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<u>Course Title</u>
Finite element method for structural dynamic
And stability analyses
Lecture – 23
Plate bending elements
(continued)

By
Prof. CS Manohar
Professor
Department of Civil Engineering
Indian Institute of Science,
Bangalore-560 012
India

Finite element method for structural dynamic and stability analyses

Module-8

Plate bending and shell elements

Lecture-23 Plate bending elements

Prof C S Manohar
Department of Civil Engineering
IISc, Bangalore 560 012 India

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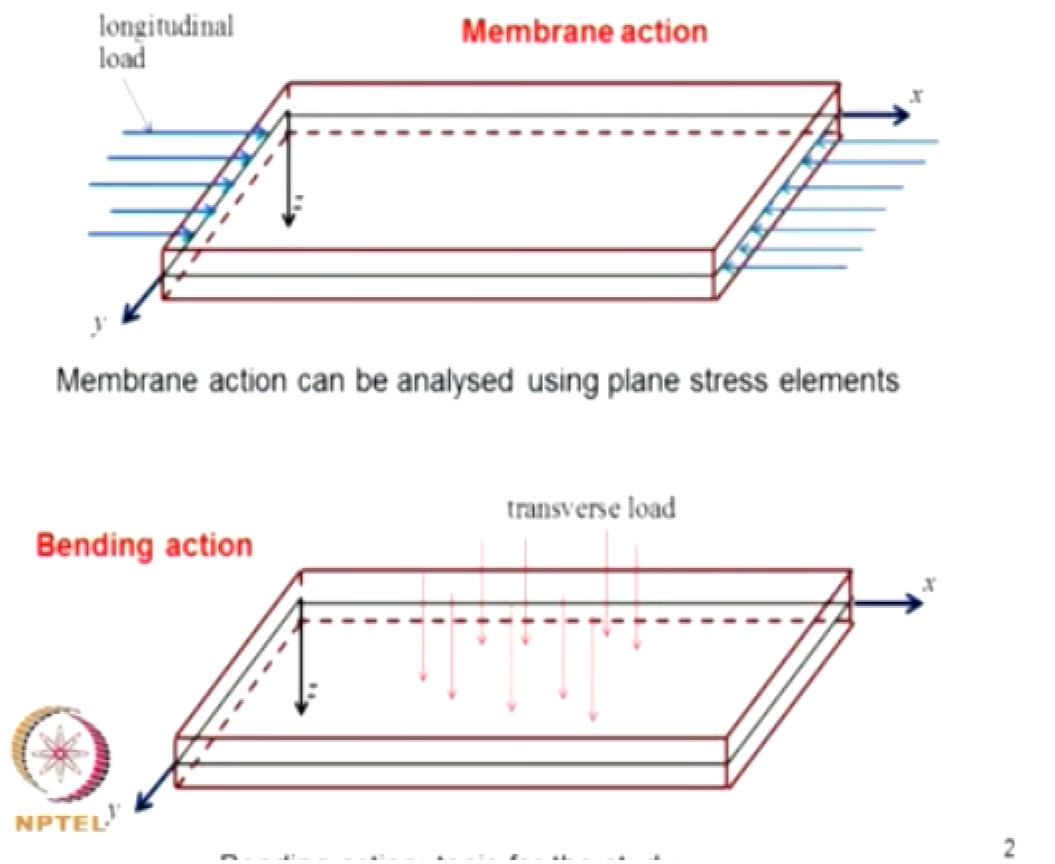
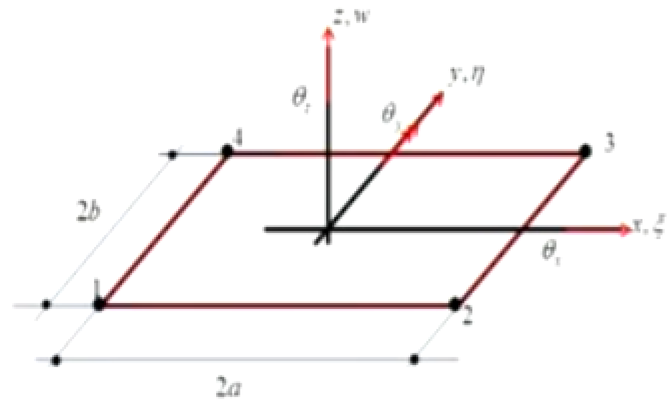


We have been discussing analysis of plate bending elements, in the last class we started discussing about the basic formulation, so we considered a thin structural member as shown here where this thickness is small compared to the least lateral dimension, and that is the essential feature of a plate and it can carry load either in its own plane or transverse to its plane, so structures carrying, plates carrying forces in its own plane are said to behave like a membrane, whereas if the structure carries loads transverse to its middle surface as shown here the plate would bend and this is known as bending action.

So right now we are focusing on analysis of the bending action, if the membrane action has to be studied the plane stress elements that we have developed earlier can be employed.

$$\xi = \frac{x}{a} \ \& \ \eta = \frac{y}{b}$$



$$\sigma_{xx} = \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right)$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}\left(\varepsilon_{yy} + \nu\varepsilon_{xx}\right)$$

$$\sigma_{xy} = 2G\varepsilon_{xy}$$

$$\varepsilon = -z\chi = -z\begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}$$

$$V = \frac{1}{2}\int_A \frac{h^3}{12}\chi^t D\chi dA; T = \frac{1}{2}\int_A \rho h\dot{w}^2 dA$$

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Now in the last class we formulated a few details so we considered a 4 noded rectangular element and at each node the degrees of freedom where translation W or slope along X and Y axis and we introduced non dimensional coordinate system XI = X/A and eta = Y/B. Now to derive the strain energy stored in the plate we invoked the Kirchhoff Love assumptions and based on that we derived this stresses, in terms of the strains so there are 3 stress components that enter our formulation, and the strains themselves are given in terms of curvatures as shown here. So the strain energy stored in the plate is obtained in terms of the curvature and the matrix that relates stress and strain, and similarly the kinetic energy is given by this expression, and in the kinetic energy only the displacement component W enters our formulation in this model.

Thin rectangular element

Field variable: $w(x,y,t)$

Order of the highest derivative present in the Lagrangian: 2

Dofs: $w(x,y,t), \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$

$\Rightarrow$ The interpolation functions must consist of complete polynomials of at least order 2.

For all elements:

$$\begin{aligned} w = &\alpha_1 \\ &+\alpha_2 x+\alpha_3 y \\ &+\alpha_4 x^2+\alpha_5 xy+\alpha_6 y^2 \\ &+\text{higher order terms} \end{aligned}$$

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So the field variable was W in our variational formulation, and order of the highest derivative present in the Lagrangian was 2, and the degrees of freedom therefore at any node must be W, dou W/dou X, and dou W/dou Y, and the interpolation function must consist of complete polynomials of at least order 2, so for any choice of elements that one can think of, the first few terms in the trial function should be this, a constant term, linear term in X and Y and quadratic terms, and the higher order terms so this completes 1, 2, 3, 4, 5, 6 remaining 6 terms we have to select depending on the theory that we employ for developing the element.

Thin rectangular element with 4 nodes, 3 dofs/node (Dofs=12)

Field variable: $w(x, y, t)$

Order of the highest derivative present in the Lagrangian: 2

Dofs: $w(x, y, t), \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$

Number of generalized coordinates: 12

$$
\begin{aligned}
w = &\alpha_1 \\
&+\alpha_2 x+\alpha_3 y \\
&+\alpha_4 x^2+\alpha_5 xy+\alpha_6 y^2 \\
&+\alpha_7 x^3+\alpha_8 x^2 y+\alpha_9 xy^2+\alpha_{10} y^3 \\
&+\alpha_{11} x^3 y+\alpha_{12} xy^3
\end{aligned}
$$

So we started discussing this thin rectangular element with 4 nodes, 3 degrees of freedom per node, so that the element has 12 degrees of freedom, the field variable was W and again the number of generalized coordinates needed based on these considerations is 12, so we use this polynomial, so after quadratic terms we use a complete cubic terms and two fourth order terms X cube Y and XY cube, we analyze this problem and in the non-dimensional coordinate system

$$w=\begin{bmatrix}1 & \xi & \eta & \xi^2 & \xi\eta & \eta^2 & \xi^3 & \xi^2\eta & \xi\eta^2 & \eta^3 & \xi^3\eta & \xi\eta^3\end{bmatrix}\{\alpha\}$$

$$\frac{\partial w}{\partial \xi}=\begin{bmatrix}0 & 1 & 0 & 2\xi & \eta & 0 & 3\xi^2 & 2\xi\eta & \eta^2 & 0 & 3\xi^2\eta & \eta^3\end{bmatrix}\{\alpha\}$$

$$\frac{\partial w}{\partial \eta}=\begin{bmatrix}0 & 0 & 1 & 0 & \xi & 2\eta & 0 & \xi^2 & 2\xi\eta & 3\eta^2 & \xi^3 & 3\xi\eta^2\end{bmatrix}\{\alpha\}$$

$$\{\bar{w}_e\}=\begin{bmatrix}w_1 & b\theta_{x1} & a\theta_{y1} & w_2 & b\theta_{x2} & a\theta_{y2} & w_3 & b\theta_{x3} & a\theta_{y3} & w_4 & b\theta_{x4} & a\theta_{y4}\end{bmatrix}^T$$

$$\{\bar{w}_e\}=\left[A_e\right]\{\alpha\}$$

$$w=\begin{bmatrix}N_1(\xi,\eta) & N_2(\xi,\eta) & N_3(\xi,\eta) & N_4(\xi,\eta)\end{bmatrix}\{\bar{w}_e\}$$

$$N_j^T=\begin{bmatrix}\frac{1}{8}(1+\xi\xi_j)(1+\eta\eta_j)(2+\xi\xi_j+\eta\eta_j-\xi^2-\eta^2)\\ \frac{b}{8}(1+\xi\xi_j)(\eta+\eta_j)(\eta^2-1)\\ \frac{a}{8}(\xi+\xi_j)(\xi^2-1)(1+\eta\eta_j)\end{bmatrix}$$

the node we expanded to determine the shape functions we represented the trial function in this form, and since the variational formulation has gradients with respect to X and Y, we evaluated this and at every node there are 4 nodes, at every node these 3 quantities are specified and using that fact we were able to determine the shape functions and were able to represent the vector of nodal displacements W bar E in terms of these shape functions, that field variable is represented in terms of N1, N2, N3, N4 and this vector of nodal coordinates. This N1, N2, N3, N4 NJ-th term has this form.

Now we carried out some analysis of this representation, and we considered the behavior along

Edge 2-3: w & θ_x depend on nodal values $w_2, w_3, \theta_{x2}, \theta_{x3}$.

θ_y along $\xi=1$ depends upon values of w and θ_x at nodes 1,2,3, and 4 as well as θ_y at nodes 2 and 3.

$\Rightarrow$ The element is not a conforming element.

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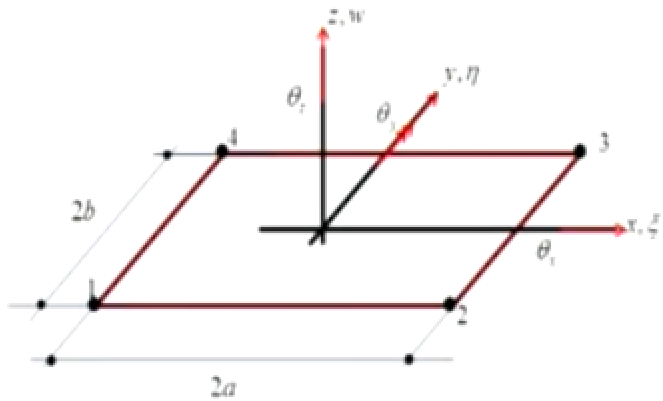
edge 2, 3, the field variable theta X and theta Y, and we were able to observe that along the edge 2, 3, W and theta X depend on nodal values of W2, W3, theta X2, and theta X3, that means along this edge 2, 3, the variable W and theta X depends on nodal values of the field variable and it is a slope with respect to X, at nodes 2 and 3. Now if we have to add 1 more

$$\xi = \frac{x}{a} \;\&\; \eta = \frac{y}{b}$$



$$\sigma_{xx} = \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right)$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}\left(\varepsilon_{yy} + \nu\varepsilon_{xx}\right) \qquad \varepsilon = -z\chi = -z\begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}$$

$$\sigma_{xy} = 2G\varepsilon_{xy}$$

$$V = \frac{1}{2}\int_A \frac{h^3}{12}\chi^t D\chi \, dA; \; T = \frac{1}{2}\int_A \rho h \dot{w}^2 \, dA$$

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element on this side here, again a point that lies on the edge 2, 3, will be shared by these 2 elements, and these nodes also would be shared by the 2 elements, so if the field variable is independent of these nodal values at nodes 1 and 4 then automatically continuity will be satisfied, continuity of the field variable and the appropriate derivative will be satisfied, but we observed that while this condition was satisfied for W and theta X we observed that theta Y which is another field variable along sai = 1 depends upon values of W and theta X at nodes 1, Edge 2-3 : w & θ_x depend on nodal values $w_2, w_3, \theta_{x2}, \theta_{x3}$.

θ_y along $\xi = 1$ depends upon values of w and θ_x at nodes 1,2,3, and 4 as well as θ_y at nodes 2 and 3.

$\Rightarrow$ The element is not a conforming element.

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2, 3, and 4, as well as theta Y at nodes 2 and 3, so this means that theta Y will not be continuous across element edges and this element is not a conforming element, and questions on convergence of solutions obtained using such elements you don't need to be carefully considered.

$$T_e = \frac{1}{2}\{\dot{w}_e\}^t [M]_e \{\dot{w}_e\}$$

$$[M]_e = \int_A \rho h N^t N dA = \rho hab \int_{-1}^{1}\int_{-1}^{1} N^t(\xi,\eta) N(\xi,\eta) d\xi d\eta$$

$$[M]_e = \frac{\rho hab}{6300}\begin{bmatrix} m_{11} & m_{21}^t \\ m_{21} & m_{22} \end{bmatrix}$$

$$m_{11} = \begin{bmatrix} 3454 & & & & & \\ 922b & 320b^2 & & & \text{sym} & \\ -922a & -252ab & 320a^2 & & & \\ 1226 & 398b & -548a & 3454 & & \\ 398b & 160b^2 & -168ab & 922b & 320b^2 & \\ 548a & 168ab & -240a^2 & 922a & 252ab & 320a^2 \end{bmatrix}$$

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But still this element is known to give acceptable results, so in view of that we will complete the formulation, so the kinetic energy is given by W dot E transpose, ME, W dot E, and elements of ME are elements of this integral, so in the non-dimensional coordinate system we obtain this, and since N consists of polynomials product of N transpose N will lead to polynomials, so this elements of mass matrix can be evaluated exactly, so that has indeed been done and we get the mass matrix a partitioned as shown here and this M11, M21, M21, M22 are themselves matrices, although the subscript may mean that it is 1, 1 element of M, it is not so, M11 is a matrix, it is 6 x 6 symmetric matrix with elements as shown here, and similarly M21 and M22 are also available, so if you put now M11, M21 transpose, M21 and M22 we construct the complete mass matrix, so this mass matrix is consistent, mass matrix it is symmetric.

$$
m_{21} = \begin{bmatrix} 394 & 232b & -232a & 1226 & 548b & 398a \\ -232b & -120b^2 & 112ab & -548b & -240b^2 & -168ab \\ 232a & 112ab & -120a^2 & 398a & 168ab & 160a^2 \\ 1226 & 548b & -398a & 394 & 232b & 232a \\ -548b & -240b^2 & 168ab & -232b & -120b^2 & -112ab \\ -398a & -168ab & 160a^2 & -232b & -112ab & -120a^2 \end{bmatrix}
$$

$$
m_{22} = \begin{bmatrix} 3454 & & & & & \\ -992b & 320b^2 & & & \text{sym} & \\ 922a & -252ab & 320a^2 & & & \\ 1226 & -398b & 548a & 3454 & & \\ -398b & 160b^2 & -168ab & -922b & 320b^2 & \\ -548a & 168ab & -240a^2 & 922a & 252ab & 320a^2 \end{bmatrix}
$$

$$
V_e = \frac{1}{2}\{w_e\}^t [K]_e \{w_e\}
$$

$$
[K]_e = \int_A \frac{h^3}{12} B^t D B dA
$$

$$
B = \begin{bmatrix} \dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial^2}{\partial y^2} \\ 2\dfrac{\partial^2}{\partial x \partial y} \end{bmatrix} [N] = \begin{bmatrix} \dfrac{1}{a^2} & \dfrac{\partial^2}{\partial \xi^2} \\ \dfrac{1}{b^2} & \dfrac{\partial^2}{\partial \eta^2} \\ \dfrac{2}{ab} & \dfrac{\partial^2}{\partial \xi \partial \eta} \end{bmatrix} [N(\xi,\eta)]
$$

$$
K_e = \frac{Eh^3}{48(1-\nu^2)ab} \begin{bmatrix} k_{11} & & \text{sym} & \\ k_{12} & k_{22} & & \\ k_{13} & k_{23} & k_{33} & \\ k_{14} & k_{24} & k_{34} & k_{44} \end{bmatrix}
$$

Similarly the strain energy can also be evaluated, this is the formula for the strain energy and this is the element stiffness matrix, so here the strain displacement matrix B, and the D matrix which relates stresses and strain appear here, so we need to evaluate the elements of this and that can be done by considering B is actually given by the action of these operators on N and we can show that this is actually this, in terms of N (sai, eta) and consequently KE can be evaluated in this form, where this K11, 1 2, 1 3, 1 4, etcetera all these are again matrices, so there are 3 x 3 matrices, so the entire matrix is 12 x 12, so these matrices have been evaluated,

$$k_{11} = \begin{bmatrix} 4(\beta^2+\alpha^2)+\frac{2}{5}(7-2\nu) & & \\ 2\left\{2\alpha^2+\frac{1}{5}(1+4\nu)\right\}b & 4\left\{\frac{4}{3}\alpha^2+\frac{4}{15}(1-\nu)\right\}b^2 & \\ 2\left\{-2\beta^2-\frac{1}{5}(1+4\nu)\right\}a & -4\nu ab & 4\left\{\frac{4}{3}\beta^2+\frac{4}{15}(1-\nu)\right\}a^2 \end{bmatrix}$$

$$k_{12} = \begin{bmatrix} -\left\{2(2\beta^2-\alpha^2)+\frac{2}{5}(7-2\nu)\right\} & 2\left\{\alpha^2-\frac{1}{5}(1+4\nu)\right\}b & 2\left\{2\beta^2+\frac{1}{5}(1-\nu)\right\}a \\ 2\left\{\alpha^2-\frac{1}{5}(1+4\nu)\right\}b & 4\left\{\alpha^2-\frac{4}{15}(1-\nu)\right\}b^2 & 0 \\ -2\left\{2\beta^2+\frac{1}{5}(1-\nu)\right\}a & 0 & 4\left\{\frac{2}{3}\beta^2-\frac{1}{15}(1-\nu)\right\}a^2 \end{bmatrix}$$

$$k_{13} = \begin{bmatrix} -\left\{2(\beta^2+\alpha^2)-\frac{2}{5}(7-2\nu)\right\} & 2\left\{-\alpha^2+\frac{1}{5}(1-\nu)\right\}b & 2\left\{\beta^2-\frac{1}{5}(1-\nu)\right\}a \\ 2\left\{\alpha^2+\frac{1}{5}(1-\nu)\right\}b & 4\left\{\frac{2}{3}\alpha^2-\frac{1}{15}(1-\nu)\right\}b^2 & 0 \\ 2\left\{-\beta^2+\frac{1}{5}(1-\nu)\right\}a & 0 & 4\left\{\frac{2}{3}\beta^2-\frac{4}{15}(1-\nu)\right\}a^2 \end{bmatrix}$$

$$k_{14} = \begin{bmatrix} \left\{2(\beta^2-2\alpha^2)-\frac{2}{5}(7-2\nu)\right\} & 2\left\{-2\alpha^2-\frac{1}{5}(1-\nu)\right\}b & 2\left\{-\beta^2+\frac{1}{5}(1+4\nu)\right\}a \\ 2\left\{2\alpha^2+\frac{1}{5}(1-\nu)\right\}b & 4\left\{\frac{2}{3}\alpha^2-\frac{1}{15}(1-\nu)\right\}b^2 & 0 \\ 2\left\{-\beta^2+\frac{1}{5}(1+4\nu)\right\}a & 0 & 4\left\{\frac{2}{3}\beta^2-\frac{4}{15}(1-\nu)\right\}a^2 \end{bmatrix}$$

$$\alpha = \frac{a}{b};\ \beta = \frac{b}{a}$$

$$I_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};\ I_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix};\ I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

here we see 2 parameters alpha and beta, this alpha and beta are ratios A/B and B/A, so we get this stiffness matrix as shown here. This alpha is A/B, and beta B/A in these matrices, so first

$$k_{22} = I_3^t k_{11} I_3$$

$$k_{32} = I_3^t k_{41} I_3 \quad k_{33} = I_1^t k_{11} I_1$$

$$k_{42} = I_3^t k_{31} I_3 \quad k_{43} = I_1^t k_{21} I_1 \quad k_{44} = I_2^t k_{11} I_2$$

Stresses in terms of nodal displacements

$$\sigma = -z[D]\{\chi\} = -z[D][B]\{w_e\}$$

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we have evaluated the elements of this first column like 1 1, 1 2, 1 3, 1 4, so 1 1, 1 2, 1 3, 1 4, and for evaluating the remaining elements we manipulate this 1 1, 2 1, 3 1, 4 1, matrices through these matrices I1, I2, I3, and these details can be worked, I leave it as an exercise for you to carry out these evaluations.

One stiffness matrix is determined, it is determined exactly in this case, the stresses in terms of nodal displacements can be used evaluated by multiplying the strain matrix with the D matrix, strain vector with the D matrix and we get this.

$$k_{22} = I_3^t k_{11} I_3$$

$$k_{32} = I_3^t k_{41} I_3 \quad k_{33} = I_1^t k_{11} I_1$$

$$k_{42} = I_3^t k_{31} I_3 \quad k_{43} = I_1^t k_{21} I_1 \quad k_{44} = I_2^t k_{11} I_2$$

Equivalent nodal forces

$$\delta W_e = \int_A f_z(x,y,t)\,\delta w(x,y,t)\,dA = \{\delta w_e\}^t \{f\}_e ; \{f\}_e = \int_A [N]^T f_z\,dA$$

$$[N] = \begin{bmatrix} N_1(\xi,\eta) & N_2(\xi,\eta) & N_3(\xi,\eta) & N_4(\xi,\eta) \end{bmatrix}$$

$$N_j^t = \begin{bmatrix} \frac{1}{8}(1+\xi\xi_j)(1+\eta\eta_j)(2+\xi\xi_j+\eta\eta_j-\xi^2-\eta^2) \\ \frac{b}{8}(1+\xi\xi_j)(\eta+\eta_j)(\eta^2-1) \\ \frac{a}{8}(\xi+\xi_j)(\xi^2-1)(1+\eta\eta_j) \end{bmatrix}$$

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Now we need to evaluate the equivalent nodal forces, so again we use this virtual work statement and based on that we get the equivalent nodal forces to be given by integral N transpose FZ DA, where N is again we have already seen this matrix, and if for example if this

$$\text{Example: } f_z(x,y,t) = \text{constant} = f_{z0}$$

$$\{f\}_e = f_{z0}\frac{ab}{3}\begin{bmatrix}3 & b & -a & 3 & b & a & 3 & -b & a & 3 & -b & -a\end{bmatrix}^t$$

Stresses in terms of nodal displacements

$$\sigma = -z[D]\{\chi\} = -z[D][B]\{w_e\}$$

$$M_{xx} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xx} z dz;\ M_{yy} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{yy} z dz;\ M_{xy} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xy} z dz$$

$$\Rightarrow$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = -\frac{h^3}{12}[I]_3[D][B]\{w\}_e$$

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forcing is a constant, a constant load then the nodal equivalent nodal forces using this formulation is obtained passionate.

Next if you want to compute bending moments, this is how bending moments are related to the stresses and we can evaluate this, we have these expression for sigma XX, sigma YY and sigma XY, and if we carry out this integration we get this stress resultants that are of interest in terms of nodal coordinates and strain matrix B and constitutive matrix of constitutive law D, I is a matrix that has been defined here in one of these places.

The element developed is not conforming.

Normal slope is not continuous across edges.

How to overcome this?

Introduce additional nodal dof

Introduce additional nodes

Ensure that normal slope varies linearly along edges

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Now the element that we have developed as a pointed out is not confirming, that is the normal slope is not continuous across the edge, now how to overcome this limitation? So how can we do it? There are various approaches in the literature, so we can introduce additional and nodal degree of freedom or introduce additional nodes or ensure that normal slope varies linearly along the edges, if normal slope varies linearly along the edges then automatically it depends on 2 nodes along the edge and therefore the continuity will be guaranteed, so we will see how that happens, so first we will consider a confirming rectangular element, the idea is basically to use

Conforming rectangular element

Idea: use products of beam shape functions in the two directions.

$$N_j^T(\xi,\eta) = \begin{bmatrix} f_j(\xi) f_j(\eta) \\ b f_j(\xi) g_j(\eta) \\ -a g_j(\xi) f_j(\eta) \end{bmatrix}$$

$$f_j(\xi) = \frac{1}{4}\left(2 + 3\xi_j \xi - \xi_j \xi^3\right)$$

$$g_j(\xi) = \frac{1}{4}\left(1 - \xi_j - \xi + \xi_j \xi^2 + \xi^3\right)$$

As before consider

$$\{\bar{w}_e\} = \begin{bmatrix} w_1 & b\theta_{x1} & a\theta_{y1} & w_2 & b\theta_{x2} & a\theta_{y2} & w_3 & b\theta_{x3} & a\theta_{y3} & w_4 & b\theta_{x4} & a\theta_{y4} \end{bmatrix}^T$$

$$w = \begin{bmatrix} N_1(\xi,\eta) & N_2(\xi,\eta) & N_3(\xi,\eta) & N_4(\xi,\eta) \end{bmatrix} \{\bar{w}_e\}$$

$$= \sum_j \left[w_j f_j(\xi) f_j(\eta) + b^2 \theta_{xj} f_j(\xi) g_j(\eta) - a^2 \theta_{y1} g_j(\xi) f_j(\eta) \right]$$

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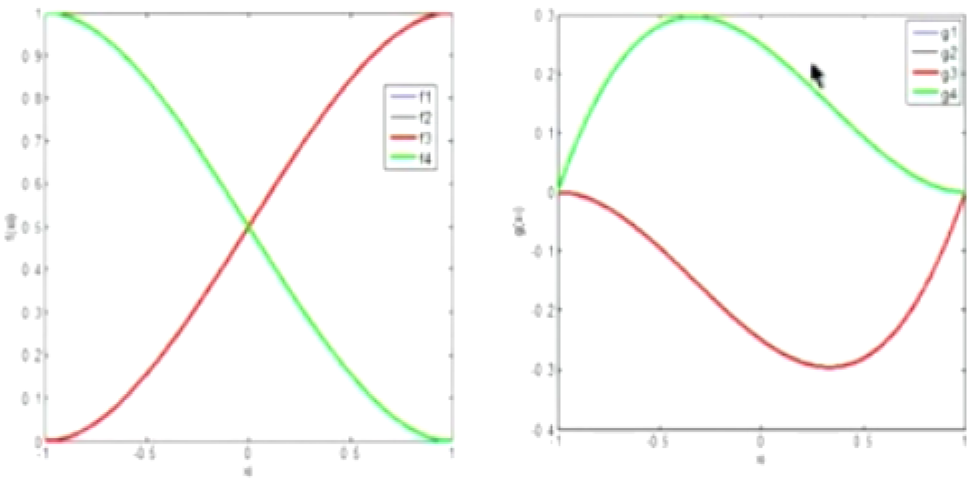
products of beam shape functions in the 2 directions, so in the n matrix that we refer to earlier the NJ (sai, eta) is taken to be in this form, there are two functions of scalar variables sai and eta, and these are the function that has been used in the cubic polynomial used in formulating the beam element, now it is in the non-dimensional coordinate system, so what we do is we consider this same degrees of freedom as we used earlier that means we will assume 3 degrees of freedom per node translation along Z rotation slopes along X and Y at the 4 nodes, suppose if you proceed with this and use these shape functions, so these shape functions I have shown



$$f_j(\xi) = \frac{1}{4}\left(2 + 3\xi_j\xi - \xi_j\xi^3\right)$$

$$g_j(\xi) = \frac{1}{4}\left(1 - \xi_j - \xi + \xi_j\xi^3 + \xi^3\right)$$

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what are these F and G functions you can see here, these are the 4 functions, this is the a cubic polynomial that we've used in formulating beam element.

$$w = \sum_{j=1}^{4}\left[w_j f_j(\xi) f_j(\eta) + b^2\theta_{xj} f_j(\xi) g_j(\eta) - a^2\theta_{yj} g_j(\xi) f_j(\eta)\right]$$

$$\frac{\partial^2 w}{\partial\xi\partial\eta} = \sum_{j=1}^{4}\left[w_j f_j'(\xi) f_j'(\eta) + b^2\theta_{xj} f_j'(\xi) g_j'(\eta) - a^2\theta_{yj} g_j'(\xi) f_j'(\eta)\right]$$

$$f_j(\xi) = \frac{1}{4}\left(2 + 3\xi_j\xi - \xi_j\xi^3\right) \Rightarrow f_j'(\xi) = \frac{1}{4}\left(3\xi_j - 3\xi_j\xi^2\right) = \frac{3\xi_j}{4}\left(1-\xi^2\right)$$

$$\Rightarrow f_j'(\xi = \pm 1) = 0$$

Similarly, $f_j'(\eta = \pm 1) = 0$

$$\Rightarrow \frac{\partial^2 w}{\partial\xi\partial\eta}(\xi = \pm 1, \eta = \pm 1) = 0$$

$\Rightarrow$ That is, the nodal value of $\frac{\partial^2 w}{\partial\xi\partial\eta} = 0$ for all nodes.

As the element becomes smaller, the plate will tend towards a zero twist condition.

Now if we now substitute this the field variable is now represented in terms of nodal values as shown here, okay, now let us put this in the details of FJ can be inserted here, and we can evaluate now the second order derivative dou square w/dou XI dou eta, this provides the twist that we observe in the plate, and if we evaluate this which is straight forward since these are polynomial so we can quickly do that, and if we evaluate the value of this gradient at the nodes it turns out to be 0, so the nodal value of dou square W/dou sai it is 0 for all the nodes, so the problem would be as the element becomes smaller the plate will tend towards a 0 twist condition, and we will not have a desired behavior incorporated into the element development, so this is a limitation, so how do we overcome that? What we do is, we will treat this quantity

Remedy: introduce $\frac{\partial^2 w}{\partial x \partial y}$ as a nodal dof.

$$\{\bar{w}_e\} = \begin{bmatrix} w_1 & \theta_{x1} & \theta_{y1} & w_{xy1} & \cdots & w_4 & \theta_{x4} & \theta_{y1} & w_{xy4} \end{bmatrix}^T$$

$$N_j^T(\xi,\eta) = \begin{bmatrix} f_j(\xi) f_j(\eta) \\ b f_j(\xi) g_j(\eta) \\ -a g_j(\xi) f_j(\eta) \\ a b g_j(\xi) g_j(\eta) \end{bmatrix}$$

$$w = \begin{bmatrix} N_1(\xi,\eta) & N_2(\xi,\eta) & N_3(\xi,\eta) & N_4(\xi,\eta) \end{bmatrix}_{1\times 16} \{\bar{w}_e\}_{16\times 1}$$

$$w = \sum_{j=1}^{4} [w_j f_j(\xi) f_j(\eta) + b^2 \theta_{xj} f_j(\xi) g_j(\eta) - a^2 \theta_{y1} g_j(\xi) f_j(\eta) + ab w_{xyj} g_j(\xi) g_j(\eta)]$$

$$\frac{\partial^2 w}{\partial \xi \partial \eta} = \sum_{j=1}^{4} [w_j f_j'(\xi) f_j'(\eta) + b^2 \theta_{xj} f_j'(\xi) g_j'(\eta) - a^2 \theta_{y1} g_j'(\xi) f_j'(\eta) + ab w_{xyj} g_j'(\xi) g_j'(\eta)]$$

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dou square W/dou X dou Y as additional nodal degree of freedom, so in addition to W theta X theta Y I will use this call it at W, X, Y, as another nodal degree of freedom, so then this element will now have 16 degrees of freedom, 4 nodes, 4 degree of freedom per element and then what we will do is if you carefully see here it has to be seen carefully, there is F term and G term, if you look at values of F at the nodes, the gradient of F at the nodes it is 0, but this is not true for G function, G is not 0 at nodes, so we introduce for WXY an interpolation in terms of GJ, XI and GJ eta function, so W now consists of as before the same format, but this now has 16 elements, and this is 16 x 1, and the 16 elements are made up of, this is a transpose, so 4 of this evaluated at J = 1, 2, 3, 4.

Now if you write in summation form, this is what it's meant by the representation as shown here. Now if you compute the second-order gradient dou square W/dou sai dou eta, dou XI dou eta we get this function, and if you now evaluate this at the nodes it turns out that this will be

$$f_j'(\xi=\pm1)=0;\ f_j'(\eta=\pm1)=0 \Rightarrow \frac{\partial^2 w}{\partial\xi\partial\eta}=\sum_{j=1}^{4} ab\,w_{xyj}\,g_j'(\xi)\,g_j'(\eta)$$

$$g_j(\xi)=\frac{1}{4}\left(1-\xi_j-\xi+\xi_j\xi^2+\xi^3\right)\Rightarrow g_j'(\xi)=\frac{1}{4}\left(-1+2\xi_j\xi+3\xi^2\right)$$

$$g_j'(\xi_i)=\frac{1}{4}\left(-1+2\xi_j\xi_i+3\xi_i^2\right)\neq 0$$

$$\Rightarrow \frac{\partial^2 w}{\partial\xi\partial\eta}(\xi=\pm1,\eta=\pm1)\neq 0$$

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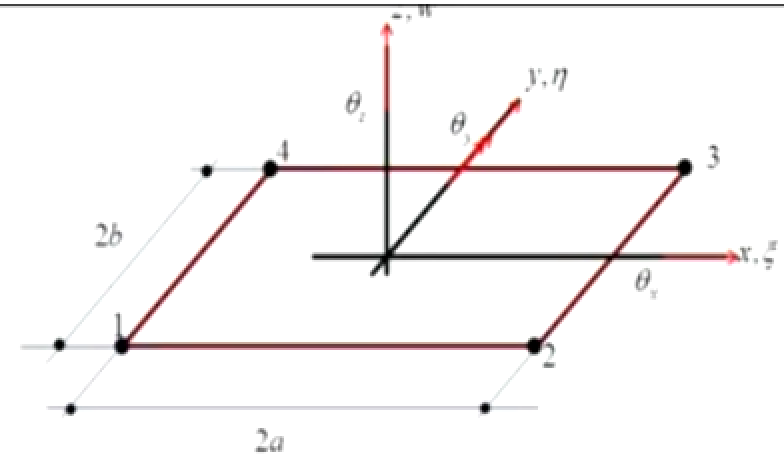
given by this function, and this function is not 0 at the nodal coordinates, so therefore this is not 0. So the limitation that we observed here that this is 0 at all the nodes is over come here.



Can the element perform rigid body motion without deformation?

$$\left.\begin{aligned} & w_1=w_3=1;\ w_2=w_4=-1 \\ & \theta_{x2}=\theta_{x3}=\frac{1}{b};\ \theta_{x1}=\theta_{x4}=-\frac{1}{b} \\ & \theta_{y1}=\theta_{y2}=\frac{1}{a};\ \theta_{y3}=\theta_{y4}=-\frac{1}{a} \\ & w_{xyj}=\frac{1}{ab} \end{aligned}\right\} \Rightarrow w(\xi,\eta)=\xi\eta\ (\text{OK})$$

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Now we can ask the question can the element perform rigid body motion without deformation, these are about some of the requirements that we initiated in one of the previous lectures when we talked about convergence and those issues. So now if we consider W1 and W3 to be 1, and W2 and W4 to be – 1, and theta X2 is theta X3, theta X1 is theta X4 as shown here which is minus of that, and this similarly theta Y has this relation, we are giving a rigid body rotation to the object, these deformations ensures that we are giving a rigid body rotation. Now if you put these values into the assumed representation here it turns out that WXYJ will be given by this and this W (sai, eta) will be sai eta, which is what we expect for such type of motion, so this representation is fine.

$$[M]_e = \int_A \rho h N^t N dA = \rho hab \int_{-1}^{1}\int_{-1}^{1} N^t(\xi,\eta) N(\xi,\eta) d\xi d\eta$$

$$[K]_e = \int_A \frac{h^3}{12} B^t D B dA$$

$$\{f\}_e = \int_A [N]^T f_z dA$$

$$[N] = \begin{bmatrix} N_1(\xi,\eta) & N_2(\xi,\eta) & N_3(\xi,\eta) & N_4(\xi,\eta) \end{bmatrix}$$

$$N_j^t(\xi,\eta) = \begin{bmatrix} f_j(\xi) f_j(\eta) \\ b f_j(\xi) g_j(\eta) \\ -a g_j(\xi) f_j(\eta) \\ ab g_j(\xi) g_j(\eta) \end{bmatrix}$$

Remarks

- Known as CR element
- Disadvantage: analysis of built-up structures using this element is difficult

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Now we can go ahead and evaluate the mass matrix, stiffness matrix, equivalent nodal forces, you know these are polynomial so it can be evaluated exactly, or you can use an appropriate quadrature rule, so I am not going to provide the details of this, this element is known as CR element, the disadvantage is that in the analysis of built-up structure using this element is difficult because of that additional node dou square w, dou X, dou Y represent at nodes, most of the other elements would not have that element and there will be problem in dealing with that, so one approximation that has been done in the literature is to represent this dou square W, dou X, dou Y, in terms of theta X and theta Y, by kind of a numerical differencing scheme, and that overcomes this difficulty, that mean we can eliminate the WXY degree of freedom but such an element again turns out to be non-conform, so there will be another penalty that we have to pay.

Thick rectangular plate bending element

Plane sections initially normal to the middle plane

- remain plane $\Rightarrow \varepsilon_{xz}(x,y,z)=\varepsilon_{xz}(x,y)$ & $\varepsilon_{yz}(x,y,z)=\varepsilon_{yz}(x,y)$

but not necessarily normal to the middle plane

- will have same length $\Rightarrow \varepsilon_{zz}(x,y,z)=0$

$$\varepsilon_{zz}(x,y,z)=0 \Rightarrow \frac{\partial w}{\partial z}=0 \Rightarrow w(x,y,z)=w(x,y)$$

Consider θ_x, θ_y to be rotations about the x- and y- axes of lines which are normal to the middle surface before deformation.

$$u(x,y,z)=z\theta_y$$

$$v(x,y,z)=-z\theta_x$$

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Now we will now consider another element, this is thick rectangular plate bending element we have been considering behavior of thin elements, now it turns out that for this element we can develop a conforming rectangular element as we will see shortly, so here what is a consequence of plate being thick, so we have assumed in the earlier theory that plane sections initially normal to the middle plane remain plane, and also normal to the middle plane, but now what we will do is we will assume that the plane sections initially normal to the middle plane remain plane, consequently epsilon XZ will be independent of Z, and epsilon YZ will be independent of Z, but not necessarily normal to the middle plane, if you insist that this should be also said, this is also satisfied and these 2 quantities identically become equal to 0, so now we are permitting a certain rotation, a constant rotation across the thickness, it will have still the same length therefore epsilon ZZ continues to be 0, so W is again function of X and Y, now what we will do is, we will consider theta X and theta Y to be the rotations about X and Y axis of the lines which are normal to the middle surface before deformation, and based on that we can evaluate U and V as Z into theta Y – Z into theta X, so now U and V also will now come into

$$u(x,y,z) = z\theta_y; \qquad v(x,y,z) = -z\theta_x$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = z\frac{\partial \theta_y}{\partial x}; \varepsilon_{yy} = \frac{\partial v}{\partial y} = -z\frac{\partial \theta_x}{\partial y}; 2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = z\left(\frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x}\right)$$

$$\varepsilon = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix} = -z\chi \text{ with } \chi = \begin{Bmatrix} -\dfrac{\partial \theta_y}{\partial x} \\ \dfrac{\partial \theta_x}{\partial y} \\ \dfrac{\partial \theta_x}{\partial x} - \dfrac{\partial \theta_y}{\partial y} \end{Bmatrix}$$

$$2\varepsilon_{xz}(x,y) = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} + \theta_y;$$

$$2\varepsilon_{yz}(x,y) = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \frac{\partial w}{\partial y} - \theta_x$$

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picture, I have U as Z theta Y, and V as - Z theta X, so strains epsilon XX is dou U/dou X it is given by Z dou theta Y/dou X, epsilon YY dou V/dou Y which is – Z dou theta X/dou Y. Shear strain dou U/dou Y, dou V/dou X this is given by this, so now if you consider the test, will partition the strains into 2 components epsilon XX, epsilon YY, and epsilon XY is one part, and the shearing strains as another part.

Now this as before is given by - Z into khi, but now this khi is given in terms of theta Y theta X, it's not curvature immediately it cannot be interpreted as curvature, so this is dou theta X, dou theta Y, dou Y. Now the shearing strains are given by this, so dou W/dou X stays dou U/dou Z

$$\gamma = \begin{Bmatrix} 2\varepsilon_{xz}(x,y) \\ 2\varepsilon_{yz}(x,y) \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial w}{\partial x} + \theta_y \\ \dfrac{\partial w}{\partial y} - \theta_x \end{Bmatrix}$$

$$\sigma = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = D \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix} \text{ with } D = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

$$\tau = \begin{Bmatrix} \sigma_{xz}(x,y) \\ \sigma_{yz}(x,y) \end{Bmatrix} = kD^s \begin{Bmatrix} 2\varepsilon_{xz}(x,y) \\ 2\varepsilon_{yz}(x,y) \end{Bmatrix} \text{ with } D^s = \frac{E}{2(1+\nu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

k = factor that accounts for variation of shear shear stresses and strains through the thickness.

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is theta Y, and dou V/dou Z is – theta X, we call this vector of shearing strain as gamma, and now we have sigma which will increase sigma XX, YY, and XY, and tau which include shearing stresses, sigma is related to epsilon XX, epsilon YY, and shearing strain with D being this. Now shearing stresses are related to shearing strains with this being the D matrix, so we will now call it as DS, and this as D. Now this factor K accounts for the variation of shear stresses and strains through the thickness, this we've already seen when we talked about D beams in one of the earlier lectures, so equipped with this now the expression for strain energy

$$V = \frac{1}{2}\int_{V_0} \varepsilon^t D \varepsilon dV_0 + \frac{1}{2}\int_{V_0} \tau^t \gamma dV_0 = \frac{1}{2}\int_A \frac{h^3}{12}\chi^t D \chi dA + \frac{1}{2}\int_A kh\gamma^t D^s \gamma dA$$

$$T = \frac{1}{2}\int_{V_0} \rho\left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right) dV_0 = \frac{1}{2}\int_A \rho\left(h\dot{w}^2 + \frac{h^3}{12}\dot{\theta}_x^2 + \frac{h^3}{12}\dot{\theta}_y^2\right) dA$$

$$\chi = \begin{Bmatrix} -\frac{\partial \theta_y}{\partial x} \\ \frac{\partial \theta_x}{\partial y} \\ \frac{\partial \theta_x}{\partial x} - \frac{\partial \theta_y}{\partial y} \end{Bmatrix} \ \& \ \gamma = \begin{Bmatrix} \frac{\partial w}{\partial x} + \theta_y \\ \frac{\partial w}{\partial y} - \theta_x \end{Bmatrix}$$

Field variables: w, θ_x, θ_y

Highest derivative of the field variables: 1

Dofs: w, θ_x, θ_y at the nodes

has contributions from sigma XX, sigma YY, and sigma XY, and contributions from shearing stresses, so I write that separately. This epsilon transpose D epsilon, this is tau transpose gamma, and carrying out the integration across the thickness I get the first term as this, and the second term as this.

Now similarly kinetic energy has now contribution from UV and W, U dot, V dot and W dot and this is expressed in terms of W theta X and theta Y dot as shown here, so khi is the vector of these quantities, and gamma is the vector of these quantities. Now let's examine the Lagrangian which is T – V, what are the field variables? We will have W and theta X and theta Y, highest derivative of the field variable is 1, see we are having dou W/dou X and dou theta Y/dou theta X, and dou theta X/dou X, right, so the degrees of freedom are W, theta X, theta Y at the nodes, and since the highest derivative of the field variable is 1, these are the degrees of

$$w(\xi,\eta,t)=\sum_{j=1}^{4}N_j(\xi,\eta)w_j(t)$$

$$\theta_x(\xi,\eta,t)=\sum_{j=1}^{4}N_j(\xi,\eta)\theta_{xj}(t)$$

$$\theta_y(\xi,\eta,t)=\sum_{j=1}^{4}N_j(\xi,\eta)\theta_{yj}(t)$$

$$N_j(\xi,\eta)=\frac{1}{4}(1+\xi\xi_j)(1+\eta\eta_j);\ j=1,2,3,4$$

$\Rightarrow w(\xi,\eta,t),\theta_x(\xi,\eta,t),\theta_y(\xi,\eta,t)$ maintain inter-element continuity.

This is a conforming element.

Named as HTK element [Hughes, Taylor, Kanoknukulcha]

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freedom, and we can represent now the field variables in terms of, W has 4 quantities we will write like this W1, W2, W3, W4, similarly theta X1, theta X2, theta X3, theta X4 so on and so forth, so we have now representation for W, theta X, theta Y and we are now going to use the interpolation function that we have encountered while dealing with rectangular plane stress elements, there also we had 4 nodal degrees of freedom, and the field variable was required to be interpolated in terms of 4 nodal values and we are encountering similar situation therefore we can use the same trial functions.

So we already seen that element maintains inter element continuity of a field variable and the required derivative, so this W, theta X, and theta Y maintain inter-element continuity, therefore this is going to be a conforming element, this is named as HTK element, named after Hughes, Taylor and Kanoknukulcha.

$$\underbrace{\begin{Bmatrix} w(\xi,\eta,t) \\ \theta_x(\xi,\eta,t) \\ \theta_y(\xi,\eta,t) \end{Bmatrix}}_{3\times 1} = \underbrace{\left[N(\xi,\eta)\right]}_{3\times 12}\underbrace{\{w(t)\}_e}_{12\times 1}$$

$$\{w(t)\}_e = \begin{bmatrix} w_1 & \theta_{x1} & \theta_{y1} & \cdots & w_4 & \theta_{x4} & \theta_{y4} \end{bmatrix}^t$$

$$[N] = \begin{bmatrix} N_1 & 0 & 0 & \cdots & N_4 & 0 & 0 \\ 0 & N_1 & 0 & \cdots & 0 & N_4 & 0 \\ 0 & 0 & N_1 & \cdots & 0 & 0 & N_4 \end{bmatrix}$$

$$T = \frac{1}{2}\int_A \rho\left(h\dot{w}^2 + \frac{h^3}{12}\dot{\theta}_x^2 + \frac{h^3}{12}\dot{\theta}_y^2\right)dA = \frac{1}{2}\int_A \rho N^t \begin{bmatrix} h & 0 & 0 \\ 0 & \frac{h^3}{12} & 0 \\ 0 & 0 & \frac{h^3}{12} \end{bmatrix} N dA$$

So we can now represent therefore the vector of field variable W, theta X, and theta Y, in terms of the nodal degrees of freedom which is 12 x 1, and this is 3 x 12 matrix of shape functions, these are the nodal degrees of freedom W1, theta X1, theta Y1 at node 1 and so on and so forth. So expression for kinetic energy now can be evaluated as shown here, this mass matrix we can

$$M = m_I + m_{II}$$

$$m_I = \frac{\rho hab}{108}\begin{bmatrix}
48 & & & & & & & & & & & \\
0 & 0 & & & & & & & & & & \\
0 & 0 & 0 & & & & & & & & & \\
24 & 0 & 0 & 48 & & & & & \text{Sym} & & & \\
0 & 0 & 0 & 0 & 0 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & & & & & \\
12 & 0 & 0 & 24 & 0 & 0 & 48 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
24 & 0 & 0 & 12 & 0 & 0 & 24 & 0 & 0 & 48 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$m_{II} = \frac{\rho h^3 ab}{108}\begin{bmatrix}
0 & & & & & & & & & & & \\
0 & 4 & & & & & & & & & & \\
0 & 0 & 4 & & & & & & & & & \\
0 & 0 & 0 & 0 & & & & & \text{Sym} & & & \\
0 & 2 & 0 & 0 & 4 & & & & & & & \\
0 & 0 & 2 & 0 & 0 & 4 & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\
0 & 1 & 0 & 0 & 2 & 0 & 0 & 4 & & & & \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 4 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 2 & 0 & 4 & \\
0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 4
\end{bmatrix}$$

evaluate in terms of 2 different components and I have, this can be evaluated exactly, so we can evaluate this, the 2 components of this matrix exactly, and thus what we do and we get the mass matrix.

$$V_e = \frac{1}{2} w_e^t k_e w_e$$

$$k_e = k_f + k_s = \int_A \frac{h^2}{12} B_f^t D B_f \, dA + \int_A k h B_s^t D B_s \, dA$$

$$B_f = \begin{bmatrix} B_{f1} & B_{f2} & B_{f3} & B_{f4} \end{bmatrix}; B_{fj} = \begin{bmatrix} 0 & 0 & -\frac{\partial N_j}{\partial x} \\ 0 & \frac{\partial N_j}{\partial y} & 0 \\ 0 & \frac{\partial N_j}{\partial x} & -\frac{\partial N_j}{\partial y} \end{bmatrix}$$

$$B_s = \begin{bmatrix} B_{s1} & B_{s2} & B_{s3} & B_{s4} \end{bmatrix}; B_{sj} = \begin{bmatrix} \frac{\partial N_j}{\partial x} & 0 & N_j \\ \frac{\partial N_j}{\partial y} & N_j & 0 \end{bmatrix}$$

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Now the strain energy is 1/2 WE transpose KE, WE, now KE itself we decompose into a flexural component and a shear component, this is the component due to flexure and this is component due to shearing, so we partition now the B matrix as shown here where BFJ here is given by this matrix, and similarly the BS for the shear, B matrix for the shear is given by this. Now there are 2 components for strains, so this will be having 2 rows, whereas this has 3 rows,

$$N_j(\xi,\eta)=\frac{1}{4}(1+\xi\xi_j)(1+\eta\eta_j);\ j=1,2,3,4 \Rightarrow$$

$$B_{fj}=\begin{bmatrix} 0 & 0 & -\frac{\partial N_j}{\partial x} \\ 0 & \frac{\partial N_j}{\partial y} & 0 \\ 0 & \frac{\partial N_j}{\partial x} & -\frac{\partial N_j}{\partial y} \end{bmatrix}=\begin{bmatrix} 0 & 0 & -\frac{\xi_j}{4a}(1+\eta\eta_j) \\ 0 & \frac{\eta_j}{4b}(1+\xi\xi_j) & 0 \\ 0 & \frac{\xi_j}{4a}(1+\eta\eta_j) & -\frac{\eta_j}{4b}(1+\xi\xi_j) \end{bmatrix}$$

$$B_{sj}=\begin{bmatrix} \frac{\partial N_j}{\partial x} & 0 & N_j \\ \frac{\partial N_j}{\partial y} & N_j & 0 \end{bmatrix}=\begin{bmatrix} \frac{\xi_j}{4a}(1+\eta\eta_j) & 0 & \frac{1}{4}(1+\xi\xi_j)(1+\eta\eta_j) \\ \frac{\xi_j}{4a}(1+\eta\eta_j) & \frac{1}{4}(1+\xi\xi_j)(1+\eta\eta_j) & 0 \end{bmatrix}$$

Use 2 x 2 Gauss quadrature : leads to exact solution.

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now we know the interpolation functions therefore we can evaluate all these gradients that appear in the B matrix, and we will be able to evaluate BFJ and BSJ matrices, so dou NJ/dou X corresponds to dou NJ/dou XI, and if we carry out the required differentiation we will be able to evaluate this without any problem. Similarly for the shear B matrix for the shear component is given by this and we get this matrix, so you can use 2 x 2 Gauss quadrature it leads to exact solution or you can evaluate this exactly as well, so if we do that again by introducing 2

$$k_f = \frac{Eh^3}{48ab(1-\nu^2)} \begin{bmatrix} k_{11}^f & & & \\ k_{21}^f & k_{22}^f & \text{sym} & \\ k_{31}^f & k_{32}^f & k_{33}^f & \\ k_{41}^f & k_{42}^f & k_{43}^f & k_{44}^f \end{bmatrix} \quad \alpha = \frac{a}{b}, \beta = \frac{b}{a}$$

$$k_{11}^f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{4}{3}\left\{\alpha^2 + \frac{1}{2}(1-\nu)\right\}b^2 & -\frac{1}{2}(1+\nu)ab \\ 0 & -\frac{1}{2}(1+\nu)ab & \frac{4}{3}\left\{\beta^2 + \frac{1}{2}(1-\nu)\right\}a^2 \end{bmatrix}$$

$$k_{21}^f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{2}{3}\left\{\alpha^2 - (1-\nu)\right\}b^2 & -\frac{1}{2}(3\nu-1)ab \\ 0 & \frac{1}{2}(3\nu-1)ab & \frac{1}{3}\left\{-4\beta^2 + (1-\nu)\right\}a^2 \end{bmatrix}$$

parameters, alpha is A/B and beta is B/A we can get the K matrix in terms of various components K11, K21, K34, K41, F means flexure, superscript F stands for flexure and by carrying out the required integration these elements of these matrices can be deduced and the details of K11F, K21F, and K31F and KF41 are given here, then the remaining matrices K22,

$$k_{31}^{f} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{2}{3}\left\{-\alpha^2 + \frac{1}{2}(1-\nu)\right\}b^2 & \frac{1}{2}(1+\nu)ab \\ 0 & \frac{1}{2}(1+\nu)ab & \frac{2}{3}\left\{-\beta^2 + \frac{1}{2}(1-\nu)\right\}a^2 \end{bmatrix}$$

$$k_{41}^{f} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{3}\left\{-4\alpha^2 + (1-\nu)\right\}b^2 & \frac{1}{2}(3\nu-1)ab \\ 0 & -\frac{1}{2}(3\nu-1)ab & \frac{2}{3}\left\{\beta^2 - (1-\nu)\right\}a^2 \end{bmatrix}$$

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$$I_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; I_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$k_{22}^{f} = I_3^t k_{11}^f I_3$$

$$k_{32}^{f} = I_3^t k_{41}^f I_3 \quad k_{33}^{f} = I_1^t k_{11}^f I_1$$

$$k_{42}^{f} = I_3^t k_{31}^f I_3 \quad k_{43}^{f} = I_1^t k_{21}^f I_1 \quad k_{44}^{f} = I_2^t k_{11}^f I_2$$

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K32, K42, etcetera are obtained in terms of this first column by doing these operations, and we will complete the K matrix.

$$k_s = \frac{Eh^3}{48ab\beta_s}\begin{bmatrix} k_{11}^s & & & \text{sym} \\ k_{21}^s & k_{22}^s & & \\ k_{31}^s & k_{32}^s & k_{33}^s & \\ k_{41}^s & k_{42}^s & k_{43}^s & k_{44}^s \end{bmatrix} \quad \beta_s = \frac{Eh^2}{12kGb^2}$$

$$k_{11}^s = \begin{bmatrix} \left(1+\alpha^2\right) & \alpha^2 b & -a \\ \alpha^2 b & \alpha^2 b^2 & 0 \\ -a & 0 & a^2 \end{bmatrix}; k_{21}^s = \begin{bmatrix} \left(-1+\alpha^2\right) & \alpha^2 b & a \\ \alpha^2 b & \alpha^2 b^2 & 0 \\ a & 0 & a^2 \end{bmatrix}$$

$$k_{31}^s = \begin{bmatrix} \left(-1-\alpha^2\right) & -\alpha^2 b & a \\ \alpha^2 b & \alpha^2 b & 0 \\ -a & 0 & a^2 \end{bmatrix}; k_{41}^s = \begin{bmatrix} \left(1-\alpha^2\right) & -\alpha^2 b & -a \\ \alpha^2 b & \alpha^2 b^2 & 0 \\ -a & 0 & a^2 \end{bmatrix}$$

$$k_{22}^s = I_3^t k_{11}^s I_3$$

$$k_{32}^s = I_3^t k_{41}^s I_3 \quad k_{33}^f = I_1^t k_{11}^s I_1$$

$$k_{42}^s = I_3^t k_{31}^s I_3 \quad k_{43}^f = I_1^t k_{21}^s I_1 \quad k_{44}^f = I_2^t k_{11}^s I_2$$

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So similarly we can carry out the exercise for the shear part, we get again KS as this matrix and this again has these components as named here and this is symmetric matrix, so the components of this first matrices in the first column are given here, and the remaining elements are computed using these operations where I1, I2, I3 have been defined earlier.

$$\delta W_e = \{\delta w\}_e^T \{f\}_e$$

$$\{f\}_e = \iint_A [N]^T \begin{Bmatrix} f_z \\ 0 \\ 0 \end{Bmatrix} dA$$

Example: f_z = constant $\Rightarrow$

$$\{f\}_e = f_z ab \{1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0\}^T$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = -\frac{h^3}{12}[I]_3 [D][B_f]\{w\}_e$$

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = kh[D_s][B_s]\{w\}_e$$

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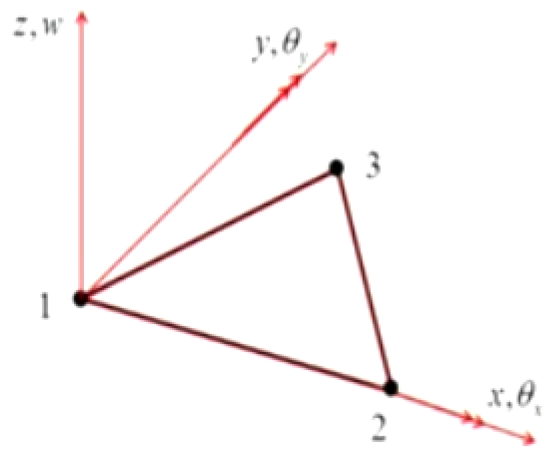
Now how about equivalent nodal forces, so again we use this formulation FE is N transpose into the applied loads, we are applying loads only in the transverse direction, so this is the formula for that, and if FZ is constant the equivalent nodal forces are obtained as shown here. Now again bending moment and shear forces can be computed as stress resultants we have evaluated all the required stresses, so by integrating across the thickness suitably we get the bending moment to string moment and the shear forces, so this completes the formulation of a thick rectangular element which is conforming and it allows for shear deformation, okay, so this is known as Mindlin plate theory.

Thin, triangular, nonconforming element



3 nodes
3 dofs/node
9 dof element

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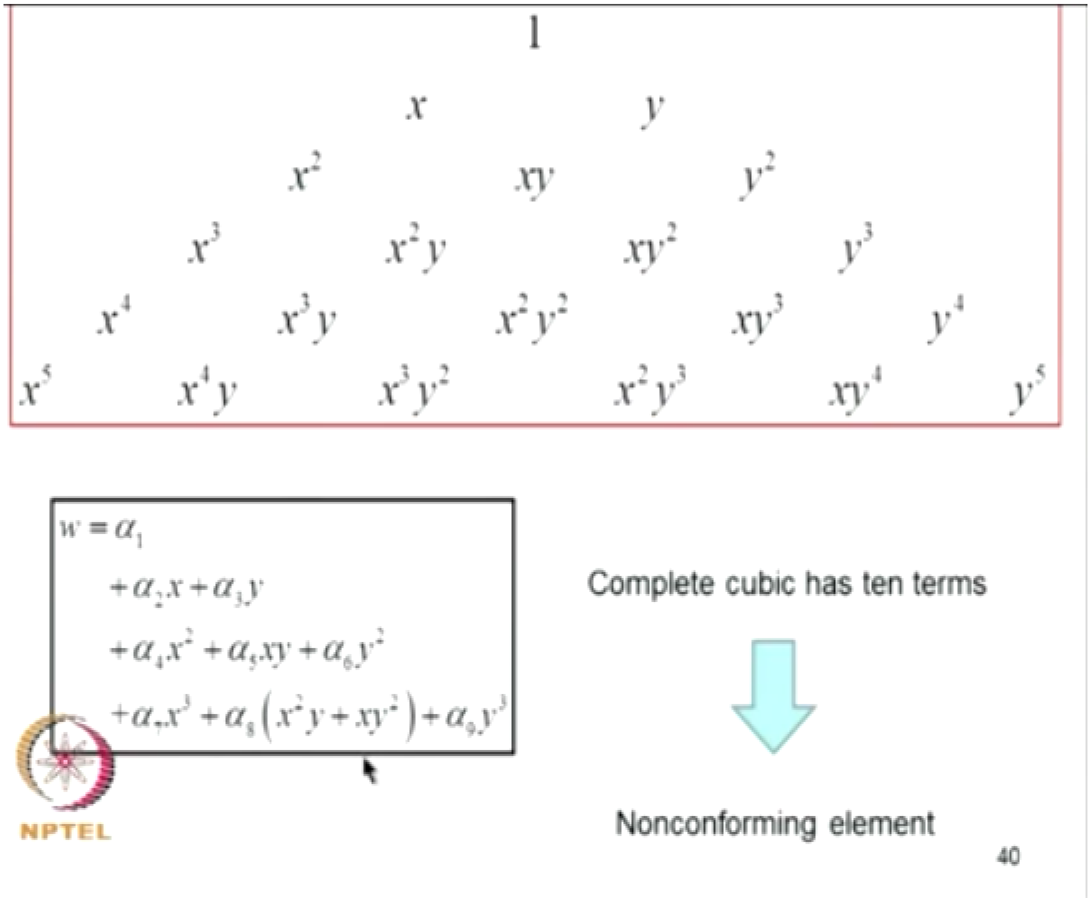
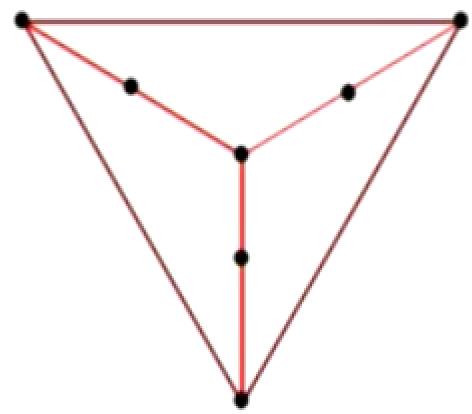
Now how about triangular elements? Now we will consider this triangular element 1, 2, 3, and if I now define X axis along one of the edges, and Y axis along orthogonal to this in the plane of the plate, and Z axis is outside the plane of this plate. Now this has 3 degrees of freedom per node, therefore it has 9 degrees of freedom, so the field variable must now be represented in terms of these 9 degrees of freedom element.

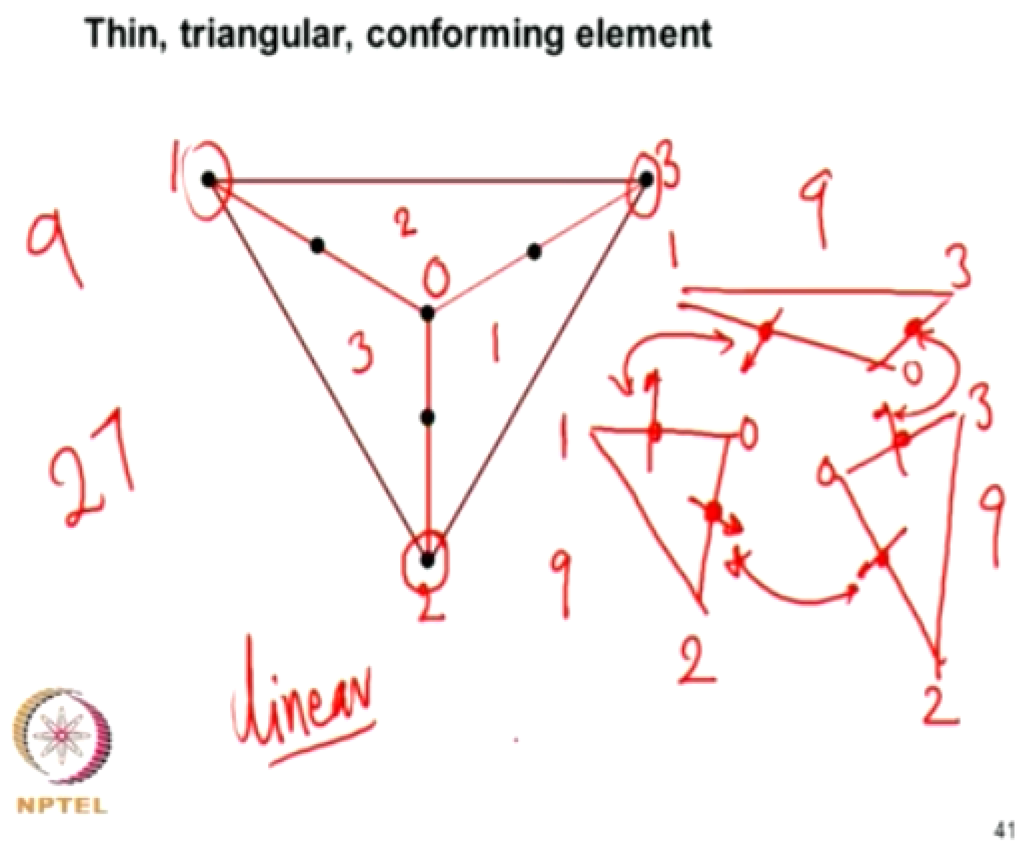


So now a complete cubic polynomial will have 10 terms, so if we now take 9 terms there will be a problem, so I can take 1, X, and Y, that is 3, X square, XY, Y square that will be 6 now, from this column we have to select 3 more terms, so the idea here is we will retain X cube, Y cube, and use a common generalized coordinates associated with some of these 2, okay, so we are retaining the complete cubic term but these are not associated with independent generalized coordinates, okay, so we can develop this element and we can show that this will not be a conforming element, again there will be problem in satisfying the continuity along of slopes normal to the surfaces on the edges, so this will again be a non-conforming element.

Thin, triangular, conforming element



Now how do we proceed here? There is one approach to develop a conforming triangular element, so what we do is suppose if we consider a triangular element 1, 2, 3 what we do is we identify an internal node, internal point and form 3 triangles, 1, 2 and 3, now we will consider now these 3 triangles separately, that means one element will be like this, other element will be like this, and yet another element will be like this, so this will be 1, 2, O, O, 2, 3, 1, 3, O, now we will formulate, each one we will formulate separately, so there will be 9 degrees of freedom here, 9 here, and 9 here, so this totally there will be 27 degrees of freedom. Now we have to now, whereas for the element 1, 2, 3, there will be 9 degrees of freedom, so we have to get now 18 relations which will eliminate this 27 degree of freedom a model reduces to 9 degree of



freedom, so what we do is obviously there will be requirements on compatibility of deformation at these 3 point nodes so that will give rise to some equations. The additional equations needed, what we will do is we will identify an intermediate point see as shown here, so this and this, this and this, and this and this, so what we will do is we will compute the normal slope at these points, for each of the elements and demand that this is equal to this, this is equal to this, and this is equal to this, if we do that we will get a conforming triangular element.

Now the way we select the trial functions in analyzing this we will ensure that the variation of the normal slope along the edges is linear, so we will be able to implement the required relations, so these 2 elements we will consider in the next class, and following that we will consider problems of stiffened plates, these are typically for example observed in say bridge decks where shell will be, a plate will be stiffened by girders, similarly these are commonly encountered in aircraft structures, automotive structures and so on and so forth, so there is a combination of beam and plate element, so there will be a plate element and a beam element, so we will have to see how we can develop a model for a stiffened plate element.

So what we will do in the following classes, we will first complete this formulation of this triangular elements and then we will consider few numerical examples and then come to the cases of stiffened plates, we will consider few numerical examples to illustrate the ideas that we

Numerical illustrations

A W Liessa, 1969, Vibration of plates, SP-160, NASA, Washington.

Classical plate theory

$$V = \frac{D}{2}\int_A \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dA$$

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

$$T = \frac{1}{2}\int_A \rho \dot{w}^2 dA$$

$$\Rightarrow D\nabla^4 w + m\ddot{w} = 0$$

$$w(x,y,t) = W(x,y)\exp(i\omega t) \Rightarrow D\nabla^4 W - \omega^2 m W = 0$$

$$\nabla^4 W - k^4 W = 0 \text{ with } k^4 = \frac{\omega^2 m}{D} \Rightarrow (\nabla^2 + k^2)(\nabla^2 - k^2) W = 0$$

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have developed so far, there is a monograph by Arthur W Liessa, it is a NASA special publication, it is on vibration of plates, and this monograph has several examples of free vibration, results of free vibration analysis for various configurations of plates, circular, rectangular, triangular, quadrilateral, polygonal, etcetera, etcetera, so it is a catalog of solutions, many of the solutions are exact, and some others are based on weighted residual approximations, so what I will do is I will pick few of these examples from this monograph and apply the finite element modeling tools on them and see how we are able to produce the answers reported in this monograph.

So let's quickly recall, this is the strain energy as per the classical plate theory, we were writing it in a different form but when expanded it will have this form, this is the kinetic energy, so the governing equation is of the form D del 4W + M double dot W dot = 0, so to find out free vibration characteristics we assume that all points on the structure vibrate harmonically at the same frequency, and we substitute this equation into the governing equation and we get this equation D del 4W - omega square MW = 0, and if we introduce the parameter K to the power of 4 as omega square M/D we get this equation and this equation itself can be rewritten in this form. Now there are few other details here, this is a governing equation and these are the stress

$$D\nabla^4 w + m\ddot{w} = 0$$

Bending and twisting moments

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right); M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right); M_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$$

Transverse shearing forces

$$Q_x = -D\frac{\partial}{\partial x}\left(\nabla^2 w\right); Q_y = -D\frac{\partial}{\partial y}\left(\nabla^2 w\right)$$

Edge reactions

$$V_x = Q_x + \frac{\partial M_{xy}}{\partial y}; V_y = Q_y + \frac{\partial M_{xy}}{\partial x}$$

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resultants in terms of the displacement field, this is the bending MX, MY, MXY, twisting moment transverse shearing forces and edge reactions, these are results from classical plate theory I will not be getting into the details of these equations, but I am stating them for sake of completeness.

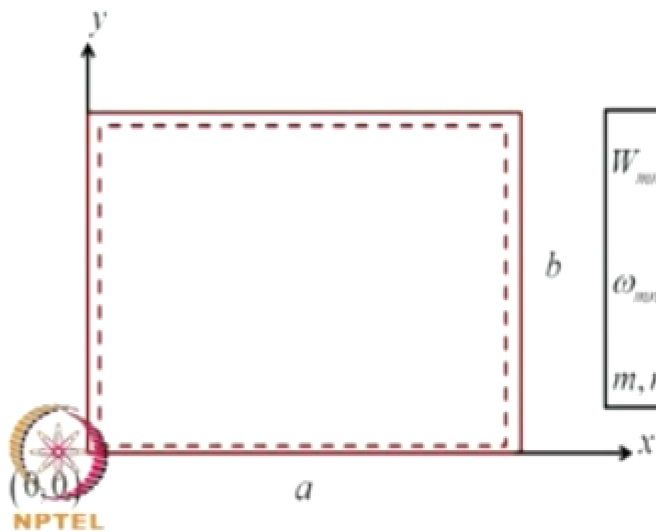
Now let us return to the problem that is free vibration problem, so we have assumed this solution harmonic solution and we’ve got this equation, and this equation in fact constitutes an

Rectangular plate with all edges simply supported

$$\nabla^4 W - k^4 W = 0$$

$$W = 0, M_x = 0 \text{ at } x = 0, x = a$$

$$W = 0, M_y = 0 \text{ at } y = 0, y = b$$



$$W_{mn}(x,y) = A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$\omega_{mn} = \sqrt{\frac{D}{m}}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]$$

$$m, n = 1, 2, \cdots$$

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Eigenvalue problem where now the operator here is a partial differential operator, and for a rectangular domain with all edges simply supported the boundary conditions will be W and MX will be 0 at X = 0, and X = A, and W and MY will be 0 at Y = 0, and Y = B. Now this dotted line is a convention used to represent simply supported edge conditions. Now as I said this is an Eigenvalue problem we need to find W = 0 is a trivial solution we satisfy this equation, but we are interested in non-trivial values of W, and we ask the question for which value of K such solutions exist, and K as you have seen is related to the frequency of harmonic excitation, so what we do is, this problem is amenable for an exact solution and the mode shapes are given in terms of sinusoidal functions, and this is the exact expression for natural frequencies.

Now the natural frequency will carry 2 indices when you are considering 2 dimensional problems, this doesn't come up in finite element solutions but in analytical solutions this feature would be present.

Numerical illustration (SSSS)

$h = 50$ mm

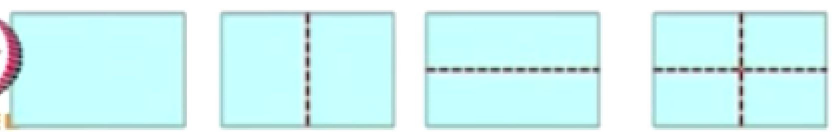
$E = 210$ GPa

$\rho = 7800$ kg/m^3

$a = 4$ m; $b = 2.5$ m

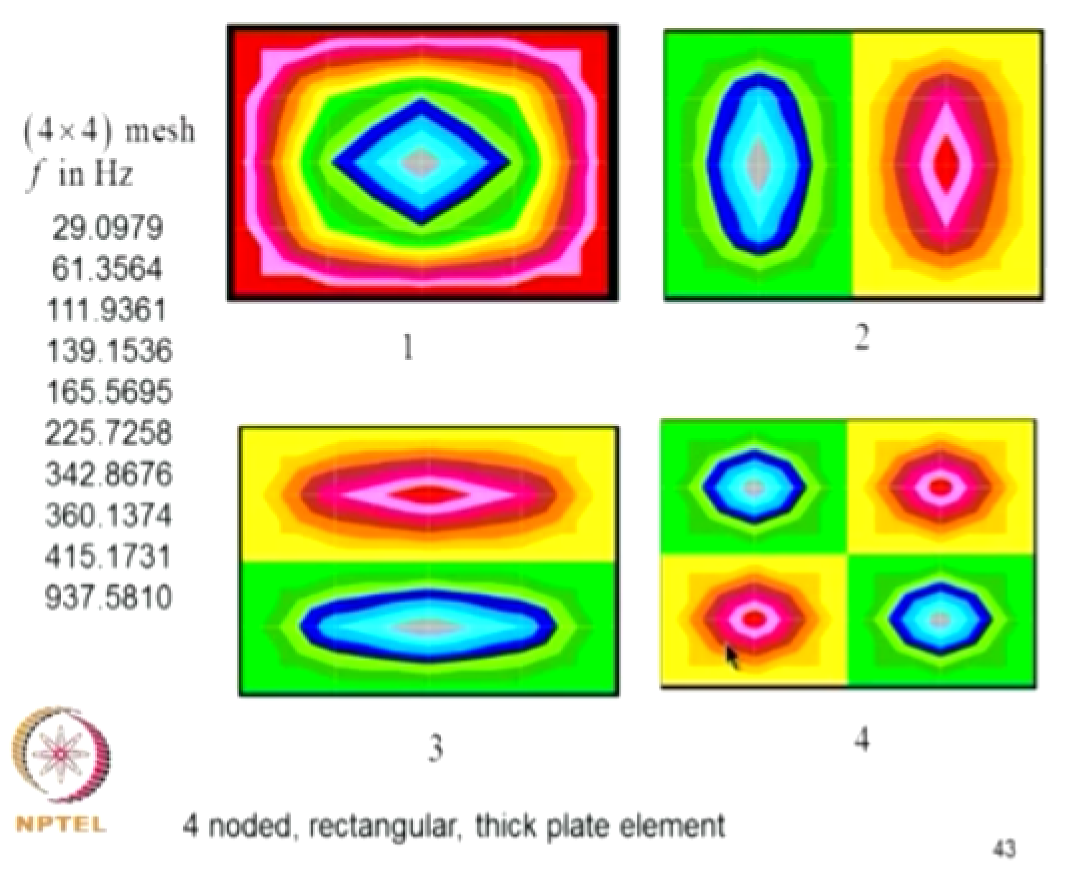
Exact natural frequencies $f(m,n)$ Hz

27.4392	86.6338	185.2915	323.4122	500.9960
50.5621	109.7567	208.4143	346.5351	524.1189
89.1002	148.2948	246.9525	385.0733	562.6571
143.0536	202.2482	300.9059	439.0267	616.6105
212.4223	271.6169	370.2746	508.3954	685.9792

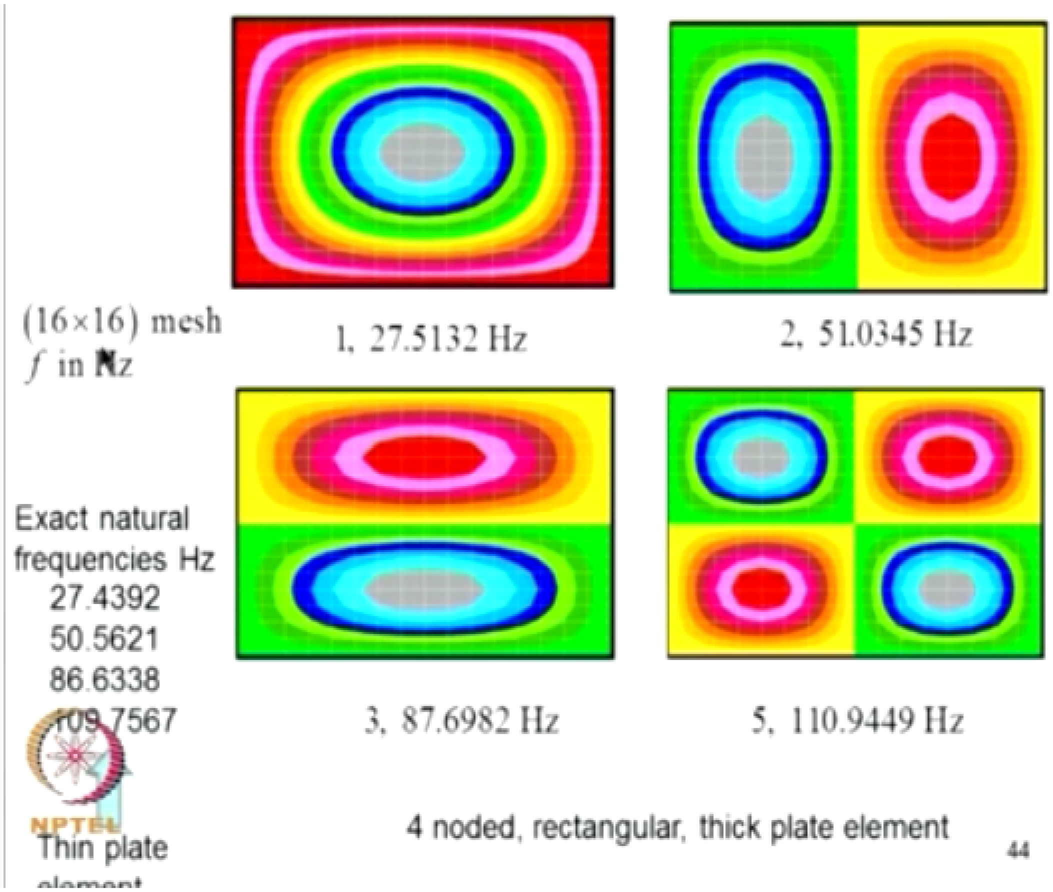


(1,1) (2,1) (1,2) (2,2)

Now for sake of numerical illustration we will consider a 50 mm thick steel plate which is 4 meter by 2.5 meter and the exact natural frequency is computed as per this formula is given here, and these frequencies are in Hertz, and these are the depiction of mode shapes, the dotted line indicates lines along which the mode shapes will be 0, so here there will be no 0 in the 1,1 mode, in 2,1 mode there will be 0 in this way, 1,2 mode like this so on and so forth, these are obtained from this exact expression for the Eigen function.



Now suppose if we use now a 4 x 4 mesh, rectangular 4 noded plate element and analyze the problem we get the frequencies to be this, first frequency has got obtained as 29.09 Hertz whereas its exact value is 27.43 hertz, in this analysis we have used thick plates whereas these results are for thin plate that also need to be borne in mind, these are the contours of mode shapes you can see here that there are no zeros here whereas this is the nodal line, this is a nodal line, and there are these nodal lines and these match with the patterns that the exact solutions depict.



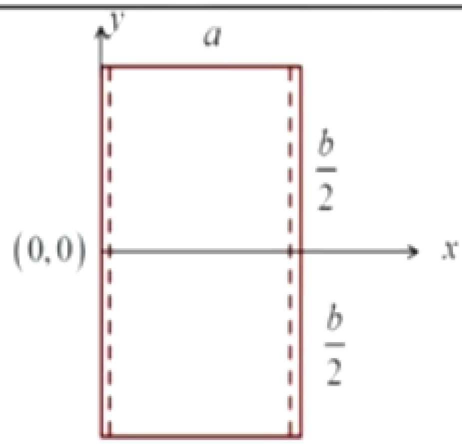
Now if we refine the mesh now instead of 4 x 4 mesh if we take 16 x 16 mesh, then the first frequency becomes 27.4392 and this is approaching the exact natural frequency, and the second, third are, these are the other frequencies, and the mode shapes these are the exact natural frequencies 27.4392 and what we have got through finite element analysis is shown in the caption here, so 27.5132 needs to be compared with 27.4392 and so on and so forth, 51 with 50, 87 with 86 and so on so forth, so as we see as we refine the mesh we are approaching the exact solutions.

Rectangular plate with two opposite edges simply supported and the other two edges having other support conditions

$$\nabla^4 W - k^4 W = 0$$

$$\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} - k^4 W = 0$$

$$W(x,y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) Y_n(y)$$



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Now for other, this is all round simply supported boundary condition for example if you have again rectangular plates with 2 opposite edges which are simply supported then it is possible to develop solutions for this, so what we do is we consider the partial differential equation and we expand this solution, in X direction we use the exact Eigen functions, and in Y direction we use this and using a Galerkin type of projections we get equations for YN(y) so we get the

$$\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} - k^4 W = 0$$

$$W(x,y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) Y_n(y)$$

$$\Rightarrow \sum_{n=1}^{\infty} \left\{ Y_n^{iv} - 2Y_n''\left(\frac{n\pi}{a}\right)^2 + Y_n\left(\frac{n\pi}{a}\right)^4 - k^4 Y_n \right\} \sin\left(\frac{n\pi x}{a}\right) = 0$$

$$\Rightarrow Y_n^{iv} - 2Y_n''\left(\frac{n\pi}{a}\right)^2 + \left\{\left(\frac{n\pi}{a}\right)^4 - k^4\right\} Y_n = 0$$

This is an eigenvlaue problem with 4 bcs specified at $y = \pm\frac{b}{2}$.

If conditions at $y = \pm\frac{b}{2}$ are identical (fixed or free, for example), then the solution can be simplified by taking advantage of the symmetry.

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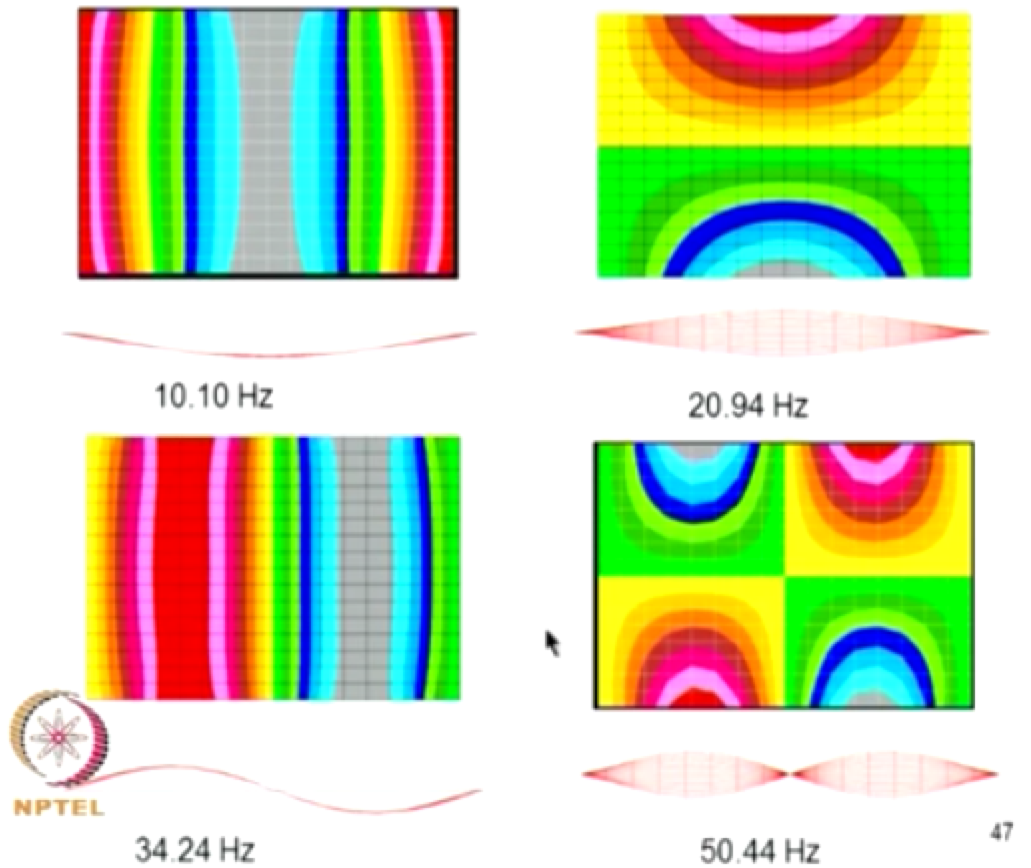
governing equation for YN is this, and this is, and again it is an Eigen value problem, it is a fourth order ordinary differential operator, and there will be 4 boundary conditions specified on Y at +-B/2. So if conditions at Y = +- B/2 are identical that for example if these 2 edges both are free or both are fixed or both are simply supported etcetera, then the solution can be simplified by taking advantage of symmetry.



Now I will not get into the analytical solution, so this is results for one such plate I have shown the cross section of the mode shape here, and this is the nodal lines and this is the 0, this again the nodal line here, and there are 2 nodal lines as shown here. Now there is one example

A Leissa (1969) Table 4.15, P.53

Fundamental natural frequency

$$\omega a^2 \sqrt{\frac{\rho h}{D}} = 18.258 \text{ for } \frac{a}{b} = 1.6$$

$$\Rightarrow f_1 = 14.2585 \text{ Hz}$$

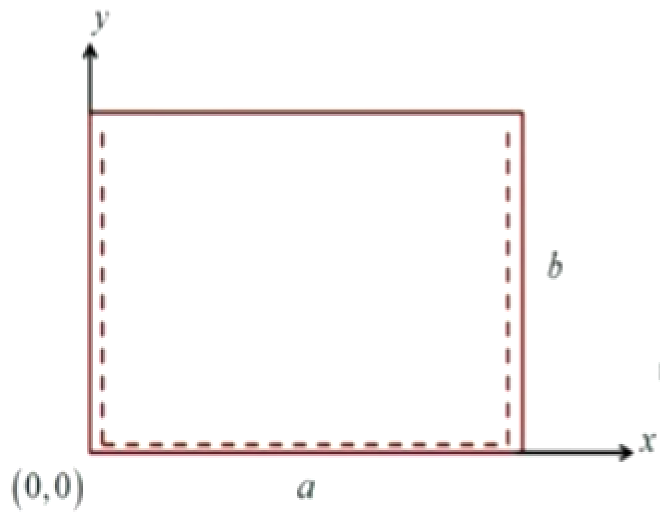
Numerical illustration (SSSS)

$h = 50$ mm

$E = 210$ GPa

$\rho = 7800$ kg/m^3

$a = 4$ m; $b = 2.5$ m



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considered by Liessa, he has considered a rectangular plate with 2 edges opposite, 2 opposing edges simply supported and the other edges, one edge is simply supported and the other edge is free, so for this case according to the data given in this book the first natural frequency is 14.2585 hertz, now we have analyzed this problem, okay, we analyze this problem and the

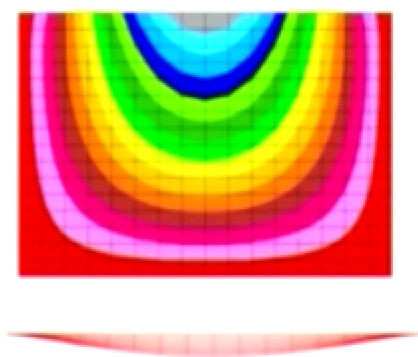
Rectangular plate with general combination of edge conditions

Use Rayleigh-Ritz/Galerkin technique with beam eigenfunctions in x and y directions

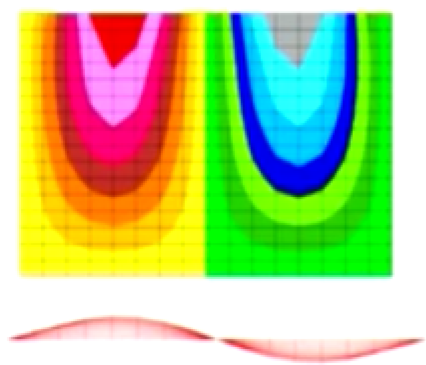
$$\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} - k^4 W = 0$$

$$W(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \alpha_{mn}\phi_n(x)\psi_m(y)$$

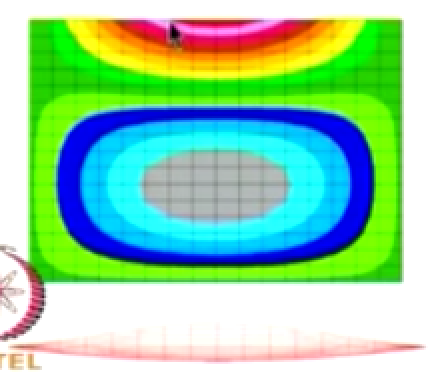
$\phi_n(x), \psi_m(y); m, n = 1, 2, \cdots$: beam eigenfunctions.



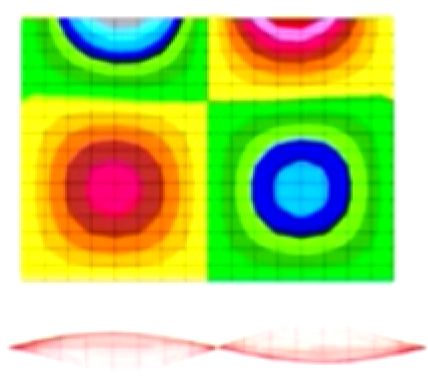
13.47 Hz



38.43 Hz



42.19 Hz



70.70 Hz

number we get for 14.2585 for the given you know measuring configuration is 13.47 hertz, so again we expect that I mean this result has to be examined by refining the mesh, and this also is based on an approximate solution, so one can only derive an order of magnitude type of comparisons here.

Now if the other edges, suppose none of the edges are simply supported, see the case that I mentioned just now is that 2 opposing edges are simply supported, but if that condition is not there so then we can use Rayleigh-Ritz or Galerkin technique by using beam Eigen functions in the 2 directions, and we can derive analytical solution which are again will be approximate so that can be done, okay. There are details of such results available in the monograph by Liessa.

Circular plate: all round clamped

$$V=\frac{D}{2}\int_A\left[\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)^2-2(1-\nu)\left\{\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)-\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right)\right)^2\right\}\right]dA$$

$$T=\frac{1}{2}\int_A\rho\dot{w}^2\,dA$$

$$D=\frac{Eh^3}{12\left(1-\nu^2\right)}$$

$$\Rightarrow D\nabla^4 w+m\ddot{w}=0 \text{ with } \nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

$$M_r=-D\left[\frac{\partial^2 w}{\partial r^2}+\nu\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)\right] \qquad Q_r=-D\frac{\partial}{\partial r}\left(\nabla^2 w\right)$$

$$M_\theta=-D\left[\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}+\nu\frac{\partial^2 w}{\partial r^2}\right] \qquad Q_\theta=-D\frac{1}{r}\frac{\partial}{\partial\theta}\left(\nabla^2 w\right)$$

$$V_r=Q_r+\frac{1}{r}\frac{\partial M_{r\theta}}{\partial\theta}$$

$$M_{r\theta}=-D(1-\nu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right) \qquad V_\theta=Q_\theta+\frac{\partial M_{r\theta}}{\partial r}$$

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Now circular plates are also other class of problems which are extensively studied in the monograph by Liessa and for that the strain here is the expression in the cylindrical polar coordinates and governing equations and stress resultants are reproduced in this view graph, this is for sake of completeness.

$$D\nabla^4 w + m\ddot{w} = 0$$

$$w(x,y,t) = W(x,y)\exp(i\omega t) \Rightarrow D\nabla^4 W - m\omega^2 W = 0$$

$$\Rightarrow \left(\nabla^2 + k^2\right)\left(\nabla^2 - k^2\right)W = 0 \text{ with } k^4 = \frac{\omega^2 m}{D}$$

$$W(r,\theta) = \sum_{n=0}^{\infty} W_n(r)\cos n\theta + W_n^* \sin n\theta$$

$$\Rightarrow \frac{d^2 W_{n_1}}{dr^2} + \frac{1}{r}\frac{dW_{n_1}}{dr} - \left(\frac{n^2}{r^2} - k^2\right)W_{n_1} = 0$$

$$\frac{d^2 W_{n_2}}{dr^2} + \frac{1}{r}\frac{dW_{n_2}}{dr} - \left(\frac{n^2}{r^2} - k^2\right)W_{n_2} = 0$$

Two similar equations for $W_{n_1}^*$ & $W_{n_2}^*$ are also obtained.

The above equations represent Bessel's equations.

Now again here if we assume for free vibration analysis all points on the structure vibrate harmonically the Eigen value problem will be again mathematically will be of this form, but this del square operator will be now quite different. Now if we expand the solution in terms of sin and cosine terms in theta we can get this set of equations for these amplitudes in R, which are depicted here, and similar equation for, this is equation for WN, and this equation for WN star can also be obtained.

$$W_{n_1} = A_n J_n(kr) + B_n Y_n(kr)$$

$$W_{n_2} = C_n I_n(kr) + D_n K_n(kr)$$

J_n & Y_n are the Bessel's function of the I and II kind

I_n & K_n are the modified Bessel's function of the I and II kind

General solution for the plate equation in cylinderical polar coordinates:

$$W(r,\theta) = \sum_{n=0}^{\infty} \left[A_n J_n(kr) + B_n Y_n(kr) + C_n I_n(kr) + D_n K_n(kr) \right] \cos n\theta$$

$$+ \sum_{n=0}^{\infty} \left[A_n^* J_n(kr) + B_n^* Y_n(kr) + C_n^* I_n(kr) + D_n^* K_n(kr) \right] \sin n\theta$$

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Now solution to this pair of equations can be obtained in terms of Bessel's functions, Bessel's function of first and second kind and the modified Bessel functions of first and second kind, so we can construct for cylindrical polar coordinate solutions in this form, so this can be used by, used to determine the characteristic equation by imposing appropriate boundary conditions.

Circular plate

(no internal holes; centre at centre of the circle)

Terms involving $I_n(kr)$ & $K_n(kr)$ are set to zero in order to avoid singular behavior at $r = 0$.

If boundary conditions are symmetric with respect to one or more diameters of the circle, $\sin n\theta$ terms are also removed. $\Rightarrow$

$$W_n = \left[A_n J_n(kr) + B_n Y_n(kr)\right]\cos n\theta$$

Ex: Plate clamped all around

$$W(a) = 0, \frac{dW}{da}(a) = 0 \text{ at } a = R.$$

$\Rightarrow$ Characteristic equation $\begin{vmatrix} J_n(\lambda) & I_n(\lambda) \\ J_n'(\lambda) & I_n'(\lambda) \end{vmatrix} = 0$

$\lambda = ka$ & a prime denotes derivative wrt the arguement.

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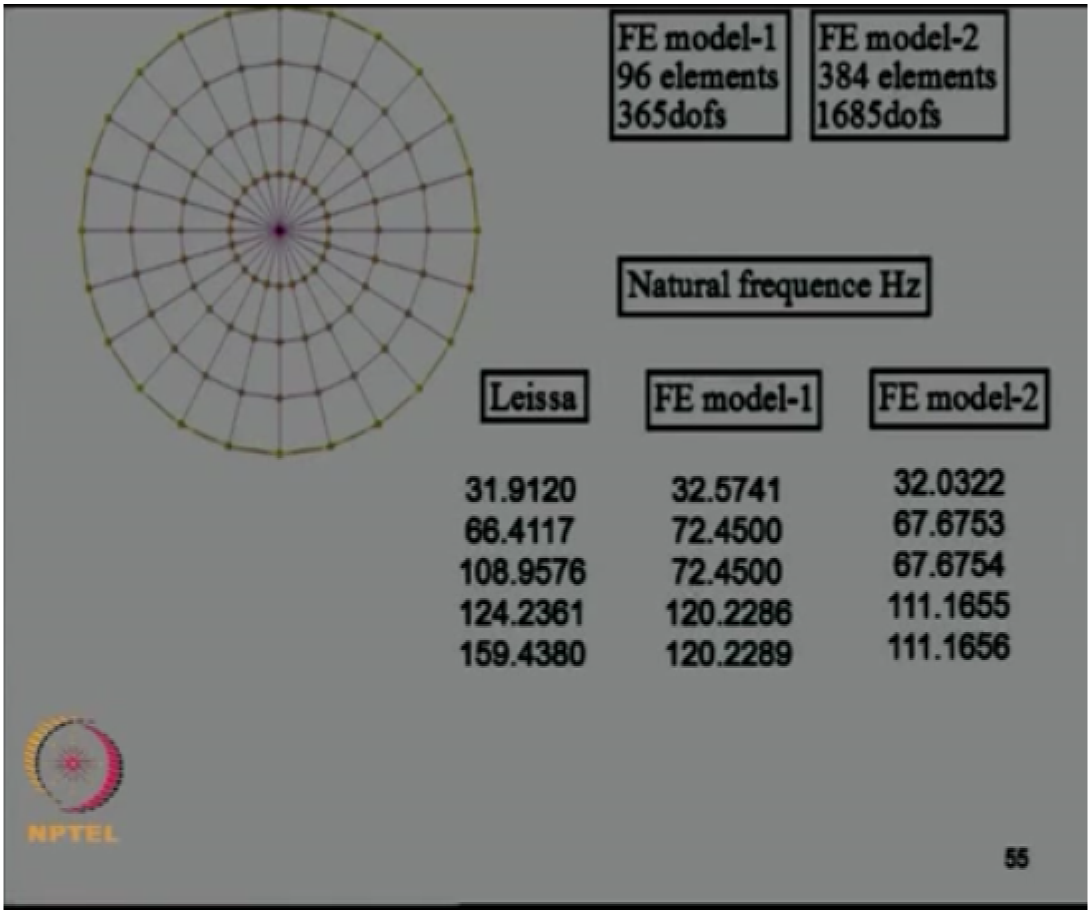
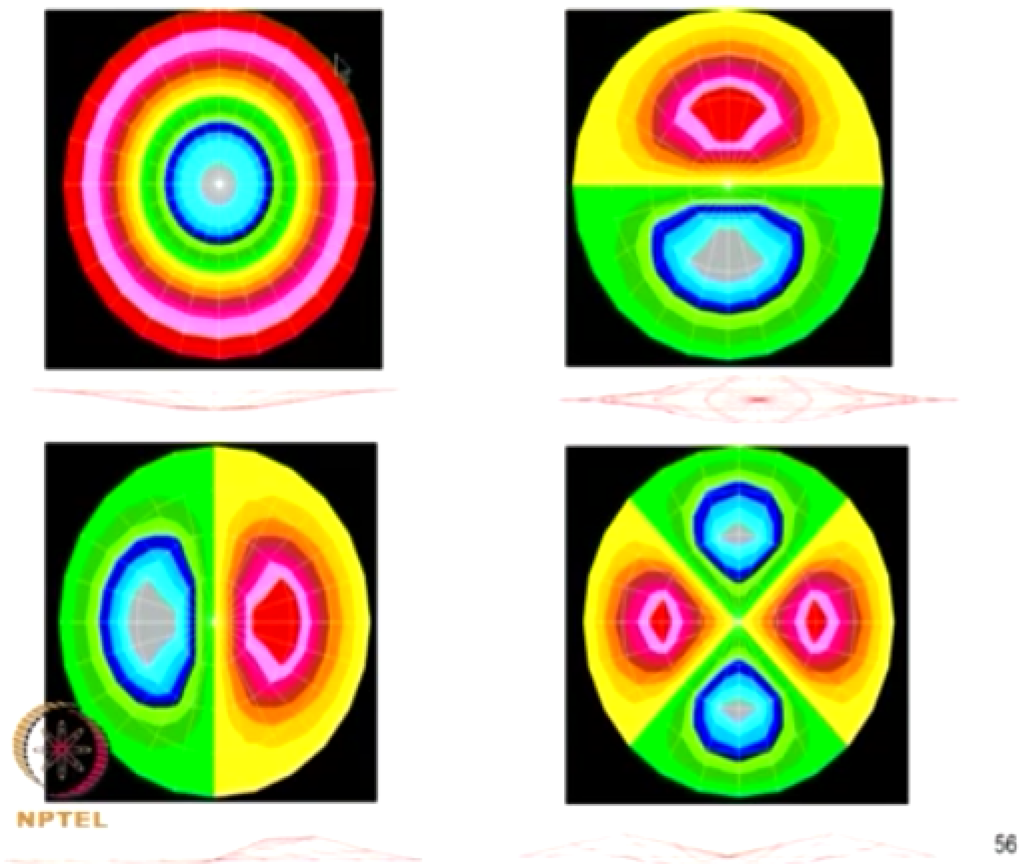
Now let's consider a circular plate with a no solid circular plate with no internal holes and at the center of the circle, and in this case terms involving IN and KN are set to 0 in order to avoid singular behavior at R = 0, so if boundary conditions are symmetric with respect to 1 or more diameters of the circle then sin N theta terms can be also be removed, and we get the representation for the Eigen function in this form.

Now if we consider plate clamped all around, the boundary conditions will be this and we can derive the characteristic equation and this problem has been solved and exact solutions are



available, and the monograph by Liessa gives that, so what we have done is we have created two FE models with the mesh details as shown here, in model 1 there are 96 elements, in model 2 there are 384 elements, so this is a more refined model, and the natural frequency is, first few natural frequencies are listed here. Now actually this system because of its symmetry the Eigen



values will be repeating, for example I will show the mode shapes, this is a first mode shape that means the plate deflects all through in this manner symmetry, in an axisymmetric manner, in the second mode the mode shape will be like this, and the nodal line is this, and a nodal line which is orthogonal to this will be another mode shape worked at the same value of the frequency, so that is what this means that if natural frequencies repeat.

FE model-1	FE model-2
96 elements	384 elements
365dofs	1685dofs

Natural frequence Hz

Leissa	FE model-1	FE model-2
31.9120	32.5741	32.0322
66.4117	72.4500	67.6753
108.9576	72.4500	67.6754
124.2361	120.2286	111.1655
159.4380	120.2289	111.1656

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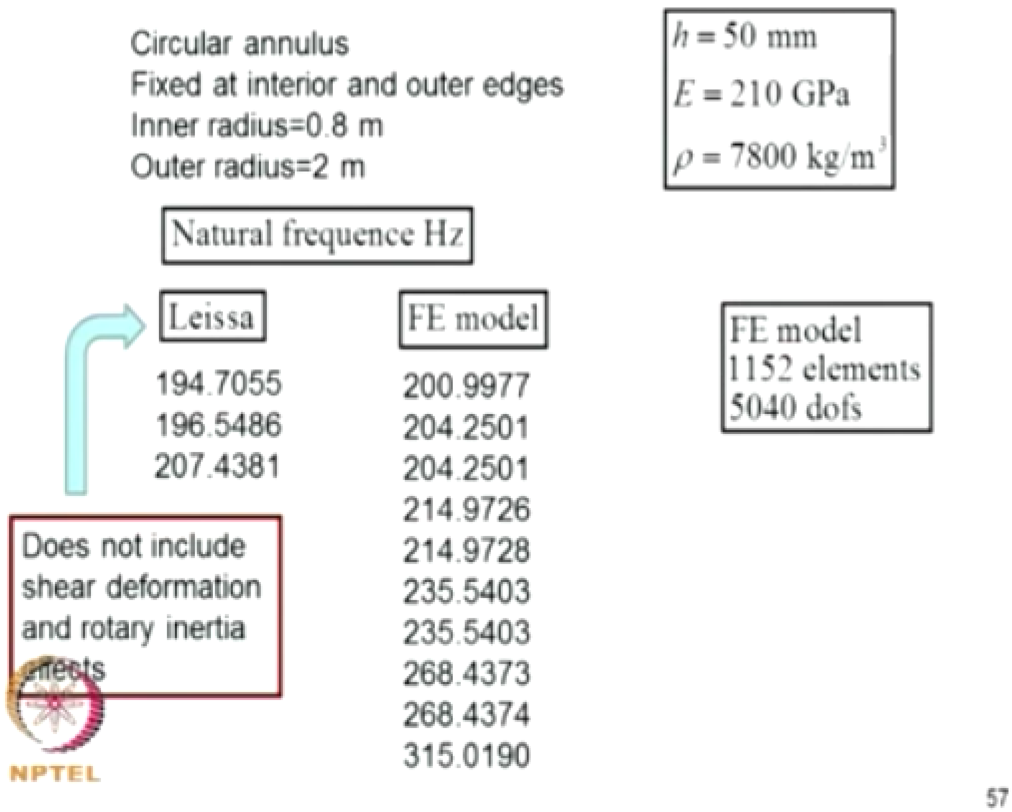
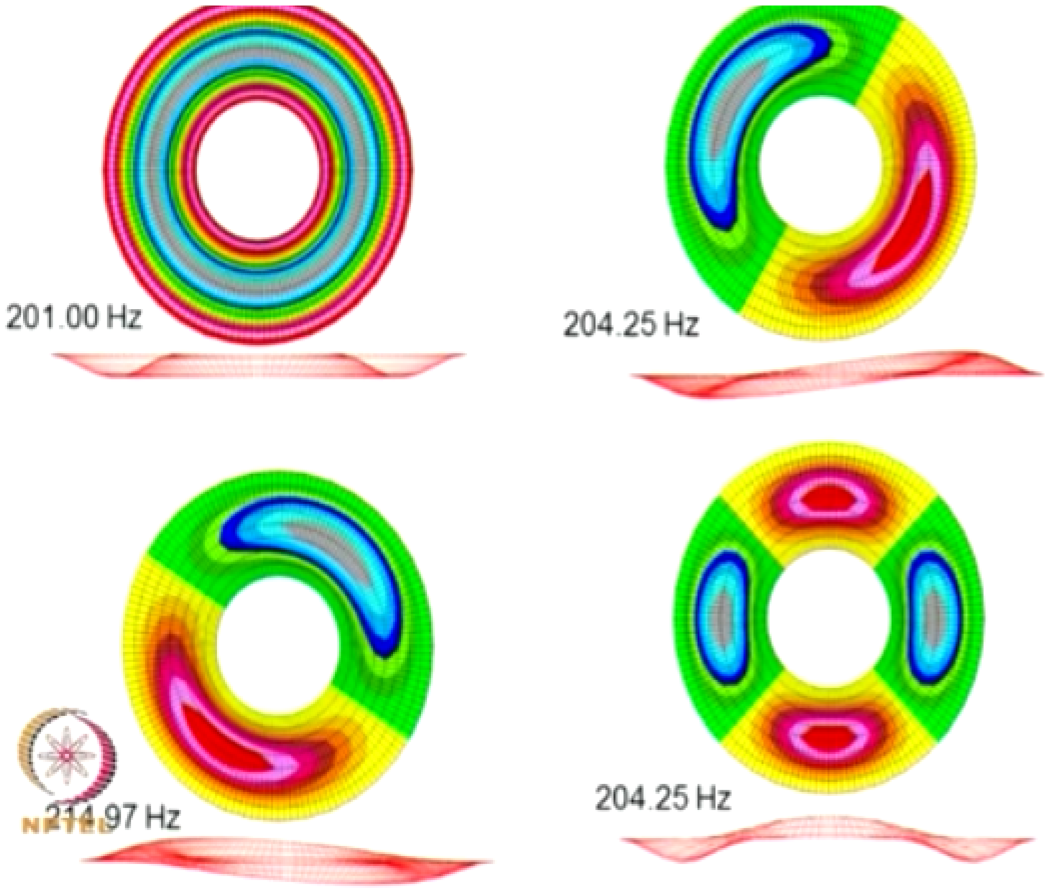
Similarly the next mode also appears like this and nodal lines are like this, and these 2 frequencies also repeat, so in a refined model we get similar features and we see that the answers are slowly moving towards the exact solutions.



Now another problem that is available in Liessa book is that of a circular annulus fixed at interior and outer edges, and inner radius of 0.8 meters and outer radius of 2 meter, this is a numerical example that we have considered the results are available for this case, and we have done an few analysis of this, and according to Liessa's monograph the 3 frequencies, first 3



frequencies are shown here, and according to the model that has been developed we get 200, 204, and 214 respectively as approximation to these 3 numbers, so these are the mode shapes for the 3 modes, this is the first mode, second mode, third mode, and the fourth mode.

Exercise

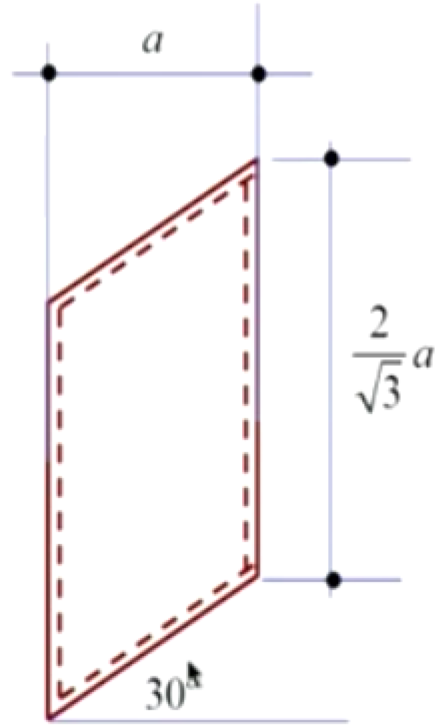
$h = 50$ mm
$E = 210$ GPa
$\rho = 7800$ kg/m^3
$a = 2$m

$x = 0$
$x = a$
$y = \dfrac{x}{\sqrt{3}}$
$y = \dfrac{x}{\sqrt{3}} + \dfrac{2a}{\sqrt{3}}$

Exact natural frequencies

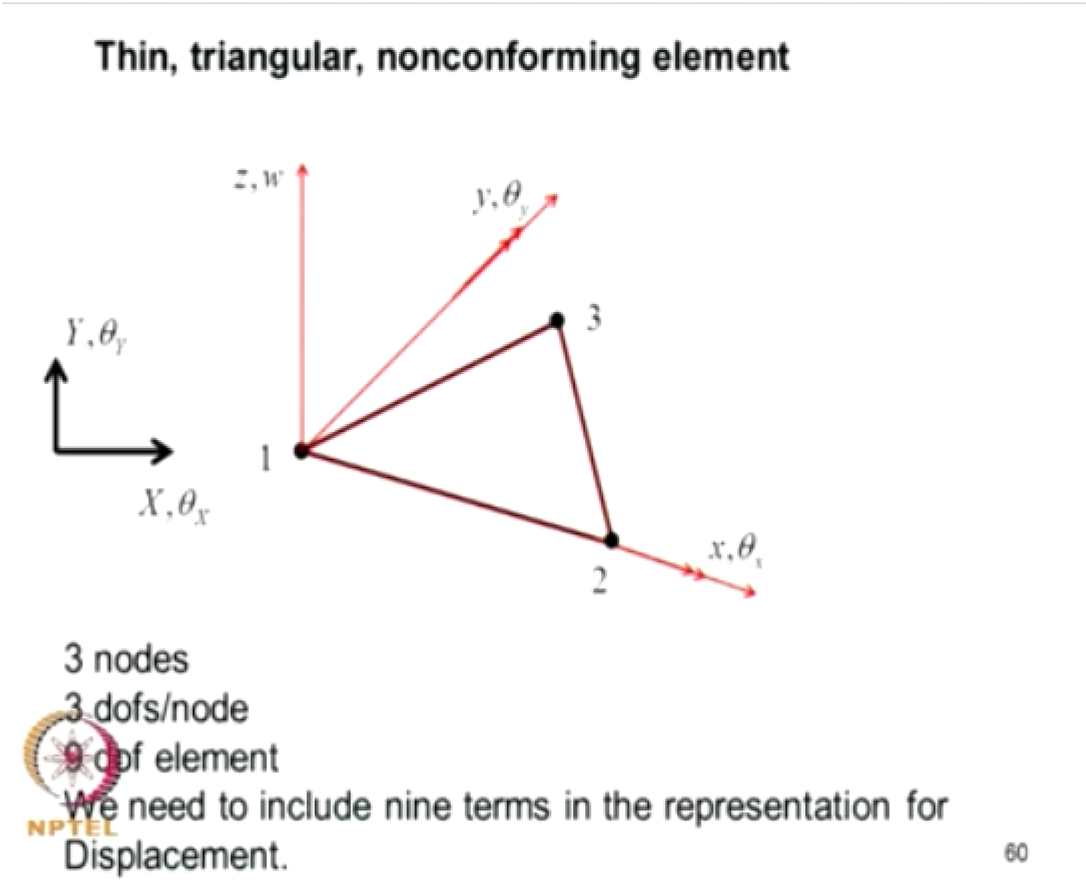
$$\omega_{mn} = \frac{\pi^2}{4a^2}\left(m^2 + mn + n^2\right)\sqrt{\frac{D}{\rho h}}$$

Leissa, 1969, pp.106-107

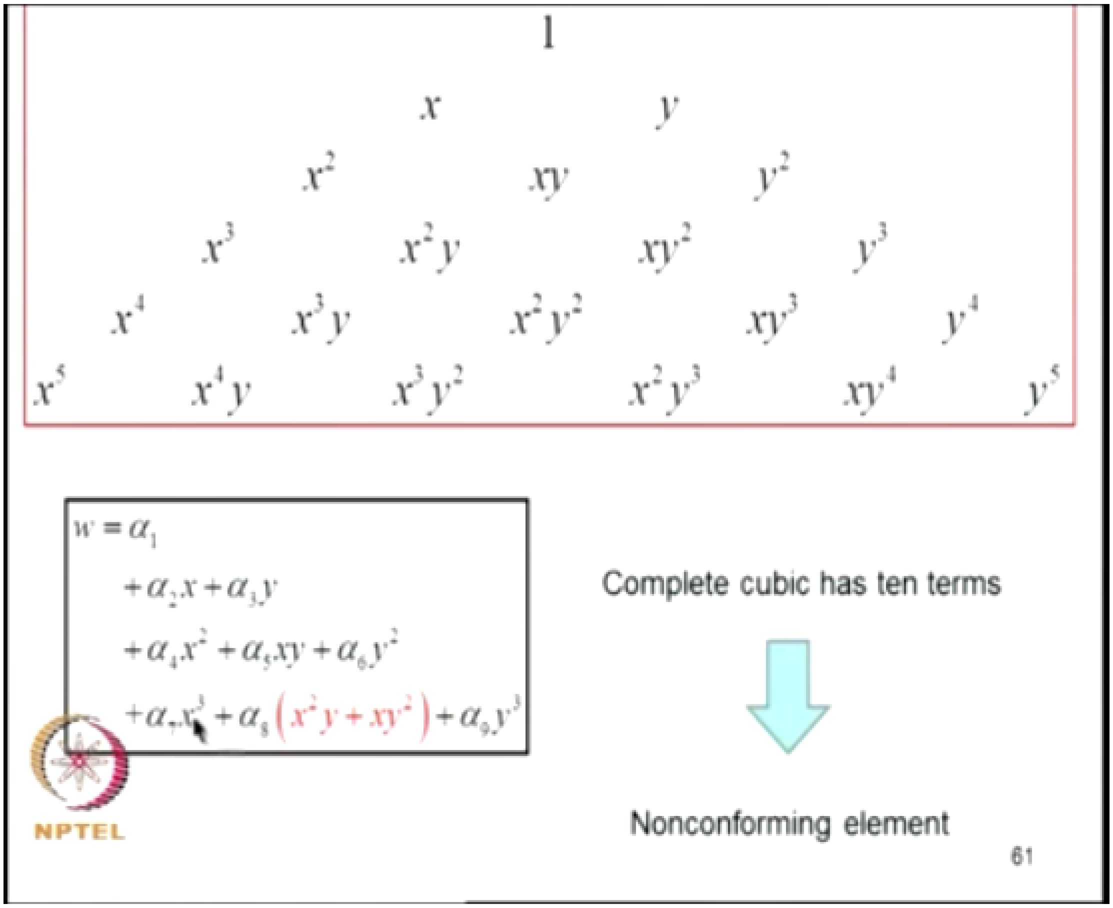


All edges simply supported

Now I'll leave it as an exercise, this is a parallelogram plate which is all round simply supported and dimensions, this angle is 30 degree, and dimensions are as shown here and exact natural frequencies for this are available, and they are given by this expression the suggested exercise is to make a finite element model for this plate and compute the natural frequencies and compare it with this exact solutions given in this monograph.



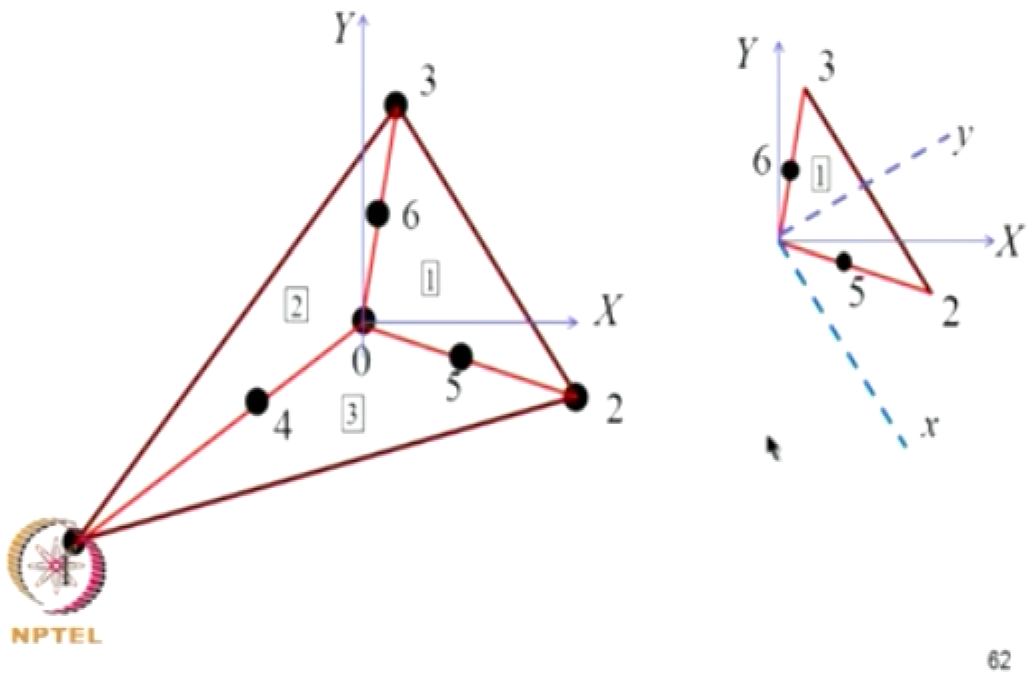
Now in the next lecture we will consider triangular geometries and develop first a non-conforming element, we will consider 3 nodes, at each node there will be 3 degrees of freedom and this will be 9 noded element, and we need to include 9 terms in the representation for the displacement, so we will develop this element by assuming a shape function of this form,



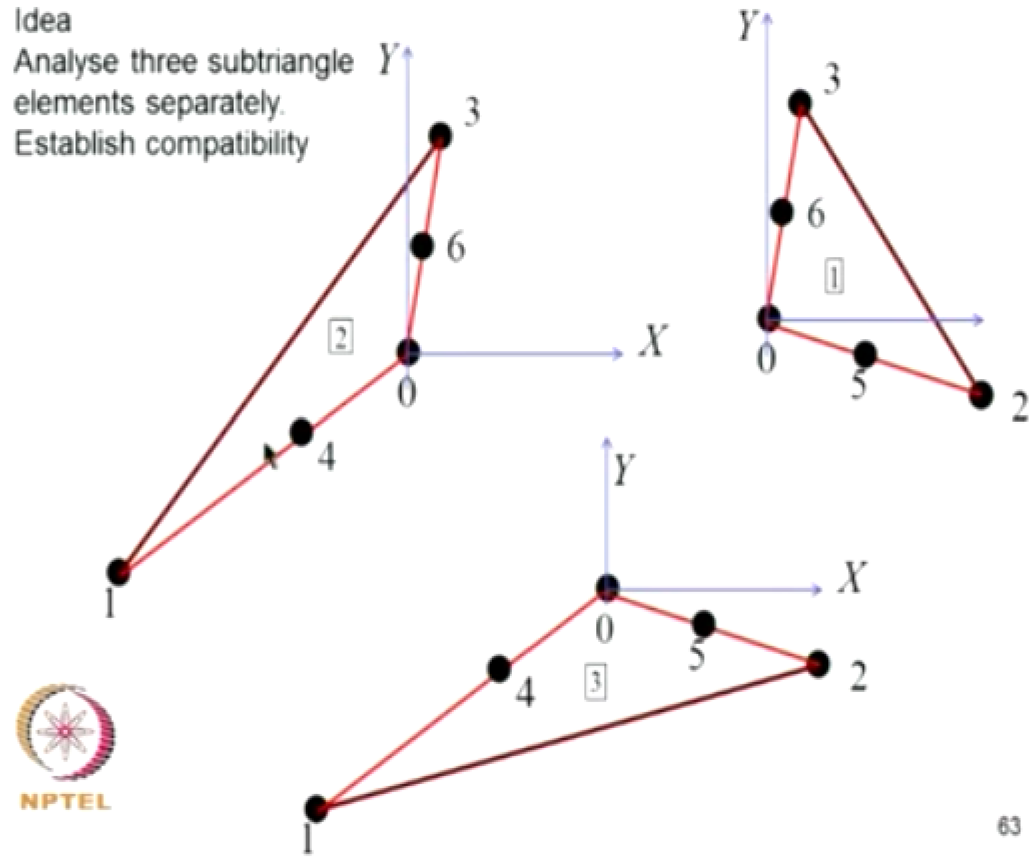
actually if we include a complete cubic it will have 10 times, but we need only 9 terms, so what we do is we club this X square Y and XY square terms and associate it with only one single generalized coordinate, and we will develop this element in the next class.

Thin triangular, conforming element

Idea: Ensure that normal slope along edges varies linearly



We will also develop another approach for and a thin conforming triangular element, where what we will do is for this triangle element 1, 2, 3, will introduce an internal point O, and divide this triangle into 3 sub triangles 1, 2, and 3, and will analyze each one separately, and



consequently there will be 27 generalized coordinates, but for this triangular element there will be 9 coordinates so 18 of them we need to eliminate, we will do so by seeking compatibility at 1, 2 and 3 and also by seeking equivalence of normal slopes at points 4 for triangle 2 and 3, at point 5 for triangles 1 and 3, and at point 6 for 1 and 2, so we will develop these elements in the next lecture, and see how this formulation develops. So at this stage we will close this lecture.

Programme Assistance
Guruprakash P
Dipali K Salokhe
Technical Supervision
B K A N Singh
Gururaj Kadloor
Indian Institute of Science
Bangalore