Indian Institute of Science
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<u>Course Title</u>
Finite element method for structural dynamic
And stability analyses
Lecture – 24
Plate bending elements
(continued)
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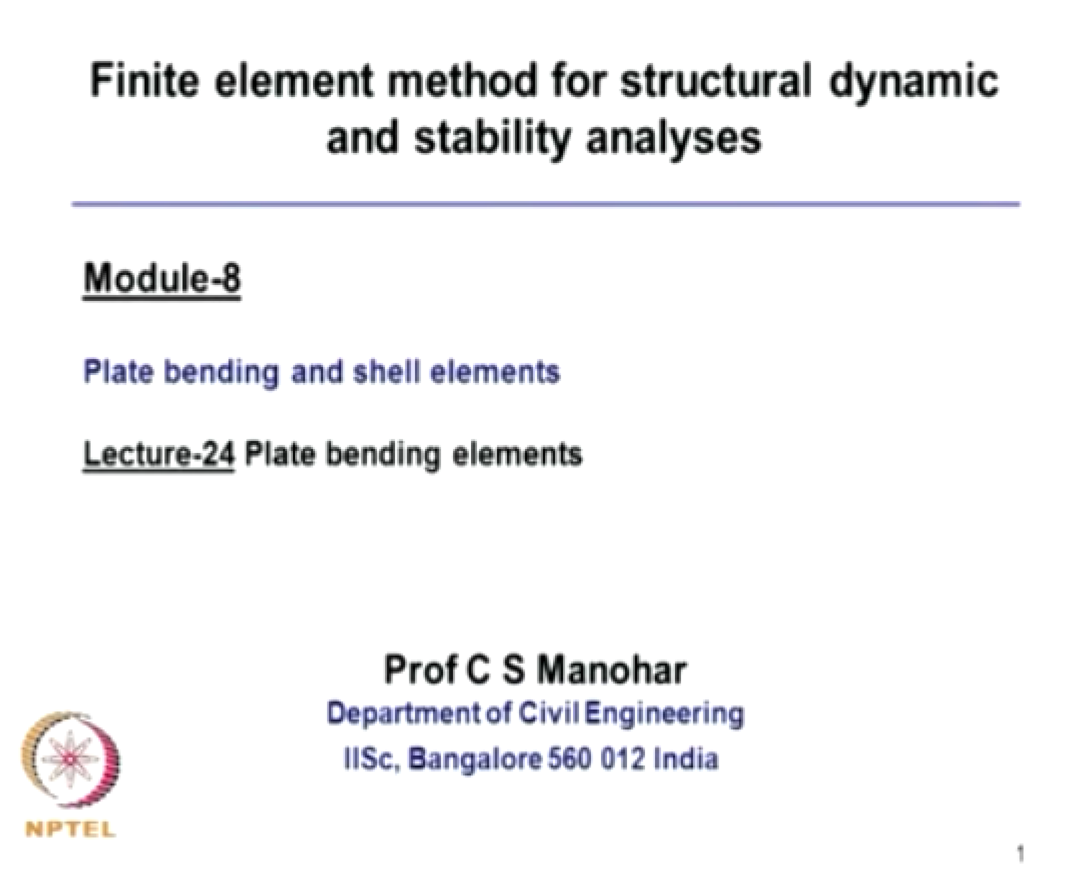
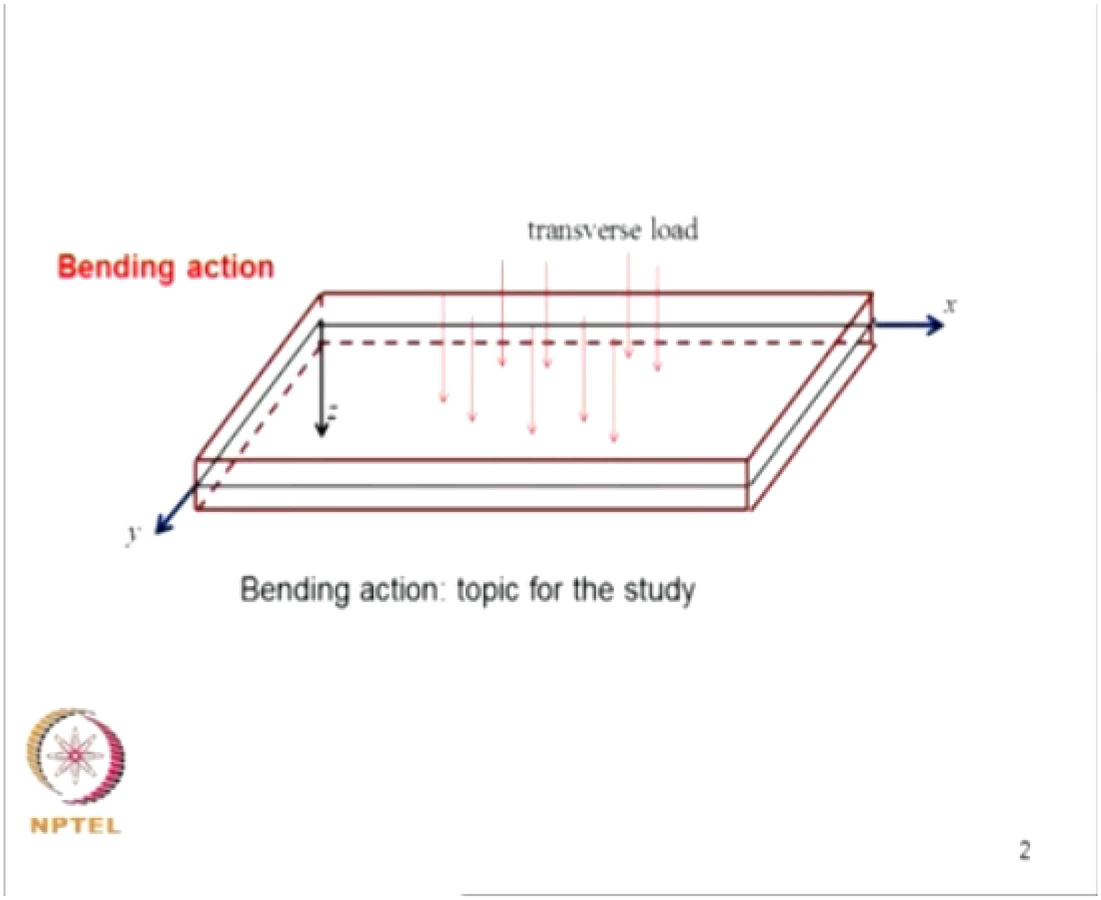


So we have been discussing modeling of plate bending elements, so we will continue with that.



So just to recapitulate this is the problem that we are considering, we are considering plate element carrying transverse loads, so the plate undergoes bending into direction and twisting, so this is what we have been studying for the last couple of classes. So we have considered two

Thin rectangular element

Field variable: $w(x, y, t)$

Order of the highest derivative present in the Lagrangian: 2

Dofs: $w(x, y, t), \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$

$\Rightarrow$ The interpolation functions must consist of complete polynomials of at least order 2.

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right)$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}\left(\varepsilon_{yy} + \nu\varepsilon_{xx}\right); \quad \varepsilon = -z\chi = -z\begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}; \quad V = \frac{1}{2}\int_A \frac{h^3}{12}\chi^t D \chi \, dA$$

$$\sigma_{xy} = 2G\varepsilon_{xy} \qquad T = \frac{1}{2}\int_A \rho h \dot{w}^2 \, dA$$

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models, one model where we assume the plate to be thin in which case the strain energy and kinetic energy expressions are of this form where khi is the vector of these curvatures dou square W/dou X square, dou square W/dou Y square, and dou square W/dou X dou Y, and so consequently if you look at the field variable that is present it is W, and then the order of highest derivative present in the Lagrangian is 2, because we are here seeing here dou square W/dou X square, so the degrees of freedom should include the field variables and all its derivatives up to order 2 – 1, which is W, dou W/dou X and dou W/dou Y, so at every node therefore there should be 3 degrees of freedom, so the interpolation functions must consist of complete polynomials of at least order 2.

Thick plate

$$V = \frac{1}{2}\int_{V_0} \varepsilon^t D \varepsilon dV_0 + \frac{1}{2}\int_{V_0} \tau^t \gamma dV_0 = \frac{1}{2}\int_A \frac{h^3}{12}\chi^t D \chi dA + \frac{1}{2}\int_A kh\gamma^t D^s \gamma dA$$

$$T = \frac{1}{2}\int_{V_0} \rho\left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right) dV_0 = \frac{1}{2}\int_A \rho\left(h\dot{w}^2 + \frac{h^3}{12}\dot{\theta}_x^2 + \frac{h^3}{12}\dot{\theta}_y^2\right) dA$$

$$\chi = \begin{Bmatrix} -\dfrac{\partial \theta_y}{\partial x} \\ \dfrac{\partial \theta_x}{\partial y} \\ \dfrac{\partial \theta_x}{\partial x} - \dfrac{\partial \theta_y}{\partial y} \end{Bmatrix} \;\&\; \gamma = \begin{Bmatrix} \dfrac{\partial w}{\partial x} + \theta_y \\ \dfrac{\partial w}{\partial y} - \theta_x \end{Bmatrix}$$

Field variables: w, θ_x, θ_y

Highest derivative of the field variables: 1

Dofs: w, θ_x, θ_y at the nodes

In a thick plate on the other hand the strain energy and kinetic energy are as shown here, in addition to this definition of khi and gamma we need to take into account, now we define in addition to W, we also define the rotation of the plane that is normal to the middle plane after deformation the angle through which it rotates that we are considering, there are theta X and theta Y and the field variables therefore if you look at it will be W, theta X, and theta Y, and the highest derivative in the field variable is 1, therefore at every node we should have W, theta X, and theta Y, as the degrees of freedom, so this is the basis for formulating thick plate elements.

Nonconforming thin rectangular element

$$w = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3 + \alpha_{11} x^3 y + \alpha_{12} xy^3$$

Conforming rectangular element

$$w = \sum_{j=1}^{4} \left[w_j f_j(\xi) f_j(\eta) + b^2 \theta_{xj} f_j(\xi) g_j(\eta) - a^2 \theta_{yj} g_j(\xi) f_j(\eta) \right]$$

Thick rectangular conforming element

$$w(\xi,\eta,t) = \sum_{j=1}^{4} N_j(\xi,\eta) w_j(t); \quad \theta_x(\xi,\eta,t) = \sum_{j=1}^{4} N_j(\xi,\eta)\theta_{xj}(t)$$

$$\theta_y(\xi,\eta,t) = \sum_{j=1}^{4} N_j(\xi,\eta)\theta_{yj}(t)$$

$$N_j(\xi,\eta) = \frac{1}{4}(1+\xi\xi_j)(1+\eta\eta_j); \; j = 1,2,3,4$$

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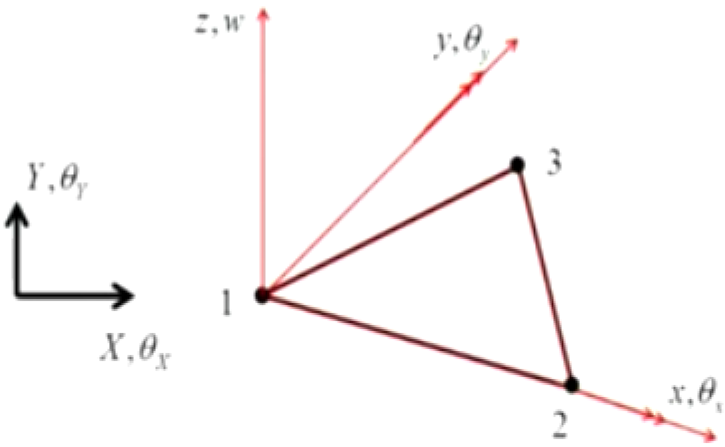
So we started with thin plates we assumed, we considered a rectangular thin plate and there are 4 nodes, 4 noded rectangular element and each node there are 3 degrees of freedom, therefore there should be 12 terms, that is 12 generalized coordinates are needed in the representation of the field variable, so we use a 12 term expansion as shown here up to cubic, all the terms in the Pascal triangle are included, the last 2 terms X cube Y and XY cube are selected so as to retain geometric invariant, this element we showed that across the element boundaries theta Y won't be continuous, therefore it is a non-conforming element. Then by considering the shape functions to be made up of products of beam shape functions cubic polynomials we developed a conforming rectangular element and in this model we showed that we need to include dou square W/dou X dou Y also has a degree of freedom, so as to ensure that if it is not done what happens is dou square W/dou X dou Y becomes 0 at all the nodes, and consequently as the element size becomes small, the plate will behave like a there won't be any twist, and to remedy that we introduced an additional degree of freedom.

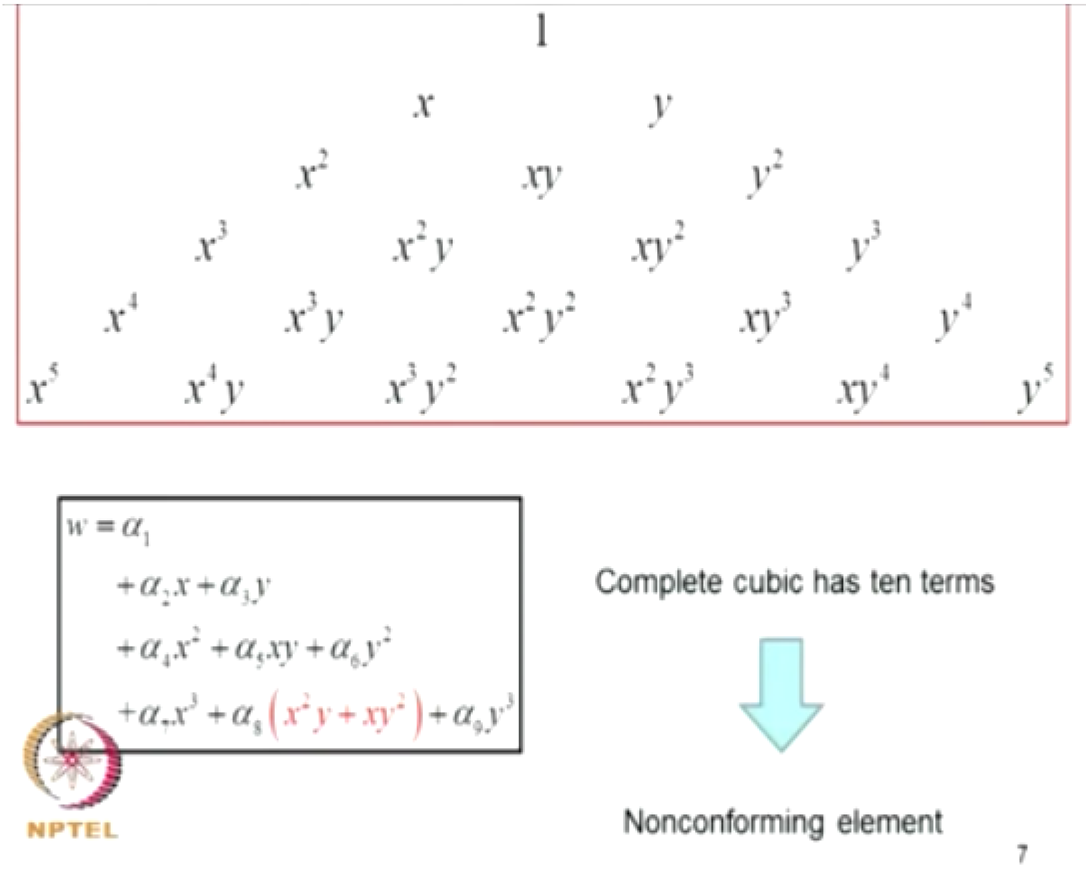
Then we consider thick rectangular plate elements and we showed that it is a conforming element, and we used these polynomials quadratics and we developed the stiffness and mass matrices and showed that this element is conforming. We have considered that rectangular geometry, now we will consider triangular geometry, we will start by considering a thin

Thin, triangular, nonconforming element



3 nodes
3 dofs/node
9 dof element
We need to include nine terms in the representation for Displacement.

triangular nonconforming element, we consider element with 3 degrees of freedom 1, 2, 3 as shown here triangular element, and we need 3 degrees of freedom at every node therefore this element will have 9 degrees of freedom, so we need to include 9 terms in the representation for the displacement, so what we do is we select a local coordinate system where X axis is taken to coincide with the edge 1, 2, and this Y axis is in the plane of the plate, orthogonal to this X axis and Z axis is out of plane, so this is local coordinates. And we also have a global coordinates and we indicate you know the variables X, theta X, and theta Y, W, theta X, theta Y, using upper case letters.



So we have to select now 9 terms, so what we do is we start with 1, then X and Y, then these 3 terms get selected, next when it comes to cubic terms to get a complete cubic we need to have 10 terms, but we need only 9 times in our representation, so what is being done in this element is this X square Y, and XY square are multiplied by a common generalized coordinate, so as we will see shortly that this will lead to, this will make the element a nonconforming element.

$$w = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 \left(x^2 y + xy^2\right) + \alpha_9 y^3$$

$$w = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & \left(x^2 y + xy^2\right) & y^3 \end{bmatrix} \{\alpha\}$$

$$= \left[P(x,y)\right]\{\alpha\}$$

$$\{\alpha\}^t = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_9 & \alpha_{10} \end{bmatrix}$$

$$\begin{Bmatrix} w \\ \theta_x = \dfrac{\partial w}{\partial y} \\ \theta_y = -\dfrac{\partial w}{\partial x} \end{Bmatrix} = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2 y + xy^2 & y^3 \\ 0 & 0 & 1 & 0 & x & 2y & 0 & x^2 + 2xy & 3y^2 \\ 0 & 0 & -1 & 0 & -y & 0 & -3x^2 & -2xy - y^2 & 0 \end{bmatrix} \{\alpha\}$$

Use this equation at

node 1 (0,0), node 2 $(x_2, 0)$, and node 3 (x_3, y_3)

So let's consider this, so W is represented as this polynomial and we can rewrite this by defining this row vector consisting of 1. X, Y, X square etcetera and the Alpha is the vector of generalized coordinates alpha 1 to alpha 9 as shown here, we call this matrix as P(x,y) and this is written as P(x,y) so this has 1 row and 9 columns, and this is 9, this is 1 x 9, this is 9 x 1. Now we need W, theta X, and theta Y, so I have W, dou W/dou Y, dou W/dou X, we differentiating this I formulate this matrix. Now what we do is, we use this equation at node 1 coordinate 0, 0, and 2 X2 0, and node X3, Y3 and determine alpha, so we have to first determine this polynomial, the shape functions, so this leads to an equation of the form, the

$$\Rightarrow \{w\}_e = [A]_e \{\alpha\}$$

with

$$[A]_e = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & x_2 & 0 & x_2^2 & 0 & 0 & x_2^3 & 0 & 0 \\ 0 & 0 & 1 & 0 & x_2 & 0 & 0 & x_2^2 & 0 \\ 0 & -1 & 0 & -2x_2 & 0 & 0 & -3x_2^2 & 0 & 0 \\ 1 & x_3 & y_3 & x_3^2 & x_3 y_3 & y_3^2 & x_3^3 & x_3^2 y_3 + x_3 y_3^2 & y_3^3 \\ 0 & 0 & 1 & 0 & x_3 & 2y_3 & 0 & x_3^2 + 2x_3 y_3 & 3y_3^2 \\ 0 & -1 & 0 & -2x_3 & -y_3 & 0 & -3x_3^2 & -\left(2x_3 y_3 + y_3^2\right) & 0 \end{bmatrix}$$

$$\Rightarrow \{\alpha\} = [A]_e^{-1} \{w\}_e$$

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element degrees of freedom, nodal degrees of freedom in terms of the generalized coordinates is related to each other through this matrix A, which is a function of nodal coordinates. So alpha can be obtained as inverse of this, so this determines the relationship between generalized coordinates and the nodal degrees of freedom, so we can write W as P alpha becomes now P,

$$w = [P(x,y)]\{\alpha\} = [P(x,y)][A]_e^{-1}\{w\}_e$$

Remark

The matrix $[A]_e$ becomes singular whenever $x_2 - 2x_3 - y_3 = 0$.

If this happens, move the nodes to avoid the satisfaction of the above condition.

Refered to as T−element (Tocher)

Evaluation of

$$\begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y + xy^2 & y^3 \\ 0 & 0 & 1 & 0 & x & 2y & 0 & x^2 + 2xy & 3y^2 \\ 0 & 0 & -1 & 0 & -y & 0 & -3x^2 & -2xy - y^2 & 0 \end{bmatrix} \{\alpha\}$$

along $y = 0$ leads to

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AE inverse WE. Now a comment needs to be made at this juncture, this matrix AE becomes singular whenever this condition is satisfied X2 - 2X3 -Y3 = 0, so elements which have the geometric property satisfying this condition, the A matrix will become singular, so if you encounter this situation you have to move the nodes, you re-mesh and move the nodes, so you have to check that.

Now the evaluation of, now let us consider what happens to this along Y = 0, so you consider this equation along Y = 0, so I get this equation, and it turns out that W will be a cubic

$$\begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} = \begin{bmatrix} 1 & x & 0 & x^2 & 0 & 0 & x^3 & 0 & 0 \\ 0 & 0 & 1 & 0 & x & 0 & 0 & x^2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -3x^2 & 0 & 0 \end{bmatrix} \{\alpha\}$$

$\Rightarrow$

$$w = \alpha_1 + \alpha_2 x + \alpha_4 x^2 + \alpha_7 x^3$$

$$\theta_x = \alpha_3 + \alpha_5 x + \alpha_8 x^2$$

$$\theta_y = -\alpha_3 - 3\alpha_7 x^2$$

It can be shown that (show it)

- w & θ_y depends upon $w_1, \theta_{y1}, w_2, \theta_{y2} \Rightarrow w$ & θ_y are continuous across element boundary
- θ_x depends upon nodal dofs at node 3 also.

$\Rightarrow$ Element is not conforming.

Question: Is geometric invariance property satisfied?

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polynomial, theta X will be a quadratic, theta Y also is quadratic, and we can show that W and theta Y depends upon W1, theta Y1, W2, and theta 2, so which is the edge that Y = 0 represents? So it is along edge 1, 2, so this is 0, 0 and 0, X2, so we have this equation. Now along edge 1, 2 the nodes are 1 and 2, so W and theta Y depends upon the nodal degrees of freedom at 1 and 2, so consequently we can show that W and theta Y will be continuous across element boundaries. But if you look at theta X you can show that it depends on nodal values at node 3 also, so therefore it means that we cannot achieve continuity of theta X across element edges and therefore this element is a non-conforming element.

Now we can I leave it as a question is the geometric invariance property satisfied in this element you can think about that, notwithstanding this limitation as we have been doing earlier

$$w = \left[P(x,y)\right]\left[A\right]_e^{-1}\{w\}_e$$

$$T_e = \frac{1}{2}\int_A \rho h \dot{w}^2 dA$$

$$= \frac{1}{2}\int_A \rho h \{\dot{w}\}_e^t \left[\left[A\right]_e^{-1}\right]^t \left[P(x,y)\right]^t \left[P(x,y)\right]\left[A\right]_e^{-1}\{w\}_e \, dA$$

$$= \frac{1}{2}\dot{w}_e^t \bar{M}_e \dot{w}_e$$

$$\bar{M}_e = \left[\left[A\right]_e^{-1}\right]^t \underbrace{\left[\int_A \rho h \left[P(x,y)\right]^t \left[P(x,y)\right] dA\right]}_{\text{This can be evaluated exactly}} \left[A\right]_e^{-1}$$

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we will proceed with the formulation, so W now is given by P AE inverse WE, the expression for kinetic energy is 1/2 area integral rho H W dot square over volume element. Now for W dot I will put this, so it will be W dot transpose, W dot, if you write in that manner we get the expression for kinetic energy to be this, and the term inside this bracket we call it as the element mass matrix in the local coordinate system, you see we should keep track of the coordinate system also, this is the mass matrix in the local coordinate system the integrals that appear here our product of polynomials so they can be evaluated in closed form, so we don't not need any special tools to evaluate this it can be evaluated exactly or a suitable Gauss quadrature can also be used.

$$w = \left[P(x,y)\right]\left[A\right]_e^{-1}\{w\}_e$$

$$V = \frac{1}{2}\int_A \frac{h^3}{12}\chi^t D\chi dA = \frac{1}{2}w_e^t \bar{K}_e w_e$$

$$\bar{K}_e = \left[\left[A\right]_e^{-1}\right]^t \left[\int_A \frac{h^3}{12}\bar{B}^t D\bar{B} dA\right]\left[A\right]_e^{-1}$$

$$\bar{B} = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 6x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2x & 6y \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 4(x+y) & 0 \end{bmatrix}$$

Again, the elements of $\left[\int_A \frac{h^3}{12}\bar{B}^t D\bar{B} dA\right]$ can be evaluated exactly.

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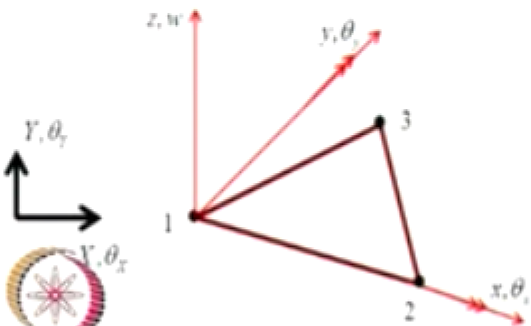
Now how about strain energy? Again let's start with W = P AE inverse WE, and strain energy is given by this H cube/12 khi transport D khi DA, and this KE inverse, KE bar which is the element stiffness matrix in the local coordinates is given by this, this B bar is the strain matrix you can show that it is given by this matrix, so you have to differentiate the shape functions through the matrix of partial derivatives that defined strain displacement relations, so again these elements of this matrix it can be evaluated exactly in this element, so there is no problem so KE can be evaluated. So what we have done right now is we have analyzed this element in its local coordinate system, so for assembling process to be implemented we should transform it to global coordinates, so how do we do that?

Structural matrices in local coordinate system

$$\bar{K}_e = \left[[A]_e^{-1}\right]^t \left[\int_A \frac{h^3}{12}\bar{B}^t D\bar{B}\,dA\right][A]_e^{-1}$$

$$\bar{M}_e = \left[[A]_e^{-1}\right]^t \left[\int_A \rho h\left[P(x,y)\right]^t\left[P(x,y)\right]dA\right][A]_e^{-1}$$

Transformation to the global coordinate system



$$\begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(x,X) & \cos(x,Y) \\ 0 & \cos(y,X) & \cos(y,Y) \end{bmatrix} \begin{Bmatrix} w \\ \theta_X \\ \theta_Y \end{Bmatrix} = [L_2]\begin{Bmatrix} w \\ \theta_X \\ \theta_Y \end{Bmatrix}$$

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So these are the structural matrices that we have derived till now in the local coordinate system, this is the element, stiffness matrix, a bar indicates that it is in local coordinates, this is element mass matrix. Now transformation to global coordinate system you can see here the degree of freedom W, theta X, theta Y in the local coordinates can be shown to be related to the values in the global coordinate system through this transformation matrix, this we have discussed earlier on many occasions, so will not get into the details, so we get this, I call this matrix as L2, so it is L2 into the nodal values in global coordinates, so equipped with this now we can go ahead,

$$\begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(x,X) & \cos(x,Y) \\ 0 & \cos(y,X) & \cos(y,Y) \end{bmatrix} \begin{Bmatrix} w \\ \theta_X \\ \theta_Y \end{Bmatrix} = [L_2] \begin{Bmatrix} w \\ \theta_X \\ \theta_Y \end{Bmatrix}$$

$$\{w_e\} = \begin{bmatrix} L_2 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_2 \end{bmatrix} \{W_e\} = [R]_e \{W_e\}$$

$$\{w_e\}^t = [w_1 \quad \theta_{x1} \quad \theta_{y1} \quad w_2 \quad \theta_{x2} \quad \theta_{y2} \quad w_3 \quad \theta_{x3} \quad \theta_{y3}]$$

$$\{W_e\}^t = [w_1 \quad \theta_{X1} \quad \theta_{Y1} \quad w_2 \quad \theta_{X2} \quad \theta_{Y2} \quad w_3 \quad \theta_{X3} \quad \theta_{Y3}]$$

$$T_e = \frac{1}{2}\{\dot{W}_e\}^t \underbrace{\left[[R]_e^t \bar{M}_e [R]_e\right]}_{M_e} \{\dot{W}_e\}^t \qquad V_e = \frac{1}{2}\{W_e\}^t \underbrace{\left[[R]_e^t \bar{K}_e [R]_e\right]}_{K_e} \{W_e\}^t$$

$$M_e = [R]_e^t \bar{M}_e [R]_e \qquad K_e = [R]_e^t \bar{K}_e [R]_e$$

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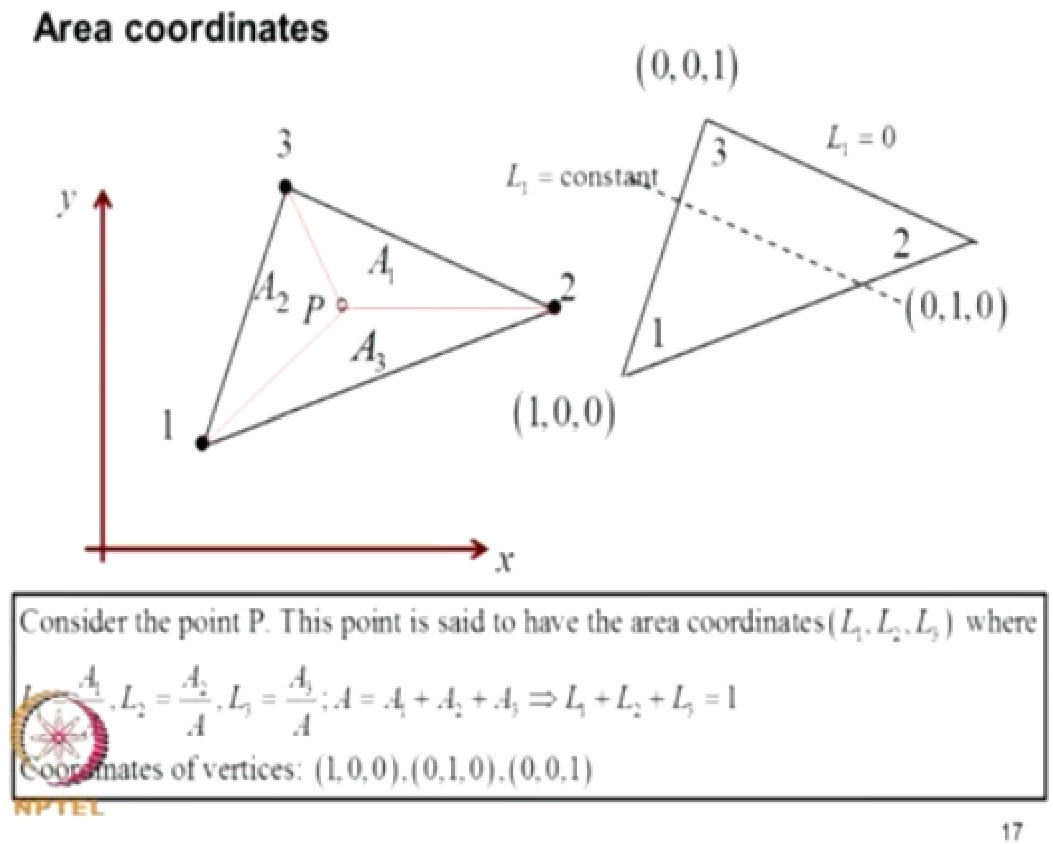
this is our L2 into this, so WE is the collection of all nodal degrees of freedom 9 x 1 in local coordinates, and this is similar quantity in global coordinates, so this is the transformation matrix 9 x 9 and this is RE, WE, so this is WET that is this transpose of that, this is the local degrees of freedom, degrees of freedom in the local coordinate system and these are the degrees of freedom in global coordinate system. So now TE which is the elements kinetic energy in the global coordinate system is you can deduce it to be this, where M bar E is the local mass matrix in the local coordinate system, so this is the element mass matrix, this is element stiffness matrix in the global coordinate system.



Before we start discussion on thick plates and other issues, we will revert back to an basic notion, that in dealing with triangular geometries we can introduce instead of Cartesian coordinate system we can also follow what is known as the area coordinate system, so to explain that you consider this triangle 1, 2, 3 and for any point within the domain I construct these 3 triangles, and the triangle opposing node 1 is called A1, now we consider the point P, this point is set to have the area coordinate is L1, L2, L3, where L1 is A1/A, L2 is A2/A, L3 is A3/A, clearly total area is A1 + A2 + A3 which is equal to A, now therefore if you add L1 + L2 + L3 it will be 1, so now you consider this node 1, suppose this point P coincides with this node 1 then clearly A1 will be the entire area of this triangle, so the coordinate, at the vertices this coordinate will be 1, 0, 0, so if I come here it will be 0, 1, 0, so similarly it is 0, 0, 1, so vertices are having this property, so this is advantageous in representing triangular geometry A1 in 3 dimensions, we can also define L1 which is the area coordinate at a node 1 to be the distance of

$$L_1 = \frac{\text{Distance from P to side 2-3}}{\text{Distance from 1 to side 2-3}}$$

Equation of side 2-3 is $L_1 = 0$.

L_1 = constant is a line parallel to edge 2-3.

Let $(x_i, y_i), i = 1,2,3$: coordinates of vertices 1,2,3.

$\Rightarrow$

$$x = x_1L_1 + x_2L_2 + x_3L_3$$

$$y = y_1L_1 + y_2L_2 + y_3L_3$$

Combining these equations with $L_1 + L_2 + L_3 = 1$ we get

$$\begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = [A]^t \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix}$$

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \left[[A]^t\right]^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} \text{ with } [A]^{-1} = \begin{bmatrix} A_1^0 & A_1^0 & A_1^0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

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point P to the side 2, 3, this distance divided by distance from 1 to side 2, 3, so that means you find the distance of 1 to this line 2, 3 and this to this, that ratio is also equal to L1, so equation of side 2, 3 is L1 = 0, right on this edge L1 is 0, so if you draw a line parallel to this line it will be a line having L1 equal to constant, so let XI, YI, I = 1, 2, 3 be the coordinates of vertices 1, 2, 3 in Cartesian coordinate system, we can show that any point in the domain with coordinates L1, L2, L3 is the x-coordinate can be obtained by using this formula, and Y can be obtained using this formula.

So now we have L1 + L2 + L3 = 1 and these 2 relations, so we can solve for L1, L2, L3 in terms of X and Y by using those 3 equations which are put together here, so this matrix I call it as A transpose, and this is the vector of L1, L2, L3, so from this the relationship between area coordinates and Cartesian coordinates can be computed and with A inverse being given by this.

$$L_i = \frac{1}{2A}\left(A_i^0 + a_i x + b_i y\right)$$

$$A_i^0 = x_j y_i - x_i y_j$$

$$a_i = y_j - y_i$$

$$b_i = x_j - x_i$$

Recall

For a triangular element $N_i = \frac{1}{2A}\left(A_i^0 + a_i x + b_i y\right) \Rightarrow N_i = L_i$.

An useful result

$$\int L_1^m L_2^n L_3^p dA = \frac{m!n!p!}{(m+n+p+2)!} 2A$$

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Now it turns out that the formulary for finding L1, L2, L3 are as given here, now if you recall when we discussed triangular element the shape functions were exactly the same as this, so NI and LI are identical for this case, an useful result which will be helpful in evaluating elements of mass and stiffness matrices is shown here and integral over area of L1 to the power of M, L2 to the power of N, L3 to the power of P is given by this, so with 0 factorial being 1, okay, so with 0 you can compute by putting M, N, R, P = 0, you can you know get various combinations.

Thick triangular plate element

$$V=\frac{1}{2}\int_{V_0}\varepsilon^t D\varepsilon dV_0+\frac{1}{2}\int_{V_0}\tau^t\gamma dV_0=\frac{1}{2}\int_A\frac{h^3}{12}\chi^t D\chi dA+\frac{1}{2}\int_A kh\gamma^t D^s\gamma dA$$

$$T=\frac{1}{2}\int_{V_0}\rho\left(\dot{u}^2+\dot{v}^2+\dot{w}^2\right)dV_0=\frac{1}{2}\int_A\rho\left(h\dot{w}^2+\frac{h^3}{12}\dot{\theta}_x^2+\frac{h^3}{12}\dot{\theta}_y^2\right)dA$$

$$\chi=\begin{Bmatrix}-\dfrac{\partial\theta_y}{\partial x}\\ \dfrac{\partial\theta_x}{\partial y}\\ \dfrac{\partial\theta_x}{\partial x}-\dfrac{\partial\theta_y}{\partial y}\end{Bmatrix}\ \&\ \gamma=\begin{Bmatrix}\dfrac{\partial w}{\partial x}+\theta_y\\ \dfrac{\partial w}{\partial y}-\theta_x\end{Bmatrix}$$

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So now equipped with this basic notion will now consider thick triangular plate, now in a thick plate the expression for strain energy and kinetic energy already we have discussed they are given by these expressions, and the field variables here will be W, theta X, and theta Y, and the highest order of the derivative is 1, therefore at any node the degree of freedom, the field

Field variables: w, θ_x, θ_y

Highest derivative of the field variables: 1

Dofs: w, θ_x, θ_y at the nodes

$$w = \sum_{i=1}^{3} L_i w_i, \quad \theta_x = \sum_{i=1}^{3} L_i \theta_{xi}, \quad \theta_y = \sum_{i=1}^{3} L_i \theta_{yi}$$

THT element

$$\begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} = [N]\{w\}_e$$

$$\{w\}_e^t = \begin{bmatrix} w_1 & \theta_{x1} & \theta_{y1} & w_2 & \theta_{x2} & \theta_{y2} & w_3 & \theta_{x3} & \theta_{y3} \end{bmatrix}$$

$$N = \begin{bmatrix} L_1 & 0 & 0 & L_2 & 0 & 0 & L_3 & 0 & 0 \\ 0 & L_1 & 0 & 0 & L_2 & 0 & 0 & L_3 & 0 \\ 0 & 0 & L_1 & 0 & 0 & L_2 & 0 & 0 & L_3 \end{bmatrix}$$

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variables are W, theta X, theta Y, highest derivative of the field variable is 1, therefore degrees of freedom are W, theta X, theta Y and the nodes, so I need now to represent any of this field variable say for example W, and there are 3 nodes I need a 3-term expansion, so I write it as a LI, WI where LI are the functions that we discussed a while before, so this is theta X, this is

Field variables: w, θ_x, θ_y

Highest derivative of the field variables: 1

Dofs: w, θ_x, θ_y at the nodes

$$w = \sum_{i=1}^{3} L_i w_i, \quad \theta_x = \sum_{i=1}^{3} L_i \theta_{xi}, \quad \theta_y = \sum_{i=1}^{3} L_i \theta_{yi}$$

THT element

$$\begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} = [N]\{w\}_e$$

$$\{w\}_e^t = \begin{bmatrix} w_1 & \theta_{x1} & \theta_{y1} & w_2 & \theta_{x2} & \theta_{y2} & w_3 & \theta_{x3} & \theta_{y3} \end{bmatrix}$$

$$N = \begin{bmatrix} L_1 & 0 & 0 & L_2 & 0 & 0 & L_3 & 0 & 0 \\ 0 & L_1 & 0 & 0 & L_2 & 0 & 0 & L_3 & 0 \\ 0 & 0 & L_1 & 0 & 0 & L_2 & 0 & 0 & L_3 \end{bmatrix}$$

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theta Y, so now we will assemble these 3 field variables and put it in a single equation, so W, theta X, theta Y is given by this, now WE is the element nodal degrees of freedom at the 3 nodes 1, 2, 3, which are W1, theta X1, theta Y1, and similar quantities at the other 2 nodes, so this N we can show that using these representations you can show that N will be having this form.

$$M_e = m_I + m_{II}$$

$$m_I = \frac{\rho h A}{144}\begin{bmatrix} 24 & & & & & & & & \\ 0 & 0 & & & & & & & \\ 0 & 0 & 0 & & & \text{SYM} & & & \\ 12 & 0 & 0 & 24 & & & & & \\ 0 & 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & & \\ 12 & 0 & 0 & 12 & 0 & 0 & 24 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

NPTEL

We can go to the mass matrix and stiffness matrix they can be evaluated exactly I will not get into all the details, so you evaluate the mass matrix you can evaluate it as sum of 2 square matrices with M1 being given by this, and M2 given by this. Now how about strain energy? So

$$\begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} = [N]\{w\}_e \;\&\; \chi = \begin{Bmatrix} -\dfrac{\partial \theta_y}{\partial x} \\ \dfrac{\partial \theta_x}{\partial y} \\ \dfrac{\partial \theta_x}{\partial x} - \dfrac{\partial \theta_y}{\partial y} \end{Bmatrix} \Rightarrow$$

$$B^f = \frac{1}{2A}\begin{bmatrix} 0 & 0 & -a_1 & 0 & 0 & -a_2 & 0 & 0 & -a_3 \\ 0 & b_1 & 0 & 0 & b_2 & 0 & 0 & b_3 & 0 \\ 0 & a_1 & -b_1 & 0 & a_2 & -b_2 & 0 & a_3 & -b_3 \end{bmatrix}; D = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

$$\frac{h^3}{12} A\left[B^f\right]^t [D]\left[B^f\right]$$

NPTEL

we have this representation W is N into WE, and khi is this, so we can using these relations we can derive the strain the matrix B for flexure and for shear separately, let’s consider flexure first, this is this, and this is a constitutive law which relates sigma XX, sigma YY, and sigma XY to the corresponding strains, and the stiffness matrix due to flexure is given by this.

$$\begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} = [N]\{w\}_e \ \& \ \gamma = \begin{Bmatrix} \dfrac{\partial w}{\partial x} + \theta_y \\ \dfrac{\partial w}{\partial y} - \theta_x \end{Bmatrix} \Rightarrow$$

$$B^s = kGh \begin{bmatrix} \dfrac{a_1}{2A} & 0 & L_1 & \dfrac{a_2}{2A} & 0 & L_2 & \dfrac{a_3}{2A} & 0 & L_3 \\ \dfrac{b_1}{2A} & -L_1 & 0 & \dfrac{b_2}{2A} & -L_2 & 0 & \dfrac{b_3}{2A} & -L_3 & 0 \end{bmatrix}$$

$$k^s = \begin{bmatrix} k_{11}^s & & \\ k_{21}^s & k_{22}^s & \\ k_{31}^s & k_{32}^s & k_{33}^s \end{bmatrix}; k_{11}^s = \begin{bmatrix} \dfrac{(a_1^2 + b_1^2)}{4A} & -\dfrac{b_1}{6} & \dfrac{a_1}{6} \\ -\dfrac{b_1}{6} & \dfrac{A}{6} & 0 \\ \dfrac{a_1}{6} & 0 & \dfrac{A}{6} \end{bmatrix}$$

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Now similarly the shear by considering gamma is this, we compute the B matrix associated with shear and this is given by this, so it has 2 rows and 8 columns, so this can be used to construct a contribution to the stiffness matrix from shearing action so based on that there is a few details of evaluation the KS is the contribution to the element stiffness matrix from shear components, and this itself is determined in terms of these quantities which are shown here,

$$k_{21}^{1} = \begin{bmatrix} \frac{(a_1a_2 + b_1b_2)}{4A} & -\frac{b_2}{6} & \frac{a_2}{6} \\ -\frac{b_1}{6} & \frac{A}{12} & 0 \\ \frac{a_1}{6} & 0 & \frac{A}{12} \end{bmatrix}; k_{31}^{1} = \begin{bmatrix} \frac{(a_3a_1 + b_3b_1)}{4A} & -\frac{b_3}{6} & \frac{a_3}{6} \\ -\frac{b_1}{6} & \frac{A}{12} & 0 \\ \frac{a_1}{6} & 0 & \frac{A}{12} \end{bmatrix}$$

$$k_{22}^{1} = \begin{bmatrix} \frac{(a_2^2 + b_2^2)}{4A} & -\frac{b_2}{6} & \frac{a_2}{6} \\ -\frac{b_2}{6} & \frac{A}{6} & 0 \\ \frac{a_2}{6} & 0 & \frac{A}{6} \end{bmatrix}; k_{32}^{1} = \begin{bmatrix} \frac{(a_3a_2 + b_3b_2)}{4A} & -\frac{b_3}{6} & \frac{a_3}{6} \\ -\frac{b_2}{6} & \frac{A}{12} & 0 \\ \frac{a_2}{6} & 0 & \frac{A}{12} \end{bmatrix}$$

okay, so the various elements are shown here it requires some manipulation so if you do that you will get this.

$$k_{33}^{s} = \begin{bmatrix} \frac{\left(a_3^2+b_3^2\right)}{4A} & -\frac{b_3}{6} & \frac{a_3}{6} \\ -\frac{b_3}{6} & \frac{A}{6} & 0 \\ \frac{a_3}{6} & 0 & \frac{A}{6} \end{bmatrix}$$

$$k_e = k^f + k^s$$

Constant force

$$\{f\}_e = \frac{f_z A}{3} \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$

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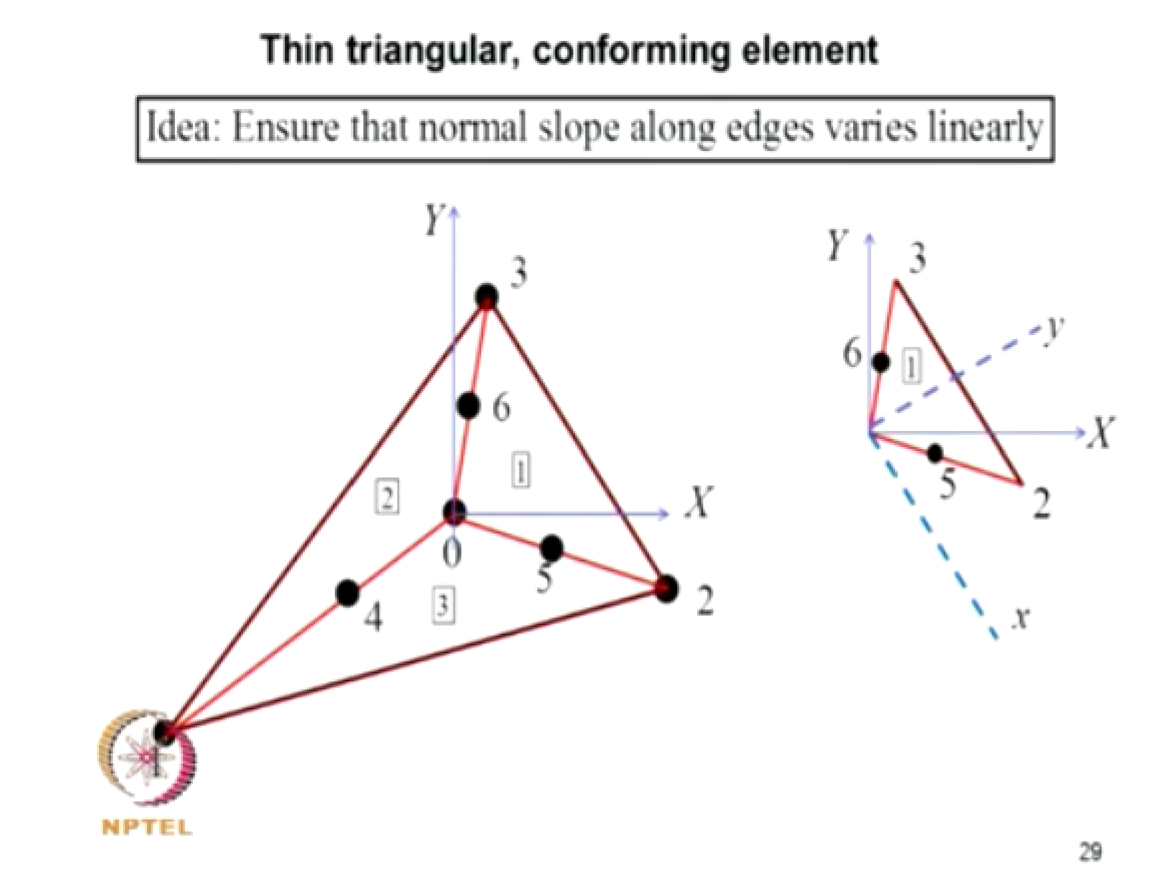
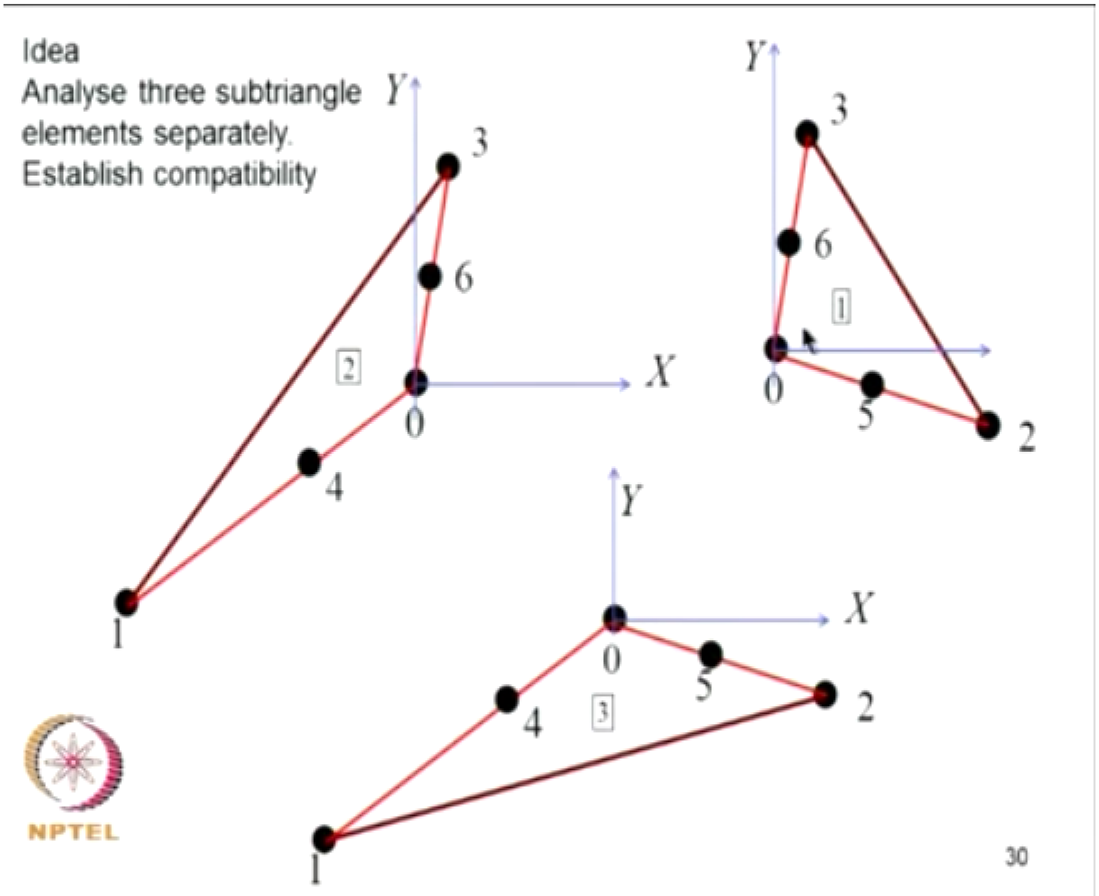
Now KE after evaluating this contribution is taken as sum of lecture contribution and shear contribution, so then the constant force suppose the element is carrying constant force, the equivalent nodal forces can be evaluated as shown here using principle of virtual, so we can show that this element a conforming element as you can see the variation of the field variables along the edges or straight lines so it will ensure the required continuity conditions to be satisfied by the all the 3 variables, that was for a thick element, thick triangular element.

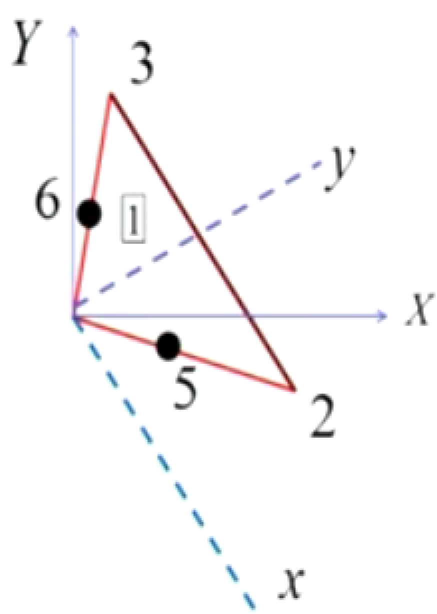


Now the thin triangular element that we formulated was non-conforming, so is there any way we can develop a thin triangular conforming element, now the idea here is the way we develop this is we will ensure that normal slope along the edges vary linearly it was the normal slope along the edge which was not satisfying the required continuity conditions in the earlier model so we want to address that by doing this, so what we do is we consider this triangle element 1, 2, 3 then introduce another node within the element at the centroid for example and construct 3 elements written as 1, 2, 3, so if you consider element 1 we will analyze these 3 elements separately, for element 1 there will be a global coordinate system and for each one of these element there will be a local coordinate system, so this is as shown here, this X-axis is taken to be parallel to edge 2, 3, so that this Y-axis is normal to the line edge 2, 3, so the idea here is we divide this plate into, this triangular plate into, plate element into 3 sub triangles 1, 2, 3 the



numbering is opposing node 1 I have element 1, opposing note 3 I have element 3, and similarly opposing node 2 I have element 2, so if you now consider these 3 elements separately the basic idea is we perform the analysis of these 3 elements separately and establish compatibility conditions. So obviously the nodal degrees of freedom at 1, 2 and 3 must match, so this 3 must be same as this 3, this 1 must be same as this 1, 2 much be same as this, similarly O, 0 at the 3 places should be the same.

Now each element will have 9 degrees of freedom, please note that 4 and 6 are not nodes, they are certain intermediate points, the nodes are 1, 0, and 3 here, and 1, 0, and 2 here, and 2, 0, and 3 here, so each one will have 9 degrees of freedom, so when we assemble this there will be 27 degrees of freedom, now by establishing the compatibility conditions at 1, 2, 3 and at 0, we will be able to derive 15 equations, now we have from 27 equations we should arrive at 9 equation, so 18 equations need to be eliminated, 15 are obtained by considering compatibility at 1, 2, 3, and 0, the remaining 3 are obtained by demanding that the normal slope along edge 0, 1 at point 4 matches with the normal slope along edge 0, 1 at 4 for element 3 and 2, and similarly the normal slope at 5 matches for elements 1 and 3, and normal slope at 6 matches for elements 1 and 2, so that 18 equations will derive and eliminate from the 27 degrees of freedom, 18 degrees of freedom, and get a 9 x 9 stiffness and mass matrices. Now the one of the detail, matter of detail that we need to consider is when we study each of these elements, we will be considering a local coordinates system which is specific to each of the individuals, and whatever answer we get for each of the element before we can establish this compatibility we should transform them to the global coordinate system, then only we will be able to do this.



$$w = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 xy^2 + \alpha_9 y^3$$
$$= \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & xy^2 & y^3 \end{bmatrix} \{\alpha\}$$
$$= \left[P(x,y) \right] \{\alpha\}$$

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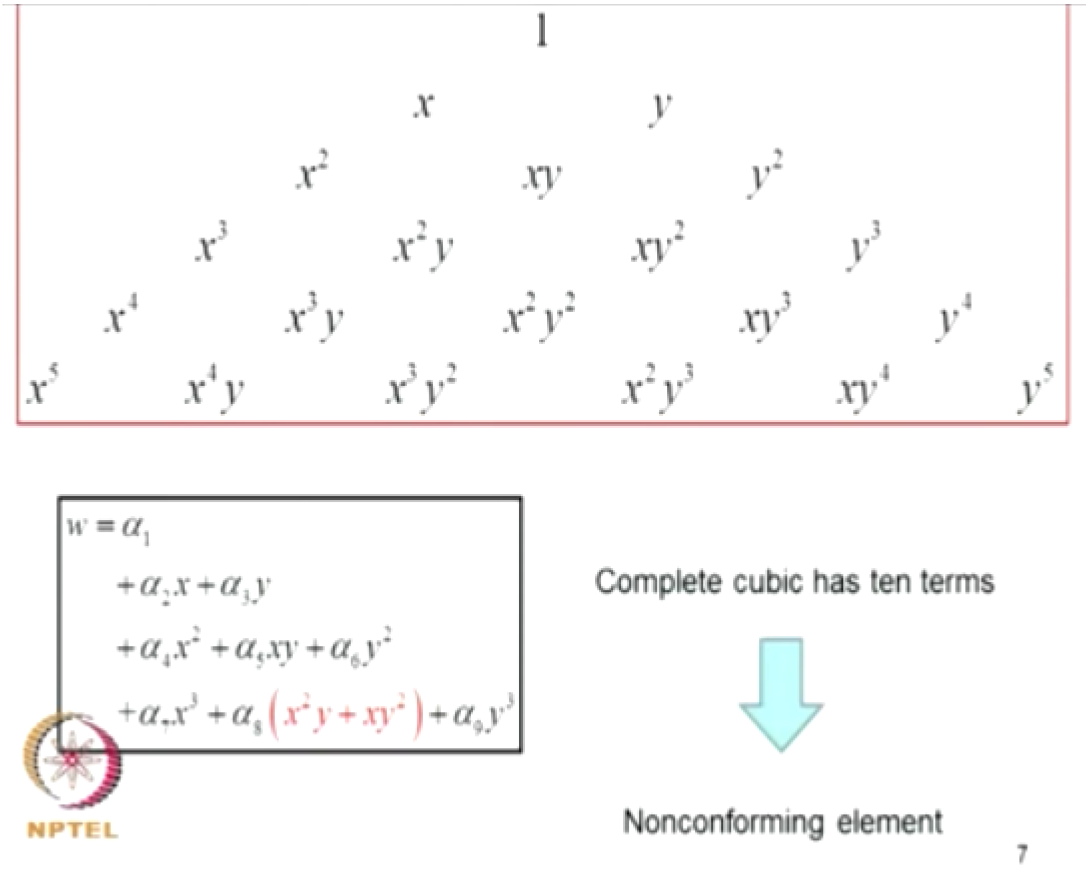
So now we will consider one of these elements, so the element 1 which has edges 0, 2, 3 and this is X axis, and this is the Y axis. Now what we do is we need 9 times in our expansion so



here what we do is we take if you go back to the Pascal's triangle we need 9 terms, so 1, 2, 3, 4, 5, 6 then what we do is 7, 8 and 9, we will eliminate, we will not include this term, okay, why is that is done? So we have to see that, the term we include is 1, XY, X square, XY, Y square, X cube, XY Square, and Y cube, the X square Y term is not included, we will come to the reason for that, we can now rewrite this in the form P into alpha where P is this matrix of these functions, and alpha is the vector of these 9 generalized coordinates.

$$w = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 xy^2 + \alpha_9 y^3$$

$$\frac{\partial w}{\partial y} = \alpha_3 + \alpha_5 x + 2\alpha_6 y + 2\alpha_8 xy + 3\alpha_9 y^2$$

Along edge 2-3, $y = c \Rightarrow$

$$\frac{\partial w}{\partial y} = \alpha_3 + \alpha_5 x + 2\alpha_6 c + 2\alpha_8 xc + 3\alpha_9 c^2$$

$$= \left(\alpha_3 + 2\alpha_6 c + 3\alpha_9 c^2\right) + 2\alpha_8 cx$$

$\Rightarrow \frac{\partial w}{\partial y}$ varies linearly in x

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Now W is given by this I want to compute now dou W/dou Y this is given by this function, so along edge 2, 3, see you should see the way the coordinate system is configured we are having X axis along, X axis here is parallel to edge 2, 3, so Y equal to constant is will be lines parallel to this edge, so if you consider Y equal to constant we will get, wherever there is Y I will get this which is dou W/dou Y, I am looking at the normal slope to the edge so I get this as a linear function, so dou W/dou Y varies linearly in X, so that ensures that the continuity conditions on

$$w^{(1)} = \left[P^{(1)}(x,y)\right]\left\{\alpha^{(1)}\right\}$$

$$T_e^{(1)} = \frac{1}{2}\left\{\dot{\alpha}^{(1)}\right\}^t\left[\bar{M}^{(1)}\right]\left\{\dot{\alpha}^{(1)}\right\}$$

$$\left[\bar{M}^{(1)}\right] = \int_{A_1} \rho h \left[P^{(1)}(x,y)\right]^t\left[P^{(1)}(x,y)\right] dA_1$$

Repeat this procedure for the other two elements and evaluate $\left[\bar{M}^{(2)}\right] \& \left[\bar{M}^{(3)}\right]$.

$$T_e = \frac{1}{2}\{\dot{\alpha}\}^t\left[\bar{M}\right]\{\dot{\alpha}\}$$

$$\bar{M} = \begin{bmatrix} \bar{M}^{(1)} & 0 & 0 \\ 0 & \bar{M}^{(2)} & 0 \\ 0 & 0 & \bar{M}^{(3)} \end{bmatrix}; \{\alpha\} = \begin{Bmatrix} \alpha^{(1)} \\ \alpha^{(2)} \\ \alpha^{(3)} \end{Bmatrix}$$

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normal slope will be satisfied, right in a linear thing there will be 2 unknowns and we have 2 nodes, so that will depend on that so it is quite satisfying. Now that is the reason why we are taking function in this form, so the strategy is to ensure that the normal slope varies linearly along the edge which enables us to achieve conformity.

Now we will now consider elements one by one, for the first element I will now add a superscript W1 is given by this, kinetic energy is given by this, and mass matrix we evaluate using this, this can be done in an exact manner because these are integrands or simple polynomials we can evaluate that. Now we can repeat this for the other 2 elements also in its own local coordinates okay, each time I get mass matrix the bar indicates the mass matrix in its own local coordinates, now so the element on the other hand, the element kinetic energy is given by this, now this M bar I will simply assemble in this form I will have to put now constraint before I do that I am simply defining M bar in this manner, this alpha, alpha A is the generalized coordinates for each element there are 9 of them, so this will be a 27 x 1 matrix, and this will be a 27 x 27 matrix each one being a 9 x 9 matrix, so now here at the element level

$$T_e = \frac{1}{2}\{\dot{\alpha}\}^T\left[\bar{M}\right]\{\dot{\alpha}\}$$

$\{\alpha\}$ has 27 elements. The finite element (123) itself would have 9 dofs.
We need to apply 18 constraints to ensure internal compatibility between triangles.
This needs to be done such that the resulting dofs would be made up of nodal displacements at nodes 1, 2, and 3.

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alpha has 27 elements, the element 1, 2, 3 what I mean is the element that is the finite element 0, 2, 3 not 1, 2, 3 itself would have 9 degrees of freedom, we need to apply 18 constraints to ensure internal compatibility between the triangles this needs to be done such that the resulting degrees of freedom would be made up of nodal displacements at nodes 1, 2, and 3, okay so that is how we have first, that means we are now talking in the language of alpha that has to be translated into the language of nodal degrees of freedom, and the elimination should be done such that in the end what remains are the nodal degrees of freedom at nodes 1, 2, and 3.

For subtriangle 1, we have

$$\begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & xy^2 & y^3 \\ 0 & 0 & 1 & 0 & x & 2y & 0 & 2xy & 3y^2 \\ 0 & -1 & 0 & -2x & -y & 0 & -3x^2 & -y^2 & 0 \end{bmatrix} \left\{ \alpha^{(1)} \right\}$$

At node 2 (that is, at (x_2, y_2) in the local coodinate system) we have

$$\begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} (x_2, y_2) = \left\{ \overline{w}^{(1)} \right\}_2 = \left[\overline{A}^{(1)} \right]_2 \left\{ \alpha^{(1)} \right\}$$

$$\left[\overline{A}^{(1)} \right]_2 = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & xy^2 & y^3 \\ 0 & 0 & 1 & 0 & x & 2y & 0 & 2xy & 3y^2 \\ 0 & -1 & 0 & -2x & -y & 0 & -3x^2 & -y^2 & 0 \end{bmatrix}_{\text{Evaluated at } (x_2, y_2)}$$

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So now for sub triangle 1 the relationship between nodal degrees of freedom and alphas are given by this, that means, at node 2 that is at X2, Y2 in the local coordinate system we have W, theta X, theta Y, at X2, Y2 is given as W bar of 1 at node 2 I write in this form, where this A bar 1 subscript 2 is this matrix.

In the global coordinate system we have

$$\begin{bmatrix} w \\ \theta_X \\ \theta_Y \end{bmatrix}_2 = \left\{ w^{(1)} \right\}_2 = \left[L_2^{(1)} \right] \left[\bar{A}^{(1)} \right]_2 \left\{ \alpha^{(1)} \right\} = \left[A^{(1)} \right] \left\{ \alpha^{(1)} \right\}$$

$\left[L_2^{(1)} \right]$ = coordinate transformation matrix

At the common nodes of the three triangles, we have

$$\left.\begin{aligned} \left\{ w^{(3)} \right\}_1 &= \left\{ w^{(2)} \right\}_1 \\ \left\{ w^{(1)} \right\}_2 &= \left\{ w^{(3)} \right\}_2 \\ \left\{ w^{(2)} \right\}_3 &= \left\{ w^{(1)} \right\}_3 \\ \left\{ w^{(2)} \right\}_0 &= \left\{ w^{(3)} \right\}_0 \\ \left\{ w^{(1)} \right\}_0 &= \left\{ w^{(2)} \right\}_0 \end{aligned}\right\} \text{These provide 15 constraint equations}$$

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In the global coordinate system now we have to use the transformation matrix L2, that we just now saw in the development of the previous element we need not get into that again, the same transformation matrix is used and we get W for the first element at node 2 is related to the nodal degrees of freedom through this matrix for the first element, so where L2 of 1 is the coordinate transformation matrix for element 1. Now at the common nodes of the 3 triangles W 3 at 1 must be W2 of 1, so that is W 3 viewed from element 1 must be W2 of this, okay that is 1, so similarly I get these conditions by demanding at 1, 2, 3, the nodes 1, 2, 3, viewed from elements 1, 2, 3 there is a compatibility of deformation. And similarly at 0 there should be elemental compatibility and would force these vectors to be equal, now this provides 15 constraint equations. So from 27 we can eliminate 15, we need as I already said 3 more need to be eliminated, so compatibility of normal slopes along the interior edges of the 3 sub triangle

Compatibility of normal slopes along the interior edges of the three subtriangles provides additional 3 equations.

These are established by equating the normal slope at midpoints 4,5,6.

Consider subtriangle 1 and midpoint 5.

Let s= direction from 0 to the node 2.

$$\left(\frac{\partial w^{(1)}}{\partial n}\right)_5 = \begin{bmatrix} 0 & \cos(s,x) & \cos(s,y) \end{bmatrix}_5 \left\{ w^{(1)} \right\}_5$$

$$= \begin{bmatrix} 0 & \cos(s,x) & \cos(s,y) \end{bmatrix}_5 \left\{ w^{(1)} \right\}_5$$

$$= \begin{bmatrix} 0 & \cos(s,x) & \cos(s,y) \end{bmatrix}_5 \left[\bar{A}^{(1)} \right]_5 \left\{ \alpha^{(1)} \right\}$$

$$= \left[A^{(1)} \right]_5 \left\{ \alpha^{(1)} \right\}$$

provides the additional 3 equations, these are established by equating the normal slope at mid points 4, 5 and 6, so now consider sub triangle 1 and midpoint 5, let S be the direction from 0 to the node 2, so dou W/dou N for the first element evaluated at node 5, we can show that this is given in terms of this transformation, so this for W1 of 5, I can use in terms of this A bar matrix that we have shown just now, in terms of generalized coordinates I get this as this, so this dou W/dou N for element 1 at node 5 is given by this.

Compatibility of normal slopes between the three subtriangles leads to

$$\left.\begin{aligned}\left(\frac{\partial w^{(3)}}{\partial n}\right)_4 &= \left(\frac{\partial w^{(2)}}{\partial n}\right)_4\\ \left(\frac{\partial w^{(1)}}{\partial n}\right)_5 &= \left(\frac{\partial w^{(3)}}{\partial n}\right)_5\\ \left(\frac{\partial w^{(2)}}{\partial n}\right)_6 &= \left(\frac{\partial w^{(1)}}{\partial n}\right)_6\end{aligned}\right\}\text{These provide 3 constraint equations.}$$

We will now be left with 9 independent values in $\{\alpha\} = \begin{Bmatrix}\alpha^{(1)}\\ \alpha^{(2)}\\ \alpha^{(3)}\end{Bmatrix}$

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Now similar quantities can be computed for other 2 elements also and at 1, 2, 3 for elements 1, 2, 3 at nodes 4, 5, 6 see I am talking about nodes 4, 5, and 6 and the normal slopes viewed from element 2, element 3, this way and this way, and this way, and this way, so this slope, this slope and this slope, now by equating that we get 3 more constraint equations. So now we have to assemble all of this and achieve the reduction from 27 degrees of freedom to 9 degrees of freedom, so we will now be left with 9 independent values in alpha, alpha 1, alpha 2, alpha 3, out of this 27 variables 9 will be left, these can be determined in terms of nodal degrees of

These can be determined in terms of nodal dofs by using the relations

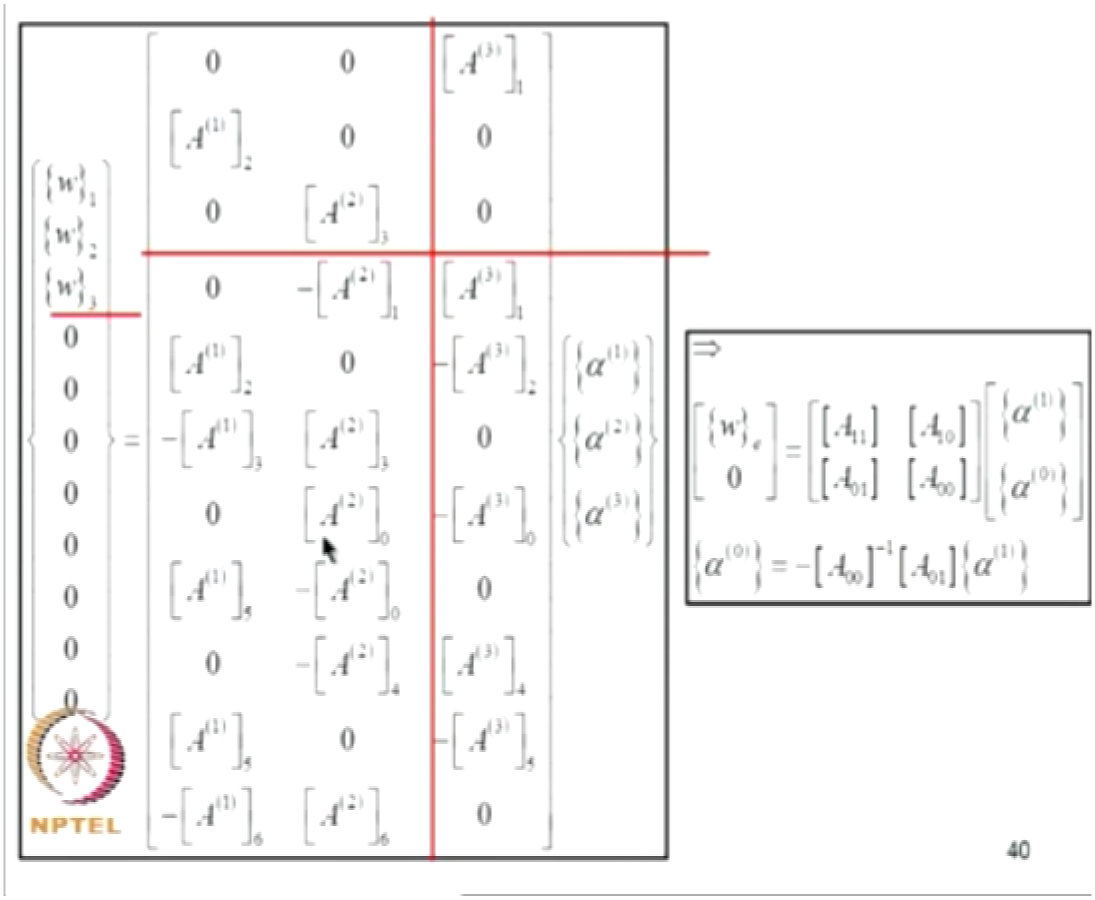
$$\{w\}_1 = \{w^{(3)}\}_1$$

$$\{w\}_2 = \{w^{(1)}\}_2$$

$$\{w\}_3 = \{w^{(2)}\}_3$$

NPTEL

freedom by using the relations W, for 1, 2, 3 this is for the element 1, 2, 3 whereas these are for the sub triangles, so these numbering equivalence is needed, so once that is done we can



assemble now all the required information W at node 1, W at node 2, W at node 3, further triangular element 1, 2, 3 in which we are primarily interested is related to these alphas through these matrices, so we are now basically putting all these equations together this, this, and these equations, so they assume this somewhat involved form it is straight forward you have to pay attention and you know get to the details, if you sit with pen and paper you will be able to follow this.

Now after getting this relation we partition this matrix as shown through these red lines here and rewrite this set of equations in this form, from this we can eliminate alpha(0), and alpha(0) is a combination of this alpha 2 and alpha 3, that we can write in terms of alpha 1 through this relation, so this is where the elimination is now taking place and consequently I get alpha

$$\begin{bmatrix} \{w\}_e \\ 0 \end{bmatrix} = \begin{bmatrix} [A_{11}] & [A_{10}] \\ [A_{01}] & [A_{00}] \end{bmatrix} \begin{bmatrix} \{\alpha^{(1)}\} \\ \{\alpha^{(0)}\} \end{bmatrix}$$

$$\{\alpha^{(0)}\} = -[A_{00}]^{-1}[A_{01}]\{\alpha^{(1)}\}$$

$$\{w\}_e = [A_{11}]\{\alpha^{(1)}\} + [A_{10}]\{\alpha^{(0)}\}$$

$$= [A_{11}]\{\alpha^{(1)}\} - [A_{10}][A_{00}]^{-1}[A_{01}]\{\alpha^{(1)}\}$$

$$= \left[[A_{11}] - [A_{10}][A_{00}]^{-1}[A_{01}]\right]\{\alpha^{(1)}\}$$

$$= [\bar{A}]\{\alpha^{(1)}\}$$

$$\{\alpha^{(1)}\} = [\bar{A}]^{-1}\{w\}_e$$

$$\begin{bmatrix} \{\alpha^{(1)}\} \\ \{\alpha^{(0)}\} \end{bmatrix} = \begin{bmatrix} [\bar{A}]^{-1} \\ -[A_{00}]^{-1}[A_{01}][\bar{A}]^{-1} \end{bmatrix} \{w\}_e$$

$$\begin{bmatrix} \{\alpha^{(1)}\} \\ \{\alpha^{(0)}\} \end{bmatrix} = [\bar{\bar{A}}]\{w\}_e$$

$$m_e = [\bar{\bar{A}}]^t [\bar{M}]_e [\bar{\bar{A}}]$$

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naught to be this, and WE therefore will be, from the first equation A11, alpha 1 + A10, alpha naught, for alpha naught I will use the second equation I get this, so the element degrees of freedom and the alphas are now related through this, W is related to this through this equation, so this is this alpha 1 therefore in terms of nodal degrees of freedom is given by this, so going back to the transformation that relates alphas to WE, we derive this matrix that relates these 2 quantities and that is the one that we are looking for so that I call it as A double bar WE. So now when I evaluate the mass matrix I have to use this transformation, okay, essentially we are eliminating the, enforcing those compatibility relations in the nodal degrees of freedom and the slopes normal to the edges and correctly assembling all the relations and this is how we get the mass matrix.

$$\begin{bmatrix} \{\alpha^{(1)}\} \\ \{\alpha^{(0)}\} \end{bmatrix} = \left[\bar{\bar{A}}\right]\{w\}_e$$

$$k_e = \left[\bar{\bar{A}}\right]^t \left[\bar{K}\right]_e \left[\bar{\bar{A}}\right]$$

$$\left[\bar{K}\right]_e = \begin{bmatrix} \left[\bar{k}^{(1)}\right] & 0 & 0 \\ 0 & \left[\bar{k}^{(2)}\right] & 0 \\ 0 & 0 & \left[\bar{k}^{(3)}\right] \end{bmatrix}$$

$$\left[\bar{k}^{(1)}\right] = \int_{A_1} \frac{h^3}{12} \left[\bar{B}^{(1)}\right]^t [D] \left[\bar{B}^{(1)}\right] dA$$

$$\left[\bar{B}\right] = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 6x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2x & 6y \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 4y & 0 \end{bmatrix}$$

$$\{f\}_e = \left[\bar{\bar{A}}\right]^t \{\bar{f}\}_e$$

$$\{\bar{f}\}_e = \begin{Bmatrix} \{\bar{f}^{(1)}\} \\ \{\bar{f}^{(2)}\} \\ \{\bar{f}^{(3)}\} \end{Bmatrix}$$

$$\{\bar{f}^{(1)}\} = \int_{A_1} \left[P^{(1)}\right]^t f_z dA$$

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How about stiffness matrix? The same transformation works, so KE is given by A double bar transpose into this matrix into this, and the elements of this can be evaluated for the shape functions that we have used, that is given by this where this strain matrix B is can be shown to be given by this. Now similarly using the virtual work principle and we can again show that equivalent nodal forces are given by this, so this I leave it as an exercise, so the main idea here is basically captured in this view graph so what we are doing is this element we are dividing into 3 sub triangles as shown here, and analyzing these 3 sub triangles separately and while analyzing each of these triangles we are enforcing, we are making a particular choice for this representation so that the slope dou W/dou Y along the edges will be linear function, so that is one important issue and then what we do is when we analyze these 3 triangle separately there will be 27 degrees of freedom and by enforcing compatibility at nodes 1, 3 and 0 among these elements we get 15 equations, and we generate 3 more equations by considering equivalence of normal slope at points 4 between elements 3 and 4, at point 5 between elements 1 and 3, and at point 6 between elements 1 and 2, so that gives 18 equations and the remaining manipulations are aimed towards implementing this you know elimination process and deriving the correct mass and stiffness matrices, so this is a 3 noded triangular conforming element.

Nonconforming thin rectangular element

$$w = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3 + \alpha_{11} x^3 y + \alpha_{12} xy^3$$

Conforming thin rectangular element

$$w = \sum_{j=1}^{4} \left[w_j f_j(\xi) f_j(\eta) + b^2 \theta_{xj} f_j(\xi) g_j(\eta) - a^2 \theta_{yj} g_j(\xi) f_j(\eta) \right]$$

Thick rectangular conforming element

$$w = \sum_{j=1}^{4} N_j w_j; \quad \theta_x = \sum_{j=1}^{4} N_j \theta_{xj}; \quad \theta_y = \sum_{j=1}^{4} N_j \theta_{yj}; N_j(\xi,\eta) = \frac{1}{4}(1+\xi\xi_j)(1+\eta\eta_j); \; j = 1,2,3,4$$

Thin, triangular, nonconforming element

$$w = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 (x^2 y + xy^2) + \alpha_9 y^3$$

Thick, triangular, conforming element: $w = \sum_{i=1}^{3} L_i w_i, \quad \theta_x = \sum_{i=1}^{3} L_i \theta_{xi}, \quad \theta_x = \sum_{i=1}^{3} L_i \theta_{yi}$

Thin, triangular, conforming element: linear variation of normal slope along edges

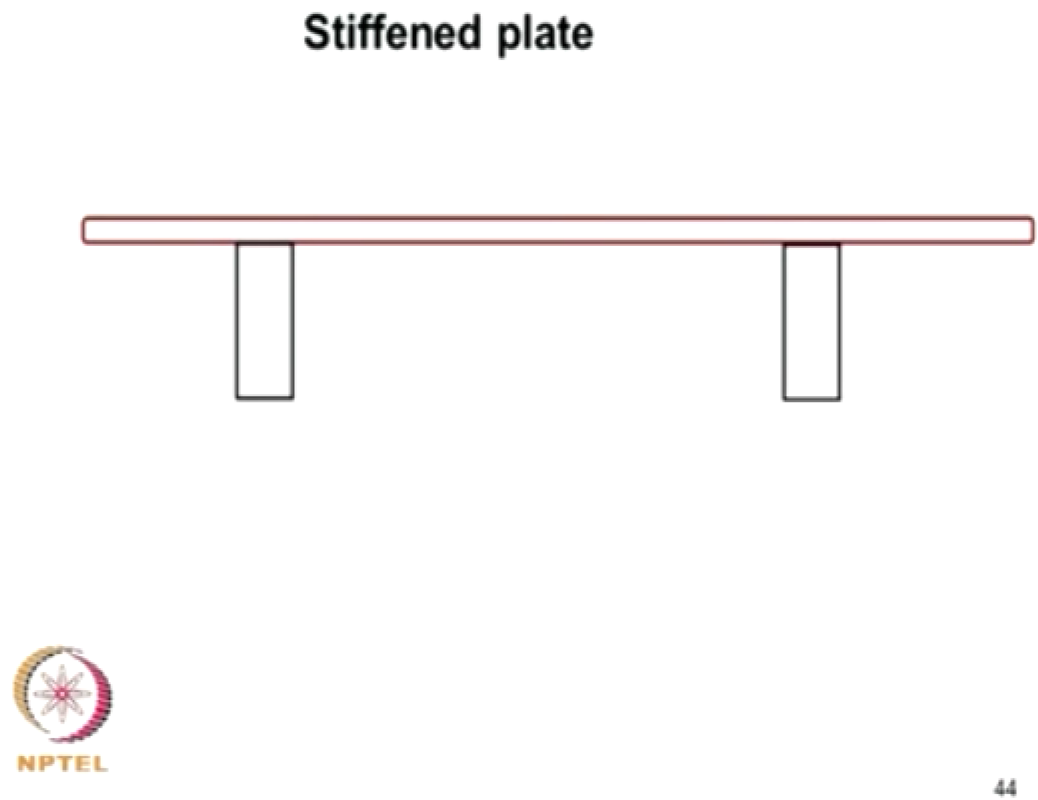
So in summary we have now developed elements for rectangular geometry and for triangular geometry, we have considered thin plate and thick plate, and for each one we have considered, we have developed a non-conforming element and a conforming element, so in the rectangular geometry we started by taking 4 noded rectangle and therefore the element had 12 degrees of freedom and we selected this as the trial function and we showed that here the normal slopes along the edges won't match therefore this becomes a non-conforming element.

Now to get a conforming thin rectangular element this is for thin, the next strategy we used was we constructed the shape functions by as products of beam trial functions, there as I pointed out the beginning of the lecture there was a need to introduce an additional degree of freedom namely dou square W/dou X dou Y at every node so that although the element becomes conforming because of that introduction of that additional degree of freedom the use of this element in conjunction with other structural elements becomes unwieldy, so that is one problem with this element, but as already as I happen to mention in the previous class that can be eliminated by approximating dou square W/dou X dou Y in terms of theta X and theta Y but if you do that the element again becomes a non-conforming element.

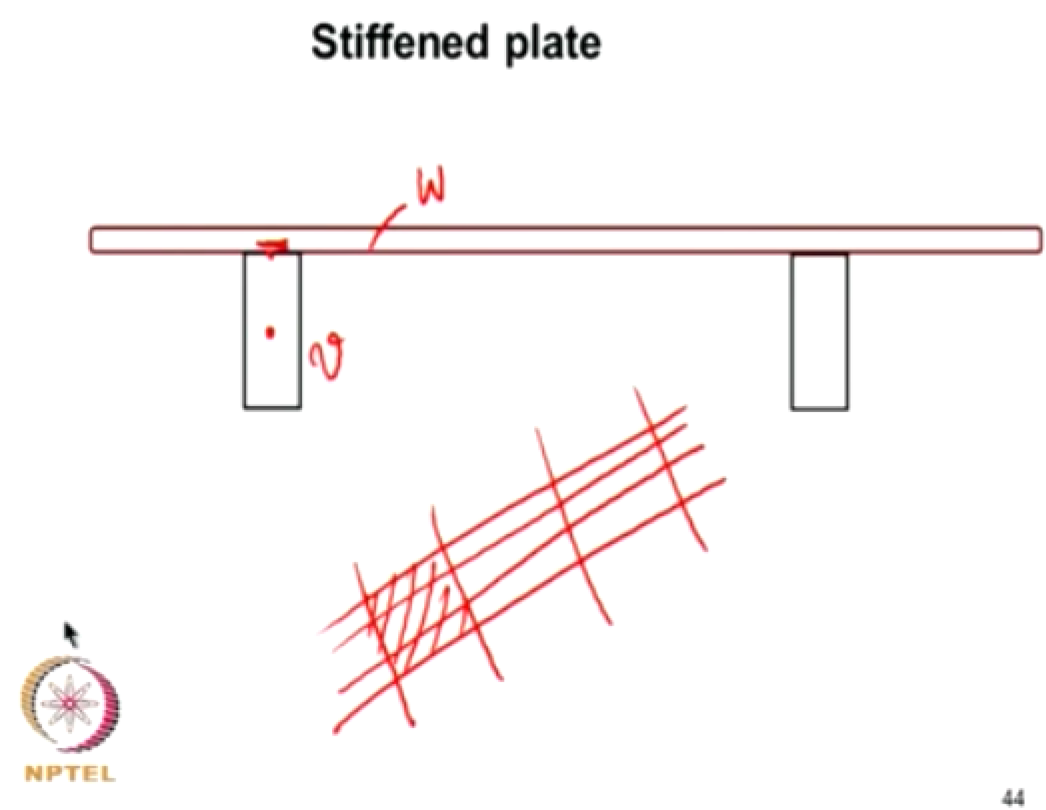
Now when we came to thick rectangular conforming elements, the field variables were in the variational equation W, theta X, and theta Y, highest derivative was 1, so in a rectangular element with 4 nodes these were the representation that we used where this trial functions were quadratic they had cross terms XI and eta, so this element we developed and showed that it is a conforming element.

Now in a thin triangular non-conforming element we started by considering this trial function actually a complete cubic polynomial would have 10 terms, but we need only 9 terms so what

we did was we combine these X square Y and XY square terms and associated with those terms a common generalize coordinate, this again had this problem of being non-conforming because of incompatibility of normal slope along edges.
Next the thick triangular conforming element we use area coordinate system to represent this, like in rectangular conforming element this was also shown to be a conforming element.
Finally we developed a thin triangular conforming element by enforcing linear variation of normal slope along edges and dividing the triangular element into 3 sub triangles and analyzing them separately and enforcing compatibility of nodal deformations and normal slope at the midpoints of the edges of the sub triangles, so this is what we have done.

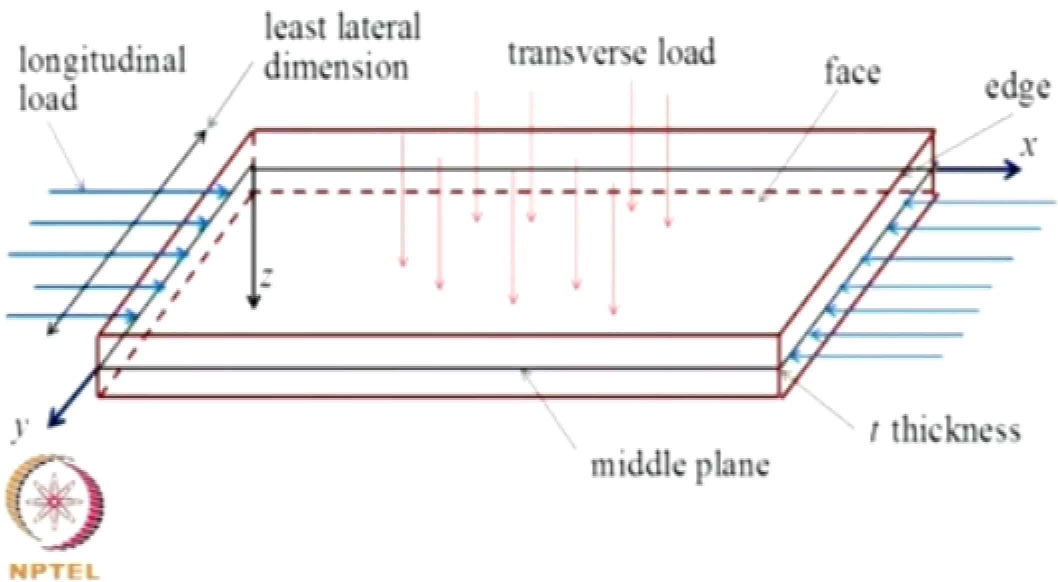


Now then we've forward now is we have to consider problems of stiffened plates, so these arise in bridge decks, so this is the slab and these are the girders and the girders scan, so if we take a planar view the girder, cross girders can be like this, and the longitudinal girders can be like this, so this rectangular element itself is a plate element so we can see that the plate element gets stiffened by the presence of these beams, so the centroidal axis of the beam will be here. the mid plane of the slab will be here, so the deformation of the beam and slab should conform to each other so when we are developing, for example a thin plate theory, we will have W as a



variable here, and in a beam equation we will have V, and this V and W we should ensure that they are constrained by the fact that the way the beam deforms along this line is compatible with the way W deforms, so we will see this in the one of the future classes.

Plate bending element



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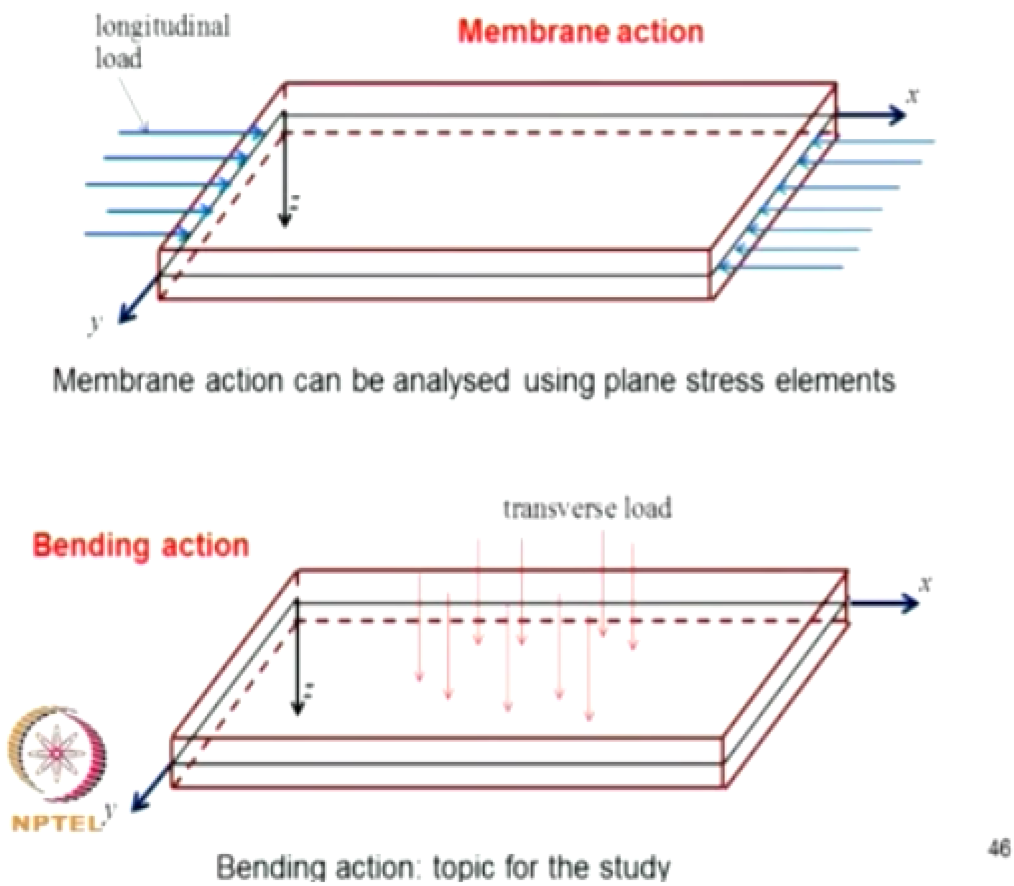
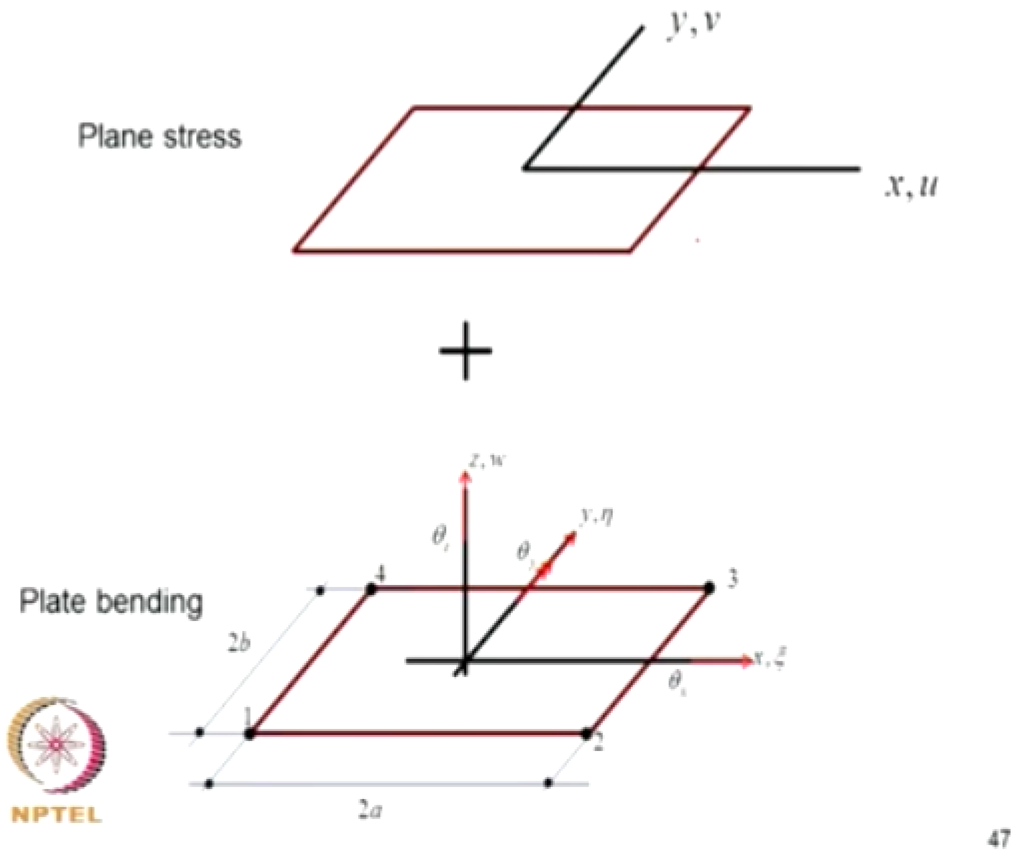
Next what we will consider is this plate bending element we have been considering only the action due to transverse loads, now in problems of shells there can be simultaneous presence of axial loads as well as transverse loads, so the shell structures, the element need not be flat initially, and that is one of the assumptions if you recall we made when we were dealing with plates, but if we agree that a curved surface can be represented by using flat surface, flat elements like this we can still use this theory that we have developed, these are called facet elements so there the issue would be we should consider the in-plane action in conjunction with the transverse action, so assuming that the deformations resulting from each one of them is



small they can be treated separately and we can consider this membrane action and the so called bending action, plate action separately and we can analyze this using plane stress model and this using one of the plate bending model that we have discussed.

Now obviously there will be many questions about conformity and issues like that, so when we combine we have to ensure that the required desirable features are imported into the built-up element, so we combine the plane stress element where we have along X and Y, U and V are



the field variables, and in plate bending we have for example in the thin plate theory we have W, theta X, and theta Y, we deal with that, so there will be additional questions on treatment of this theta Z is called drilling degree of freedom that needs to be studied separately, so we will try to see how to combine these 2 for flat elements, and maybe if time permits will also consider curved elements subsequently.

So in the next class what we will do is we will try to work out few numerical examples, so certain class of problems like rectangular plates which are all-round simply supported have exact solutions, so we can you know check the accuracy of some of these formulations against the exact solutions, and then we can see the influence of boundaries and so on and so forth, so we can solve a set of you know examples to illustrate the various features of this triangular and rectangular elements, and I will also develop the theory for the stiffened plates and illustrate them with some examples. So with that our discussion on plate and shell elements would be completed, so with that we will conclude the present lecture.

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