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Course Title
Finite element method for structural dynamic
And stability analyses
Lecture – 25
Plate bending elements
(continued)

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Finite element method for structural dynamic and stability analyses

Module-8

Plate bending and shell elements

Lecture-25 Plate bending elements

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Plate elements

- Thick/Thin plates
- Conforming/Non conforming elements

Discussions

How do thick elements behave when applied to thin elements?

How to examine convergence characteristics when nonconforming elements are used?

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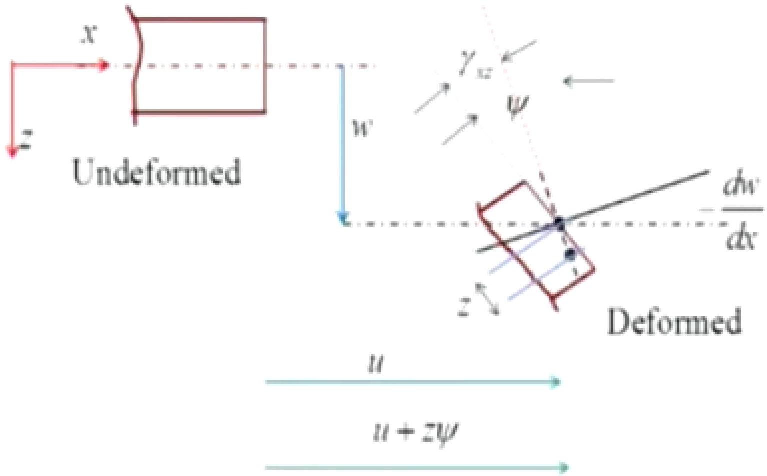
In today's lecture we will continue our discussion with plate bending elements, so far what we have done is we have developed various types of elements for both theory based on thin plate hypothesis and thick plate hypothesis and the element that we have developed where of, you know some of the elements were conforming, somewhere non-conforming.

Now before I start discussing a topic for today that is stiffened plates, I would like to make few observations on some of the issues related to using of these elements. Now two questions I would like to raise, first question is how do you think elements behave when used for thin elements? Then how to examine convergence characteristics when non-conforming elements are used? Both these questions are widely discussed in the literature, so I will just give some broad indications on what are the issues and how some of the difficulties can be resolved, so to

Timoshenko beam revisited: what happens if beam depth reduces?



$$U = \frac{1}{2}\int_0^l \left\{ EI\left(\frac{\partial \psi}{\partial x}\right)^2 + kAG\left(-\psi + \frac{\partial w}{\partial x}\right)^2 \right\} dx$$

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enable this discussion we will revisit the problem of Timoshenko beam so ask the question what happens if beam depth reduces, so if you recall in the Timoshenko beam this is undeformed geometry, and in the deform geometry the plane section here remains plane, but it rotates by an angle sai, it doesn’t remain normal to the neutral axis there will be shearing strain that we need to, we wish to take into account.

$$U = \frac{1}{2}\int_0^l \left\{ EI\left(\frac{\partial \psi}{\partial x}\right)^2 + kAG\left(-\psi + \frac{\partial w}{\partial x}\right)^2 \right\} dx$$

Consider a two-noded element with dofs w and ψ at each node

$$\psi(x) = \psi_1 \frac{x}{l} + \psi_2 \left(1 - \frac{x}{l}\right)$$

$$w(x) = w_1 \frac{x}{l} + w_2 \left(1 - \frac{x}{l}\right)$$

$$\frac{\partial w}{\partial x} = \frac{w_1 - w_2}{l} = \text{constant for the element.}$$

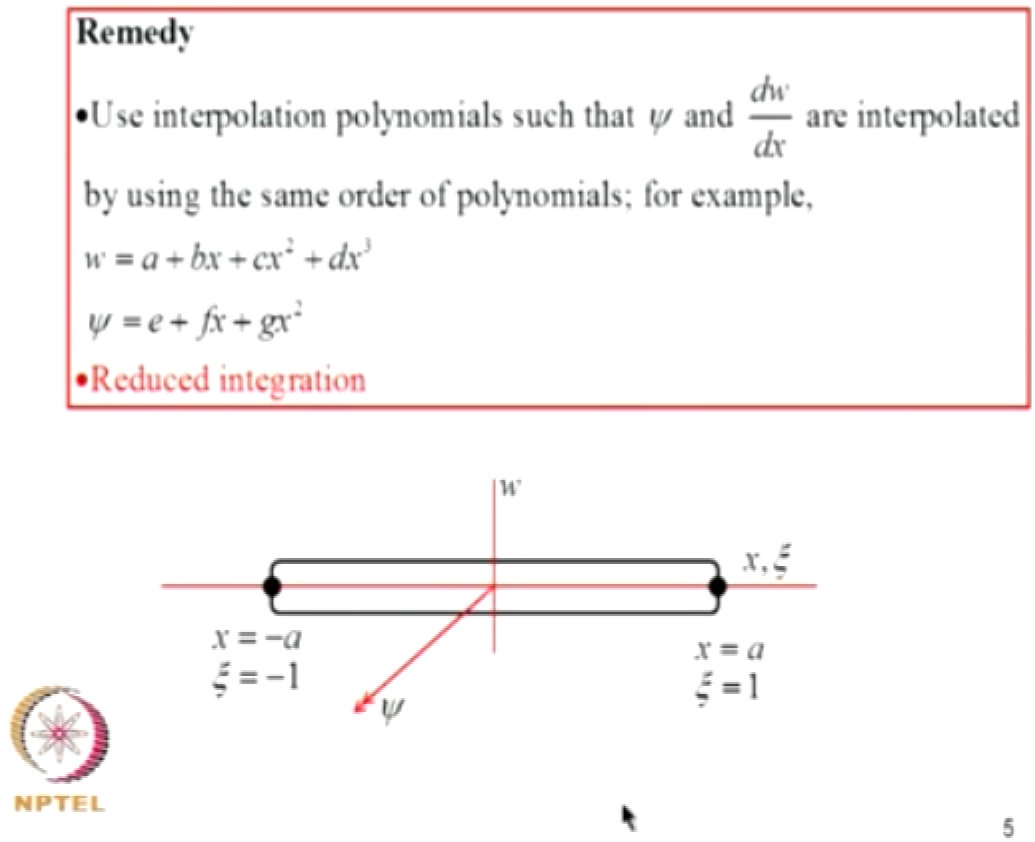
As beam depth becomes small, $\psi(x) = \frac{\partial w}{\partial x}$.

This becomes constant. But this is not admissible since the bending energy given by $U = \frac{1}{2}\int_0^l EI\left(\frac{\partial \psi}{\partial x}\right)^2 dx$ becomes 0.

This problem is known as shear locking.

The expression for strain energy we've already derived will be of this form, now if you examine the integrand in this expression for the energy we see that the field variables are sai and W, and the order of the highest derivative is 1 therefore we can use linear interpolation functions for both representing sai and W, so if we consider 2 noded element with degrees of freedom W and sai at each node we can write sai(x) as this and W(x) as this. Now if I differentiate W(x) I will get W1 - W2/L, this is a constant for the element.

Now as beam depth becomes small the sai (x) you know that the sai (x) should go to dou W/dou X this is what we assume in thin plate theory, so in the thick plate theory as beam depth becomes small sai (x) we can expect that it will go to this, according to this representation becomes constant, but this is not admissible since the bending energy is given by this term and this becomes 0, if sai (x) is constant, so this problem is known as the shear locking, so this is undesirable, so how can we remedy this situation? So there are 2 possible remedies, one is we



can use interpolation polynomials such that sai and DW/DX are interpolated by using the same order of polynomials, for example W I will use a cubic polynomial, and for sai quadratic so that dou W/dou X and sai will have the same order of representation or there is an alternative fixed known as reduced integration, I will try to explain what these things mean in the following slides.

So let's consider a beam element, now I will use the non-dimensional coordinate sai = X/L and I will locate the origin here, so the beam lies between XI = - 1, and XI = 1, and its length is actually 2A, X varying from - A to + A, so this axis is X, sai and this rotation is a sai, and this is W.

Field equations

$$kAG\frac{d^2w}{dx^2}-kAG\frac{d\psi}{dx}=0 \qquad \frac{d^4w}{dx^4}=0$$

$$kAG\frac{dw}{dx}+EI\frac{d^2\psi}{dx^2}-kAG=0 \qquad \frac{d^3\psi}{dx^3}=0$$

$$\xi=\frac{x}{a} \Rightarrow \begin{array}{l} w=a_1+a_2\xi+a_3\xi^2+a_4\xi^3 \\ \psi=b_1+b_2\xi+b_3\xi^2 \end{array}$$

$a_1, a_2, a_3, a_4, b_1, b_2, \& b_3$ are not independent.

$$kAG\frac{dw}{dx}+EI\frac{d^2\psi}{dx^2}-kAG=0\Rightarrow$$

$$b_1=\frac{1}{a}a_2+\frac{6\beta}{a}a_4, b_2=\frac{2}{a}a_3, b_3=\frac{3}{a}a_4 \text{ with } \beta=\frac{EI}{kAGa^2}$$

Note: $\beta=\frac{EI}{kAGa^2}\rightarrow 0$ as depth of the beam reduces.

We have only four independent constants.

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Now the field equations are given by this and these imply that D4W/DX4 is 0, D cube sai/DX cube is 0. Now accordingly using sai = X/A, W can be represented as a cubic polynomial and sai can be represented a quadratic polynomial. Now in this representation these A1, A2, A3, A4, B1, B2, B3 are not independent, they are connected by this equation and if we impose that constraint we get the constraint equations as B1 given by this, B2 given by this, and so on and so forth, and we introduce a parameter beta which is EI/KA GA square, now as beta goes to 0, beta goes to 0 as depth of the beam reduces so this is one handle that we will have to examine the behavior of the element as depth becomes small. Now we have only 4 independent constants, because of this constraint equation that is fine.

$$w = \begin{bmatrix} N_1(\xi) & aN_2(\xi) & N_3(\xi) & aN_4(\xi) \end{bmatrix} \{w\}_e$$

$$\psi = \begin{bmatrix} \frac{1}{a}N_5(\xi) & N_6(\xi) & \frac{1}{a}N_7(\xi) & N_8(\xi) \end{bmatrix} \{w\}_e$$

$$\{w\}_e = \begin{bmatrix} w_1 & \psi_1 & w_2 & \psi_2 \end{bmatrix}$$

$$N_1(\xi) = \frac{1}{4(1+3\beta)}\left\{2+6\beta-3(1+2\beta)\xi+\xi^3\right\};$$

$$N_2(\xi) = \frac{1}{4(1+3\beta)}\left\{2+3\beta-\xi-(1+3\beta)\xi^2+\xi^3\right\}$$

$$N_3(\xi) = \frac{1}{4(1+3\beta)}\left\{2+6\beta+3(1+2\beta)\xi-\xi^3\right\}$$

$$N_4(\xi) = \frac{1}{4(1+3\beta)}\left\{-(1+3\beta)-\xi+(1+3\beta)\xi^2+\xi^3\right\}$$

$$N_5(\xi) = \frac{1}{4(1+3\beta)}\left\{-\xi+3\xi^2\right\}; N_6(\xi) = \frac{1}{4(1+3\beta)}\left\{-1+6\beta-(2+6\beta)\xi+3\xi^2\right\}$$

$$N_7(\xi) = \frac{1}{4(1+3\beta)}\left\{3-3\xi^2\right\}; N_8(\xi) = \frac{1}{4(1+3\beta)}\left\{-1+6\beta+(2+6\beta)\xi+\xi^2\right\}$$

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So now consequently we can represent W in terms of this shape function N1, N2, N3, N4 and WE is the nodal coordinate W1, sai 1, W2, sai 2, sai itself is represented in terms of this quantities, interpolation function N5, N6, N7, N8 but they are all given by this there are only enough independent constants as mentioned here, as beta see one thing we should notice is if we now consider the representation for displacement field W, as beta goes to 0 this functions N1, N2, N3, N4 reduces to the cubic polynomial that we've used in analyzing Euler-Bernoulli beam.

As $\beta \to 0$ (slender beam), the functions $N_i(\xi), i = 1,2,3,4$ reduce to

$$N_1(\xi) = \frac{1}{4}\{2 - 3\xi + \xi^3\};$$

$$N_2(\xi) = \frac{1}{4}\{2 - \xi - \xi^2 + \xi^3\}$$

$$N_3(\xi) = \frac{1}{4}\{2 + 3\xi - \xi^3\}$$

$$N_4(\xi) = \frac{1}{4}\{-1 - \xi + \xi^2 + \xi^3\}$$

which are the cubic polynomials used in analysing Euler-Bernoulli beam element.

$$\psi = \frac{1}{a}\frac{\partial w}{\partial \xi} = \frac{1}{a}\left[\frac{\partial N_1}{\partial \xi} \quad a\frac{\partial N_2}{\partial \xi} \quad \frac{\partial N_3}{\partial \xi} \quad a\frac{\partial N_4}{\partial \xi}\right]\{w\}_e$$

It can be verified that $N_i(\xi), i = 5,6,7,8$ satisfy these requirements.

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Similarly if I now take sai as dou W/dou sai I get this, and it can be verified that the functions that are listed here N1 to N8 satisfy these requirements, that means in the limit of, the analysis being applied to a slender beam we recover what we have analyzed, what we considered for Euler-Bernoulli beam, so that is embedded in the way we are interpolating here.

$$m_e = \frac{\rho A a}{210(1+3\beta^2)}\begin{bmatrix} m_1 & & & \\ m_2 & m_5 & & \text{sym} \\ m_3 & -m_4 & m_1 & \\ m_4 & m_6 & -m_2 & m_5 \end{bmatrix} + \frac{\rho I_z}{30a(1+3\beta^2)}\begin{bmatrix} m_7 & & & \\ m_8 & m_9 & & \text{sym} \\ -m_7 & -m_8 & m_7 & \\ m_8 & m_{10} & -m_8 & m_9 \end{bmatrix}$$

$m_1 = 156 + 882\beta + 1260\beta^2$

$m_2 = (44 + 231\beta + 315\beta^2)a$

$m_3 = 54 + 378\beta + 630\beta^2$

$m_4 = (-26 - 189\beta - 315\beta^2)a$

$m_5 = (16 + 84\beta + 126\beta^2)a^2$

$m_6 = (-12 - 84\beta - 126\beta^2)a^2$

$m_7 = 18$

$m_8 = (3 - 45\beta)a$

$m_9 = (8 + 30\beta + 180\beta^2)a^2$

$m_{10} = (-2 - 30\beta + 90\beta^2)a^2$

$$k_e = \frac{EI}{2a^3(1+3\beta)}\begin{bmatrix} 3 & & & \\ 3a & (4+3\beta)a^2 & & \text{sym} \\ -3 & -3a & 3 & \\ 3a & (2-3\beta)a^2 & -3a & (4+3\beta)a^2 \end{bmatrix}$$

For slender beams $(\beta = 0)$, these matrices reduce to the matrices valid for Euler-Bernoulli beam element.

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Then the procedure for deriving structural matrices is straightforward, we have the energy expressions and field variables are represented in terms of the nodal values and interpolation functions, so we can do the evaluation of structural matrices and we get the structural matrix, mass matrix as given here, it has 2 parts and they are listed separately and it is given here. Similarly the stiffness matrix is given by, it can be shown that it will be given by this. Now in this case as beta becomes small it can be verified that the mass matrix and stiffness matrix converge to the mass and stiffness matrices that we’ve already derived for Euler-Bernoulli beam, so here there is no problem, here the thick beam element remains applicable as a beam becomes thinner and no special precautions need to be taken to use this element. But on the

Reduced integration

$$U = \frac{1}{2}\int_0^l \left\{ EI\left(\frac{\partial \psi}{\partial x}\right)^2 + kAG\left(-\psi + \frac{\partial w}{\partial x}\right)^2 \right\} dx$$

$$w = \begin{bmatrix} N_1(\xi) & N_2(\xi) \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix}; \psi = \begin{bmatrix} N_1(\xi) & N_2(\xi) \end{bmatrix} \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}$$

$$N_1(\xi) = \frac{1}{2}(1-\xi); N_2(\xi) = \frac{1}{2}(1+\xi)$$

$$w_e = \begin{bmatrix} w_1 & \psi_1 & w_2 & \psi_2 \end{bmatrix}^t$$

$$\begin{Bmatrix} w \\ \psi \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_1 & 0 \\ 0 & N_2 & 0 & N_2 \end{bmatrix} \{w_e\}$$

$$\frac{\partial \psi}{\partial x} = \begin{bmatrix} 0 & -\frac{1}{2a} & 0 & \frac{1}{2a} \end{bmatrix} \{w_e\}$$

$$\frac{\partial w}{\partial x} - \psi = \begin{bmatrix} -\frac{1}{2a} & -\frac{(1-\xi)}{2} & \frac{1}{2a} & -\frac{(1+\xi)}{2} \end{bmatrix} \{w_e\}$$

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other hand suppose if we start with the argument that sai and W are the field variables, and the highest order of the derivative that is present in the integrand of the variational formulation is 1, therefore I am at liberty to use linear interpolation for both W and sai, that is fine, so we can start with that, so W is interpolated like this and sai is interpolated like this, and where N1, N2 are linear functions as shown here.

Now the element, and nodal degrees of freedom will be W1 sai 1, W2 sai 2, and they can be assembled in terms of this N matrix and the nodal values as shown here. Now we can compute dou sai/dou X, and dou W/dou X which are needed in the evaluation of the strain energy, and

$$U = \frac{1}{2}\int_0^l \left\{ EI\left(\frac{\partial \psi}{\partial x}\right)^2 + kAG\left(-\psi + \frac{\partial w}{\partial x}\right)^2 \right\} dx$$

$$\frac{\partial \psi}{\partial x} = \frac{\psi_2 - \psi_1}{2a}$$

$$\frac{\partial w}{\partial x} - \psi = \frac{w_2 - w_1}{2a} - \frac{(1-\xi)}{2}\psi_1 - \frac{(1+\xi)}{2}\psi_2$$

$$U = U_b + U_s$$

$$U_b = \frac{1}{2}\int_0^l EI\left(\frac{\psi_2 - \psi_1}{2a}\right)^2 dx$$

$$U_s = \frac{1}{2}\int_0^l kAG\left\{\left(\frac{w_2 - w_1}{2a} - \frac{(1-\xi)}{2}\psi_1 - \frac{(1+\xi)}{2}\psi_2\right)^2\right\} dx$$

U_b can be integrated exactly using 1 point Gauss quadrature.

U_s can be integrated exactly using 2 point Gauss quadrature.

As has been already noted, this results in shear locking as beam becomes slender.

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we see that the strain energy due to bending and shear can be evaluated, we get for the bending this is the expression, that is EI dou sai/dou X whole square will be this, and this is a constant and we're shearing function shearing, energy due to shearing will be given by this, so this integral can be evaluated with one term Gauss quadrature, and this to evaluate exactly I need a 2-point Gauss quadrature, because this will be a fourth order in XI, so but if we do this then as I already been pointed out we will face the problem of shear locking, okay, because in this representation as the beam depth becomes small the representation doesn't automatically converge to the Euler-Bernoulli beam limit, so there will be a problem if we do this, what is

Remedy:

Replace linear shear strain variation by a constant in the sense of minimizing the mean square error.

Following this the integrand in equation for U_s becomes constant and hence can be evaluated using a 1 point Gauss quadrature.

This method of overcoming the problem of locking is called the method of reduced integration.

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proposed is, we replace the linear shear strain variation by a constant in the sense of minimizing the mean square error across the depth, so following this the integrand equation for US, integrand in equation for US becomes constant and hence can be evaluated using a 1 point Gauss quadrature, moment we do this then the locking problem will be eliminated, and this method of overcoming the problem of locking is called method of reduced integration.

So this amounts to, by doing this integration we are virtually ensuring that DW/DX and dou sai are of the same order, okay, so then by using a reduced quadrature as mentioned here we are achieving basically that, that is why the shear locking problem gets eliminated. Now this is as

Returning to the 4-noded thick rectangular plate element

$$V = \frac{1}{2}\int_A \frac{h^3}{12}\chi^t D\chi dA + \frac{1}{2}\int_A kh\gamma^t D^s\gamma dA = \frac{1}{2}w_e^t k_e w_e$$

$$k_e = k_f + k_s = \int_A \frac{h^2}{12}B_f^t DB_f dA + \int_A khB_s^t DB_s dA$$

The problem of shear locking is possible here too.

This can be overcome by using reduced integration.

Both $\int_A \frac{h^2}{12}B_f^t DB_f dA$ & $\int_A khB_s^t DB_s dA$ can be evaluated exactly using 2×2 Gauss quadrature. This provides acceptable results for thick plates. For thin plates, however, the locking problem occurs.

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far as Timoshenko beam is concerned, now let us return to the 4 noded thick rectangular plate element, we saw that energy expression is given by this and we have found out the element stiffness matrix in terms of flexure and shear and by using this formula. Now the problem of shear locking is possible here to, again this can be overcome by using reduced integration, see if we examine the details of this integrand shear, these integrals can be evaluated exactly using 2 x 2 Gauss quadrature, this provides acceptable result for thick plates there is no problem here, but however for thin plates the locking problem occurs if we do this, so what is suggested is we

Remedy

Evaluate $\int_A \frac{h^2}{12} B_f^t D B_f \, dA$ using 2×2 Gauss quadrature

and

Evaluate $\int_A k h B_s^t D B_s \, dA$ using 1×1 Gauss quadrature

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evaluate the flexure component by using 2 x 2 Gauss quadrature, but for evaluating the shear part we use a 1 x 1 Gauss quadrature, so this indirectly ensures that the locking problem is eliminated.

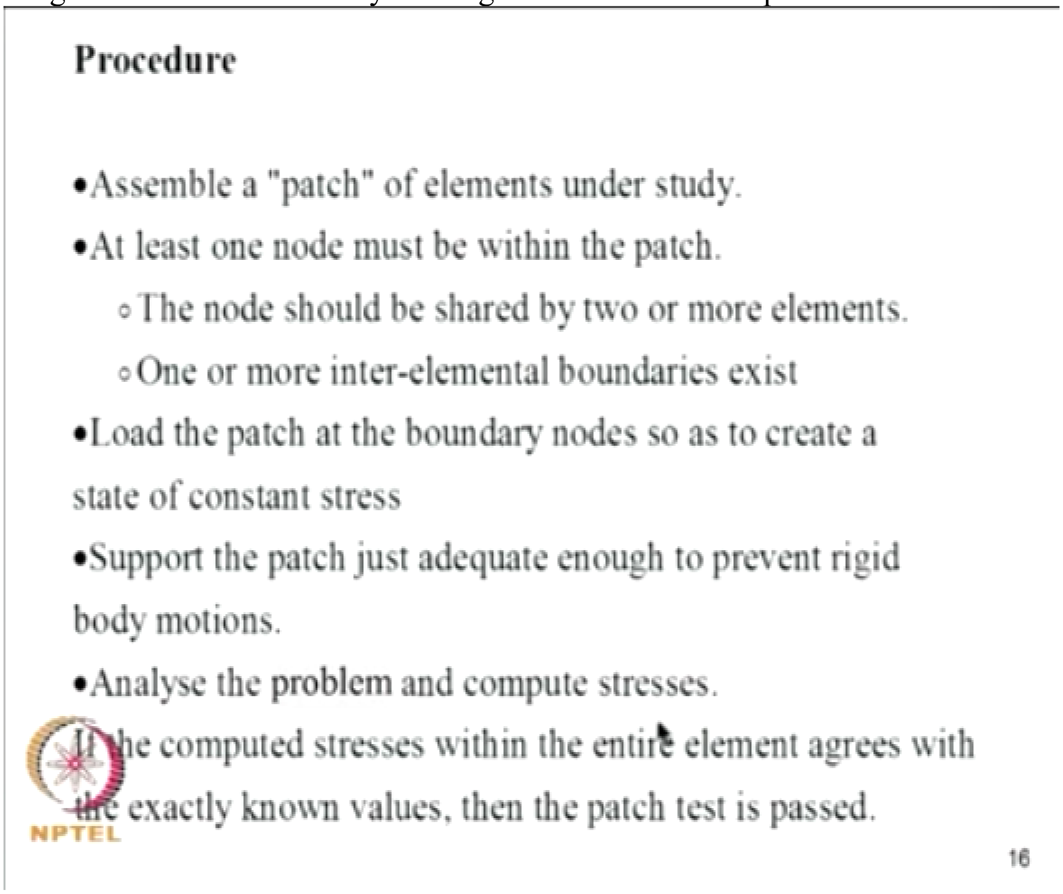
Comments on convergence : patch test

- Has the element been formulated correctly?
- Has the computer code been developed correctly?
- How do we know that an element available in readily available software is corretly formulated and coded?
- When nonconforming elements are used, how do we verify if convergence would be achieved by refining the mesh?

So this is one question that we post about behavior of deep beam and plate elements as the thickness becomes small. Now I wish to make few comments on issues related to convergence, in the using any finite element formulation we can ask several questions after we formulate the element of the program, at the stage of completing the formulation of the element and when we are ready to use it we can have several questions, has the element been formulated correctly?

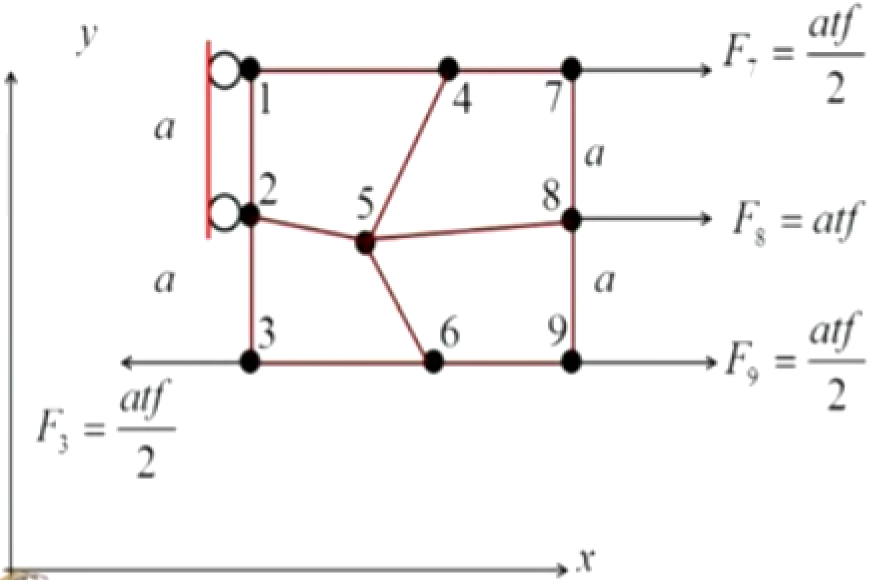
And has the computer codes have been developed correctly? How do we know that an element available in readily available software is correctly formulated and coded? Suppose even if we are not the people who have developed the element, we may be using production version finite element software's and we may like to use a particular element in a given model, how do we know that either we are using that element correctly as intended or whether the element itself has been formulated correctly and coded correctly.

Now other question we can ask is when non-conforming elements are used how do we verify if the converges should be achieved by refining the mesh? So these questions can be answered by



considering what is known as a patch test, the story is it goes something like this, what we do is we assemble a patch of elements under study, the assembling of this patch of element should be such that at least 1 node must be within the patch, and the node should be shared by 2 or more elements, and 1 or more inter-elemental boundaries must exist. Now we load this patch at the boundary nodes so as to create a state of constant stress, so this state of constant stress can be with respect to one of the stress components, for example in a 2-dimensional problem it can be, state of constant stress in sigma XX, or sigma YY, or sigma XY. Now we support the patch just adequate enough to prevent rigid body motions, now we analyze the problem using the element develop and we compute the stresses, if the computed stresses within the entire element agrees with the exactly known value of constant stress and perhaps 0 stress for other element, I mean some of the components then the patch test is passed by the element. An example for that

A patch of plane stress elements



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Exact stresses

$$\sigma_{xx} = f, \sigma_{xy} = 0, \sigma_{yy} = 0$$

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is suppose you are using plane stress quadrilateral element so we make a patch, this is a node here and we create 4 quadrilateral plane stress elements and we load this edge by a constant surface traction, so this surface traction is a constant, and for this constant surface traction we evaluate the equivalent nodal forces using the correct formulation, and these are the forces, and this also they another force that we have applied, and this is the support conditions, you can verify that under this state of loading and the way it is supported, this patch would be having constant value of stress sigma XX = F, where F is the magnitude of this constant value, and sigma XY and sigma YY must be equal to 0, so we can analyze suppose you have developed these elements you can analyze and find out whether these conditions are met or not.

Remarks

- The example considered depicted the case of σ_{xx} = constant.
  For a plane stress element, we need to repeat the test for other loading configurations corresponding to constant states of σ_{yy} & σ_{xy}.
- If the element passes the patch test, it is ensured that a finite element model which uses this element converges to the correct solution as the mesh is refined repeatedly.
- It is assumed that the element being tested is stable. A stable element does not exihibit zero energy modes when it is adequately supported so as to avoid rigid body motions. One can study stability of the element by considering the eigenvalues of the stiffness matrix.

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So the example that I mentioned, I consider the case of sigma XX being constant, for a plane stress element we need to repeat the test for other loading configurations corresponding to constant states of sigma YY and sigma XY before we can pass the test, pass the element for further use, if the element passes the patch test it is ensured that a finite element model which uses this element converges to the correct solution as the mesh is refined repeatedly, if an element fails to pass the patch test I mean we should be very careful in using those elements, it is assumed that the element being tested is stable, so what is the meaning of a stable element? A stable element does not exhibit 0 energy modes when it is adequately supported so as to avoid rigid body motions, one can study stability of the element by considering the Eigenvalues of the stiffness matrix, there is another numerical way to do that, so what we do is we assemble a

Procedure to test stability of an element

Assemble the patch and load as in the patch test.

Peturb one of the nodal loads by a small amount.

If the computed stresses change by a large amount, then the patch has failed the stability test.

If the test is applied at the single element level, then the element has failed the stability test.

Weak patch test

The element approaches towards passing the patch test as the mesh of the patch is refined successively.

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patch as in the patch test, and apply the load as in the patch test.
Now assuming that the element has passed the patch test, what we do is we perturb one of the nodal loads by a small amount, if the computed stress is changed by a large amount because of the small perturbation then the patch has failed the stability test, if the test is applied at the single element level then the element has failed the stability test that means if at the element level we see that a small perturbation to the loads as depicted here produces large changes in the stress values, again we say that particular element has failed the stability test, what is known as weak patch test where the element approaches towards passing the patch test as a mesh of the patches refined successively, that means for a given mesh configuration like this the element may not pass the patch test, but if we go and refining this as the, as you repeatedly refine the mesh the patch test tends towards you know success, so I mean the test is passed and then we say that the element passes the weak patch test.

Higher order patch test

Example:

Modeling of pure bending in plane elements: an element with quadratic expansion for u and v must be able to represent exactly a field of pure bending.

Robustness

Example:

A state of pure bending in plane mesh must not be affected by Poisson ratio.

R D Cook, D S Malkus, and M E Plesha, Concepts and applications of finite element analysis, 3rd edition, John Wiley and Sons, 1989.

O C Zienkiewicz, and R L Taylor, The finite element method, Vol. 1, 4th edition, McGraw-Hill, London.

There are other issues like higher-order patch test, for example we have considered in the patch test, the states of constant stress there can be other issues, for example modeling of pure bending in plane elements, an element with quadratic expansion for U and V must be able to represent exactly a field of pure bending, so pure bending is not a state of constant stress, but it has a specific distribution and this can be thought of as higher-order patch test. Similarly there are other issues like robustness, for example a state of pure bending in plane mesh must not be affected by the Poisson ratio of the element, so some of this can be used to verify whether the element has been formulated correctly or are we using the element as intended as that, so these issues are discussed at a tutorial level in these textbooks, there are many research papers on this so this is a subject of research, if you're interested there are many research papers on this.

Remark

Numerically integrated elements lead to the correct convergent results as the mesh is refined, if the order of integration chosen leads to the correct evaluation of the volume of the element.

Why? (Hint)

As the mesh is refined and constant strain is reached, we get

$$\frac{1}{2}\int_V \varepsilon^t D \varepsilon dV = \int_V [\text{constant}] |J| d\xi d\eta d\zeta$$

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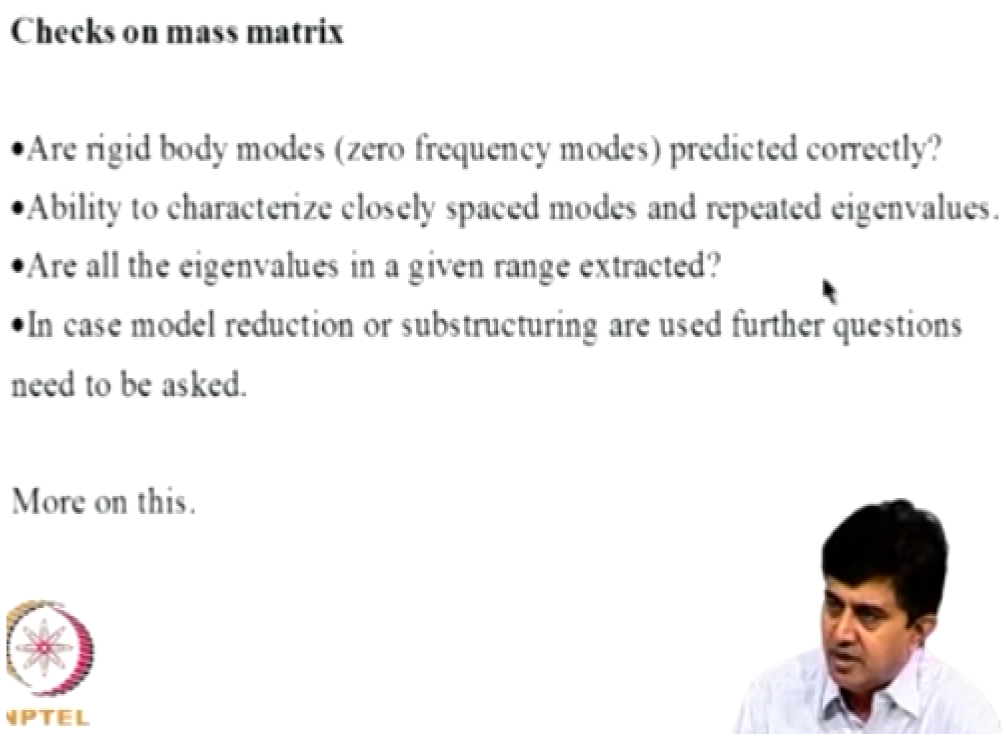
Now one more remark that can be made in the same context, this numerically integrated elements lead to the correct convergent results as the mesh is refined, if the order of integration chosen leads to the correct evaluation of the volume of the element, why is that so? The question we are asking is suppose you have evaluated stiffness matrix using Gauss quadrature, and we have seen that the Gauss quadrature would not integrate the elements of KE exactly because there will be ratios of polynomials in the integrand, so the question is if we are using such elements how do we know that we are going to get the correct convergent results as mesh is refined.

Now the suggestion that is made here is if the order of integration chosen leads to the correct evaluation of the volume of the element then we are okay,, I want you to think about this and answer, but I will give an hint as the mesh is refined and constant strain is reached as mesh becomes finer within an element strain tends to become constant we get the stiffness matrix to be of this form, so this term becomes constant and what remains is the volume of the element, so the statement made here is that if this volume is corrected correctly then this required condition on convergence is met.

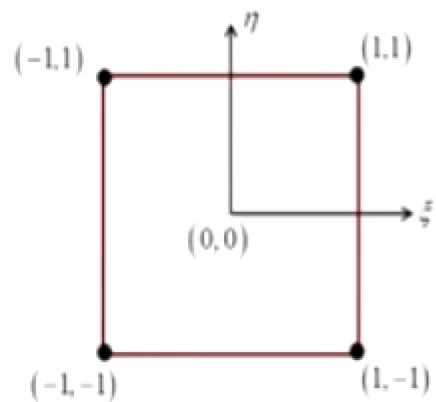


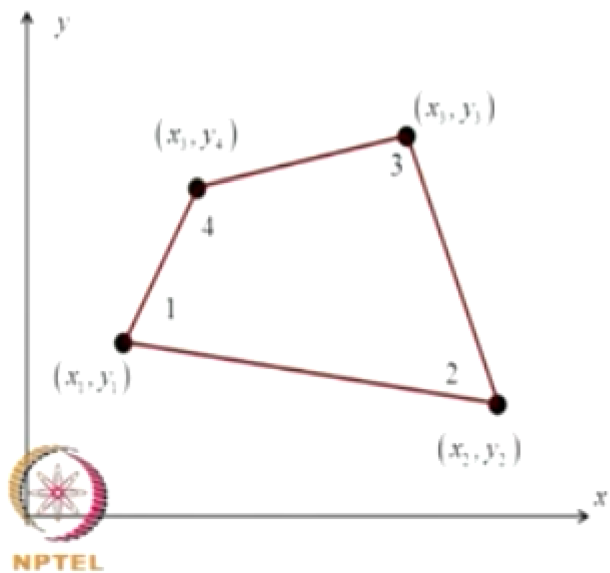
Now all these checks have been on static behavior of the system, so how about problems of dynamics? So there are few checks on mass matrix I will return to this sometime later in the course, but I will just indicate a few points here this is something to do with Eigenvalue analysis, the question we can ask is are rigid body modes predicted correctly, if I know that a structure is supported in a way that suppose it admits 2 rigid body modes the Eigenvalue analysis should reveal that.

Next the question of ability to characterize closely spaced modes and repeated Eigenvalues we have seen that in structural displaying symmetry in stiffness and mass characteristics, special symmetry the Eigenvalues can repeat, for example in a circular plate we saw in the previous lecture which is fixed all around, there are 2 mode shapes for a given Eigenvalue, so how these things are handled, and if the natural frequencies are closely spaced will my Eigenvalue solver be able to handle that correctly. Next are all the Eigenvalues in a given range extracted, that means within a frequency range we should not miss any Eigenvalues, so that is also another question, and then again if you’ve used model reduction or substructuring there will be further questions on choice of masters and slaves and so on and so forth, so the questions on dynamics require further consideration maybe we'll return to some of these issues later provided the time permits.

Four noded linear quadrilateral thick plate bending element

$$V = \frac{1}{2}\int_A \frac{h^3}{12}\chi^t D\chi dA + \frac{1}{2}\int_A kh\gamma^t D^s\gamma dA$$





$$x(\xi,\eta) = \sum_{i=1}^{4} x_i N_i(\xi,\eta)$$

$$y(\xi,\eta) = \sum_{i=1}^{4} y_i N_i(\xi,\eta)$$

Now in the previous lecture we considered 4 noded rectangular quadratic thick plate element, I leave it as an exercise for you to repeat that exercise for the quadrilateral element, the strain energy expressions are given here and this is the geometry of the quadrilateral element, and we use the isoparametric representation as shown here, and this element is mapped to this master element, and we use this interpolation functions NI(sai, eta) is this, and X and Y are represented

$$x(\xi,\eta)=\sum_{i=1}^{4}x_i N_i(\xi,\eta); y(\xi,\eta)=\sum_{i=1}^{4}y_i N_i(\xi,\eta)$$

$$w(\xi,\eta)=\sum_{i=1}^{4}w_i N_i(\xi,\eta)$$

$$\theta_x(\xi,\eta)=\sum_{i=1}^{4}\theta_{xi} N_i(\xi,\eta)$$

$$\theta_y(\xi,\eta)=\sum_{i=1}^{4}\theta_{yi} N_i(\xi,\eta)$$

$$N_i(\xi,\eta)=\frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta), i=1,2,3,4$$

Exercise

Develop the mass and stiffness matrices for this element

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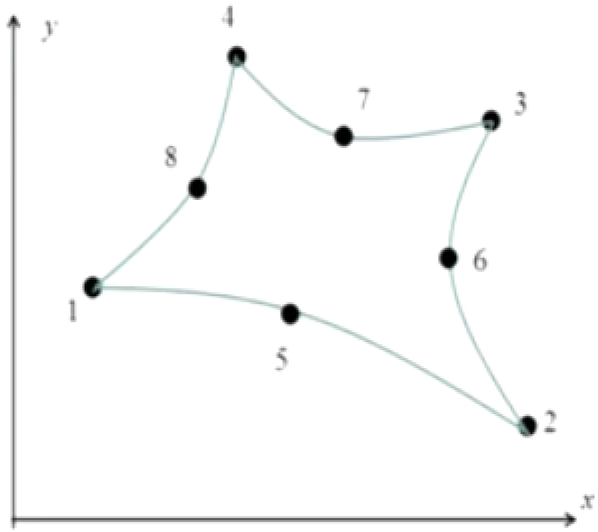
using the same interpolation functions and the displacement field theta X, theta Y, are represented in this form, so the exercise is to develop the mass and stiffness matrices for this element.

Eight noded curved quadrilateral thick plate bending element



$$V = \frac{1}{2}\int_A \frac{h^3}{12}\chi^t D\chi dA + \frac{1}{2}\int_A kh\gamma^t D^s \gamma dA$$

Another exercise you take a 8 noded curved quadrilateral thick plate bending element and this is the expression for strain energy, and using these representations there are 8 trial functions and the trial functions have been defined here for 1, 2, 3, 4 this is the trial function, and for 5,7 and 6, 8 these are given here, and we again use isoparametric formulation where the coordinates

$$x(\xi,\eta)=\sum_{i=1}^{8} x_i N_i(\xi,\eta); y(\xi,\eta)=\sum_{i=1}^{8} y_i N_i(\xi,\eta)$$

$$w(\xi,\eta)=\sum_{i=1}^{8} w_i N_i(\xi,\eta)$$

$$\theta_x(\xi,\eta)=\sum_{i=1}^{8} \theta_{xi} N_i(\xi,\eta)$$

$$\theta_y(\xi,\eta)=\sum_{i=1}^{8} \theta_{yi} N_i(\xi,\eta)$$

$$N_i(\xi,\eta)=\frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta)(\xi_i\xi+\eta_i\eta-1), i=1,2,3,4$$

$$N_i(\xi,\eta)=\frac{1}{4}(1-\xi^2)(1+\eta_i\eta), i=5,7$$

$$N_i(\xi,\eta)=\frac{1}{4}(1+\xi_i\xi)(1-\eta^2), i=6,8$$

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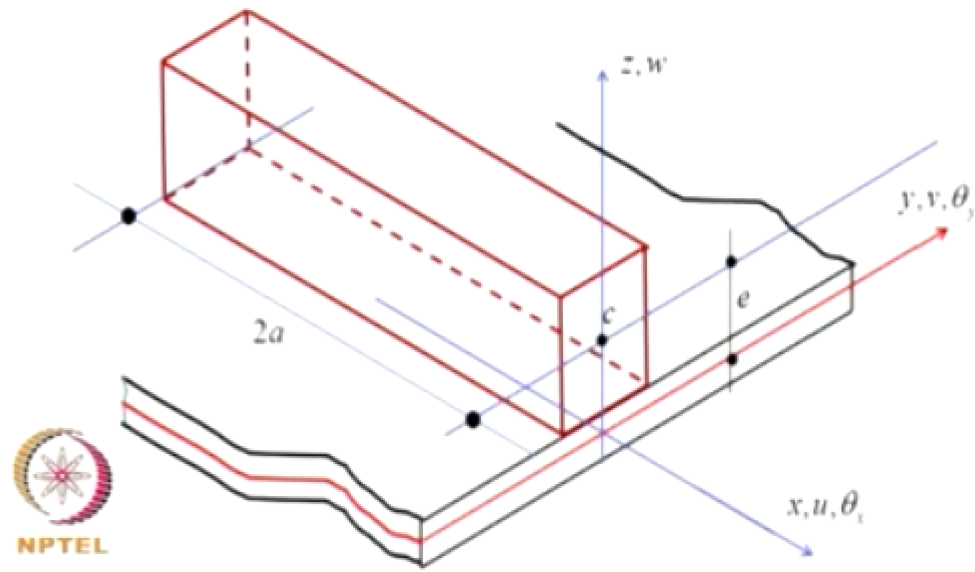
Exercise

Develop the mass and stiffness matrices for this element

X and Y and the field variable W, theta X, and theta Y are interpolated using nodal values using the same set of interpolation functions. Now the problem is to develop the mass and stiffness matrices for this element, so this is suggested as an exercise, if you complete these two exercises you would have learnt a great deal about formulation of the structural matrices for plate elements.

Plate stiffened by beam elements

- Examples: Bridge deck, building floors, ship hulls,⋯
- Beam centroidal axis is placed eccentrically to the middle surface of the plate.
- Membrane and bending action gets coupled



Now we will now consider, next class of problems associated with plate bending here we are considering plate structures which are stiffened by beam elements like this, so these are typically observed in bridge decks, building floors, ship hulls, aircraft, structures so on and so forth, now the beam central axis that is a line passing through this point C here is eccentrically placed with respect to the middle surface of the plate which it aims to stiffen, now obviously the membrane and bending action of the plate and the flexural action of the beam all of them get coupled, and correct formulation should be able to handle these details with care, so how do we do this? So the coordinate systems here we assume that the coordinate origin passes through the mid surface of the plate element and the centroidal axis of the beam is at a eccentricity E, and I consider this to be the X-axis, this is Y axis, and this is Z axis, and the degrees of freedom that we are considering is U and theta X, V and theta Y, and W, okay, so a first cut model for

Assumptions:

•Ignore membrane displacements of the plate

$$T = \frac{1}{2}\int_{-a}^{a} \rho A\left(\dot{u}_c^2 + \dot{v}_c^2 + \dot{w}_c^2\right)dx + \frac{1}{2}\int_{-a}^{a} \rho I_x \dot{\theta}_x^2 dx$$

$$V = \frac{1}{2}\int_{-a}^{a} AE\left(\frac{\partial u_c}{\partial x}\right)^2 dx + \frac{1}{2}\int_{-a}^{a} EI_y\left(\frac{\partial^2 w_c}{\partial x^2}\right)^2 dx + \frac{1}{2}\int_{-a}^{a} GJ\left(\frac{\partial \theta_x}{\partial x}\right)^2 dx + \underbrace{\frac{1}{2}\int_{-a}^{a} EI_y\left(\frac{\partial^2 v_c}{\partial x^2}\right)^2 dx}_{\text{This term is ignored}}$$

$$u_c = e\theta_y = e\frac{\partial w}{\partial x}, \quad v_c = -e\theta_x, \quad w_c = w$$

The beam displacements need to be **compatible with** the plate displacements along the attachment line

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this problem we can begin by ignoring membrane displacements of the plate, so we will consider only the flexural action and of the plate and the beam.

So now the strain energy stored in the beam is given by this expression, we are assuming a deep beam element so we are including rotary moment of inertia as well, so this is the equation. The strain energy is due to axial deformation, bending, twisting, bending about Y, bending in the two directions, what we are going to do is we are going to ignore this bending contribution for the time being, this can be you know, this bending in this direction can be taken to be ignored. Now based on the geometry of this configuration we can see that UC is given by E into theta Y, which is E dou W/dou X, and VC is - eta X and WC is W, the beam displacements actually need to be compatible with the plate displacements along the attachment line. The expression for kinetic energy I will use these representations, so this is the expression for kinetic energy, and UC, VC, WC are given in terms of W, theta X, through these equations and if I substitute,

$$T=\frac{1}{2}\int_{-a}^{a}\rho A\left(\dot{u}_c^2+\dot{v}_c^2+\dot{w}_c^2\right)dx+\frac{1}{2}\int_{-a}^{a}\rho I_x\dot{\theta}_x^2dx$$

$$=\frac{1}{2}\int_{-a}^{a}\rho Ae^2\left(\frac{\partial\dot{w}}{\partial x}\right)^2dx+\frac{1}{2}\int_{-a}^{a}\rho Ae^2\dot{\theta}_x^2dx+\frac{1}{2}\int_{-a}^{a}\rho A\dot{w}^2dx+\frac{1}{2}\int_{-a}^{a}\rho I_x\dot{\theta}_x^2dx$$

$$=\frac{1}{2}\int_{-a}^{a}\rho Ae^2\left(\frac{\partial\dot{w}}{\partial x}\right)^2dx+\frac{1}{2}\int_{-a}^{a}\rho A\dot{w}^2dx+\frac{1}{2}\int_{-a}^{a}\rho\dot{\theta}_x^2\left(I_x+Ae^2\right)dx$$

$$V=\frac{1}{2}\int_{-a}^{a}AE\left(\frac{\partial u_c}{\partial x}\right)^2dx+\frac{1}{2}\int_{-a}^{a}EI_y\left(\frac{\partial^2 w_c}{\partial x^2}\right)^2dx+\frac{1}{2}\int_{-a}^{a}GJ\left(\frac{\partial\theta_x}{\partial x}\right)^2dx$$

$$=\frac{1}{2}\int_{-a}^{a}AEe^2\left(\frac{\partial^2 w}{\partial x^2}\right)^2dx+\frac{1}{2}\int_{-a}^{a}EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)^2dx+\frac{1}{2}\int_{-a}^{a}GJ\left(\frac{\partial\theta_x}{\partial x}\right)^2dx$$

compatible with

$$=\frac{1}{2}\int_{-a}^{a}E\left(I_y+Ae^2\right)\left(\frac{\partial^2 w}{\partial x^2}\right)^2dx+\frac{1}{2}\int_{-a}^{a}GJ\left(\frac{\partial\theta_x}{\partial x}\right)^2dx$$

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you make these substitutions I get for UC dot for example I will write E dou W dot/ dou X so that is what is here, and similarly other terms are represented and I get the expression for kinetic energy in this form.

Similarly strain energy is given by these 3 terms, again for UC, WC, and theta X, UC and WC I will use these terms, and using that I will get the expression for strain energy in terms of W and

$$T=\frac{1}{2}\int_{-a}^{a}\rho Ae^2\left(\frac{\partial\dot{w}}{\partial x}\right)^2dx+\frac{1}{2}\int_{-a}^{a}\rho A\dot{w}^2dx+\frac{1}{2}\int_{-a}^{a}\rho\dot{\theta}_x^2\left(I_x+Ae^2\right)dx$$

$$V=\frac{1}{2}\int_{-a}^{a}E\left(I_y+Ae^2\right)\left(\frac{\partial^2 w}{\partial x^2}\right)^2dx+\frac{1}{2}\int_{-a}^{a}GJ\left(\frac{\partial\theta_x}{\partial x}\right)^2dx$$

Along the attachment line, the beam and plate must deform in a compatible manner.

Recall

Along the edges, many of the plate bending elements have cubic variation for *w*.

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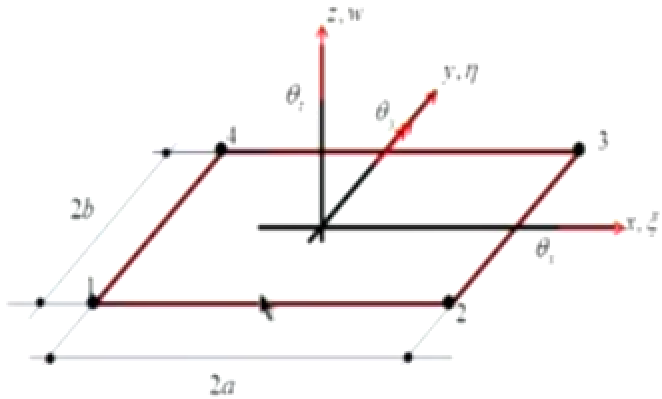
theta X to be this, so kinetic energy is also obtained in terms of W and theta X, so these are the expressions for kinetic energy and strain energy, as I already said along the attachment line the beam and plate must deform in a compatible management. Now if we recall along the edges

Recall



$$\xi = \frac{x}{a} \;\&\; \eta = \frac{y}{b}$$

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right)$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}\left(\varepsilon_{yy} + \nu\varepsilon_{xx}\right)$$

$$\sigma_{xy} = 2G\varepsilon_{xy}$$

$$\varepsilon = -z\chi = -z\begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}$$

$$V = \frac{1}{2}\int_A \frac{h^3}{12}\chi^t D\chi dA; T = \frac{1}{2}\int_A \rho h\dot{w}^2 dA$$

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many of the plate bending elements of cubic variation for W, so if you recall if we consider this 4 noded beam element, thin plate element we had this expression for strain energy, and we saw

Thin rectangular element with 4 nodes, 3 dofs/node (Dofs=12)

Field variable: $w(x, y, t)$

Order of the highest derivative present in the Lagrangian: 2

Dofs: $w(x, y, t), \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}$

Number of generalized coordinates: 12

$$\begin{aligned} w = {} & \alpha_1 \\ & + \alpha_2 x + \alpha_3 y \\ & + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 \\ & + \alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3 \\ & + \alpha_{11} x^3 y + \alpha_{12} xy^3 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} w(a, y) = {} & (\alpha_1 + \alpha_2 a + \alpha_4 a^2 + \alpha_7 a^3) \\ & + (\alpha_3 + \alpha_5 a + \alpha_8 a^2 + \alpha_{11} a^3) y \\ & + (\alpha_6 + \alpha_9 a) y^2 \\ & + (\alpha_{10} + \alpha_{12} a) y^3 \end{aligned}$$

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that the field variable in this expression for strain energy is W and the highest derivative present in the Lagrangian is 2, and therefore degrees of freedom are W, dou W/dou X and dou W/dou Y and there are 4 nodes and number of generalized coordinates were 12, because we needed 3 degrees of freedom at every node and we use this representation. Now suppose if you consider this representation along one of the edges you see here that if I put X = A, I get this cubic polynomial in Y, so that is what I am saying here along the edges many of the plate bending elements of cubic variation for W.

$$V = \frac{1}{2}\int_{-a}^{a} E\left(I_y + Ae^2\right)\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx + \frac{1}{2}\int_{-a}^{a} GJ\left(\frac{\partial \theta_x}{\partial x}\right)^2 dx$$

$$w = w_1 N_{w1}(\xi) + \theta_{y1} N_{w2}(\xi) + w_2 N_{w3}(\xi) + \theta_{y2} N_{w4}(\xi)$$

$$N_{w1}(\xi) = N_1(\xi); N_{w2}(\xi) = -aN_2(\xi); N_{w3}(\xi) = N_3(\xi); N_{w4}(\xi) = -aN_4(\xi)$$

$$N_1(\xi) = \frac{1}{4}\left(2 - 3\xi + \xi^3\right)$$

$$N_2(\xi) = \frac{1}{4}\left(1 - \xi - \xi^2 + \xi^3\right)$$

$$N_3(\xi) = \frac{1}{4}\left(2 + 3\xi + \xi^3\right)$$

$$N_4(\xi) = \frac{1}{4}\left(-1 - \xi + \xi^2 + \xi^3\right)$$

$$w = [N_w]\{w\}_e$$

$$[N_w] = \left[N_1(\xi) \quad -aN_2(\xi) \quad N_3(\xi) \quad -aN_4(\xi)\right]$$

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$$\{w\}_e = \left[w_1 \quad \theta_{y1} \quad w_2 \quad \theta_{y2}\right]^t$$

So now I will use this representation for W, W1, NW1 sai, theta Y1, and W2 sai etcetera, etcetera as shown here, where NW1 is N1, NW2 is – AN2 and other details are as here, and N1, N2, N3, N4 are the Hermite polynomials that we have used, so using this I will write W as NW into WE, so where NW is this and WE is given by this, for twisting theta X the second term I

$$V=\frac{1}{2}\int_{-a}^{a}E\left(I_y+Ae^2\right)\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx+\frac{1}{2}\int_{-a}^{a}GJ\left(\frac{\partial\theta_x}{\partial x}\right)^2 dx$$

$$\theta_x=N_{x1}(\xi)\theta_{x1}+N_{x2}(\xi)\theta_{x2}$$

$$N_{x1}(\xi)=\frac{1}{2}(1-\xi)$$

$$N_{x2}(\xi)=\frac{1}{2}(1+\xi)$$

$$\theta_x=\left[N_{x1}(\xi)\quad N_{x2}(\xi)\right]\begin{Bmatrix}\theta_{x1}\\ \theta_{x2}\end{Bmatrix}=\left[N_x(\xi)\right]\{\theta_x\}_e$$

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use linear interpolation functions that we have seen NX1 sai, theta X1 + NX2 sai theta X2 and these are linear interpolation function, for theta X I get this, so I have this representation for W, and this representation for theta, and I can combine the 2, and write the expression for kinetic

$$T = \frac{1}{2}\int_{-a}^{a} \rho A e^2 \left(\frac{\partial \dot{w}}{\partial x}\right)^2 dx + \frac{1}{2}\int_{-a}^{a} \rho A \dot{w}^2 dx + \frac{1}{2}\int_{-a}^{a} \rho \dot{\theta}_x^2 \left(I_x + Ae^2\right) dx$$

$$w = [N_w]\{w\}_e \;\&\; \theta_x = [N_x(\xi)]\{\theta_x\}_e$$

$$\frac{1}{2}\int_{-a}^{a} \rho A \dot{w}^2 dx = \frac{1}{2}\{\dot{w}\}_e^t \left[\int_{-a}^{a} \rho A [N_w]^t [N_w] dx\right] \{\dot{w}\}_e$$

$$= \frac{1}{2}\{\dot{w}\}_e^t \begin{bmatrix} 78 & & & \\ -22a & 8a^2 & & \\ 27 & -13a & 78 & \\ 13a & -6a & 22a & 8a^2 \end{bmatrix} \{\dot{w}\}_e$$

$$\frac{1}{2}\int_{-a}^{a} \rho \dot{\theta}_x^2 \left(I_x + Ae^2\right) dx = \frac{1}{2}\{\dot{\theta}_x\}_e^t \rho A\left(r_x^2 + e^2\right)\frac{a}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\{\dot{\theta}_x\}_e$$

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energy, W is this, and theta X is this and substitute and we can carry out the necessary integrations, and we can show that the mass matrix will have first integral can be shown to be, will lead to this form, and the second integral will lead to this form. So we have these

$$\frac{1}{2}\int_{-a}^{a}\rho Ae^2\left(\frac{\partial\dot{w}}{\partial x}\right)^2 dx\frac{1}{2}=\frac{1}{2}\{\dot{w}\}_e^T\frac{\rho Ae^2}{30}\begin{bmatrix}18 & & & \\ -3a & 8a^2 & & \\ -18 & 3a & 18 & \\ -3a & -2a^3 & 3a & 8a^2\end{bmatrix}\{\dot{w}\}_e$$

Define $\{u\}_e=\begin{bmatrix}w_1 & \theta_{x1} & \theta_{y1} & w_2 & \theta_{x2} & \theta_{y2}\end{bmatrix}^T$

$$\Rightarrow T_e=\frac{1}{2}\{\dot{u}\}_e^T[m]_e\{\dot{u}\}_e \text{ with } [m]_e=\begin{bmatrix}m_{11} & m_{12}\\ m_{12}^t & m_{22}\end{bmatrix}$$

$$m_{11}=\frac{\rho Aa}{210}\begin{bmatrix}282 & 0 & -63a\\ 0 & 140e_x^2 & 0\\ -63a & 0 & 72a^2\end{bmatrix}\quad\left(e_x^2=e^2+r_x^2\right)$$

$$m_{12}=\frac{\rho Aa}{210}\begin{bmatrix}72 & 0 & 5a\\ 0 & 70e_x^2 & 0\\ -5a & 0 & -26a^2\end{bmatrix};m_{22}=\frac{\rho Aa}{210}\begin{bmatrix}282 & 0 & 63a\\ 0 & 140e_x^2 & 0\\ 63a & 0 & 72a^2\end{bmatrix}$$

expressions for the 3 terms in the expression for kinetic energy, and we now define UE as element nodal degrees of freedom W1, theta X1, theta Y1, W2, theta X2, theta Y2.

Now we can write the expression for kinetic energy and assemble these matrices we can show that this ME can be given by these 4 matrices, and these 4 matrices are as shown here, this M11 is here, and M12 is here, and M22 will be this.

$$V = \frac{1}{2}\int_{-a}^{a} E\left(I_y + Ae^2\right)\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx + \frac{1}{2}\int_{-a}^{a} GJ\left(\frac{\partial \theta_x}{\partial x}\right)^2 dx$$

$$\frac{1}{2}\int_{-a}^{a} E\left(I_y + Ae^2\right)\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx$$

$$= \frac{1}{2}\{w\}_e^t AE\left(r_y^2 + e^2\right)\frac{1}{2a^3}\begin{bmatrix} 3 & & & \\ -3a^2 & 4a^2 & & \\ -3 & 3a & 3 & \\ -3a & 2a^2 & 3a & 4a^2 \end{bmatrix}\{w\}_e \quad \left(r_y^2 = \frac{I_y}{A}\right)$$

$$\frac{1}{2}\int_{-a}^{a} GJ\left(\frac{\partial \theta_x}{\partial x}\right)^2 dx = \frac{1}{2}\{\theta_x\}_e^t \frac{GJ}{2a^3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\{\theta_x\}_e$$

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Now how about strain energy? We have this expression for strain energy and again we will use the representation for W and theta X and we evaluate these 2 terms separately, first we consider the first integral and this leads to a mass matrix of this kind, and the stiffness matrix of this kind, and second one leads to stiffness matrix of this kind, again we can assemble in terms of

$$V = \frac{1}{2}\{u\}_e^t [k]_e \{u\}_e$$

$$[k]_e = \begin{bmatrix} k_{11} & k_{12} \\ k_{12}^t & k_{22} \end{bmatrix}$$

$$k_{11} = \frac{AE}{4a^3}\begin{bmatrix} 6e_y^2 & 0 & -6ae_y^2 \\ 0 & \frac{a^2 r_j^2}{1-\nu} & 0 \\ -6ae_y^2 & 0 & 8ae_y^2 \end{bmatrix} \quad \left(r_j^2 = \frac{J}{A}\right)$$

$$k_{12} = \frac{AE}{4a^3}\begin{bmatrix} -6e_y^2 & 0 & -6ae_y^2 \\ 0 & -\frac{a^2 r_j^2}{1-\nu} & 0 \\ 6ae_y^2 & 0 & 4ae_y^2 \end{bmatrix}; k_{22} = \frac{AE}{4a^3}\begin{bmatrix} 6e_y^2 & 0 & 6ae_y^2 \\ 0 & \frac{a^2 r_j^2}{1-\nu} & 0 \\ 6ae_y^2 & 0 & 8ae_y^2 \end{bmatrix}$$

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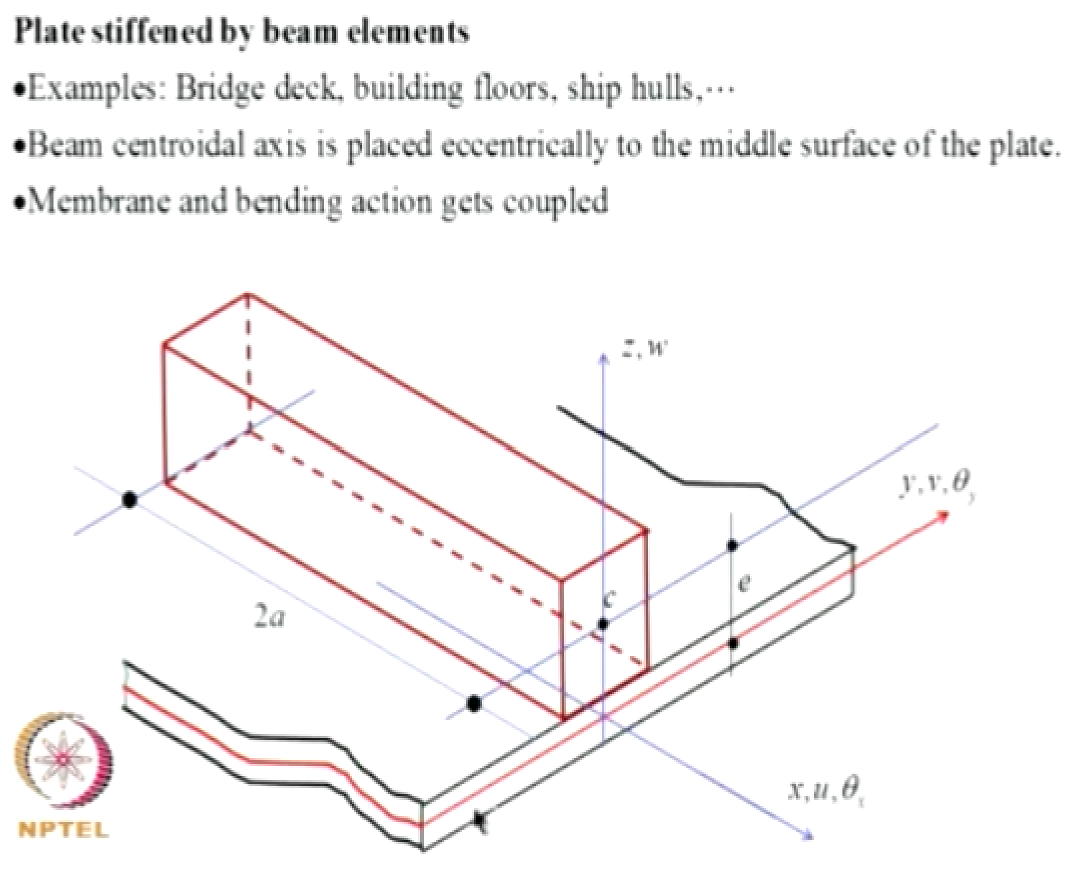
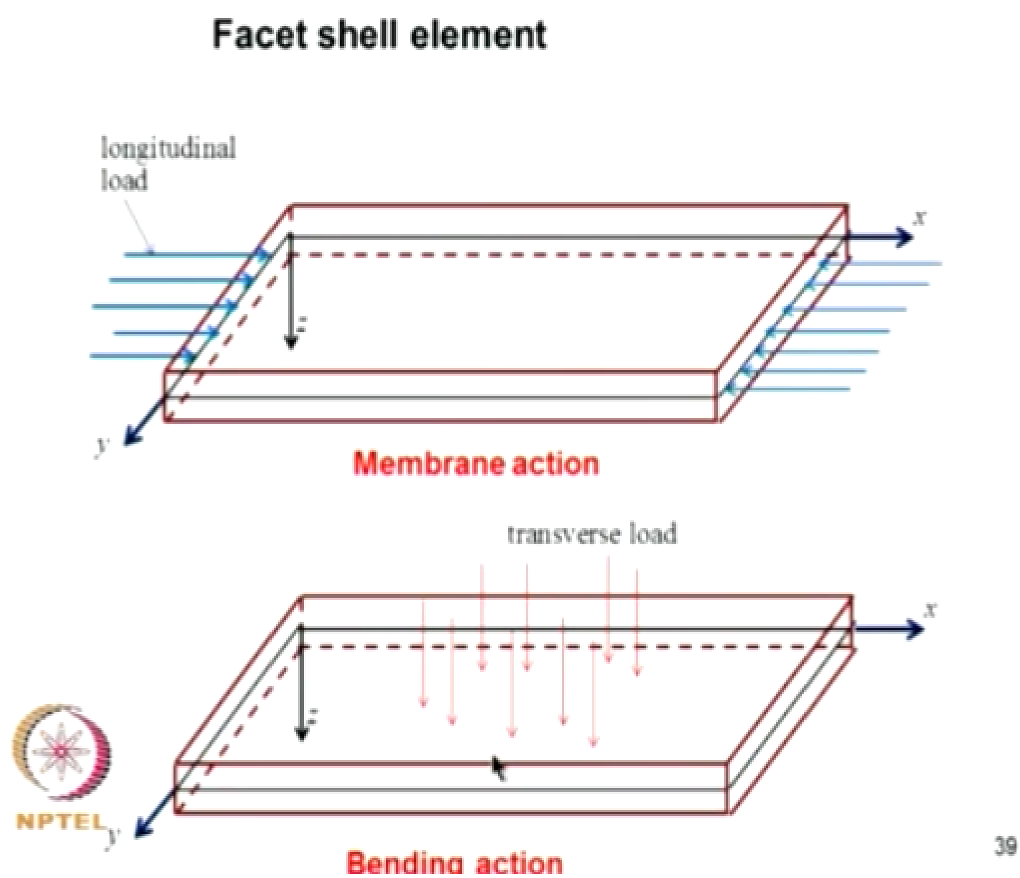
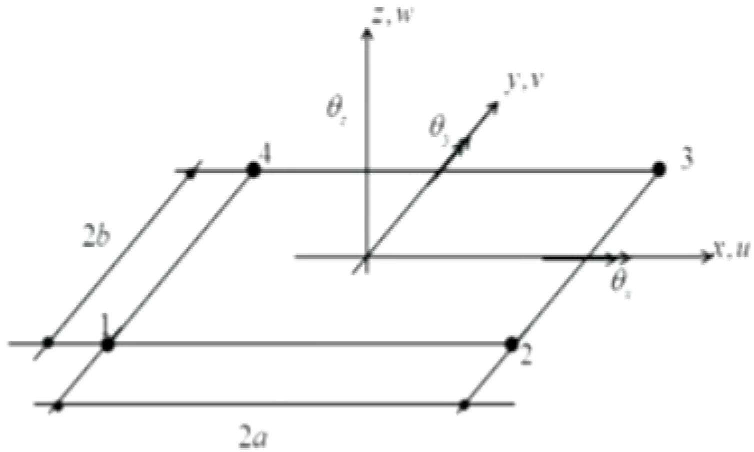
nodal degrees of freedom the element stiffness matrix in terms of these 4 matrices K11, K12, K 12 transpose, and K22, and the details of this are given here. In this formulation we have not



included the effect of membrane action of the shell element on possible behavior of this beam. Now if we want to include that first we need to develop a element for the plate itself where we combine the flexural and membrane actions, so that type of elements are known as facet



elements, this combines the action of the membrane action and bending action, so the element configuration will be as shown here for a 4 noded element, so the nodes are 1, 2, 3, 4, and at



$$u = \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{bmatrix}^t$$

$$w = \begin{bmatrix} w_1 & \theta_{x1} & \theta_{y1} & w_2 & \theta_{x2} & \theta_{y2} & w_3 & \theta_{x3} & \theta_{y3} & w_4 & \theta_{x4} & \theta_{y4} \end{bmatrix}^t$$

$$u_s = \begin{bmatrix} u_1 & v_1 & w_1 & \theta_{x1} & \theta_{y1} & \cdots & u_4 & v_4 & w_4 & \theta_{x4} & \theta_{y4} \end{bmatrix}^t$$

$$T = T_m + T_b; V = V_m + V_b$$

4 noded element, 5 dofs/node; 20 dofs

every node I will have, for membrane action I will have 2 degrees of freedom, and for flexural action I will have 3 degrees of freedom, so therefore W will be this, and S is subscript for shell and this will be this, so this will be a 20 degrees of freedom model, there are 4 nodes, at each node there are 5 degrees of freedom, the 5 degrees of freedom are U1, V1, W1, theta X1, theta Y1 at node 1, so we need to now assemble these matrices and write the correct expression for energies and derive the matrices, this is straightforward because we already done the problem it is now matter of just bookkeeping and assembling the matrices correctly.

$$T = T_m + T_b$$

$$T_m = \frac{1}{2}\{\dot{u}\}^T [m]^m \{\dot{u}\}$$

$$u = \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{bmatrix}^T$$

$$[m]^m = \begin{bmatrix} m_{11}^m & & & \\ m_{21}^m & m_{22}^m & & \\ m_{31}^m & m_{32}^m & m_{33}^m & \\ m_{41}^m & m_{42}^m & m_{43}^m & m_{44}^m \end{bmatrix}; \text{ Each } m_{ij}^m \text{ is } 2\times 2$$

$$T_b = \frac{1}{2}\{\dot{w}\}^T [m]^b \{\dot{w}\}$$

$$w = \begin{bmatrix} w_1 & \theta_{x1} & \theta_{y1} & w_2 & \theta_{x2} & \theta_{y2} & w_3 & \theta_{x3} & \theta_{y3} & w_4 & \theta_{x4} & \theta_{y4} \end{bmatrix}^T$$

$$[m]^b = \begin{bmatrix} m_{11}^b & & & \\ m_{21}^b & m_{22}^b & & \\ m_{31}^b & m_{32}^b & m_{33}^b & \\ m_{41}^b & m_{42}^b & m_{43}^b & m_{44}^b \end{bmatrix}; \text{ Each } m_{ij}^b \text{ is } 3\times 3$$

So we will write the kinetic energy in terms of membrane action and bending action, the membrane action is given by this and this is U1 V1, U2 V2, U3 V3, U4 V4, and I get this mass matrix for membrane action where each of these sub matrices is a 2 x 2 matrix, this we have derived earlier so we need not have to get into the details once again. For the bending action again the nodal degrees of freedom are 12 which are W1, theta X1, theta Y1 and similarly the other, so the mass matrix for bending action again is given by this and each of these matrices sub matrices is 3 x 3, so now if I write the expression for T I need to appropriately assemble these matrices.

$$u_s = \begin{bmatrix} u_1 & v_1 & w_1 & \theta_{x1} & \theta_{y1} & \cdots & u_4 & v_4 & w_4 & \theta_{x4} & \theta_{y4} \end{bmatrix}^t$$

$$T = T_m + T_b$$

$$T = \frac{1}{2}\{\dot{u}_s\}^t [m]^s \{\dot{u}_s\}$$

$$[m]^s = \begin{bmatrix} m_{11}^s & & & \\ m_{21}^s & m_{22}^s & & \\ m_{31}^s & m_{32}^s & m_{33}^s & \\ m_{41}^s & m_{42}^s & m_{43}^s & m_{44}^s \end{bmatrix}; \text{ Each } m_{ij}^s = \begin{bmatrix} m_{ij}^m & 0 \\ 0 & m_{ij}^b \end{bmatrix} \text{ is } 5\times 5$$

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So now if I define the nodal coordinates for the shell element subscript S is for shell, I will have 20 nodal degrees of freedom and I need a 20 x 20 mass matrix and in terms of the element mass matrix, shell element mass matrix this is given in terms of this nodal, you know kinetic energies in terms of US dot transpose MS US as shown here, and this MS itself it can be assembled in this form where each of these matrix is a 5 x 5 matrix consisting of 2 sub matrices 2 x 2 membrane matrix and a 3 x 3 bending matrix, so this assembled mass matrix now is the mass matrix for the so called facet shell element.

$$u_s = \begin{bmatrix} u_1 & v_1 & w_1 & \theta_{x1} & \theta_{y1} & \cdots & u_4 & v_4 & w_4 & \theta_{x4} & \theta_{y4} \end{bmatrix}^t$$

Similarly,

$$V = V_m + V_b$$

$$T = \frac{1}{2}\{u_s\}^t [k]^s \{u_s\}$$

$$[k]^s = \begin{bmatrix} k_{11}^s & & & \\ k_{21}^s & k_{22}^s & & \\ k_{31}^s & k_{32}^s & k_{33}^s & \\ k_{41}^s & k_{42}^s & k_{43}^s & k_{44}^s \end{bmatrix}; \text{ Each } k_{ij}^s = \begin{bmatrix} k_{ij}^m & 0 \\ 0 & k_{ij}^b \end{bmatrix} \text{ is } 5\times 5$$

$k_{ij}^m = 2\times 2$ submatrix of the membrane stiffness matirx

$k_{ij}^b = 3\times 3$ submatrix of the bending stiffness matirx

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Similar arguments can be used for stiffness, we have strain energy contribution from membrane action and bending action again the nodal degrees of freedom for US is same as what we use, what was mentioned for kinetic energy, and KS is the shell element stiffness matrix and this again is written in this form, all these matrices are symmetric.

Stiffened plate : a refined model

Include membrane displacements of the plate in the analysis.

$$T=\frac{1}{2}\int_{-a}^{a}\rho A\left(\dot{u}_c^2+\dot{v}_c^2+\dot{w}_c^2\right)dx+\frac{1}{2}\int_{-a}^{a}\rho I_x\dot{\theta}_x^2dx$$

$$V=\frac{1}{2}\int_{-a}^{a}AE\left(\frac{\partial u_c}{\partial x}\right)^2dx+\frac{1}{2}\int_{-a}^{a}EI_y\left(\frac{\partial^2 w_c}{\partial x^2}\right)^2dx+\frac{1}{2}\int_{-a}^{a}GJ\left(\frac{\partial \theta_x}{\partial x}\right)^2dx$$

$$u_c=u+e\theta_y=u-e\frac{\partial w}{\partial x}$$

$$v_c=v-e\theta_x$$

$$w_c=w$$

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Now so we have now derived the mass and stiffness matrix for the facet shell element, now we'll return to the problem of, now the stiffened plate, so we will now try to construct a refined model where we include the membrane displacements of the plate in the analysis, the expression for energies will remain the same, but the details of displacement will be different, see earlier we use UC as E theta Y, now we are including the membrane displacement the 2 quantities shown in the red are the new terms, so consequently now the form of the Lagrangian will change now, so we need to now include these new terms.

$$T=\frac{1}{2}\int_{-a}^{a}\rho A\left(\dot{u}_c^2+\dot{v}_c^2+\dot{w}_c^2\right)dx+\frac{1}{2}\int_{-a}^{a}\rho I_x\dot{\theta}_x^2dx$$

$$V=\frac{1}{2}\int_{-a}^{a}AE\left(\frac{\partial u_c}{\partial x}\right)^2dx+\frac{1}{2}\int_{-a}^{a}EI_y\left(\frac{\partial^2 w_c}{\partial x^2}\right)^2dx+\frac{1}{2}\int_{-a}^{a}GJ\left(\frac{\partial \theta_x}{\partial x}\right)^2dx$$

$$u_c=u+e\theta_y=u-e\frac{\partial w}{\partial x}$$

$$v_c=v-e\theta_x$$

$$w_c=w$$

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$$T=\frac{1}{2}\int_{-a}^{a}\rho A\left(\dot{u}^2+\dot{v}^2+\dot{w}^2\right)dx+\frac{1}{2}\int_{-a}^{a}\rho Ae^2\left(\frac{\partial \dot{w}}{\partial x}\right)^2dx$$

$$+\frac{1}{2}\int_{-a}^{a}\rho\left(I_x+Ae^2\right)\dot{\theta}_x^2dx-\frac{1}{2}\int_{-a}^{a}\rho Ae\dot{u}\frac{\partial \dot{w}}{\partial x}dx-\frac{1}{2}\int_{-a}^{a}\rho Ae\dot{v}\dot{\theta}_x dx$$

$$V=\frac{1}{2}\int_{-a}^{a}AE\left(\frac{\partial u}{\partial x}\right)^2dx+\frac{1}{2}\int_{-a}^{a}E\left(I_y+Ae^2\right)\left(\frac{\partial^2 w}{\partial x^2}\right)^2dx$$

$$+\frac{1}{2}\int_{-a}^{a}GJ\left(\frac{\partial \theta_x}{\partial x}\right)^2dx-\int_{-a}^{a}AE\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2}dx$$

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So this is the expression for kinetic energy, this is the expression for strain energy, and these are the displacement fields in terms of W, dou W/dou X theta X and theta Y, so now if we consider the expression for kinetic energy you have to substitute this into this expression and rearrange the terms I will get the expression for kinetic energy to be given by this. There are 5 integrals now because of the cross terms and other things these new terms will be present. Similarly strain energy will have 5 contributing terms, 4 contributing terms as shown here, and now the field variables are U, V, W, W theta X, and so we have U, V, W, theta X, these are the field variables.

$$u = \begin{bmatrix} N_1(\xi) & N_2(\xi) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} N(\xi) \end{bmatrix} \{u\}_e$$

$$v = \begin{bmatrix} N_1(\xi) & N_2(\xi) \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{bmatrix} N(\xi) \end{bmatrix} \{v\}_e$$

$$N_1(\xi) = \frac{1}{2}(1-\xi); N_2(\xi) = \frac{1}{2}(1+\xi)$$

$$\{u\}_b = \begin{bmatrix} u_1 & v_1 & w_1 & \theta_{x1} & \theta_{x1} & u_2 & v_2 & w_2 & \theta_{x2} & \theta_{x2} \end{bmatrix}^t$$

$$T_e = \frac{1}{2}\{\dot{u}\}_b^t [m]^b \{\dot{u}\}_b$$

$$[m]^b = \begin{bmatrix} m_{11} & m_{12} \\ m_{12}^t & m_{22} \end{bmatrix}$$

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So now we represent them in terms of interpolation functions and nodal values, first the membrane action U is U1 U2, these are the interpolation function so for bending action I have these degrees of freedom and I get the mass matrix for bending given by this, this is for the

$$m_{11} = \frac{\rho A a}{210}\begin{bmatrix} 140 & 0 & -105\frac{e}{a} & 0 & -35e \\ 0 & 140 & 0 & 140e & 0 \\ -105\frac{e}{a} & 0 & 282 & 0 & -63a \\ 0 & 140e & 0 & 140\left(r_x^2+e^2\right) & 0 \\ -35e & 0 & -63a & 0 & 72a^2 \end{bmatrix}$$

$$m_{12} = \frac{\rho A a}{210}\begin{bmatrix} 70 & 0 & 105\frac{e}{a} & 0 & 35e \\ 0 & 70 & 0 & 70e & 0 \\ -105\frac{e}{a} & 0 & -72 & 0 & 5a \\ 0 & 70e & 0 & 70\left(r_x^2+e^2\right) & 0 \\ 35e & 0 & -5a & 0 & -26a^2 \end{bmatrix}$$

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beam element, the B is for beam element, and this is the elements for M11, M12, and M22 which appear in this matrix.

$$m_{22} = \frac{\rho A a}{210}\begin{bmatrix} 140 & 0 & 105\frac{e}{a} & 0 & -35e \\ 0 & 140 & 0 & 140e & 0 \\ 105\frac{e}{a} & 0 & 282 & 0 & 63a \\ 0 & 140e & 0 & 140\left(r_x^2+e^2\right) & 0 \\ -35e & 0 & 63a & 0 & 72a^2 \end{bmatrix}$$

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$$u = \begin{bmatrix} N_1(\xi) & N_2(\xi) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} N(\xi) \end{bmatrix} \{u\}_e$$

$$v = \begin{bmatrix} N_1(\xi) & N_2(\xi) \end{bmatrix} \begin{Bmatrix} v_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} N(\xi) \end{bmatrix} \{v\}_e$$

$$N_1(\xi) = \frac{1}{2}(1-\xi); N_2(\xi) = \frac{1}{2}(1+\xi)$$

$$\{u\}_b = \begin{bmatrix} u_1 & v_1 & w_1 & \theta_{x1} & \theta_{x1} & u_2 & v_2 & w_2 & \theta_{x2} & \theta_{x2} \end{bmatrix}^t$$

$$V_e = \frac{1}{2}\{u\}_b^t [k]^b \{u\}_b$$

$$[k]^b = \begin{bmatrix} k_{11} & k_{12} \\ k_{12}^t & k_{22} \end{bmatrix}$$

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So basically we are considering this Lagrangian made up of this and using this interpolation functions and constructing the solution. Stiffness matrix again given in this form U transpose KB, UB and this is assembled in a form K11, K12 and K22, each one is a 5 x 5 matrix as shown

$$k_{11} = \frac{AE}{4a^3}\begin{bmatrix} 2a^2 & 0 & 0 & 0 & -2ea^2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6e_y^2 & 0 & -6ae_y^2 \\ 0 & 0 & 0 & \dfrac{a^2 r_j^2}{1+\nu} & 0 \\ -2ea^2 & 0 & -6ae_y^2 & 0 & 8a^2 e_y^2 \end{bmatrix}$$

$$k_{12} = \frac{AE}{4a^3}\begin{bmatrix} -2a^2 & 0 & 0 & 0 & 2ea^2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -6e_y^2 & 0 & -6ae_y^2 \\ 0 & 0 & 0 & -\dfrac{a^2 r_j^2}{1+\nu} & 0 \\ 2ea^2 & 0 & 6ae_y^2 & 0 & 4a^2 e_y^2 \end{bmatrix}$$

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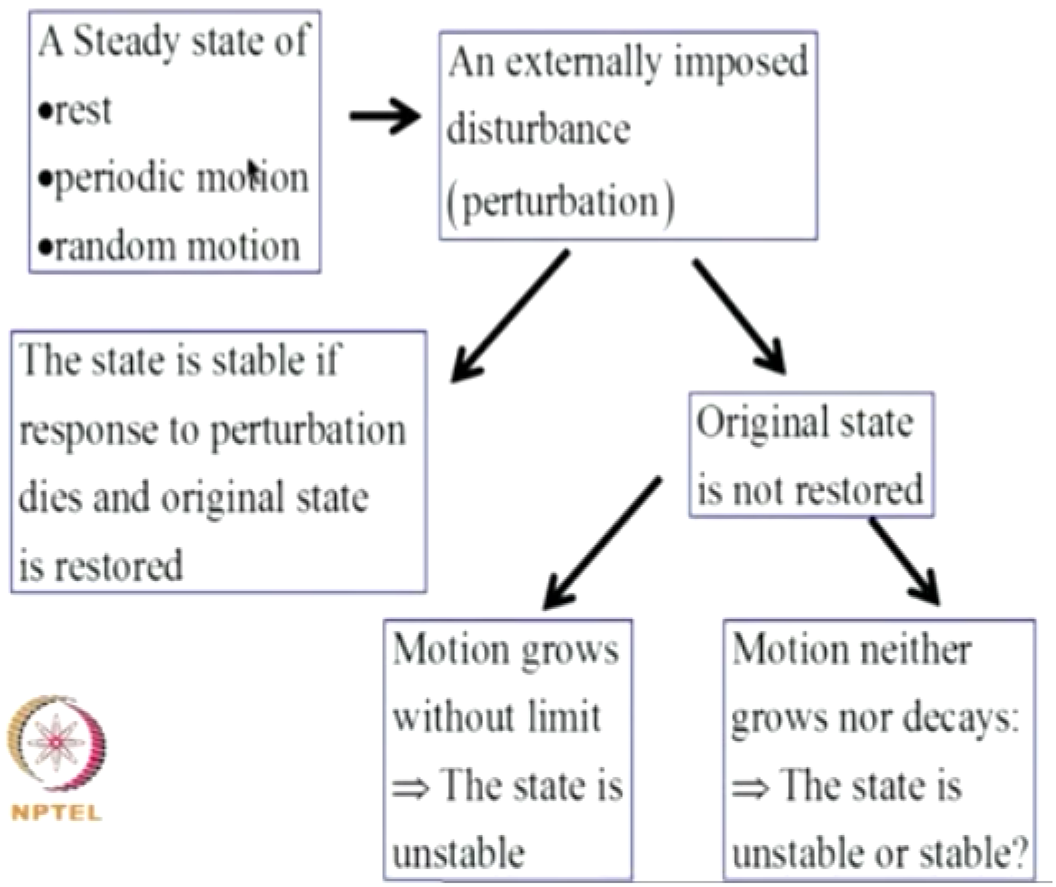
here, K11, K12, K22, so all these details need to be worked out and I leave that as an exercise, this is K22.

$$k_{22} = \frac{AE}{4a^3}\begin{bmatrix} 2a^2 & 0 & 0 & 0 & -2ea^2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6e_y^2 & 0 & 6ae_y^2 \\ 0 & 0 & 0 & \dfrac{a^2 r_j^2}{1+\nu} & 0 \\ -2ea^2 & 0 & 6ae_y^2 & 0 & 8a^2 e_y^2 \end{bmatrix}$$

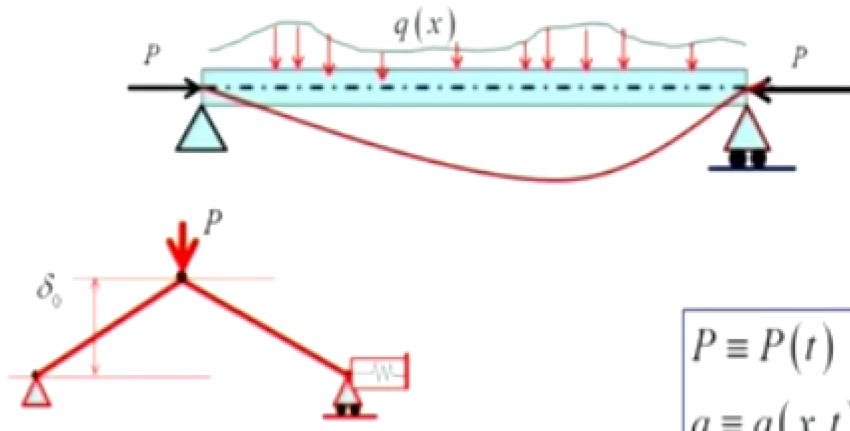
Now so far in the course we have focused on vibration problems, and we started by discussing simple axially vibrating bar, then flexural given plane and 3-dimensional beam element, and with that we constructed planer frames, grids, 3D frames, and we have considered several aspects of vibration analysis including time integration methods and substructuring and model reduction, and then we moved on to study of 2 dimensional elements, plane stress and plane strain elements, and plate bending elements, and we have seen today the facet shell element, and the stiffened plate element.

Now in the next part of module of this course we will take up a new topic that is related to stability of structures, later on after completing the discussion on stability we will return to problems of vibration analysis again, and we will consider problems of finite element model updating and some issues about nonlinear vibrations, and some questions on combining numerical experimental models in problems of you know finite element model updating as well as structural testing using hybrid simulation, so we will return to those topics after we address few issues related to stability of the structure.

Now in the subject of stability of structures we basically consider certain states of the system which could be state of rest, or state of periodic motion, or it could be as well state of random



motion. The idea here is we ask the question whether this, what happens to these states if we perturb these states by a small perturbation. Now such perturbations can occur in engineering practice because of various reasons so suppose these perturbations occur because of this perturbation the response of the structure would change, now we say that these states are stable if the response to perturbation dies and the original state is restored, if not if original state is not restored, then if the motion grows without limits then we say that the state is unstable, the issue remains unresolved if motion neither grows nor decays, we need to consider further you know approaches to analyze this type of problems.



$$P \equiv P(t)$$
$$q \equiv q(x,t)$$

How to analyse problems of stability using FEM?

Role of

- Imperfections
- Interactions between imperfections and nonlinearities
- Parametric excitations and parametric resonance

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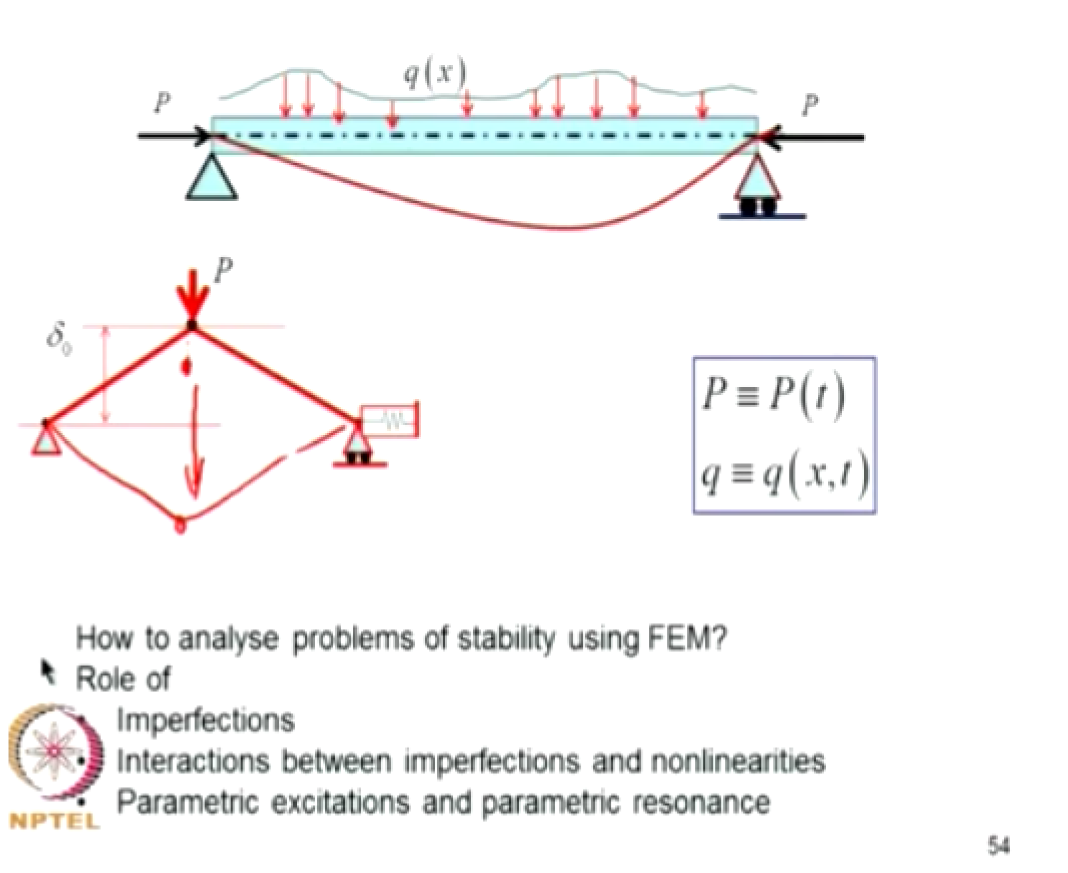
So in the next few lectures we will consider questions related to this and we will begin by considering some familiar problems like problems of beam column, suppose if you consider a simply supported beam carrying transverse load Q(x) and also axial loads P, in absence of axial loads P we have studied how the structure vibrates, how it displaces so on and so forth. Now the question we wish to consider is what would be the influence of simultaneous presence of transverse load and actual loads like P like this, we will be showing that there exist certain critical values of P for which slightest transverse load will produce huge responses in the system, so the equilibrium state in the neighborhood of P being close to the value of those critical values is unstable, in the sense a small perturbation will produce large responses.



There are two types of issues here, the new equilibrium position that the structure takes could be in the neighborhood of the original equilibrium position or we can have certain types of problems as schematically shown here, this is a problem where there are 2 rigid links hinged at these places and loaded as shown here, as the load P increases this point starts moving downwards and for a certain critical value of P, after the structure displaces and reaches this point it will snap. So from this the equilibrium position in the neighborhood of this somewhere here is unstable because a small perturbation pushes the response to a faraway equilibrium position, so this phenomena is known as snap through.

Now in a structure that is loaded by external loads it is always a question that we should consider, what would happen if there is a slight perturbations in the external loads, if the structure is in the neighborhood of being you know on the verge of losing its stability any slight perturbations we will create traumatic increase in the response and that virtually means in an engineering system the structure would fail, that is highly undesirable, so we would like to know how close we are to a critical state and in a good design we wish to be sufficiently far away from those states that is what we do in design of metal structures, so the question that we will be addressing is not so much on the phenomenological aspects of stability analysis, but more specifically how to analyze problems of stability within finite element method, specifically we will be considering what is the role of imperfections in the structure, what is the role of interactions between imperfections and nonlinearities, and if there are dynamic excitations for example if this P and Q that I have shown here are depicted to be static loads here, but they can as well be functions of time. Suppose Q is a static loader but P is a dynamic load. Now we can show that for certain types of loads, for example if P is something like P naught + epsilon cos omega T, if it is a harmonic load then the structure can get into resonance even when this driving frequency capital Omega need not coincide with any of the natural frequencies of the system, so these are not the traditional resonances but these are special types of resonance or parametric resonances, there are many systems in engineering practice where we get problems of this kind, and problems of this kind are characterized by structural matrices which vary in time, so for these time varying structural matrices the questions on parametric excitations and stability of the structure also needs to be considered.

So in the lectures to follow what we will do is we will first look at certain conceptual issues related to study of beam columns, and we will be showing that the problem of determining critical loads can be tackled using again in Eigenvalue problem will derive the elastic stiffness matrix, and we will also derive a new matrix associated with the structure known as geometric stiffness matrix, so that an Eigenvalue analysis are associated with those matrices will be able to help us to determine the critical axial loads. Then we will consider built up structures like continuous beams and frames and so on and so forth, which are laterally which carry you know axial loads or lateral loads and so on and so forth. The question we will ask is for a given loading configuration if the entire loading configuration is increased in its magnitude by keeping the relative values of loads at different places the same, at want value of the increasing the parameter that increases the load will the structure lose stability, so that type of questions we will consider and we will begin by addressing that type of problems for simple structures like single span beams and then generalize it to built-up structures. So with this we will close the present lecture.

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