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<u>Course Title</u>
Finite element method for structural dynamic
And stability analyses
Lecture – 26
Structural stability analysis – Introduction
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Finite element method for structural dynamic and stability analyses

Module-9

Structural stability analysis

Lecture-26 Introduction

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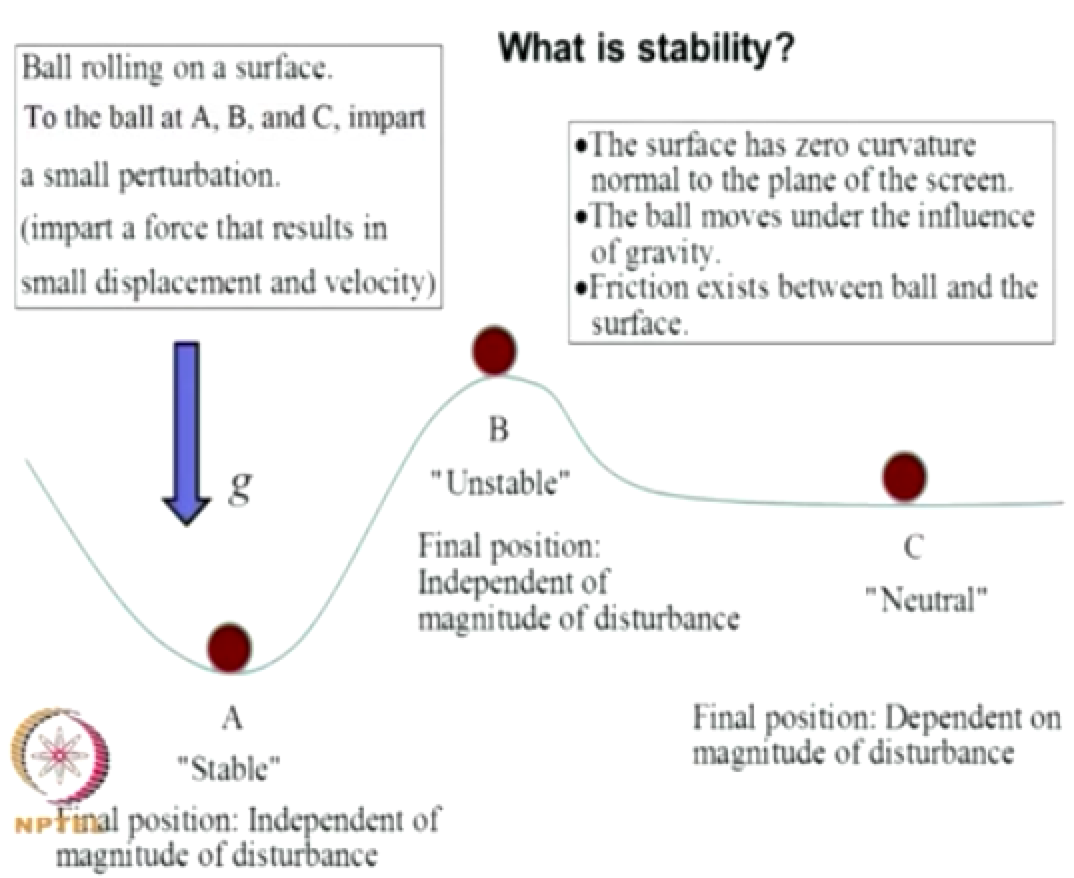
In today's lecture we will start discussing topics related to the second topic listed in the title of the course, so we will start this module on structural stability analysis.
So in today's class what I wish to do is to introduce certain basic notions and recall some results from basically strength of material type approach to analyzing stability of beam columns, so then we will move on to finite element formulation and development of analytical results based on which such models are developed, so that will come in due course.

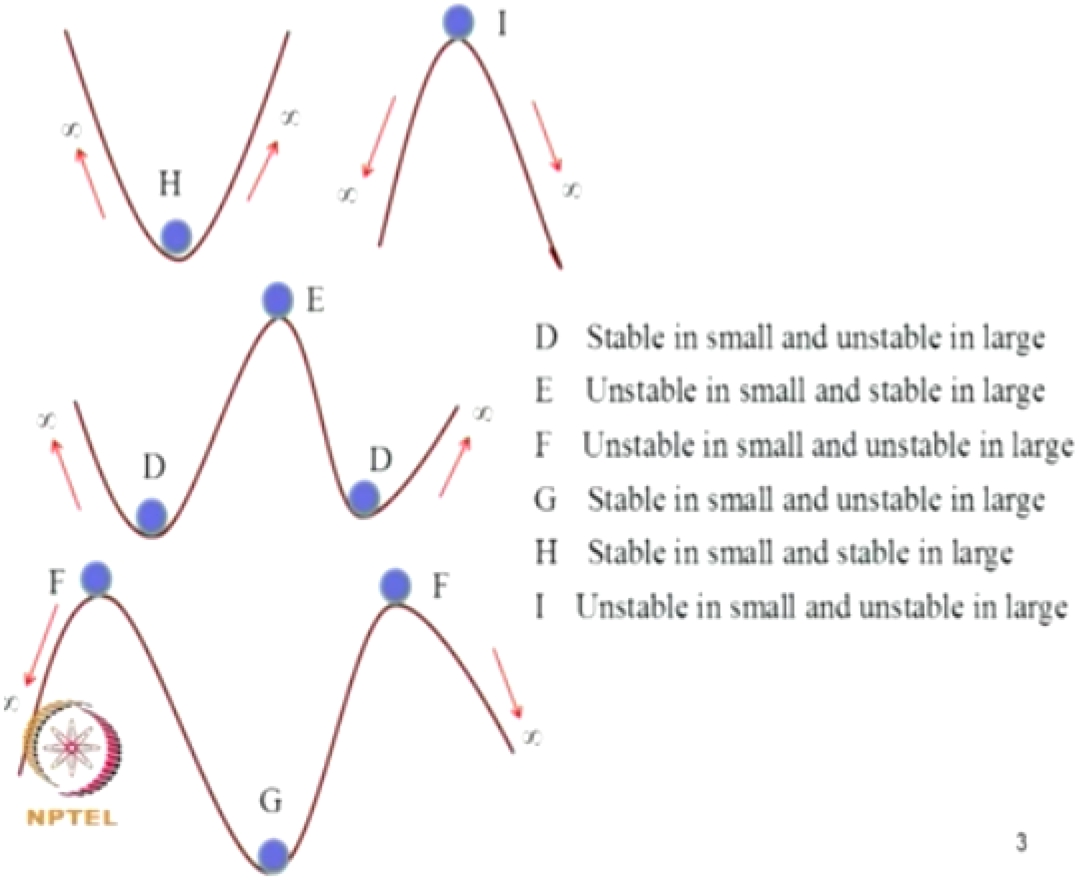


So first let us try to develop some notion about what is stability, to do that we will consider a curved surface which is prismatic, that is perpendicular to the plane of the screen it is a constant, it doesn't have, it doesn't change the curvature, and we imagine that this is basically a valley and a mountain and things like that, so we imagine that the ball is rolling on this surface, so initially ball can be placed here at A or at B or at C, so there is gravity that is present, so which drives the motion of the ball, so what we wish to do is we will first place the ball in at point A, and impart a slight disturbance to it, so we will assume that there is friction in the between the ball and the surface so this ball will now oscillate, and because of friction it will come back to this position.

Now if you now consider the ball being placed perched on this top and any slight disturbance that you will give, it will make the ball to roll away from this point either on this side it will end up here, or it will start rolling on this and it will stop when the friction overcomes the energy in the ball, so whereas if you imagine ball is placed on this flat surface at this point C if you import a small perturbation to this, so the ball will roll and occupy this position, so the amount by which the ball rolls depends on the magnitude of the perturbations given.

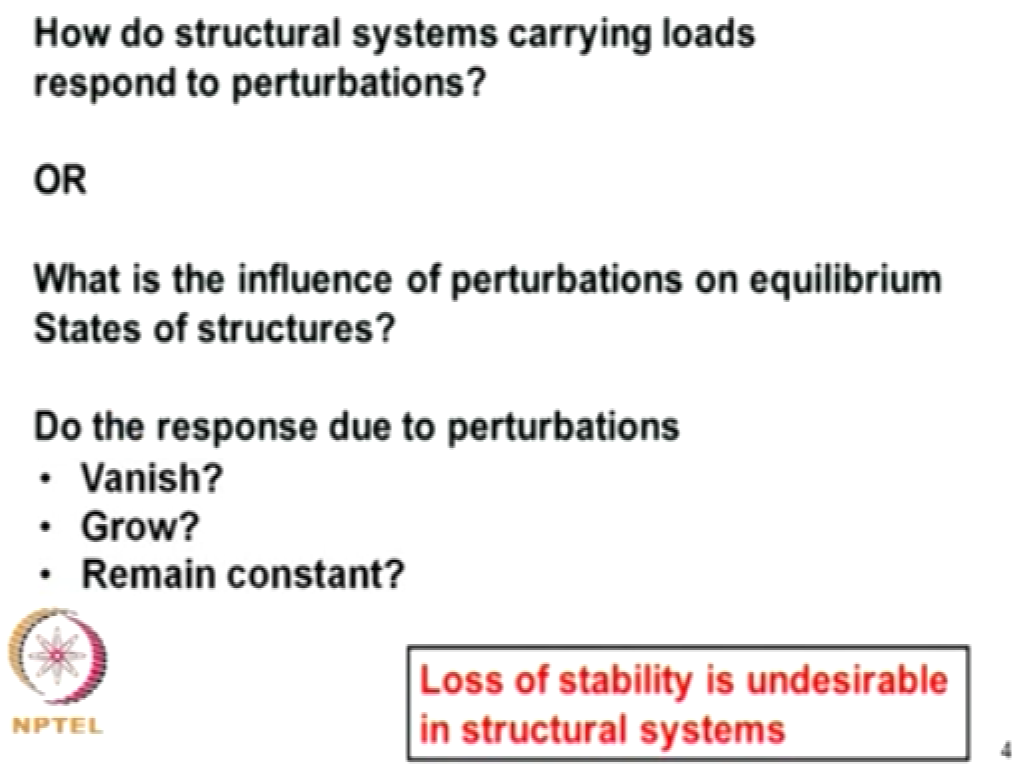
Here as long as perturbation is not very large, suppose if I give such a very large perturbation the ball may negotiate this curve and end up somewhere else. Now assuming that the perturbation that we provide is within certain range, the final position occupied by the ball is the same. Here also the final position is independent of the magnitude of the perturbation, even if you give a slightest perturbation either it will end up here or somewhere on this surface, so the final position here is independent of magnitude of the disturbance, here the final position is independent of magnitude of the disturbance as long as the disturbance is small by small, I mean the displacement and the velocity of this ball is small.

Now we say that position A is stable, and position B is unstable, and position C is neutral, we will have to refine these statements to be able to do that we will consider more situations,

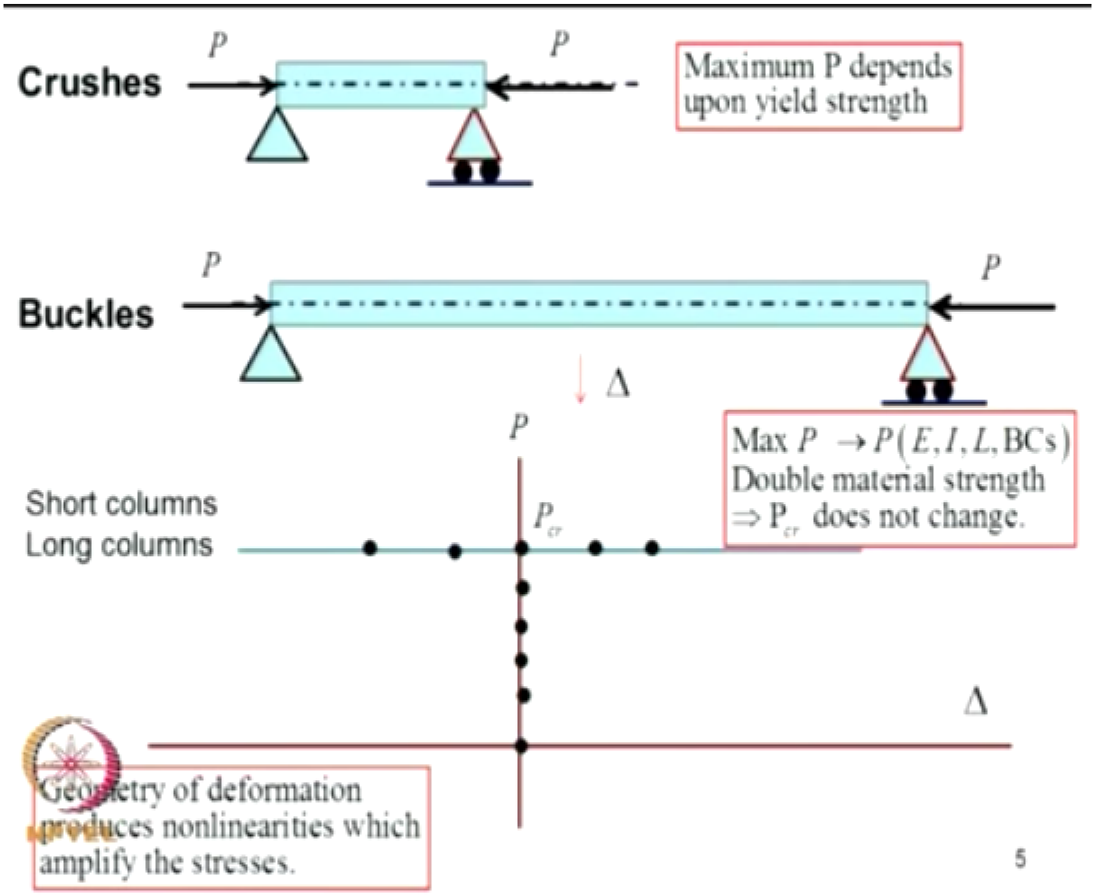


suppose there is a valley like this which is infinitely deep so it goes up to infinity in this way a ball is placed here. Now any perturbation that you give no matter how large it is, the ball will finally end up here because of presence of friction, so we say that this position H is stable in small and stable in large, whereas if you now consider this imagine that this goes to infinity, this goes to infinity any slight perturbation no matter how big or small it is the ball will end up somewhere else, so it is unstable in small as well as in large. Here position D it is stable in small, small perturbations which keeps the ball within this valley that means the ball should not negotiate this curve and end up here, the ball will be stable, so D we say is stable in small and unstable in large, but the final position either it is here or here if it negotiates this it will oscillate in this well and maybe end up here or here.

Now if you consider now this position this is unstable in small and we can say it is stable in large because it won't move away infinitely away from this place, so this is you know issues about defining stability. Now F is unstable in small and unstable in large, because any perturbation here ends up far away, G is stable in small and unstable in large, F is unstable in small and unstable in large.

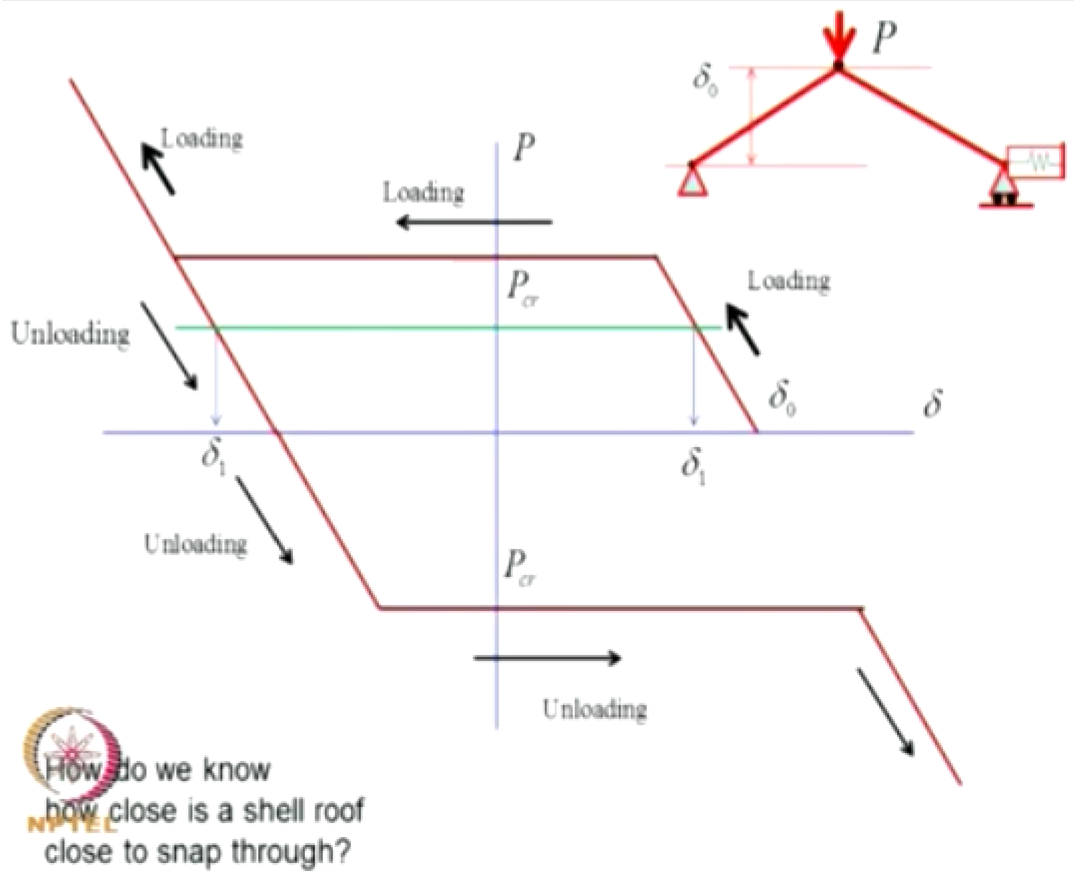


Now we are talking about now motion of a ball in a valley, how do structural system which carry loads respond to perturbations, or what is the influence of perturbations on equilibrium states of the structures, on account of the perturbation that we give to the structure do the responses, due to perturbation vanish, grow, or remain constant. Now this is a type of questions that we are interested in addressing, now if because of small perturbations the motion of the system grows and so structure doesn't return to its original state then we can think that structure has lost its stability, and loss of stability is undesirable in structural systems, because any departure from equilibrium configuration for which it is designed may result in failure.



Now to elaborate this let us consider the case of an axially loaded beam element as shown here, so in this case as P increases, the maximum P that we can apply on this structure probably depends upon the compressive strength, that means P depends on the yield strength of this material, we can go on increasing P till this material here starts yielding, whereas if the length of this column becomes large, this element becomes large and we go on increasing P, if we ask now what is the maximum P that this structure can carry, it depends on as we will see shortly it depends on not the yield strength of the material, but on Young's modulus area moment of inertia, length, boundary conditions, and so on and so forth. Now for this type of structure, suppose if you double the strength of the material the highest load that the structure can carry may not change, okay, so suppose if you do now show a kind of a conceptual experiment, what we will do is we will apply P, the load P in increments, suppose you start with 0 and look at transverse displacement of this beam.

Now we will start with 0, so delta is 0, now we will increment P and after that P is incremented will pluck this beam and allow it to move, so assume that it oscillates and assume that the beam has damping in it, it will undergo a free vibration and it will settle down, then delta is 0, so you keep on increasing P at one stage what happens the slight perturbation that you give the beam doesn't return to its original position, but occupies a position which is proportional to the perturbation that you have given, so the load deflection diagram in this case climbs up like this and either it can move to the left or to the right depending on whether the beam deflects upwards or downwards, so at this value of load P we say that structure has lost stability, so this P critical this phenomena is due to geometry of deformation which produces nonlinearities which amplify the stresses, I mean this is the mechanism that we have to now understand, so those columns which behave like this are called short columns, and those columns which behaves like this are called long columns, and this type of behavior is known as bucking. This is something that you would have studied in your strength of materials now let us see how we can develop this idea further.



Now we will do another thought experiment, imagine there is a structure consisting of 2 members, both are rigid, it is hinged here and it is supported on a roller with a spring here, and when P is not applied the deflection of this point or this distance is delta naught, so we will now draw the load deflection diagram, so as you go on increasing load P this distance from this point to this point goes on reducing, so we will be tracing this path. For some value of P what happens is the structure snaps and it may start after this point, that is this point, it will immediate next position that it occupies this, so this happens for example in umbrella in a windy day the sign of the curvature of the umbrella can suddenly change, this is what is known as snap through buckling, so the loading increases, and at this point there is a dramatic increase in the deflection and further increase in load this fellow will start deflecting in this direction. Now if you start unloading that means from this value of the load you are going reducing the load, the unloading path will not trace this path, the unloading path will be different so that means suppose when we are moving in this direction, at this point suppose the structure has come here so that is this point, so upon unloading when this point is reached it won't revert back to this in this manner, it will start unloading in a different way and at some other point it will again snapback, so in this kind of problems the difference between this problem and this problem is that when the structure loses stability there will be a equilibrium position in the neighborhood of the original position, whereas here the new equilibrium position that this structure occupies upon loss of stability will be far from the original configuration, so for example if we are now dealing with a shell roof structure which is a shell, shell roof structure suppose it is carrying sulfide, wind load, etcetera, how do we know that, how close is the shell roof close to a snap throw? Obviously we cannot deal with snap through in roof structures so how do we know that, what is the margin of, so to say the safety again such occurrence of such a phenomena, so this is a type of question that we would be interested.

Stability of steady state motions

$$m\ddot{x} + c\dot{x} + kx + \alpha x^3 = P\cos\Omega t; x(0) = x_0; \dot{x}(0) = \dot{x}_0$$

$$\lim_{t\to\infty} x(t) = X\cos(3\Omega t - \theta) = x^*(t)$$

$$x(t) \rightarrow x^*(t) + \Delta(t) \quad [\Delta(t) \text{ is "small"}]$$

$$\Rightarrow$$

$$m\ddot{\Delta} + c\dot{\Delta} + k\Delta + 3\alpha x^{*2}(t)\Delta = 0$$

$$\lim_{t\to\infty} |\Delta(t)| \rightarrow ?$$

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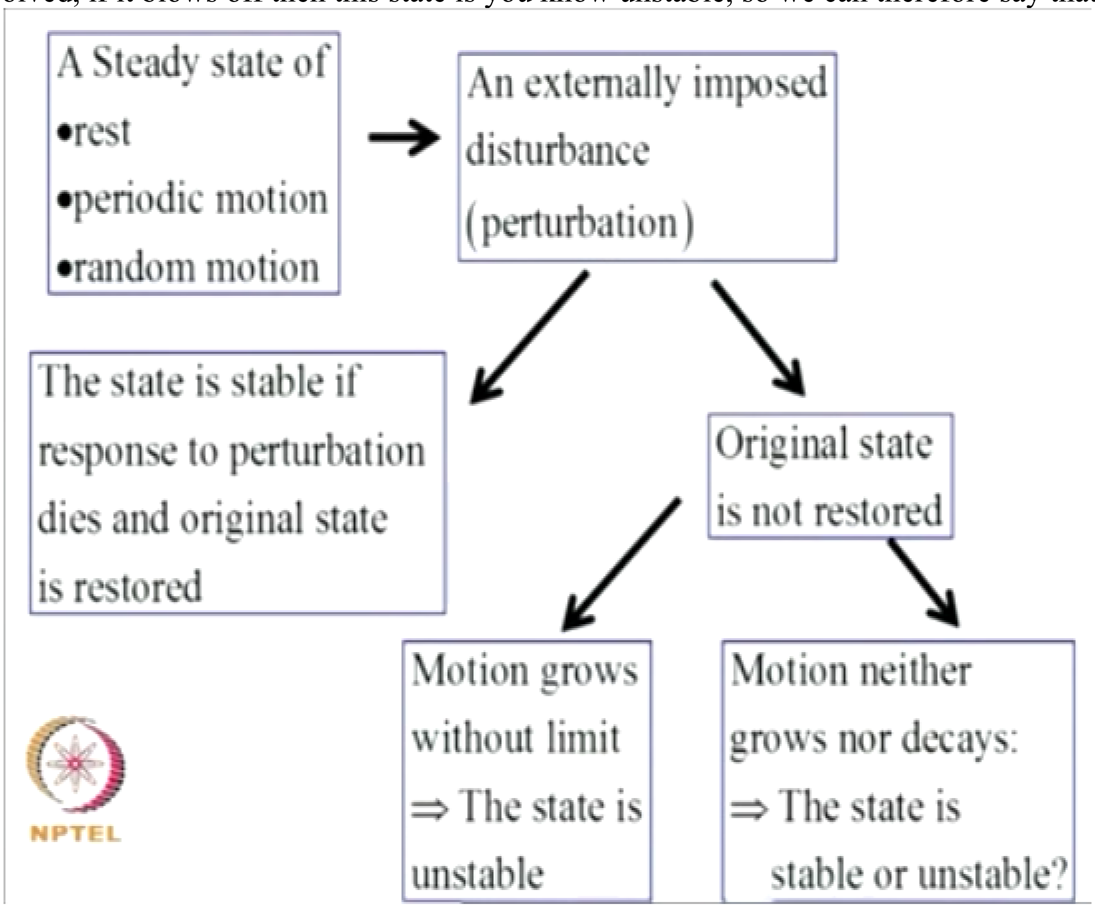
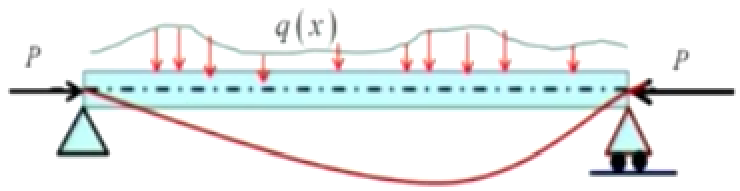
The other type of questions about stability is stability of motions, for example if you consider a single degree freedom system which has say non-linear terms, cubic nonlinear stiffness, suppose it is driven harmonically every certain initial conditions, now based on certain logic I propose that as T tends to infinity, one of the steady-state solutions will be of this form, X is amplitude and because of non-linearity I want to examine whether a solution of the form 3, omega T - theta is possible, I call this solution as X star of T, now I want to know whether this motion can be realized? In the sense motion can be realized if a slight perturbation given to that you know goes to 0 as time passes, right if there any slight perturbation that we give to X star of T results in a growth of response in an unbounded manner then that particular steady state solution cannot be realized, so what we do is we change X(t) to X star(t) + delta(t) and assuming that X star (t) satisfy this equation we will be able to get the so called variational equation for delta(t) which will be of this form. What drives delta(t) is the solution X star square(t) which appears now as a parameter in the problem, so the forcing function multiplies the state, so X star(t) can be viewed as forcing function for the system that governs delta, this X star(t) is in fact a solution to this problem, but that is not very you know that doesn't make any difference here as far as interpreting X star square(t) as a load for this system, so this type of systems are known as parametrically excited systems.

Now in this type of system what we would like to know is as time becomes large whether delta T goes to 0 or not, I don't want to know details of how solution delta T behaves, I simply want to answer this question whether delta T goes to 0 or not, if it goes to 0 then we say that this solution is stable, if it doesn't go to 0, suppose it becomes constant then the question will be an unresolved, if it blows off then this state is you know unstable, so we can therefore say that we



consider steady state response of systems which could be state of rest or periodic motion or even random motion, on this we impose an external disturbance we call it as perturbation, because we applied this perturbation the structures will again start you know its position will change, we say that this state of steady state is stable if response to perturbation dies and original state is restored, if origin state is not restored there are 2 possibilities, the motion grows without limit, then we say that this state, this state that we are talking about is unstable, but if motion neither grows nor decays then the state is stable or unstable this question remains unresolved, and we have to go back and see for example in this context if we end up with a situation where delta(t) as T tends to infinity neither grows to infinity nor decays to 0, in that case we will not be able to answer the question whether the system is stable or not, so the idea here is you see there is an approximation here when I substitute this into this I am assuming delta(t) to be small, then terms like alpha, delta, cube, and 3 delta square, the 3 alpha X square delta and other one is delta square term there have been ignored here based on the assumption the delta is small, so maybe you may have to include and do something else, so these are questions which are widely discussed in literature on stability, and there is a great deal of diversity of definitions, so as we go along hopefully we will get familiar with some of these issues.

Beam columns



What are the new issues that crop up due to the action of additional compressive loads?

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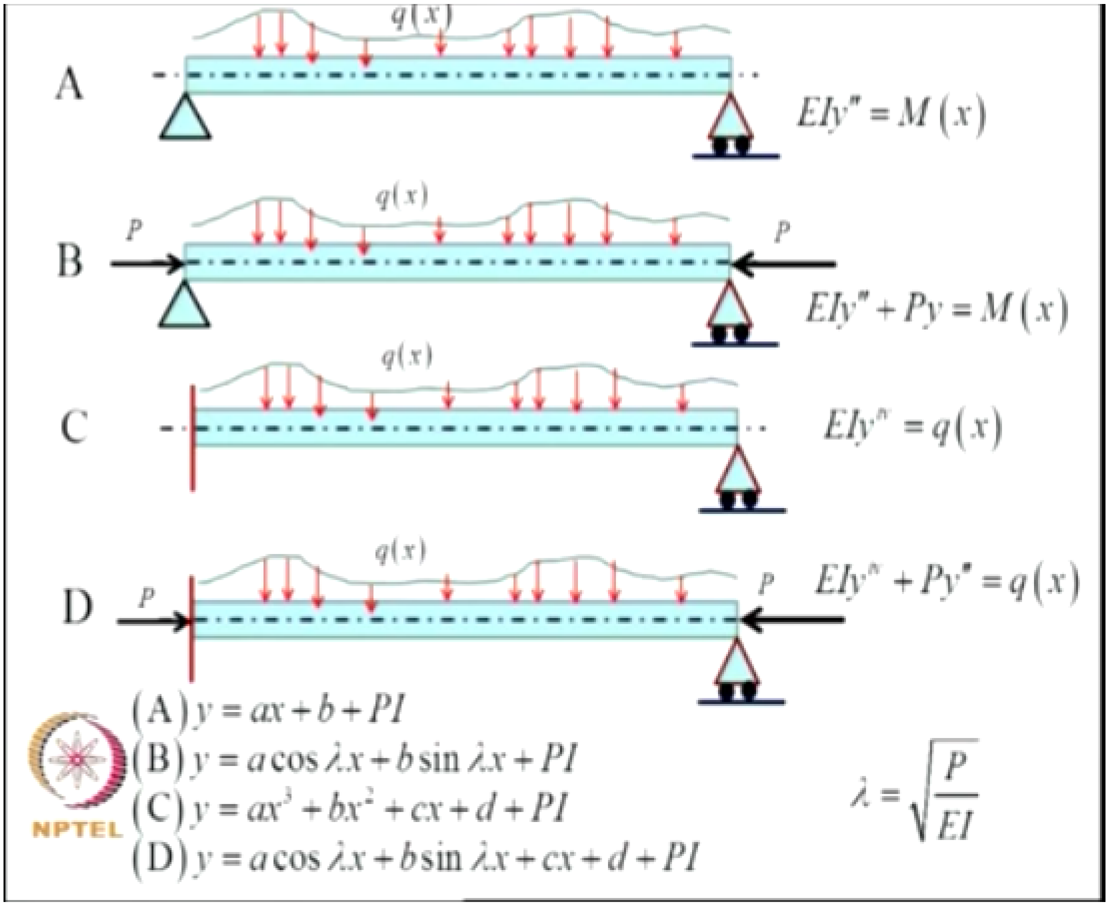
Now what we will do now is we will start with an archetypal problem that can be understood by you know students who have done a course in strength of materials, so consider a so-called beam column, a beam column carries an in-plane load as well as a transverse load Q(x) and in-plane load P. Now in absence of P we know how this structure behaves, now the question that we are asking is what is the influence of P on the transverse behavior of this structure, not the



axial deformation, axial deformation is understood due to P, but there will be an interaction between effect due to P and Q that leads to newer issues and this is what we need to understand, suppose we start set up that as a problem.

Now we will consider to understand the issues some simple situations suppose the beam is carrying load Q(x) it is simply supported to keep the discussion simple, so the governing equilibrium equation will be EIY double dot is = M(x). Now the beam now carries additional load P, so on the deflected profile so when I take bending moment at any point P, in addition to the bending moment due to Q(x) there will be a bending moment due to P into this deflection, so that I call as PY, now there is something very interesting the way we write this term I will come to that as we go along.

Next we will consider a statically indeterminate structure carrying load Q(x) and here we write the equation EI D4Y/DX4 is equal to applied load Q(x), suppose this beam now carries an axial load, so there will be EI D square Y/DX square + P, D square Y/DX square = Q(x), so you need to understand the nature of these equations, now if we quickly solve these equations, the first equation situation A, solution will be like AX + B + particular integral which involves the bending moment and hence Q(x). In the second case suppose if I call square root P/EI as lambda now I will get A cos lambda X + B sin lambda X + particular integral, in this case the complementary function will be AX cube + BX square + CX + D because there is fourth order derivative plus a particular integral which depends on Q(x). In the last case we will have A cos lambda X + B sin lambda X + CX + D + a particular integral, so what you need to focus on is the change in the nature of this complimentary function for these 4 different situations. The

Remarks

- The nature of complementary function changes dramatically due to presence of axial load P. Trigonomentric terms involving P appear. $y(x)$ becomes nonlinear function of P.
  The PI continues to be linear function of $q(x)$.
- (C) and (D) more general since BCs other than slope and deflection can be specified. Suitable for statically indeterminate structures.
- Beam-columns: beams which carry lateral loads and at the same time carry axial compressive loads.

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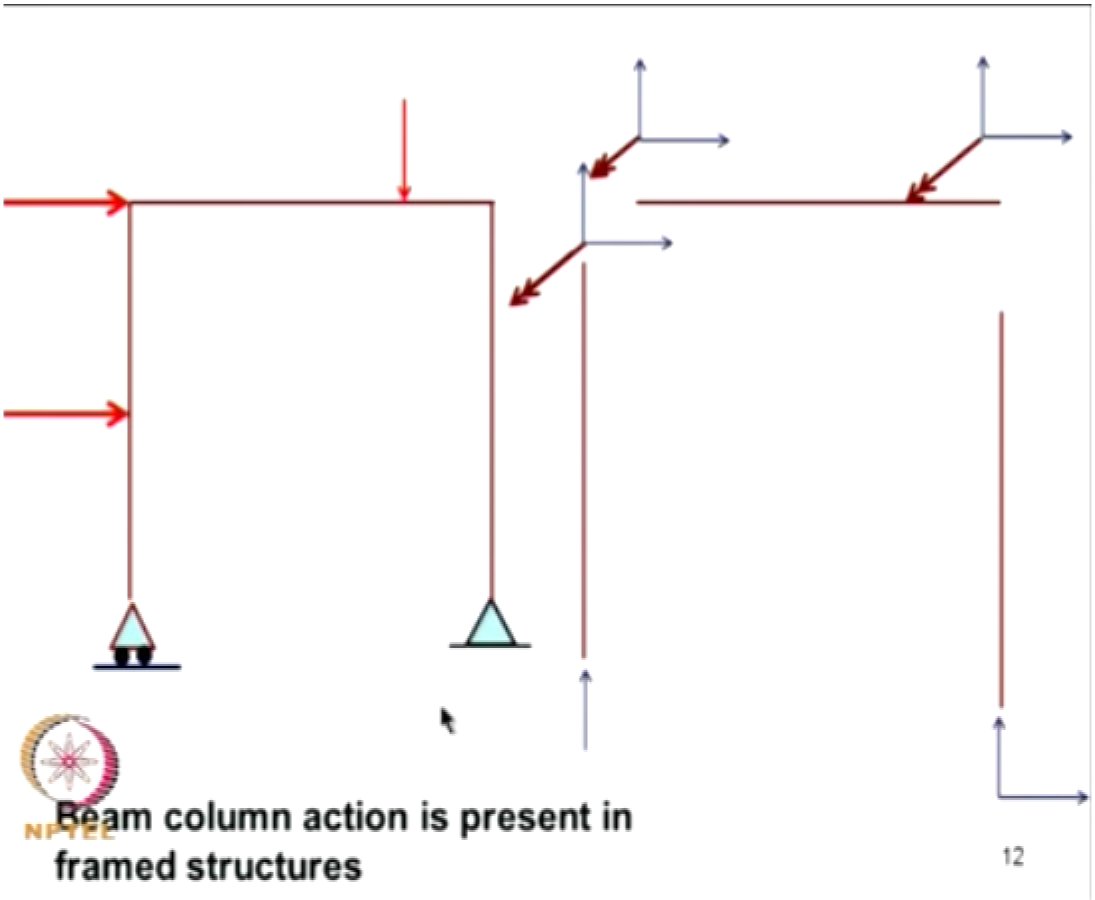
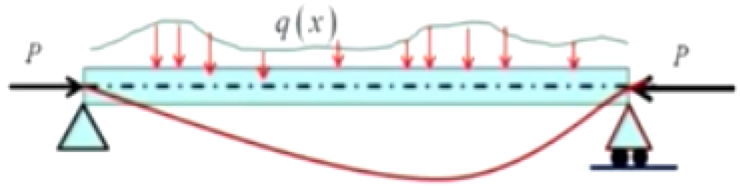
nature of complementary function changes dramatically due to presence of axial load P, trigonometric terms involving P appear in our solution, therefore Y(x) become nonlinear function of P, the particular integral continues to be linear function of Q(x), the particular integral is not influenced by P it will be influenced by P, but here you can see here the dependence between Y and P in the complementary function is through lambda, lambda is square root P/EI, the particular integral will be a linear function of Q(x) it may be function of axial load P, but still it will be a linear function of Q(x).

Now in this case again there are trigonometric terms, so trigonometric terms means non-linearity okay in P. Now C and D where we write fourth order equations are more generally applicable, these are applicable for this also so we can start with applied load and write the equation, this helps us to write the boundary conditions like fixed end, free end, and things like that, they are suitable for statically indeterminate structures, now this type of structure are known as beam columns which carry lateral loads and at the same time carry axial compressive loads.



Now this beam action you must understand that it is quite often present in many application, for example if we have a single wave portal frame carrying these loads, if you perform the analysis this member will be subjected to axial load and the bending moment, this member will be subjected to axial loads, shear, and bending-moment, so each member behaves like a beam column, so it is not that beam columns are rare things, they are the often the rule than the exception in frame structure.



Remarks

• $y(x) = y(x, q, P)$.

Double $q \Rightarrow y(x)$ doubles. Double P $\Rightarrow y(x)$ does not double.

First casualty: principle of superposition.

• $EIy'' + Py = M(x)$

Py: this force is written based on deformed geometry

$M(x)$: this force is independent of deformed geometry.

Thus produces dramatic effects.

$$y(x) = \begin{Bmatrix} \text{Transverse load} \\ \text{parameter. Linear.} \end{Bmatrix} \begin{Bmatrix} y(x) \text{ computed based on} \\ \text{undeformed geometry} \end{Bmatrix} \begin{Bmatrix} \text{Amplification factor} \\ \text{calculated based on} \\ \text{deformed geometry} \end{Bmatrix}$$

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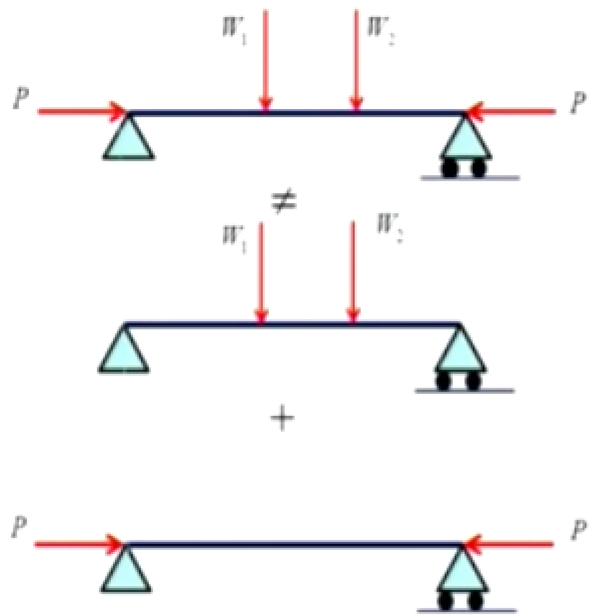
Now we will come to the equation governing this and will solve the equation, but right now we will try to you know assuming that you are familiar with some of this, we will try to interpret these solutions. Now the solution Y(x) for this type of situation is function of X, applied load Q and as well as the axial load P applied transverse load Q and axial load P, now if you double Q okay Y(x) doubles, but if you double P, Y(x) does not double, so that means the principle of superposition is valid with respect to Q in some sense but it is invalid with respect to P, so the effect of including axial loads or considering problems of beam column the first casualty is the principle of superposition which is bread and butter for linear analysis, so that has to be sacrificed.

Now the other thing that we should notice is when we write this equilibrium equation EIY double prime + PY = M(x), when I am writing M(x) or when I am computing bending moment, drawing the bending moment diagram I would not consider how the beam deforms, that means the bending moment is established with respect to the undeformed geometry of this structure, but whereas when it comes to computing the contribution to bending moment by the axial load we are considering the deformed configuration and we are writing P into Y with respect to the deform configuration, so there is a contradiction in the way we write these 2 terms, but this leads to dramatic effects as you will see, so a generic form of the response will be, the response will be a product of 3 terms, transverse load parameter which will be linear in this, suppose Q(x) is Q naught this can be Q naught. Then Y(x) computed based on undeform geometry, that means in absence of P whatever response we got that. Then there is an amplification factor calculated based on deform geometry, so this will be the structure of the solution and we are interested in the behavior of this amplification factor, if this becomes very large then we will be worried in designs.



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Traditional principle of superposition breaks down

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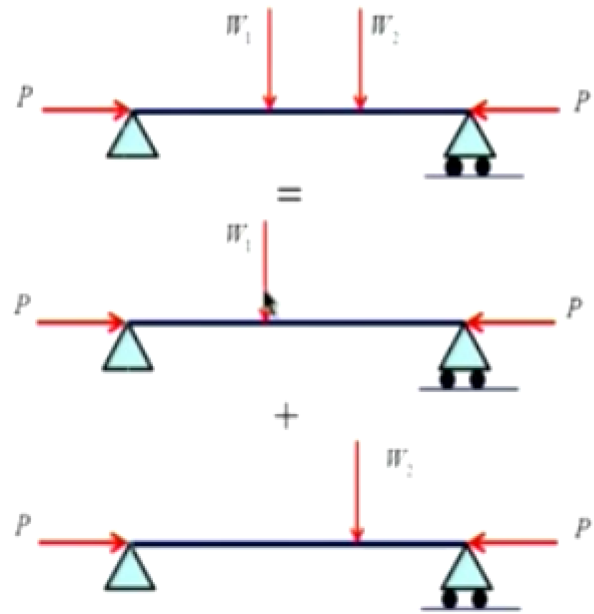
Now I mentioned about a principle of superposition, so if you consider a beam carrying say transverse load W1 and W2 and axial load P, if we assume that principle of superposition is applicable I can first apply load W1 and W2 and then consider the effect of axial load P, but if you sum up the response due to this and this, we would not get the beam column action, so the principle of superposition is not applicable in this form, it breaks down, but what remains applicable is a modified version of principle of superposition which is as follows, so we have



A new version of principle of superposition

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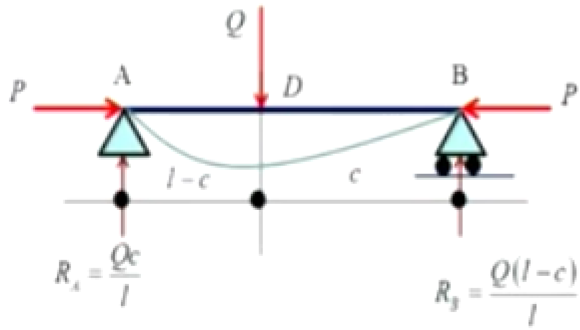
W1 and W2, we first apply W1 and retain these load P, then we apply W2 and retaining load P, now if this Y is a solution here and Y1 is a solution here, Y2 is a solution here we can show that Y = Y1+ Y, slightly later in this lecture we will come to that stage, so a new version of principle of superposition emerges that needs to be taken into account.

Analysis of a beam column



$$\text{AD}: EIy_1'' = -\frac{Qcx}{l} - Py_1 \Rightarrow y_1'' + \lambda^2 y_1 = -\frac{Qcx}{EIl}$$

$$\text{DB}: EIy_2'' = -\frac{Q(l-c)(l-x)}{l} - Py_2 \Rightarrow y_2'' + \lambda^2 y_2 = -\frac{Q(l-c)(l-x)}{EIl}$$

$$\sqrt{\frac{P}{EI}}$$

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Now to fix the ideas what we will do is we will analyze one of the simple structure like this, and see what is the consequence, suppose I have a simply supported beam, hinge here and a roller here, span is L and at a distance C I am applying a load Q, and this structure carries axial load P, so we can find out the reactions based on un-deformed geometry, this are the reactions, so now for the part of the beam lying in AD, the equilibrium equation is given by this EIY1 double prime – QCX/L that is QC X into L that is the bending moment, - P into Y1, so this leads to this equation Y1 double prime + lambda square Y1 is equal to the applied load. Similarly for the second part we can write the equation and I get Y2 double prime lambda square Y2 is equal to this, lambda square as before is square root P/EI.

$$y_1'' + \lambda^2 y_1 = -\frac{Qcx}{EIl}; 0 \le x \le l - c$$

$$y_2'' + \lambda^2 y_2 = -\frac{Q(l-c)(l-x)}{EIl}; l - c \le x \le l$$

$$y_1(x) = A\cos\lambda x + B\sin\lambda x - \frac{Qcx}{\lambda^2 EIl}; 0 \le x \le l - c$$

$$y_2(x) = C\cos\lambda x + D\sin\lambda x - \frac{Q(l-c)(l-x)}{\lambda^2 EIl}; l - c \le x \le l$$

$$y_1(0) = 0 \Rightarrow A = 0 \Rightarrow y_1(x) = B\sin\lambda x - \frac{Qcx}{\lambda^2 EIl}$$

$$y_2(l) = 0 \Rightarrow C\cos\lambda l + D\sin\lambda l \Rightarrow C = -D\tan\lambda l$$

$$y_2(x) = D(-\tan\lambda l\cos\lambda x + \sin\lambda x) - \frac{Q(l-c)(l-x)}{\lambda^2 EIl}$$

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Now I can solve these equations so for the part X lying between 0 to L – C, I have this equation and for X between L - C2L I have this equation, the first part the solution is A cos lambda X + B sin lambda X, and particular integral is this, this is valid over this region. Second part C cos lambda X + D sin lambda X + this valid over the region L - C2L. Now at the end A Y1(0) is 0, therefore Y1(x) I get this, A becomes 0, at Y2 X = L, Y2L is 0 that leads to this, so we eliminate C in terms of D, and I get solution in this form, so 2 constants we have eliminated, 2 are still remaining and they can be established by demanding continuity at this point, so continuity conditions are this is Y1 and this is Y2, there is a constant B, and there is a constant D floating around.

$$y_1(x) = B\sin\lambda x - \frac{Qcx}{Pl};\ y_2(x) = D(-\tan\lambda l\cos\lambda x + \sin\lambda x) - \frac{Q(l-c)(l-x)}{Pl}$$

Continuity conditions

$$y_1(l-c) = y_2(l-c)$$

$$y_1'(l-c) = y_2'(l-c)$$

$$y_1'(x) = B\lambda\cos\lambda x - \frac{Qc}{Pl};\ y_2'(x) = D(\lambda\tan\lambda l\sin\lambda x + \lambda\cos\lambda x) + \frac{Q(l-c)}{Pl}$$

$$\Rightarrow D = -\frac{Q}{P\lambda}\sin\lambda(l-c)\ \&\ B = \frac{Q}{P\lambda}\frac{\sin\lambda c}{\sin\lambda l}$$

$$\Rightarrow y_1(x) = \frac{Q}{P\lambda}\frac{\sin\lambda c}{\sin\lambda l}\sin\lambda x - \frac{Qcx}{Pl}$$

$$y_2(x) = -\frac{Q}{P\lambda}\sin\lambda(l-c)(-\tan\lambda l\cos\lambda x + \sin\lambda x) - \frac{Q(l-c)(l-x)}{Pl}$$

$$= \frac{Q}{P\lambda\sin\lambda l}\sin\lambda(l-c)\sin\lambda(l-x) - \frac{Q(l-c)(l-x)}{Pl}$$

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So Y1(L-C) must be equal to Y2(L-C) that deflection should match, and Y1 prime of L - C must be equal to Y2 prime of L – C, so if we impose those conditions and simplify we will be able to determine this constant B and D, and I will get Y1(x) to be given by this, and Y2(x) to be given by this, to understand what this specifically mean we will consider a special case the

Special case

$$c = \frac{l}{2}; x = \frac{l}{2}$$

$$y_1(x) = \frac{Q}{Pl}\frac{\sin\lambda c}{\sin\lambda l}\sin\lambda x - \frac{Qcx}{Pl}$$

$$= \frac{Q}{Pl}\frac{\sin\frac{\lambda l}{2}}{\sin\lambda l}\sin\frac{\lambda l}{2} - \frac{Qcl}{2Pl} = \frac{Q}{Pl}\frac{\sin^2\frac{\lambda l}{2}}{2\sin\frac{\lambda l}{2}\cos\frac{\lambda l}{2}} - \frac{Qcl}{2Pl}$$

$$= \frac{Q}{2P\lambda}\left\{\tan\frac{\lambda l}{2} - \frac{\lambda l}{2}\right\}$$

Put $u = \frac{\lambda l}{2}; y\left(\frac{l}{2}\right) = \delta$

$$\Rightarrow \delta = \frac{Q}{2P\lambda}\{\tan u - u\}$$

Recall $P = 0, \delta = \delta_0 = \frac{Ql^3}{48EI}$

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point of application of the load is at L/2 and we are interested in response at L/2, so Y1(x) is this, therefore at X = L/2 I get this, and I need to manipulate a bit and if I do this I will get Y1(x) to be given by Q/2P lambda, tan lambda L/2 -lambda L/2, now what I do is this parameter lambda L/2 I will call it as U, so lambda itself is square root P/EI, so lambda is a nonlinear function of P, and U is this, so now I also call Y(L/2) is delta, so I get delta as Q/2P lambda into tan U – U, so as far as the transverse load is concerned there is a linear parameter, but the axial load parameter is nonlinear at P, I mean this function is nonlinear in that parameter P, so when P = 0, we know that the deflection is QL cube/48EI, so what I will do is I will

$$\delta = \frac{Ql^3}{48EI}\frac{48EI}{Ql^3}\frac{Q}{2P\lambda}\{\tan u - u\}$$

$$= \delta_0 \frac{24}{l^3}\frac{1}{\lambda^3}\{\tan u - u\}$$

$$= \delta_0 \frac{24}{l^3}\frac{l^3}{8u^3}\{\tan u - u\} \qquad \left(u = \frac{\lambda l}{2} \Rightarrow \lambda = \frac{2u}{l}\right)$$

$$\boxed{\delta = \delta_0 \frac{3\{\tan u - u\}}{u^3} = \delta_0 \chi(u) \quad \text{with } \chi(u) = \frac{3\{\tan u - u\}}{u^3}}$$

$$\tan u = u + \frac{u^3}{3} + \frac{2u^5}{15} + \cdots$$

$$\Rightarrow \frac{\tan u - u}{u^3} = \frac{1}{3} + \frac{2u^2}{15} + \cdots$$

$$\lim_{u \to 0} \frac{\tan u - u}{u^3} \to \frac{1}{3}$$

$$\lim_{u \to 0} \chi(u) \to 1$$

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rearrange these terms, I will multiply and divide by that delta naught I call that as delta naught and rearrange these terms, and after I do this I get the mid-span deflection as delta naught, that is response with P = 0 into a amplification factor we call it as chi(U), and this chi(U) is 3 tan U – U/U cube, this is known as a stability function.

Now how does it behave suppose we take a look at what happens to this function as U goes to 0, U goes to 0 means the P going to 0, as P goes to 0 delta should go to delta naught, so chi(U) should go to 1, as U goes to 0, so that we can verify by expanding tan U in the Taylor's expansion and rearranging the term we can show that chi(U) to 1 as U goes to 0, so which is as it should be.

$$\delta = \delta_0 \frac{3\{\tan u - u\}}{u^3} = \delta_0 \chi(u) \quad \text{with } \chi(u) = \frac{3\{\tan u - u\}}{u^3}$$

$$\lim_{u \to \frac{\pi}{2}} \chi(u) \to \infty$$

$$u = \frac{\pi}{2} \Rightarrow \frac{\lambda l}{2} = \frac{\pi}{2} \Rightarrow \sqrt{\frac{P}{EI}} \frac{l}{2} = \frac{\pi}{2}$$

$$\Rightarrow P_{cr} = \frac{\pi^2 EI}{l^2}$$

$$y(x) = \frac{Q}{P\lambda} \frac{\sin \lambda c}{\sin \lambda l} \sin \lambda x - \frac{Qcx}{Pl}$$

$$\Rightarrow \frac{dy}{dx} = \frac{Q}{P} \frac{\sin \lambda c}{\sin \lambda l} \cos \lambda x - \frac{Qc}{Pl}$$

$$c = \frac{l}{2} \;\&\; x = \frac{l}{2}$$

$$\frac{dy}{dx}\left(\frac{l}{2}\right) = \frac{Q}{P} \frac{\sin \frac{\lambda l}{2}}{\sin \lambda l} \cos \frac{\lambda l}{2} - \frac{Ql}{2Pl} = 0$$

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Now on the other hand what happens to this function as U goes to phi/2, we know tan phi/2 there is a problem, so as chi(U) goes to infinity, I mean as U goes to phi/2, chi(U) goes to infinity that means the condition U = phi/2 is a critical condition, so if we evaluate for value of P we will reach this condition U = phi/2 I will get this value P critical as phi square EI/L square, so this is a well-known Euler's buckling load, we will arrive at this by other arguments later but this present argument leads to the same solution. So if now P is close to this value then the amplification factors will be very large, so that is where you know designers need to be concerned about.

Now let us consider behavior of other parameters like slope and bending moment, suppose Y(x) is given by this, and DY/DX we can compute, and again if I evaluated load at mid-span and response at mid-span, we expect the mid-span slope to be 0 and that happens to be 0 you can verify that.

$$\frac{dy}{dx}(0)=\frac{Q}{P}\frac{\sin\frac{\lambda l}{2}}{\sin\lambda l}-\frac{Q}{2P}=\frac{Q}{2P}\frac{\sin\frac{\lambda l}{2}}{2\sin\frac{\lambda l}{2}\cos\frac{\lambda l}{2}}-\frac{Q}{2P}$$

$$\Rightarrow\frac{dy}{dx}(0)=\frac{Q}{2P}\left(\frac{1}{\cos\frac{\lambda l}{2}}-1\right)$$

For P=0, $\frac{dy}{dx}(0)=\frac{Ql^2}{16EI}=\theta_0$

$$\frac{dy}{dx}(0)=\frac{Ql^2}{16EI}\frac{16EI}{Ql^2}\frac{Q}{2P}\left(\frac{1}{\cos\frac{\lambda l}{2}}-1\right)=\theta_0\frac{8}{\lambda^2 l^2}\left(\frac{1-\cos u}{\cos u}\right)=\theta_0\frac{8}{l^2}\frac{l^2}{4u^2}\left(\frac{1-\cos u}{\cos u}\right)$$

$$\theta(0)=\theta_0\varepsilon(u)\ \text{with}\ \varepsilon(u)=\frac{2(1-\cos u)}{u^2\cos u}$$

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On the other hand if you look at the slope at the supports if you manipulate these expressions we can show that the slope is given by this, and again following the same argument that we did for mid span deflection, first we will find out what should be the slope at with P = 0 I know this, I call it as theta naught and I will scale this value by theta naught and rearrange the terms, and I will get the slope at X = 0 given by theta naught into epsilon U, where epsilon U is this function 2 into 1 - cos U/U square cos U, this is the stability function associated with slope. Again you can show that as U goes to 0, this goes to 1 and so on and so forth so that can be done.

$$\frac{d^2y}{dx^2} = -\frac{Q\lambda \sin\lambda c \sin\lambda x}{P\sin\lambda l}$$

$$c = \frac{l}{2} \;\&\; x = \frac{l}{2}$$

$$\Rightarrow EI\frac{d^2y}{dx^2} = -EI\frac{Q\lambda \sin^2\frac{\lambda l}{2}}{P\sin\lambda l} = -\frac{Q}{2\lambda}\tan\frac{\lambda l}{2}$$

$$M\left(\frac{l}{2}\right) = -\left(\frac{Ql}{4}\right)\frac{2}{\lambda l}\tan\frac{\lambda l}{2} = M_0\frac{\tan u}{u}$$

$$\boxed{M\left(\frac{l}{2}\right) = M_0\xi(u) \text{ with } \xi(u) = \frac{\tan u}{u}}$$

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Now how about bending moment? So we differentiate Y once again, and again apply the load at mid-span and response at mid-span, so if we arrange these terms we know the maximum bending moment at mid-span is QL/4 when there is no load P, and how does this bending moment get modified, it get modified by a function tan U/U, I call it as XI(U).

Summary

$$\delta = \delta_0 \frac{3\{\tan u - u\}}{u^3} = \delta_0 \chi(u)$$

$$\theta(0) = \theta_0 \frac{2(1-\cos u)}{u^2 \cos u} = \theta_0 \varepsilon(u)$$

$$M\left(\frac{l}{2}\right) = M_0 \frac{\tan u}{u} = M_0 \xi(u)$$

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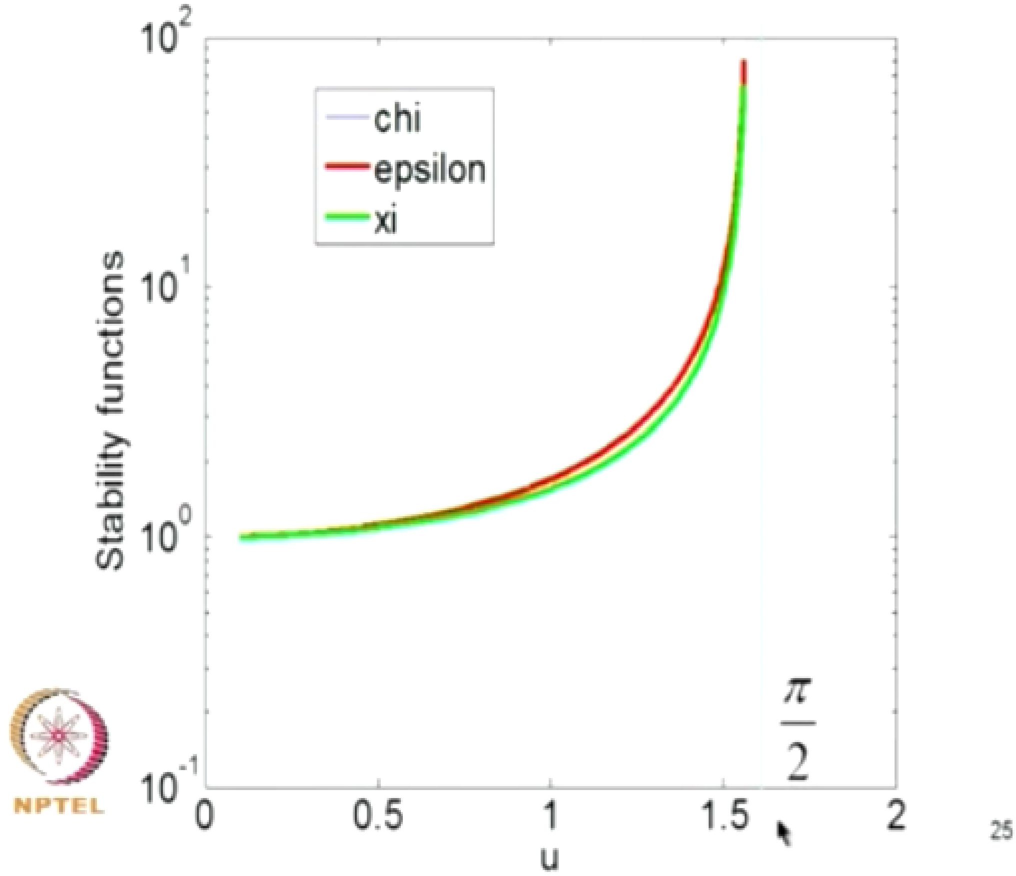
So now in summary we have mid span deflection equal to, the mid-span deflection in terms of axial load into an amplification factor, slope at the left support, slope in the absence of axial load into a magnification factor, bending moment, maximum bending moment, bending moment in absence of axial load into this amplification factor, how these factors look like? As a function of U, if you plot U on X axis and this stability functions on Y axis the 3 function chi,



epsilon, and XI are shown here, they show similar trend and at phi/2 they go to infinity and these lines are asymptotic to that, and these points go to 1 as U becomes 0, so these are the stability functions, these functions are tabulated, it is for example the book by Timoshenko on elastic stability has an appendix where these functions are tabulated.

Remarks

- $\chi(u), \varepsilon(u), \& \xi(u)$ are known as the stability functions.
- As $u \to 0$, that is, as $P \to 0$, $\chi(u), \varepsilon(u), \& \xi(u) \to 1$. This means that as $P \to 0$there would be no modification to the response- which is as it should be.
- As $u \to \frac{\pi}{2}, P \to P_{cr} = \frac{\pi^2 EI}{l^2} \& \chi(u), \varepsilon(u), \& \xi(u) \to \infty$
- Responses are not linear function of P. The main feature of linear analysis thus breaks down.
- $EIy'' + Py = M(x)$

Py: this force is written based on deformed geometry

$M(x)$: this force is independent of deformed geometry.

- $y(x) = y_0(x, P = 0)\phi(x, P)$

$\phi(x, P)$ = Modification factor

Nonlinear function of P. Not well behaved for all P. At some P, MF $\to \infty$

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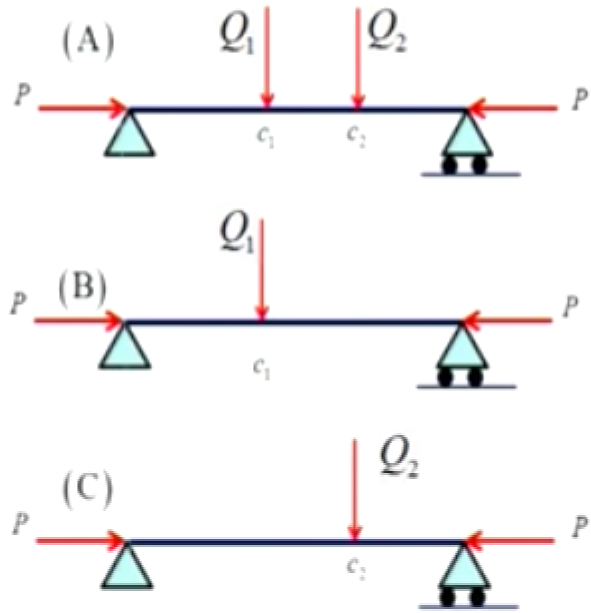
Now we can make some remarks on the behavior of these functions, this chi, epsilon, and XI as already mentioned are known as stability functions, as U goes to 0 that is as axial load go to 0 this stability function goes to 1, so this means that as P tends to 0 there would be no modification to the response which is as it should be, as U goes to 0 and P going to P critical these I think as U goes to phi/2, P goes to P critical, and these functions tend to infinity that means this function go to infinity as U goes to phi/2. Now the responses are not linear functions of P here, because U is a non-linear function of U obviously responses are non-linear function of U, the main feature of linear analysis thus break down.

Now this I already mentioned when I write this term PY this force is written based on deform geometric, whereas in writing M(x) this force is independent of deformed geometry, the response is obtained as Y(x) as Y naught (x, P= 0) and a modification factor which is function of X and applied load, so it is a nonlinear function of P it is not well behaved for all P and at some P the magnification factor goes to infinity. Now I mentioned that there is a new form of principle of superposition we can verify that by writing the relevant equations, suppose I



Remarks (continued)

New form of principle of superposition

$$(A)\ EIy'' + Py = -\frac{Q_1c_1}{l}x - \frac{Q_2c_2}{l}x$$

$$(B)\ EIy_1'' + Py_1 = -\frac{Q_1c_1}{l}x$$

$$(C)\ EIy_2'' + Py_2 = -\frac{Q_2c_2}{l}x$$

Add (B)&(C)

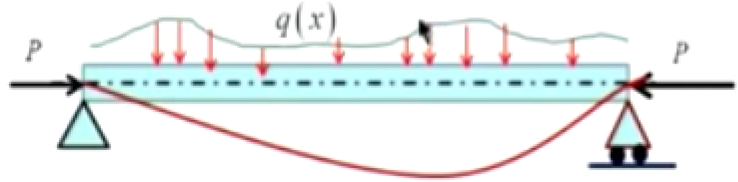
$$EI(y_1'' + y_2'') + P(y_1 + y_2) = -\frac{Q_1c_1}{l}x - \frac{Q_2c_2}{l}x$$

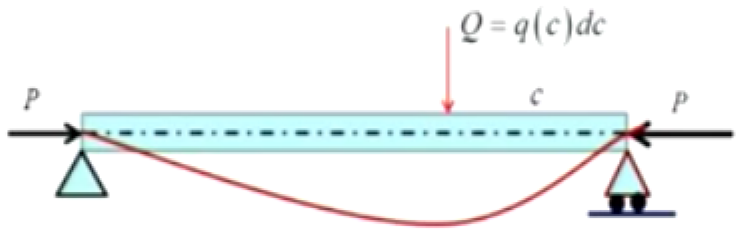
$$\Rightarrow y = y_1 + y_2$$

Case A can be handled by using solutions for the case of single load.

consider a situation A where the beam carries axial load P and loads W1, transverse load W1 and W2, in situation B in addition to P there is W1, and in situations C in addition to P there is W2, so system equation governing A is EIY double prime + PY + the bending moment due to this, system B there is only one contribution, system C there is again only one contribution here due to Q2, here due to Q1.

Beam carrying distributed load





$$dy(x) = q(c)\,dc\,\alpha(x,c) \quad \text{for } 0 \le x \le l - c$$
$$dy(x) = q(c)\,dc\,\beta(x,c) \quad \text{for } l - c \le x \le l$$

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Now if I add this B and C I get EIY1 double prime + Y2 double prime + PY1 + Y2 equal to the sum of this, now if I call Y as Y1 + Y2 I see that this Y satisfy this equation, therefore if I combine these 2 problems I get solution to this problem, so this would mean that if we use this modified version of principle of superposition knowing the solution to this problem I can synthesize solution for this problem, so I can still use principle of superposition but I have to take care to interpret in this form as displayed here. So now equipped with that insight we can now consider a few other problems, for example instead of a concentrated load if beam is carrying a distributed load Q(x), so what I will do I will consider a constant response due to an incremental concentrated load Q(c) into DC and I know already this solution and I call it as DYX, for X lying in this region and for X in L – C2L this is this, this is what we have derived already.

Now the solution to this problem is obtained by integrating over the relevant regions, and I am keeping track of the correct form of a principle of superposition, so if I now consider special

$$dy(x) = q(c)\,dc\,\alpha(x,c) \text{ for } 0 \le x \le l - c$$

$$dy(x) = q(c)\,dc\,\beta(x,c) \text{ for } l - c \le x \le l$$

$$y(x) = \int_0^{l-x} q(c)\alpha(x,c)\,dc + \int_{l-x}^{l} q(c)\beta(x,c)\,dc$$

Special case: $q(x) = q_0$ & $x = \frac{l}{2}$

$$\delta_{\max} = \left(\frac{5q_0 l^4}{384EI}\right)\left[\frac{12(\sec u - 2 - u^2)}{5u^4}\right]$$

$$\theta_{\max} = \left(\frac{q_0 l^3}{24EI}\right)\left[\frac{3(\tan u - u)}{u^3}\right]$$

$$BM_{\max} = \left(\frac{q_0 l^2}{8}\right)\left[\frac{2(1-\cos u)}{u^2 \cos u}\right]$$

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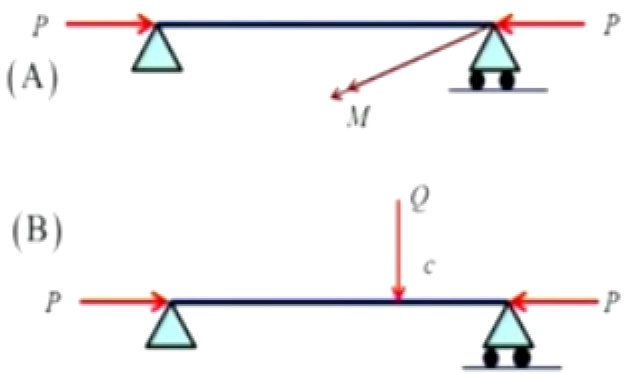
case where Q(x) is Q naught, that is a beam carrying UDL and an axial load P, I know that mid span deflection is 5Q naught L to the power of 4/384EI, now in presence of axial load this is the amplification factor, similarly the maximum slope at the support is given by this and this function which is shown in the red is the amplification factor, maximum bending moment I know it is Q naught L square/8 and this is amplification factor, we can verify for these 3 functions that as U goes to 0, they go to 1, and as U goes to phi/2 they all become unbounded, okay, so in a problem where there is a transverse axial load if there were to be an axial load transverse load Q(x), if there were to be an axial load P, and if this P happens to be in the neighborhood of the critical load then this frame is going to, the response of this frame due to Q(x), that response will be substantially amplified, and this is undesirable therefore our interest in this subject.

Bending of beam with end couple



$$\lim_{\substack{c \to 0 \\ Qc \to M}} (B) \to (A)$$

Now we will consider some derivatives of the problem that we have solved, for example I have a beam column with the end moment, so this M is nothing but this couple shown here, how do we solve this problem? Now I will synthesize the solution to this problem by considering this problem, I have the solution to this problem there is a load at a distance C, now what I will do is I will take limit of C going to 0 and at the same time Q into C goes to M, so in that case the solution in situation B will go to solution to the problem in A, so that can be quickly done, so

Consider (B)

$$y(x)=\frac{Q\sin\lambda c}{P\lambda\sin\lambda l}\sin\lambda x-\frac{Qc}{Pl}x;\ 0\le x\le l-c$$

As $c\to 0,\ \sin\lambda c\to\lambda c$

$$\lim_{\substack{c\to 0\\ Qc\to M}} y(x)=\frac{M\lambda\sin\lambda x}{P\lambda\sin\lambda l}-\frac{Mx}{Pl}=\frac{M}{P}\left(\frac{\sin\lambda x}{\sin\lambda l}-\frac{x}{l}\right)$$

$$\frac{dy}{dx}=\frac{M}{P}\left(\frac{\lambda\cos\lambda x}{\sin\lambda l}-\frac{1}{l}\right)$$

$$\frac{dy}{dx}(0)=\theta_a=\frac{M}{P}\left(\frac{\lambda}{\sin\lambda l}-\frac{1}{l}\right)$$

$$-\frac{dy}{dx}(l)=\theta_b=-\frac{M}{P}\left(\frac{\lambda\cos\lambda l}{\sin\lambda l}-\frac{1}{l}\right)$$

Recall, for P=0, $\theta_a=\frac{Ml}{6EI}$ & $\theta_b=-\frac{Ml}{3EI}$

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you consider the second situation I have Y(x) to be this, and now if I put C = 0, sin lambda C will go to lambda C, and the limit of this function as C goes to 0, and QC going to M becomes this, okay, so this is a deflection. Now how do you get slope? By differentiating this I get slope, so therefore at X = 0, this is the maximum slope.

Similarly at X = L, theta A and theta B are found out, we know that for P = A, theta A and theta

$$\Rightarrow \theta_a = \theta_{a0}\,\frac{3}{u}\left[\frac{1}{\sin 2u} - \frac{1}{2u}\right] \;\&\; \theta_b = \theta_{b0}\,\frac{3}{2u}\left[\frac{1}{2u} - \frac{1}{\tan 2u}\right]$$

Denote

$$\phi(u) = \frac{3}{u}\left[\frac{1}{\sin 2u} - \frac{1}{2u}\right]$$

$$\psi(u) = \frac{3}{2u}\left[\frac{1}{2u} - \frac{1}{\tan 2u}\right]$$

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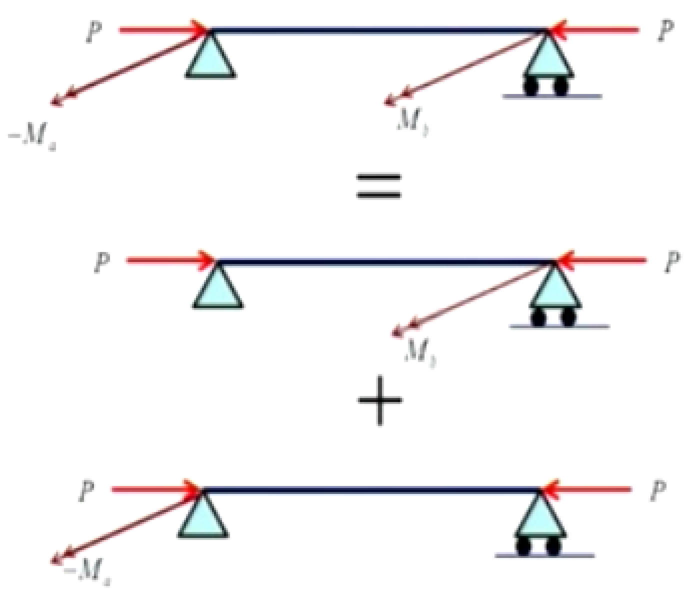
B are given by this, therefore now the new amplification factor and slopes are given by these functions, we denote these functions as phi(u) and sai(u). Now how about a system carrying 2



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couples? Now I can use the principle of superposition, first I will solve the problem with MA and then solve the problem with, this is MB and then with MA and solve the problem, so when

$$y(x) = \frac{M_b}{P}\left[\frac{\sin \lambda x}{\sin \lambda l} - \frac{x}{l}\right] + \frac{M_a}{P}\left[\frac{\sin \lambda (l-x)}{\sin \lambda l} - \frac{l-x}{l}\right]$$

$$\theta_a = \frac{M_a l}{3EI}\psi(u) + \frac{M_b l}{6EI}\phi(u)$$

$$\theta_b = \frac{M_b l}{3EI}\psi(u) + \frac{M_a l}{6EI}\phi(u)$$

$$\begin{Bmatrix}\theta_a \\ \theta_b\end{Bmatrix} = \begin{bmatrix}\frac{l}{3EI}\psi(u) & \frac{l}{6EI}\phi(u) \\ \frac{l}{6EI}\phi(u) & \frac{l}{3EI}\psi(u)\end{bmatrix}\begin{Bmatrix}M_a \\ M_b\end{Bmatrix}$$

$$S = \begin{bmatrix}\frac{l}{3EI}\psi(u) & \frac{l}{6EI}\phi(u) \\ \frac{l}{6EI}\phi(u) & \frac{l}{3EI}\psi(u)\end{bmatrix} = \text{Modified flexibility matrix}$$

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I do this I get the superposition principle leads to this solution, so if I now consider theta A, that is differentiate this and put X = 0 the solution will be in this form, where sai(u) and phi(u) are the stability functions given here. Now I can assemble this in a matrix form and the response theta A and theta B are related to the applied forces MA and MB through this matrix therefore this is a stiffness matrix, so this stiffness matrix is given by this and this is a modified flexibility, sorry it is not stiffness it is the flexibility matrix, it is a modified flexibility matrix, the inverse of this is the stiffness matrix, so that is the stiffness matrix which takes into account the presence of axial loads.

$$M_a = M_b = Pe = M_0$$

$$y(x) = \frac{M_0}{P}\left[\frac{\sin \lambda x}{\sin \lambda l} + \frac{\sin \lambda (l-x)}{\sin \lambda l} - 1\right] \Rightarrow y\left(\frac{l}{2}\right) = \delta_0 \frac{2(1-\cos u)}{u^2 \cos u}$$

$$\frac{dy}{dx} = \frac{M_0}{P}\left[\frac{\lambda \cos \lambda x}{\sin \lambda l} - \frac{\lambda \sin \lambda (l-x)}{\sin \lambda l}\right] \Rightarrow \frac{dy}{dx}(0) = \frac{M_0 \lambda}{P}\left[\frac{(1-\cos u)}{\sin 2u}\right]$$

$$\frac{d^2 y}{dx^2} = \frac{M_0}{P}\left[-\frac{\lambda^2 \sin \lambda x}{\sin \lambda l} + \frac{\lambda^2 \cos \lambda (l-x)}{\sin \lambda l}\right] \Rightarrow \frac{d^2 y}{dx^2} = -\frac{M_0}{EI}\sec u$$

$$BM_{max} = -EI\frac{d^2 y}{dx^2} = M_0 \sec u$$

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Now suppose this axial load is applied with an eccentricity E, what happens to this problem? So this can be thought of as this problem where MA and MB are equal to PE which is M naught, so we can use the same solutions and we will be able to you know simplify the problem because M1 = M2, and we will get mid-span deflection to be given by this, slope at X = 0 given by this, and the maximum bending moment is given by M naught into secant U, okay, so U if you recall lambda L/2, where lambda itself is square root P/EI, so maximum bending moment is

$$BM_{max} = -EI\frac{d^2y}{dx^2} = M_0 \sec u$$

$$\sigma_{max} = \frac{P}{A} + \frac{Pe}{I}\sec\left\{\frac{l}{2}\sqrt{\frac{P}{EI}}\right\} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left\{\frac{l}{2}\sqrt{\frac{P}{EI}}\right\}\right]$$

Remarks

- Axial stress in presence of eccentrically applied axial loads
- In the neighbourhood of critical load, slightest eccenctricity would dramatically amplify the stresses
- The amplification factor can be approximated by $\left\{\frac{1}{1-\frac{P}{P_{cr}}}\right\}$

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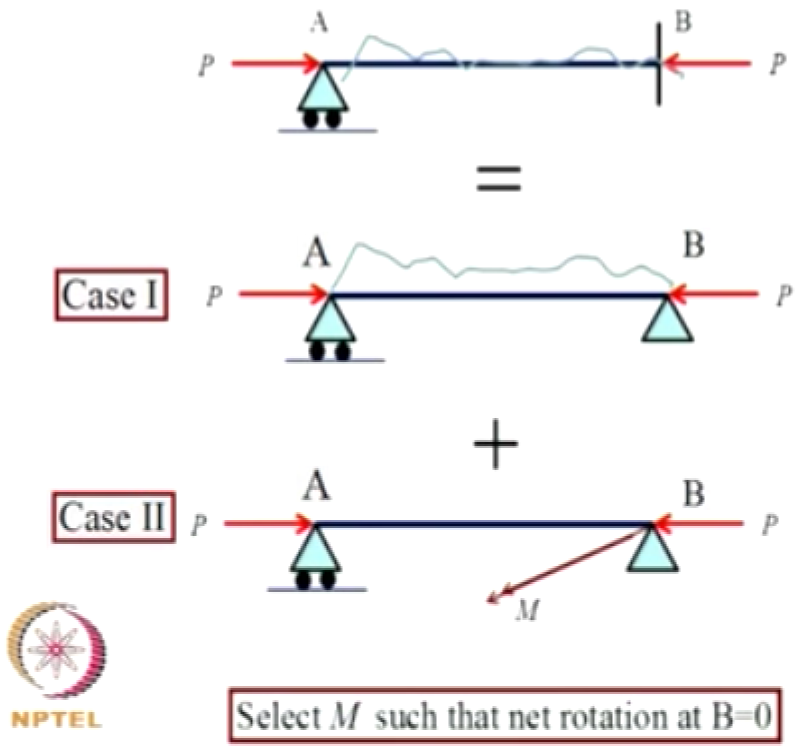
this. Now if I compute the maximum stress, bending stress P/A contribution due to axial load and this is a contribution due to the moment, so this becomes P/A 1 + EC/R square secant to this, so this is a well-known secant formula that is used in design of axial members especially in metal structures, so axial stress in presence of eccentrically applied axial loads so this is a formula for the axial stress in presence of eccentrically applied axial loads. If P is close to critical load then slightest eccentricity, here there is no lateral load but if this load there is a slightest eccentricity then the response will be amplified by this factor, and such eccentricities are inevitable in you know built up structures, so we need to be ensure that we need to determine what should be this critical load with care. Now this magnification factor can be in fact be approximated by 1 - 1/B/P critical by expanding this trigonometric term, and this gives reasonable accuracy for you know acceptable ranges of P close to P critical in which we are interested.



How about statically indeterminate structures? Now we can, supposed to illustrate that we can consider a propped cantilever and end B is now clamped, now I apply load P here and this is some transverse load Q(x) on that and what I will do is I will consider 2 problems, one in which this fixed end is, in both the cases the fixed end is replaced by a hinge, in one case I analyze this problem under axial loads and this transverse loads as shown here, and then I will analyze this problem with a bending moment here, because there will be reaction which is the bending moment which I would not know, this bending moment is selected in such a way that the rotation that we get in this case is nullified so that we know that rotation here is 0, so by analyzing this problem will not be getting rotation as 0, so therefore the correction that we apply to this is through finding out this bending moment which nullifies that rotation, so this is the idea, now how does it work? First rotation at B in case 1, so at B the rotation we've analyze

Rotation at B in case I= $\dfrac{q_0 l^3}{24EI}\dfrac{3(\tan u - u)}{u^3} = \dfrac{q_0 l^3}{24EI}\chi(u)$

Rotation at B in case II= $\dfrac{M_0 l}{3EI}\dfrac{3}{2u}\left(\dfrac{1}{2u} - \dfrac{1}{\tan 2u}\right) = \dfrac{M_0 l}{3EI}\psi(u)$

M_0 must be such that $\dfrac{q_0 l^3}{24EI}\chi(u) + \dfrac{M_0 l}{3EI}\psi(u) = 0$

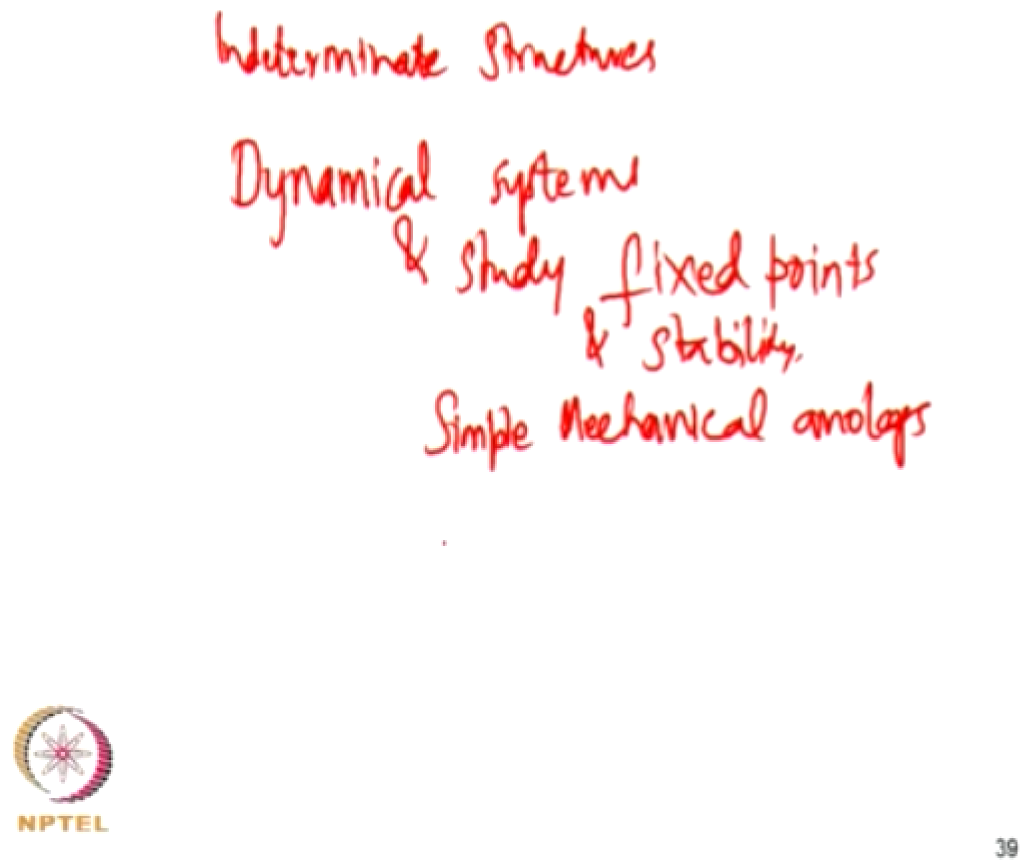
$\Rightarrow$

$$M_0 = -\frac{q_0 l^3}{24EI}\frac{3EI}{l}\frac{\chi(u)}{\psi(u)} = -\frac{q_0 l^3}{8}\frac{\chi(u)}{\psi(u)}$$

this problem so we have this expression for rotation and this, it is Q naught LQ/24EI which is the rotation in absence of axial load, and this is the magnification factor and this we have called it a chi(u).

Now rotation at B in case 2 here again we have solved this problem, and we have obtained this as M naught L/3I into this, we call it as sai(u), so M naught must be such that the sum of this must be equal to 0, so that gives me an equation for M naught which is this, so this M naught we find out as Q naught LQ/8 into ratio of chi(u) sai(u) using this M naught I can now, I have the solutions I can superpose and obtain the solution to this built up case.

Now the way we will proceed further now is that we will complete the discussion on indeterminate structures following the strength of material logic that we have developed, then what we will do is we will consider treat the systems as dynamical systems, and investigate what are known as fixed points and there stability this discussion will conduct with some simple mechanical systems which are analogues of larger structural system in which we are interested and gain an understanding of this problem, this will also help us to understand the relationship between imperfection and nonlinear deformation, that is large deformation, so once we get that we will be able to you know we will get a complete picture of the range of behavior,



then we will introduce certain axiomatic notions about stability based on energy concepts, those axioms would help us to formulate the problem of stability analysis through optimization arguments, and we will be able to develop weighted residual and finite element formulations for that, so we will show that the question of elastic stability of engineering systems can be handled through an Eigen value analysis, and as a prelude to that during the course of this discussion we will consider Eigenvalue problems associated with system governed by differential equations, so then we will be able to translate that formulation to suitable logic for when we bring in you know discretization and matrix algebra, so these things we will cover in the lectures to follow, and at this juncture we will close this lecture

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