a it is a fixed point, so Y* is also 0, so I get X dot(t) as eta dot(t), which is F of this, and Y dot is XI dot is equal to this. Now what I will do is I will perform a Taylor's expansion around the fixed point and retain only the first order terms,

$$\dot{\eta}(t) = f\left[x^* + \eta(t), y^* + \xi(t)\right] \approx f\left[x^*, y^*\right] + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x^* \\ y=y^*}} \eta(t) + \left.\frac{\partial f}{\partial y}\right|_{\substack{x=x^* \\ y=y^*}} \xi(t)$$

$$\dot{\xi}(t) = g\left[x^* + \eta(t), y^* + \xi(t)\right] \approx g\left[x^*, y^*\right] + \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x^* \\ y=y^*}} \eta(t) + \left.\frac{\partial g}{\partial y}\right|_{\substack{x=x^* \\ y=y^*}} \xi(t)$$

$$\dot{\eta}(t) = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x^* \\ y=y^*}} \eta(t) + \left.\frac{\partial f}{\partial y}\right|_{\substack{x=x^* \\ y=y^*}} \xi(t)$$

$$\dot{\xi}(t) = \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x^* \\ y=y^*}} \eta(t) + \left.\frac{\partial g}{\partial y}\right|_{\substack{x=x^* \\ y=y^*}} \xi(t)$$

$$\begin{Bmatrix} \dot{\eta}(t) \\ \dot{\xi}(t) \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{bmatrix}_{\substack{x=x^* \\ y=y^*}} \begin{Bmatrix} \eta(t) \\ \xi(t) \end{Bmatrix} \quad \text{(Linear homogeneous set of ODE-s)}$$

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because I am assuming that the perturbation that I am providing are small. And mind you my question, the type of question that I will answer is not to determine the time histories of eta(t) and XI(t) in all detail, the only question that I will ask is at T becomes large will eta(t) and XI(t) grow or decay or remain you know unbound neither growing nor decaying, so these are the questions that we want to answer, in which manner they decay to 0, or in which manner they grow is not of primary concern, so we are asking therefore a qualitative question about the system behavior, so we are not so much particular about performing a complete time history analysis and answering this question, so if we now perturb, use Taylor's expansion F(x*,y*) + the two gradients with respect to X and Y for the function F and G.

Now we know that for fixed point F and G are 0, F(x*,y*) is 0, G(x*,y*) is 0, so I am now left with this pair of equation eta dot is this gradient into eta(t), and this gradient into XI(t) and similarly another equation for XI(t). Now this 2 equations I can put in the matrix form, I write it as eta dot XI dot into this matrix Jacobian or gradient matrix evaluated at the fixed points and this is a vector of eta and XI(t), so these are linear homogeneous set of ordinary differential equation.

$$\begin{Bmatrix} \dot{\eta}(t) \\ \dot{\xi}(t) \end{Bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \begin{Bmatrix} \eta(t) \\ \xi(t) \end{Bmatrix} \text{ (Omit subscripts)}$$

Seek the solution in the form

$$\begin{Bmatrix} \eta(t) \\ \xi(t) \end{Bmatrix} = \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} \exp(st)$$

$$\Rightarrow s \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} \exp(st) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} \exp(st)$$

$$\Rightarrow \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = s \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} \text{ (Eigenvalue problem)}$$

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Now for sake of simplicity in writing I will drop writing this subscript that we are evaluating at X = X*, and Y = Y*, let's assume that we are doing, we need not have to write explicitly. Now that would mean this is really a constant, now we seek the solution in the form, this is a linear system homogeneous therefore an exponential is always a solution, so we seek an exponential form of solution where the parameter S is unknown, so we want to know if this solution is permissible if so for what value of S it is permissible, so I substitute into this, so I get S, alpha, beta, exponential E raise ST into this matrix alpha beta exponential of ST. Now exponential of ST cannot be 0, so I get this equation which is nothing but an eigenvalue problem associated with this matrix, so the eigenvalues are S and alpha and beta are the eigenvectors, so clearly alpha beta = 0 is a solution to this problem for all values of S, but we are interested in those values of S for which alpha and beta are nonzero, so we perform the Eigenvalue analysis, in this

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = s \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} \Rightarrow (s_1, s_2): \text{ eigenvalues; complex, in general.}$$

$$s = a + ib$$

$$\Rightarrow$$

$$\eta(t) = \alpha \exp(at + ibt) = \alpha \exp(at)(\cos bt + i \sin bt)$$

$$\xi(t) = \beta \exp(at + ibt) = \beta \exp(at)(\cos bt + i \sin bt)$$

$$\text{If } a > 0, \lim_{t \to \infty} \begin{Bmatrix} \eta(t) \\ \xi(t) \end{Bmatrix} \to \infty \Rightarrow \text{the fixed point } (x^*, y^*) \text{ is unstable}$$

$$\text{If } a < 0, \lim_{t \to \infty} \begin{Bmatrix} \eta(t) \\ \xi(t) \end{Bmatrix} \to 0 \Rightarrow \text{the fixed point } (x^*, y^*) \text{ is stable}$$

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case we get 2 eigenvalues S1 and S2, they will be complex and therefore there will be complex conjugates, so I can write the roots as S = A + IB.

Now the solution therefore can be written as some alpha into exponential AT + IBT, and this I can write it as alpha into E raise to exponential AT + into cos BT + I sine BT, similarly I will write a solution for XI(t), now this modulus of, this exponential function E raise to IBT is 1, so this also modulus is 1, therefore as time becomes large if we are interested in observing whether eta(t), X(t) grow or not we should look at how this multiplier behaves, E raise to AT will grow to infinity if A is positive, so the condition if A is greater than 0 then this limit of eta and XI, ST tending to infinity becomes infinity, and then we say that the fixed point X*,Y* is unstable, mind you this matrix is evaluated at this X*,Y*, so if A is less than 0 on the other hand, as T tends to infinity the fixed point is the perturbation die, we say that fixed point is stable.

Classification of fixed points

Node

- both eigenvalues are real and are of the same sign
- the fixed point can be stable (if roots are <0) or unstable (if roots are >0)

Saddle

- both roots are real and of different signs
- the fixed point is unstable

Focus

- the roots are complex conjugates (but not pure imaginary)
- the fixed point could be stable or unstable

Center

- roots are pure imaginary
- linearized stability analysis is inadequate to answer the question on whether the fixed point is stable or unstable

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Now depending on how the nature of these Eigen solutions, that is also a solution, we have different nomenclatures for the fixed point we say that fixed point is a node if both eigenvalues are real and are of the same sign, they can be positive or negative but both should be real that means B will be 0, the fixed points can be stable or unstable depending on the sign of these 2 roots, both roots are negative then the node is stable, if one root is positive another root is negative we get another situation. Now if both roots are real with one root real and other root imaginary, other root negative, the fixed point is unstable.

Now on the other hand the roots are complex conjugates and the fixed point could be stable or unstable depending on the sign of the real part, so if on the other hand, if roots are pure imaginary that means real part is 0 then the linearized stability analysis is inadequate to answer the question on whether the fixed point is stable or unstable, okay, we have to carry out further analysis.

$$\ddot{x} + 2\eta\omega\dot{x} + \omega^2 x = 0; x(0) = x_0, \dot{x}(0) = x_0$$

$$\dot{y}_1 = y_2 = f(y_1, y_2)$$

$$\dot{y}_2 = -2\eta\omega y_2 - \omega^2 y_1 = g(y_1, y_2)$$

Fixed point: $(0,0)$

$$\frac{\partial f}{\partial y_1} = 0; \frac{\partial f}{\partial y_2} = 1; \frac{\partial g}{\partial y_1} = -\omega^2; \frac{\partial f}{\partial y_2} = -2\eta\omega$$

Eigenvalues

$$\begin{vmatrix} -\lambda & 1 \\ -\omega^2 & -2\eta\omega - \lambda \end{vmatrix} = 0 \Rightarrow \lambda(2\eta\omega + \lambda) + \omega^2 = 0$$

$$\lambda^2 + 2\eta\omega\lambda + \omega^2 = 0 \Rightarrow \lambda = -\eta\omega \pm \sqrt{\eta^2\omega^2 - \omega^2}$$

$\eta = 0 \Rightarrow \lambda = \pm i\omega \Rightarrow$ origin is a center

$0 < \eta < 1 \Rightarrow \lambda = -\eta\omega \pm i\omega\sqrt{1-\eta^2} \Rightarrow$ origin is a stable focus

$\eta > 1 \Rightarrow \lambda = -\eta\omega \pm \omega\sqrt{\eta^2 - 1} \Rightarrow$ origin is a stable node

$\eta < 0 \Rightarrow$ origin is unstable (node/focus?)

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Now let's see what we learn from I mean if I apply these ideas to simple oscillators, what is that we can learn? So let's take the damped single degree freedom system X double dot + 2 eta omega X dot + omega square X = 0, and I introduce Y1 dot is Y2, Y2 dot is this, and this is my F and this is G, the fixed point is 0 0, and dou F/dou Y1 is 0, dou F/dou Y2 is 1, dou G/dou Y1 is - omega square, dou G/dou Y2 is – 2 eta omega, so the Eigen values of this matrix are given by this characteristic equation, and if I write this I will get lambda as - eta omega + - square root this. Now depending on the nature of this eta there are different possibilities, now if eta = 0, then lambda will be + - I omega, then we say that origin is the center, because it is a pure imaginary number. On the other hand if eta lies between 0 and 1, lambda will be given by - eta omega + - I Omega 1 - eta square, and since eta is, we were assuming that eta is positive, damping is positive it lies between 0 and 1 therefore origin is a stable focus, if eta is greater than 1 this will be, both routes will be negative and we say that origin is a stable node. If eta is less than 0, I leave it an exercise origin is unstable if negatively damped system origin will be unstable, and you can verify whether, what is the range of eta for which this would be a saddle, and node, or a focus, okay.

$$\ddot{x}+\dot{x}-x+x^3=0; x(0)=x_0, \dot{x}(0)=x_0$$

$$\dot{y}_1 = y_2 = f(y_1, y_2)$$

$$\dot{y}_2 = -y_2 + y_1 - y_1^3 = -y_2 + y_1\left(1-y_1^2\right) = g(y_1, y_2)$$

Fixed points: $(0,0), (\pm 1, 0)$

$$\frac{\partial f}{\partial y_1} = 0; \frac{\partial f}{\partial y_2} = 1;$$

$$\frac{\partial g}{\partial y_1} = \left(1-3y_1^2\right); \frac{\partial f}{\partial y_2} = -1$$

Consider $(0,0)$

Eigenvalues

$$\begin{vmatrix} 0-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} \Rightarrow \lambda(1+\lambda)-1=0$$

$\lambda = -\frac{1}{2} \pm \frac{\sqrt{5}}{2} \Rightarrow$ Origin is unstable and is a saddle

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Now let's consider now the nonlinear system, a simple nonlinear system I have X double dot + X dot - X + X cube = 0, now again I introduced Y1 dot as Y2, Y2 dot as -Y2 + Y1 - Y1 cube, where Y1 is X, and Y2 is X dot, so I get this equation, so my F is Y2 and G is - Y2 + Y1 into 1 - Y1 square, the fixed points you can see Y2 = 0, if Y2 = 0 then Y1 into 1 - Y1 square is 0, therefore Y1 equal to 0 is a fixed point, so 0,0 becomes a fixed point, and 1 - Y1 square = 0 is also a fixed point so the fixed points are origin and + - 1, 0. Now we can evaluate dou F/dou Y1 and other functions, and this will be now function of Y therefore we have to assign the right values for the equilibrium points and then evaluate the Eigenvalues, so at the origin we have the Eigenvalues to be given by this equation, and we get lambda to be this, and we can conclude that origin is unstable and is the saddle.

$$\frac{\partial f}{\partial y_1} = 0; \frac{\partial f}{\partial y_2} = 1; \frac{\partial g}{\partial y_1} = \left(1 - 3y_1^2\right); \frac{\partial f}{\partial y_2} = -1$$

Consider $(\pm 1, 0)$

Eigenvalues

$$\begin{vmatrix} 0-\lambda & 1 \\ -2 & -1-\lambda \end{vmatrix} \Rightarrow \lambda(1+\lambda) + 2 = 0$$

$$\lambda = -\frac{1}{2} \pm i\frac{\sqrt{7}}{2} \Rightarrow \text{Origin is stable focus}$$

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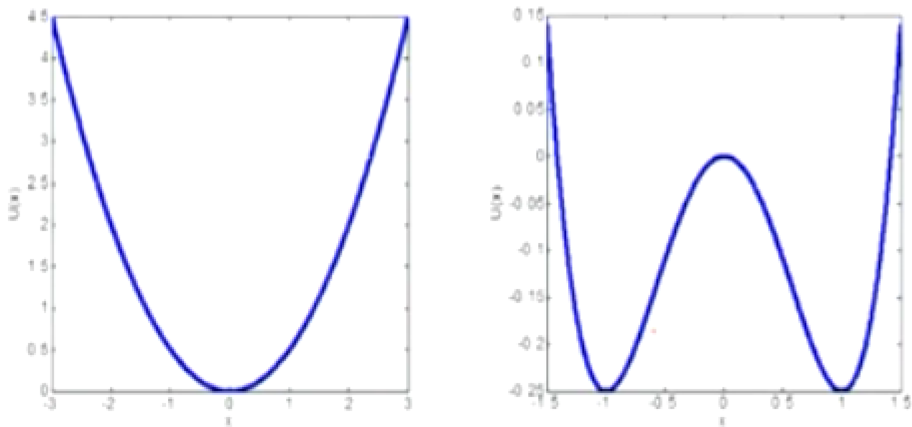
We now consider the other pair of equilibrium points, this is + - 1 and 0, and we see that there is a term Y1 square here, so it doesn't really matter whether we take plus or minus, so the Jacobian will be, this determinant will be the same, so again the characteristic equation is this and we get the lambda to be a pair of conjugate, complex conjugate numbers with negative real part, therefore we save origin is a stable focus.

$$\ddot{x} + 2\eta\omega\dot{x} + \omega^2 x = 0$$

$$V = \frac{\omega^2 x^2}{2}$$

$$\ddot{x} + \dot{x} - x + x^3 = 0$$

$$V = -\frac{1}{2}x^2 + \frac{1}{4}x^4$$



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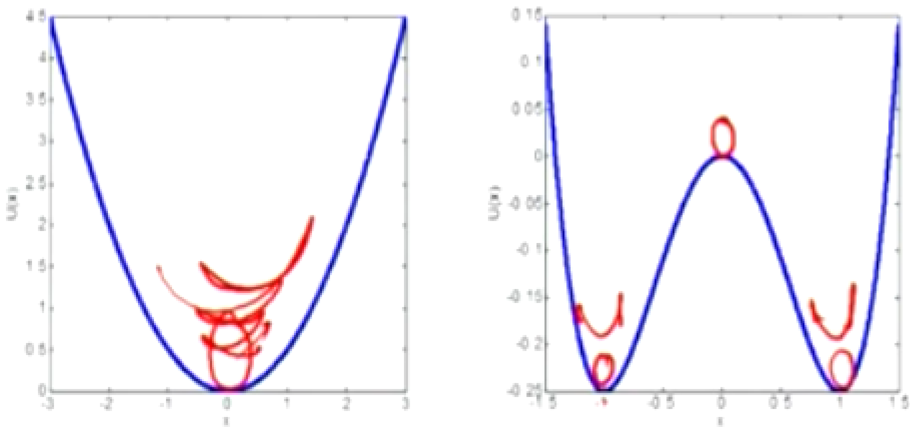
Now we can see here if I now plot this potential energy associated with this system, so that will be omega square X square/2, so if I plot U of excess versus X you get this type of a valley, so the motion of the system can be perceived as a ball rolling in this potential, okay, so what we are concluding here is the origin is stable, there is damping therefore this oscillation will continue and again it will come to rest, that means this equilibrium point I give a slight

$$\ddot{x} + 2\eta\omega\dot{x} + \omega^2 x = 0$$

$$V = \frac{\omega^2 x^2}{2}$$

$$\ddot{x} + \dot{x} - x + x^3 = 0$$

$$V = -\frac{1}{2}x^2 + \frac{1}{4}x^4$$



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perturbation, the system oscillates and it decays to 0. Whereas here origin is here and if you place an object here any slight perturbation will move the ball away from this point, so the origin is unstable, whereas if you are here small perturbations you know motion will be stable, okay.

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

$$\dot{\theta}_1 = \theta_2 = f(\theta_1, \theta_2)$$

$$\dot{\theta}_2 = -\frac{g}{l}\sin\theta_1 = h(\theta_1, \theta_2)$$

Fixed points: $(n\pi, 0), n = 0, \pm 1, \pm 2, \cdots, \pm\infty$

Eigenvalues

$$\begin{vmatrix} 0-\lambda & 0 \\ -\frac{g}{l}\cos n\pi & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + \frac{g}{l}\cos n\pi = 0$$

$n = 0, 2, 4, \cdots \lambda = \pm i\sqrt{\frac{g}{l}} \Rightarrow$ Fixed points are centers

$n=1,3,5,\cdots \lambda = \pm\sqrt{\frac{g}{l}} \Rightarrow$ Fixed points are saddles

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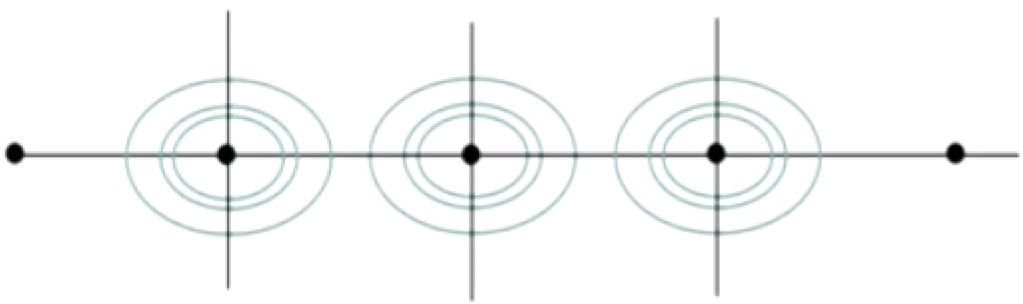
Now we will consider a simple pendulum, the governing equation is theta double dot + G/L sine theta = 0, so again theta 1 dot is theta 2, theta2 dot is – G/L sine theta 1, so this F is this, and H is this, the fixed points are theta 2 = 0, and sine theta 1 = 0, sine theta 1 = 0 means I have N pi, 0 where N varies from 0 + - 1, + - 2 etcetera, so this system has an infinite set of fixed points. Now the Eigenvalues you can evaluate all the fixed points it can be written as N pi, and by assigning different values to N we can approach different fixed points, so we will keep it as N pi, and if we perform the analysis we get this to be the characteristic equation, and for even N I get the fixed points are centers okay. Then for 0 and even numbers, so for example origin is a center, for N = 1, 3, 5 etcetera the fixed points are saddles, okay, so we can plot on this axis, the



Phase portrait

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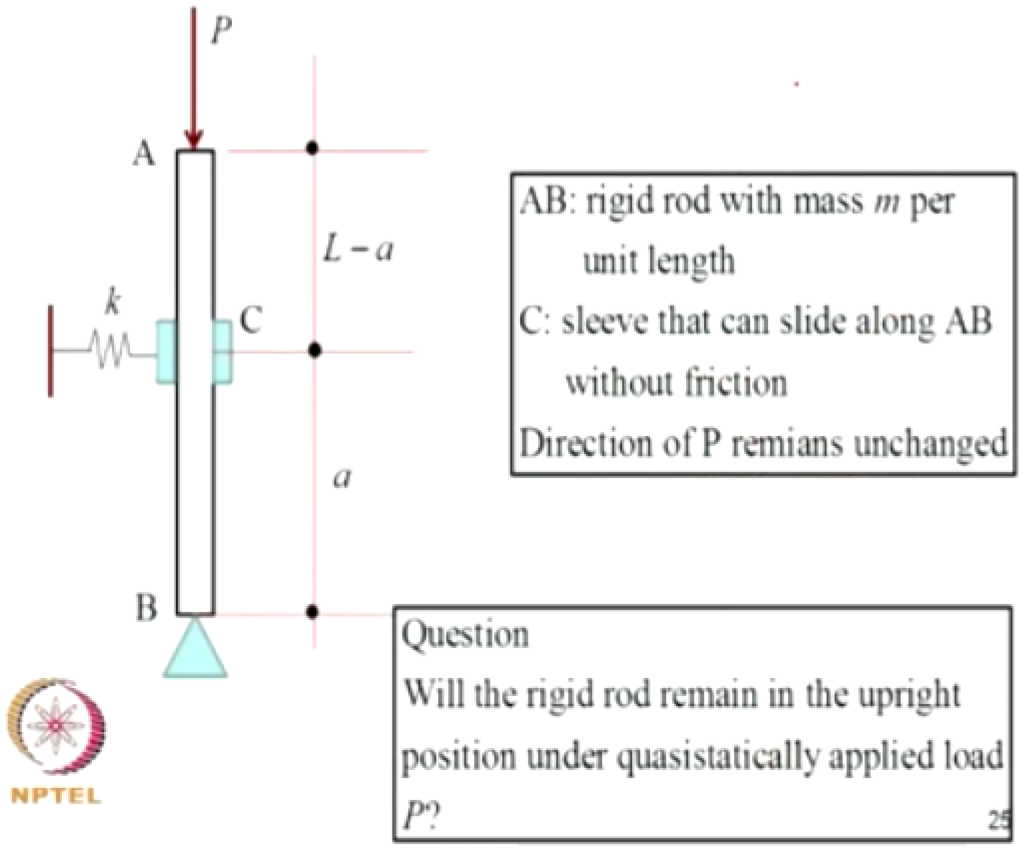
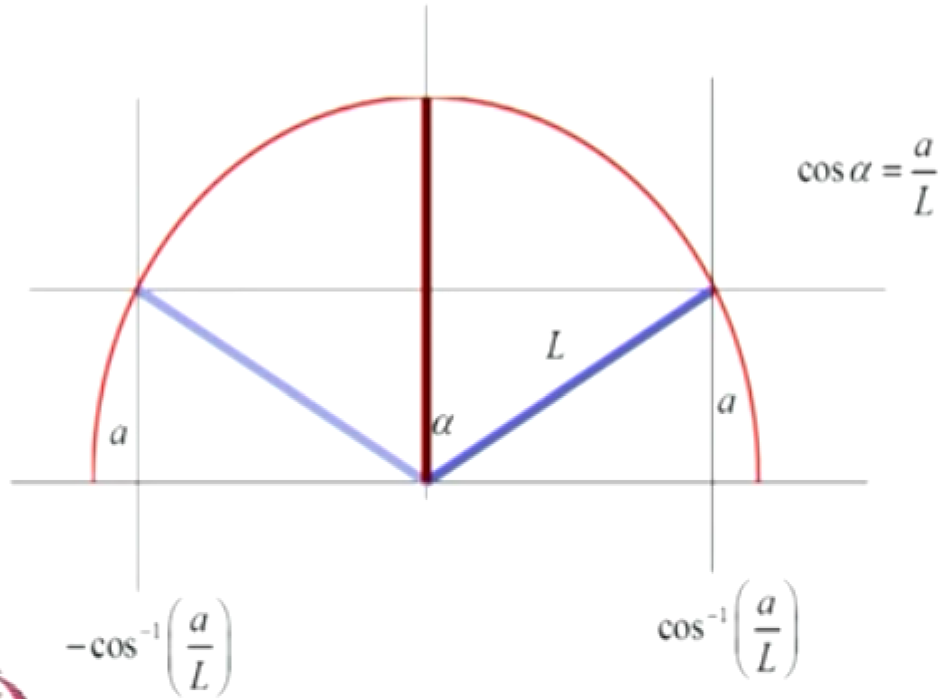
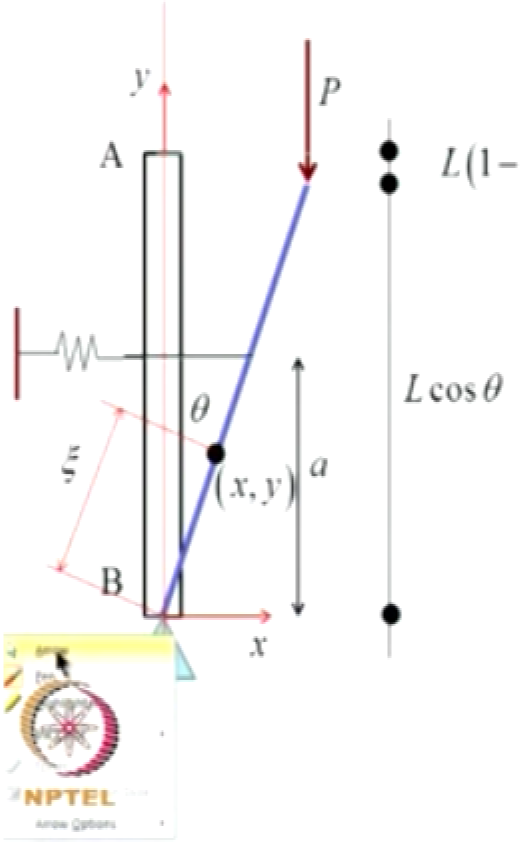
fixed point suppose this origin, suppose this is origin, now the pi will be here, - pi will be here, so if we now plot the trajectory passing through that, it will be like this, so these are, the pendulum if you see what we are saying is origin is a fixed point, and if the pendulum occupies the vertical position it is an unstable equilibrium that this is this point, since the pendulum can rotate or several you know complete motion is also permitted there are several fixed points okay, so these points that you are seeing are origin and after it completes one revolution there is a 2 pi, another revolution 4 pi so on and so forth, they are the fixed points. Whenever the pendulum returns to this vertical position we come across an unstable fixed point and they are here, so these are this, these are centers, these are saddles, and this is known as separatrix.



Now we will consider, we will apply these ideas to a mechanical system, a simple mechanical system which is made up of a rigid bar AB and there is a sleeve here and it is supported on a spring to this wall or some fixed condition here, so the position of this spring doesn't change but the sleeve can slide on this, so as a system oscillate this slide, this sleeve will be sliding on this, so now the question that we are asking is will the rigid rod remain in the upright position under quasistatically load P, okay, so for some value of P can a neighboring equilibrium position exist, is almost like asking the question on, the kind of questions we asked on beam columns we are asking here, but mind you AB is a rigid bar, and the flexibility is accounted by this presence of this discrete spring, so what we will do is we will formulate this problem as a problem in dynamics and see what we learn from this, that means will basically consider the oscillation of this bar around this point.



Now before we do that we should understand that if this bar start deflecting to the right, moment this sleeve crosses this edge our interest in this system ceases, right because that bar would have escaped from the sleeves and we are not interested in that, so we'll limit our interest to this range - cos inverse A/L to cos inverse A/L, so this range of angle, so we will



$$x = \xi \sin\theta;\ y = \xi\cos\theta$$

$$dT = \frac{1}{2} m d\xi\left(\dot{x}^2 + \dot{y}^2\right)$$

$$= \frac{1}{2} m d\xi\left\{\left(\xi\dot{\theta}\cos\theta\right)^2 + \left(-\xi\dot{\theta}\sin\theta\right)^2\right\}$$

$$= \frac{1}{2} m\xi^2\dot{\theta}^2 d\xi$$

$$T = \int_0^L \frac{1}{2} m\xi^2\dot{\theta}^2 d\xi = \frac{mL^3}{6}\dot{\theta}^2$$

$$V = \frac{1}{2} k a^2 \tan^2\theta - PL\left(1-\cos\theta\right)$$

consider the deform, one of the snapshot of the deform geometry of the bar, suppose at some point in during its oscillation it occupies this position, the load P is acting here then the sleeve would have you know, it would be originally at this distance now it would have moved up, okay, so now I want to construct the expressions for kinetic energy and strain energy and write the Lagrangian and derive have the equation of motion, so what we will do is we will consider a point X, Y and along this bar I choose a coordinate XI, so I will call X as XI sine theta, sine theta is this angle, but the axis of the deformed rod to the vertical, XI sine theta and Y is XI cos theta, so X is here, and Y is there.

So now DT is the incremental strain energy stored in a chunk D sai here at a located at a distance XI, so this is 1/2 MD sai, M is the mass per unit length X dot square + Y dot square, for X and Y I will substitute this, so I will get this equation and the kinetic energy stored in the D sai element is given by this, and the total energy will be I have to integrate from 0 to L and I will get it a ML cube theta dot square. Now the strain energy is given by the energy stored in the spring and the work done by this load, so the change in length of the spring if we include and compute this will be 1/2 KA square tan square theta, and this is contribution to the load is - PL 1 - cos theta, so I can now construct the Lagrangian T - V and if I now put it in the

$$L = \frac{mL^3}{6}\dot{\theta}^2 - \frac{1}{2}ka^2\tan^2\theta + PL(1-\cos\theta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \frac{ml^3}{3}\ddot{\theta} + \left(\frac{ka^2}{\cos^3\theta} - PL\right)\sin\theta = 0$$

Consider damped system

$$\frac{ml^3}{3}\ddot{\theta} + c\dot{\theta} + \left(\frac{ka^2}{\cos^3\theta} - PL\right)\sin\theta = 0$$

$$\dot{\theta}_1 = \theta_2$$

$$\dot{\theta}_2 = -\frac{3c}{ml^3}\theta_2 - \frac{3}{ml^3}\left(\frac{ka^2}{\cos^3\theta_1} - PL\right)\sin\theta_1$$

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Lagrange's equation and perform the necessary differentiations I get this equation.
Now you see here inertia term is linear but thus stiffness terms is highly nonlinear, there is cos cube theta here, sine theta here so it's highly nonlinear system, so for sake of discussion we will also include that there is dissipation in the system which is notionally represented through a viscous damper, so the governing equation for free vibration of this system is given by this, now I will again introduce the state space form theta 1 dot is = theta 2, theta 2 dot I will derive from this - 3C/ML cube I will divide by ML cube/3 and I get this equation, so what I wish to do is we will identify the fixed points of the system and study their stability, how many fixed points are there we will determine that first and around each of the fixed point will apply a small perturbation and see whether perturbation grows in time or not, so this can be written as

$$\dot{\theta}_1 = \theta_2 = f(\theta_1, \theta_2)$$

$$\dot{\theta}_2 = -\frac{3c}{ml^3}\theta_2 - \frac{3}{ml^3}\left(\frac{ka^2}{\cos^3\theta_1} - PL\right)\sin\theta_1 = g(\theta_1, \theta_2)$$

Fixed points

$$\theta_2 = 0, \left(\frac{ka^2}{\cos^3\theta_1} - PL\right)\sin\theta_1 = 0 \Rightarrow \theta_2 = 0, \theta_1 = 0, \sec^{-1}\left\{\frac{PL}{ka^2}\right\}^{1/3}$$

$$\frac{\partial f}{\partial \theta_1} = 0; \frac{\partial f}{\partial \theta_2} = 1;$$

$$\frac{\partial g}{\partial \theta_1} = -\frac{3}{ml^3}\left(\frac{ka^2}{\cos^3\theta_1} - PL\right)\cos\theta_1 - \frac{3}{ml^3}\sin\theta_1\left(\frac{ka^2}{\cos^6\theta_1}3\cos^2\theta_1\sin\theta_1\right)$$

$$\frac{\partial g}{\partial \theta_2} = -\frac{3c}{ml^3}$$

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theta 1 dot is theta 2, I call F theta 1, theta 2, and theta 2 dot is this and this is some G(theta 1, theta 2) the fixed points are roots of F = 0 and G = 0, if you do that theta 2 is 0, once theta 2 is 0 this term drops off and I will be getting KA square/cos cube theta 1 - PL sine theta 1 = 0, so sine theta 1 = 0 is a possibility and theta 1 is root of this equation which is secant inverse of this to the power 1/3, we are not taking N pi here because as I said already we are not interested in the bar rotating about this, our limitation, our interest is only in the range of theta lying in this, so that is why we are taking only the one of the roots of this equation theta 2 = 0, and theta 1 is this.

Consider the fixed point $(0,0)$

$$\frac{\partial f}{\partial \theta_1} = 0; \frac{\partial f}{\partial \theta_2} = 1;$$

$$\frac{\partial g}{\partial \theta_1} = -\frac{3}{ml^3}\left(\frac{ka^2}{\cos^3\theta_1} - PL\right)\cos\theta_1 - \frac{3}{ml^3}\sin\theta_1\left(\frac{ka^2}{\cos^6\theta_1}3\cos^2\theta_1\sin\theta_1\right)$$

$$= -\frac{3}{ml^3}\left(ka^2 - PL\right)$$

$$\frac{\partial g}{\partial \theta_2} = -\frac{3c}{ml^3}$$

Eigenvalues

$$\begin{vmatrix} 0-\lambda & 1 \\ -\left(ka^2 - PL\right) & -c-\lambda \end{vmatrix} = 0$$

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So now we will evaluate the matrix with Eigenvalues we need to study, so dou F/dou theta 1 is 0, dou F/dou theta 2 is this, and dou G/dou theta 1 and dou G/dou theta 2 are given by this. Now consider the fixed point 0, 0 first, so I get the gradients in this case to be dou F/dou theta 1 is 0 1, and dou G/dou theta 1 will be this quantity, and dou G/dou theta 2 is this, so the Eigenvalues of interest are to be evaluated by finding roots of this equation, if you do this I get the Eigen values as – C/2 + - 1/2 square root of this etcetera.

$$\lambda = -\frac{c}{2} \pm \frac{1}{2}\sqrt{c^2 - 4\left(ka^2 - PL\right)}$$

Suppose $c = 0$

$\left(ka^2 - PL\right) > 0 \Rightarrow \text{Re}\left(\lambda\right) > 0 \Rightarrow$ fixed point is unstable

$\left(ka^2 - PL\right) < 0 \Rightarrow \text{Re}\left(\lambda\right) < 0 \Rightarrow$ fixed point is a center

Condition for stability of $\left(0,0\right)$ is $P < \dfrac{ka^2}{L}$

Consider $\left(\sec^{-1}\left\{\dfrac{PL}{ka^2}\right\}^{1/3}, 0\right)$

When $\left(\dfrac{ka^2}{\cos^3\theta_1} - PL\right) = 0, \dfrac{\partial g}{\partial \theta_1} = -\dfrac{3ka^2 \sin^2\theta_1}{\cos^4\theta_1} = -\alpha^2$

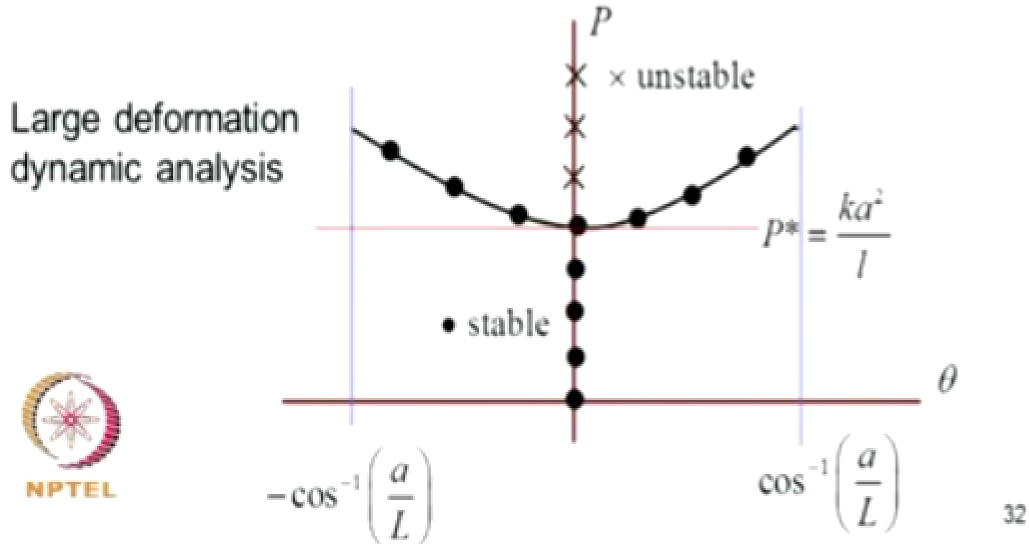
Other gradients remain the same.

Now suppose for sake of discussion damping is 0, next if we see that if KA square - PL is less than 0, okay then real lambda will be greater than 0, therefore fixed point is unstable.
On the other hand if KA square - PL is less than 0, real lambda will be less than 0, and fixed point is stable, so therefore condition for stability of 0, 0 is P must be less than this, so this is the condition on value of P for origin to be stable. Now let us consider the other fixed points which are given by this. Now when deriving this we use this condition to find the roots KA square by cos cube theta 1 - PL = 0, and consequently dou G/dou theta 1 will be given by this quantity, and you have to see here all these quantities are real, there is cos to the power of 4, sine to the power of 2, A square etcetera, K is positive, this is alpha square. - alpha square or alpha square is a positive number, and all other gradients are as before, so the Eigenvalues are given by this where alpha square of course is this messy term and I get now this equation for lambda and from this we can see that the fixed points are always stable, okay.

Eigenvalues: $\begin{vmatrix} 0-\lambda & 1 \\ -\alpha^2 & -c-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 + c\lambda + \alpha^2 = 0$

$\lambda = -\dfrac{c}{2} \pm \dfrac{1}{2}\sqrt{c^2 - 4\alpha^2} \Rightarrow$ Fixed points are always stable



So now we can plot the load deflection diagram, so on this axis I plot theta and load is plotted here and first we look at origin 0 0 until P crosses this, this will be the only fixed point that system has, and beyond this value of P the origin becomes unstable, so this cross indicates unstable points and the dot blog represent stable point, but at this point 2 branches emerged, and allowing each one of these brands a solution stable, okay, that is given by this, that is what this analysis of the second branch of fixed point shows.

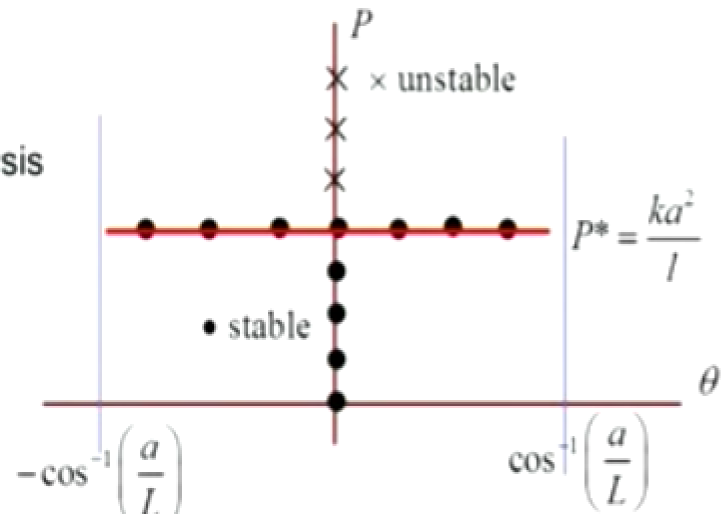
So P* which is the value of the load at which origin becomes unstable is given by KA square/L, so this is the complete load deflection analysis of the system by including large deformation and dynamics, okay, now what we'll do subsequently is will assume small deformation and we will do a static analysis and see what happens, and then ask the question what are the type of problems where we need to include dynamic effects, what are the type of problem where we need to include large deformation effects, okay, so now if I were to do a linear dynamic

Linear dynamic analysis

$$\frac{ml^3}{3}\ddot{\theta} + c\dot{\theta} + \left(ka^2 - PL\right)\theta = 0$$

$$\left.\begin{aligned} \dot{\theta}_1 &= \theta_2 \\ \dot{\theta}_2 &= -\frac{3c}{ml^3}\theta_2 - \frac{3}{ml^3}\left(ka^2 - PL\right)\theta_1 \end{aligned}\right\} \quad \begin{aligned} &\text{Fixed point: } (0,0) \\ &\text{Stable if } \left(ka^2 - PL\right) > 0 \Rightarrow P^* = \frac{ka^2}{l} \end{aligned}$$

Linear
dynamic analysis



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analysis that means after formulating the equation of motion I will linearize this term, sine theta 1 is written as theta 1, cos cube theta 1 is written as 1, and this equation becomes linear and I get this equation, so again I can find out fixed points for this linear system and perform the stability analysis, we will now only find the origin the other fixed points do not figure in this analysis, so we get the origin to be stable if this condition is satisfied afterwards the origin will be unstable, so this is stable and later on this is unstable.

Does the linearized stability analysis give qualitatively correct picture of the phase portrait?
Yes, provided the fixed point for linearized system is not a border line case (as in center)
If the linearized system predicts a saddle, node, or a focus, then the fixed point is really is a saddle, node, or focus.

Example to follow:
H S Strogatz, 1994, Nonlinear dynamical systems and chaos, Westview, Cambridge, MA.

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Now we can ask the question, does the linearized stability analysis give qualitatively the correct picture of the phase portrait? For example if a linearized analysis shows that a point is a center irrespective of value of certain system parameters can we believe that, the answer to this question is yes provided the fixed point for linearized system is not a borderline case as in a center, if the linearized system predicts a saddle node or a focus then the fixed point is really a saddle node or a focus, but if it predicts it as a center we are not sure, so I will present an example to substantiate this argument and this is taken from this book by Strogatz, the system

$$\dot{x} = -y + ax\left(x^2 + y^2\right)$$

$$\dot{y} = x + ay\left(x^2 + y^2\right)$$

Origin is a fixed point

Eigenvalue problem

$$\begin{vmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

Origin is a center.

Let us analyse the governing equation and see how the solutions behave near the origin.

$$x = r\cos\theta, \; y = r\sin\theta$$

$$x^2 + y^2 = r^2 \Rightarrow x\dot{x} + y\dot{y} = r\dot{r}$$

$$r\dot{r} = x\left[-y + ax\left(x^2 + y^2\right)\right] + y\left[x + ay\left(x^2 + y^2\right)\right]$$

$$= a\left(x^2 + y^2\right)^2 = ar^4 \Rightarrow \dot{r} = ar^3$$

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here is given by X dot is - Y + AX into X square + Y square and Y dot is X + AY, X square + Y square, though we can easily see that origin is a fixed point by equating this to 0, and the Eigenvalue problem analysis shows that the origin is the center, because there are pure imaginary roots.

Now let us analyze, now what we will do is we will solve this equation, it so happens that in this case it is possible to construct this solution, so what we will do will make the substitution X = R cos theta, and Y = R sine theta, and X square + Y square is R square, so if I differentiate this it becomes 2X X dot + 2Y Y dot = 2R R dot, so I get XX dot + YY dot is RR dot, so for RR dot I will write X into X dot, for X dot I will write the governing term from the governing equation, and for Y dot similarly I would write the second term here I get this. Now after simplifying this I get the equation for R as R dot = AR cube, so this can be solved.

$$\dot{x} = -y + ax\left(x^2 + y^2\right); \dot{y} = x + ay\left(x^2 + y^2\right)$$

$$x = r\cos\theta, y = r\sin\theta$$

$$\dot{r} = ar^3$$

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} \Rightarrow \dot{\theta} = 1$$

$$\dot{r} = ar^3$$

$$\dot{\theta} = 1$$

$$a < 0 \Rightarrow \lim_{t \to \infty} r(t) \to 0 \text{ [origin is a stable focus]}$$

$$a = 0 \Rightarrow r(t) = r_0 \text{ [origin is a center]}$$

$$a > 0 \Rightarrow \lim_{t \to \infty} r(t) \to \infty \text{ [origin is an unstable focus]}$$

Linearized system: origin is center for all values of a.

Exact solution: does not support this.

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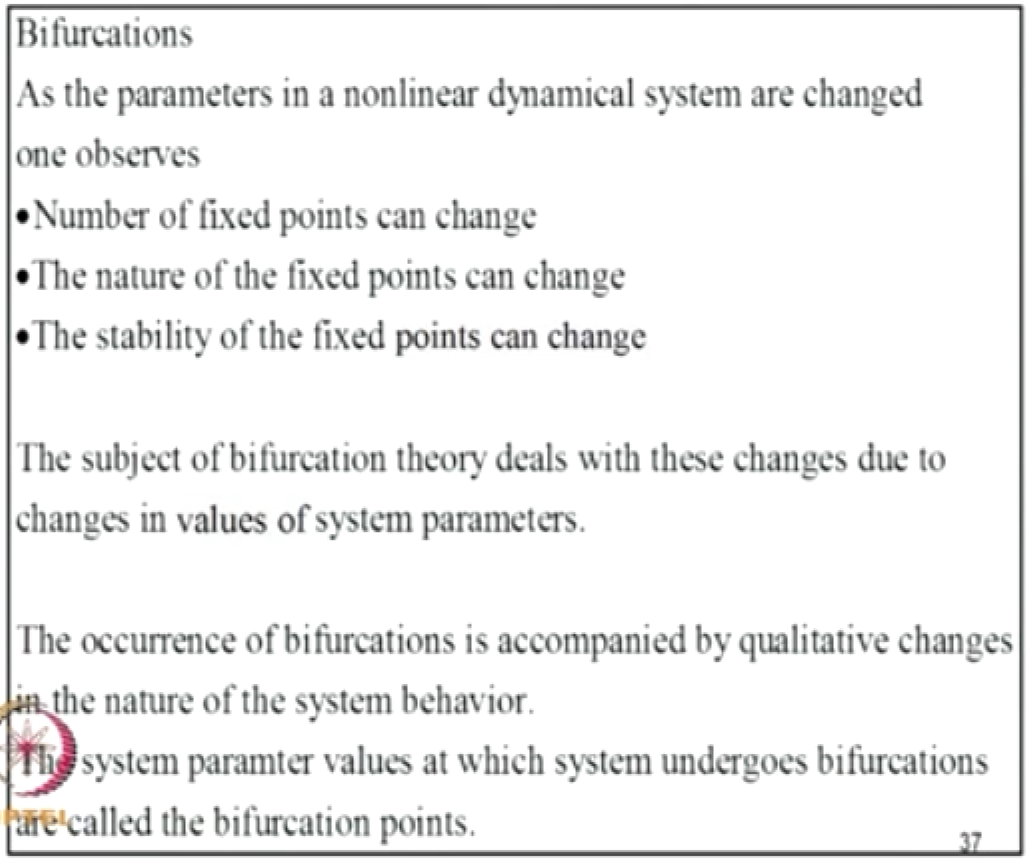
Similarly the equation for theta dot can also be obtained and we can show that theta dot = 1, so the governing equation in polar coordinates corresponding to the given equations in Cartesian coordinates is given by R dot = AR cube, and theta dot = 1. Now we can analyze this problem and show that if A is less than 0 as limit T tends to infinity R(t) goes to 0, that means origin is a stable focus. Now if A = 0 origin is a center, if A is greater than 0, R(t) tends to infinity and origin is an unstable focus. Now whereas the linearized system analysis showed that origin is a center for all values of A, so exact solution does not support this observation, so if you get a center as a answer to your question on stability then you need to, you cannot be decisive, so what is advisable is to carry out a nonlinear analysis.



Now we use the term bifurcations, what does this word mean? Now nonlinear systems are or dynamical systems are characterized by fixed points and their stability characteristics, and also the nature of this fixed point, for example nodes, saddle, or focus, etcetera, and they being stable or not, so and this a dynamical system is characterized by set of system parameters, so as a system parameter a dynamical system are change one observes that the number of fixed points can change, the nature of fixed points can change, and the stability of the fixed points can change, the subject of bifurcation theory deals with these changes due to changes in the values of system parameters, bifurcation therefore is change in the number of fixed point, their nature and their stability, as a parameter of the system is changed.

Now the occurrence of bifurcation is accompanied by qualitative changes in the nature of system behavior, if suppose a fixed point becomes from being stable it becomes unstable the system will behave differently, the system parameter values at which system undergoes bifurcations are called the bifurcation points, so Euler buckling phenomena can be interpreted as a phenomena of bifurcation, so at P, less than P critical the system will have certain types of fixed points and P greater than P critical it will have different types of fixed points, so we can

$$\ddot{x} + \alpha x + \beta x^3 = 0$$

$\alpha > 0, \beta > 0 \Rightarrow$ origin is a center

$\alpha < 0, \beta < 0 \Rightarrow$ center+saddle

$\alpha < 0, \beta > 0 \Rightarrow$ 2 centers and one saddle

consider to illustrate that a system with 2 parameters alpha and beta, X dot + alpha X + beta X cube = 0, now if alpha is greater than 0, beta is greater than 0, we can show that origin is the center, if alpha is less than 0 and beta is less than 0, this is center and saddle, if alpha less than 0 beta greater than 0 there will be 2 centers and 1 saddle, so you can analyze this problem and see if these statements are right.

Limit cycles

$$\ddot{x} - \varepsilon \dot{x}\left(1 - x^2\right) + x = 0; x(0) = x_0, \dot{x}(0) = \dot{x}_0$$

$x^2 < 1$

Isolated periodic free vibration solutions of nonlinear systems.

Stable

Unstable

$$\left.\begin{aligned} \dot{r} &= \mu r\left(r^2 - 1\right)\sin\left(\frac{1}{1-r^2}\right) \\ \dot{\theta} &= -1 \end{aligned}\right\} \text{this system has infinite number of limit cycles}$$

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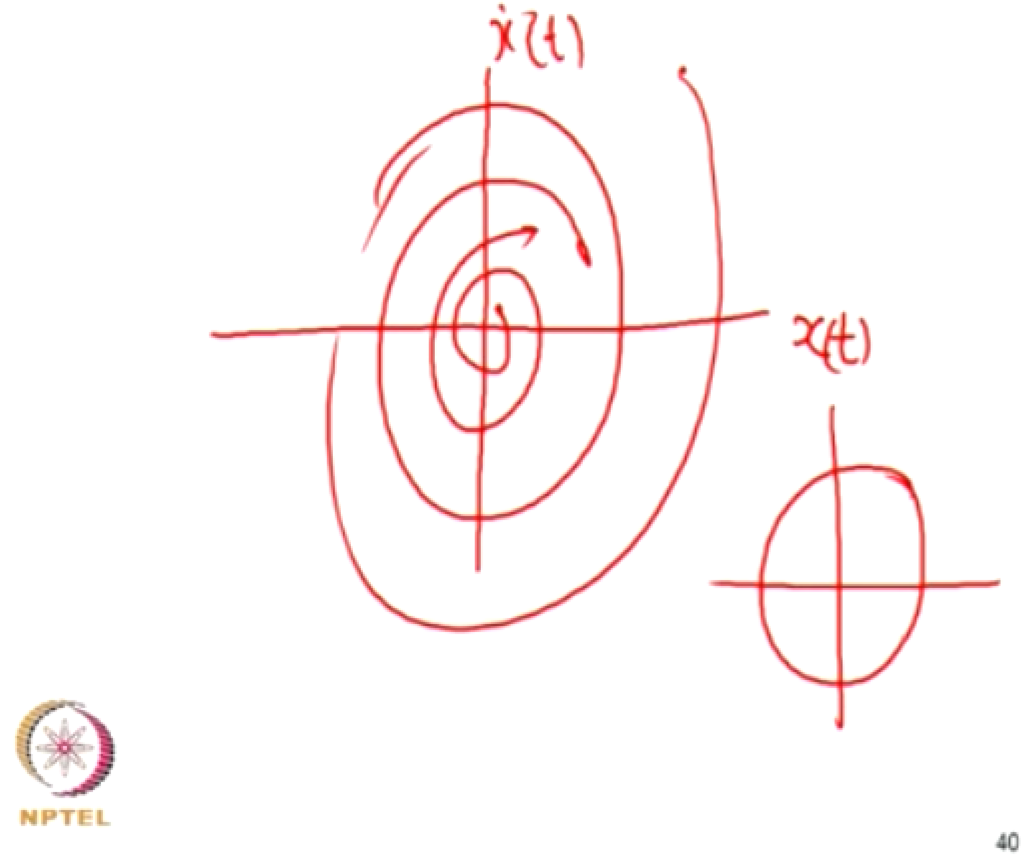
Now there is another kind of system behavior that we should be aware of, what are known limit cycles? Now let's consider an equation of the form X double dot - epsilon X dot multiplied by 1 - X square + X = 0, so here the nonlinearity term is associated with dissipation characteristics, we have been considering nonlinearity associated with stiffness, they can be associated with dissipation also, this type of nonlinearity occurs for example in fluid structure interaction problems.

Now if we now consider this system if X square is less than 1 then this 1 - X square will be positive, and this system will have negative damping, for small amplitudes of oscillation the system behaves as if it has negative damping, on the other hand if X is large, larger than 1, X square is larger than 1 then this term will be negative, and negative into a negative term becomes positive, and the system behaves as if it is positively damped, so if we now look at that in the phase plane, suppose if I plot a motion that is originating near the origin that is small



oscillation we will tend to grow, whereas a motion which is originated away from the origin will decay, so in between that there will be a periodic orbit to which the response will move towards in the steady state, okay, so that these periodic orbits are known as limit cycles, so they're isolated periodic free vibration solutions of nonlinear systems, that means I am looking at free vibration solutions and irrespective of the initial conditions the same steady-state periodic solution is obtained, and in the neighborhood of this closed curve there cannot be another closed curve, a system can have several limit cycles, for example if you consider a dynamical system as given here you can show that this system has infinite number of limit

Limit cycles

$x^2 < 1$

$$\ddot{x} - \varepsilon\dot{x}\left(1 - x^2\right) + x = 0; x(0) = x_0, \dot{x}(0) = \dot{x}_0$$

Isolated periodic free vibration solutions of nonlinear systems.

Stable

Unstable

$$\left.\begin{aligned}\dot{r} &= \mu r\left(r^2 - 1\right)\sin\left(\frac{1}{1-r^2}\right)\\ \dot{\theta} &= -1\end{aligned}\right\}\text{this system has infinite number of limit cycles}$$

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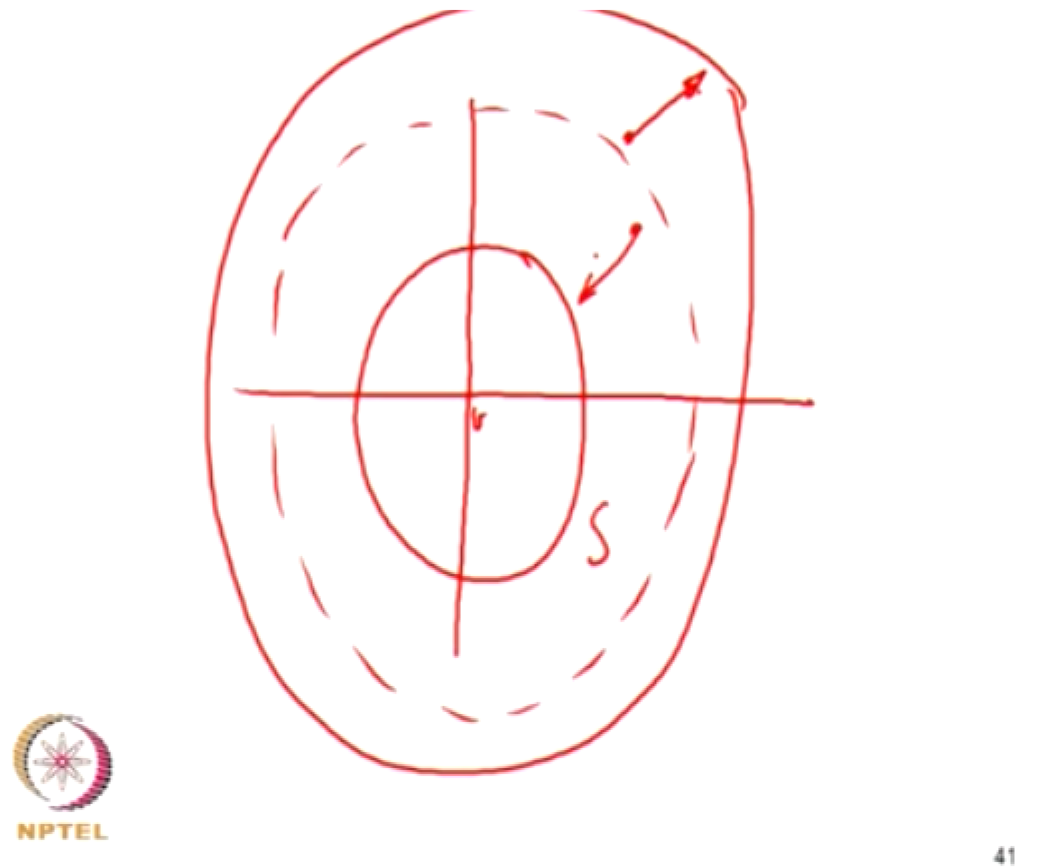
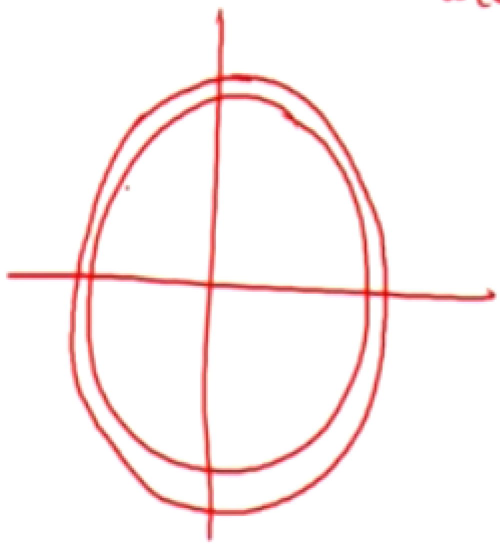
cycle, so for example if you have the origin is unstable, then small motions will start growing, so there will be a stable limit cycle, any motion which is initiated within this will be repelled to this, whereas if there is another stable limit cycle, suppose this is another stable limit cycle at another distance, now if a trajectory gets initiated here to which orbit will it move towards, if both are stable there will be a contradiction so there will be an unstable limit cycle sandwich between 2 stable limit cycle, that means any motion within this region will go here, any motion



in this region will go to that, so in systems is multiple limit cycles the unstable and stable limit cycles alternate.

$$\ddot{x} + \omega^2 x = 0 \qquad x(0) = x_0$$
$$\dot{x}(0) = \dot{x}_0$$

$$x_0 \rightarrow x_0 + \Delta x_0$$
$$\dot{x}_0 \rightarrow \dot{x}_0 + \Delta \dot{x}_0$$



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Now I said that limit cycles are isolated, close trajectories, if we now consider for example X double dot, undamped single degree freedom vibrations if I plot the phase plane plot for this this will be one orbit for a given set of initial conditions, say X naught and X naught dot, if I now change X naught by some small initial, small change and similarly make a similar change on this I can realize in the neighborhood of this solution, another closed trajectory, this is not a limit cycle. Any closed orbit in the phase plane is not a limit cycle, in the neighborhood of a limit cycle there cannot be another closed trajectory, so where is in linear systems it is possible but systems exhibiting limit cycles it is not possible, because for example this oscillator that I discuss is known as Van Der Pol oscillator it has one limit cycle here, so if a motion is initiated here close to this it has to be observed by this, if it is here it is observed by this, so in the neighborhood there cannot exist another closed orbit that is a major feature of a limit cycle oscillation.

Energy methods for stability analysis

Axiom - 1

A stationary value of the total potential energy with respect to the generalized coordinates is necessary and sufficient condition for the equillibrium state of the system.

Axiom - 2

A complete relative minimum of the total potential energy with respect to the generalized coordinates is necessary and sufficient for the stability of an equillibrium state of the system.

J M T Thompson and G W Hunt, 1973, A general theory of elastic stability, John Wiley, London

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Now in the next part of our lecture what we will do now is, we will consider energy methods for stability analysis, this analysis will be based on 2 axioms, here I will simply state the axioms we will consider there what it means in the subsequent next lecture, so according to the first axiom is stationary value of the total potential energy with respect to the generalized coordinates is necessary and sufficient condition for the equilibrium state of the system, this is first axiom tells you condition for getting an equilibrium state. The next axiom tells a complete relative minimum of the total potential energy with respect to the generalized coordinates is necessary and sufficient for the stability of an equilibrium state of the system, so we will use these 2 axioms and study dynamical system or I mean here we are not considering dynamical system we are considering basically statical systems as we will see, and we will apply these problems to some of the simpler system that we have studied so far. And then we will see that this can form the basis for developing stability analysis of elastic systems to start with we can use Rayleigh-Ritz type of approximations, and then it will graduate to the finite element method. So a discussion on this will be taken up in the next lecture.

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