Indian Institute of Science
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<u>Course Title</u>
Finite element method for structural dynamic
And stability analyses
Lecture – 21
3D Solid element
By

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Previous lecture we have considered a 2 dimensional continuum, problems of 2 dimensional continuum we have considered plane stress and plane strain problems. Now we will continue that discussion and extend our discussion to 3 -dimensional solid elements.

Finite element method for structural dynamic and stability analyses

Module-7

Analysis of 2 and 3 dimensional continua

Lecture-21 3D solid element

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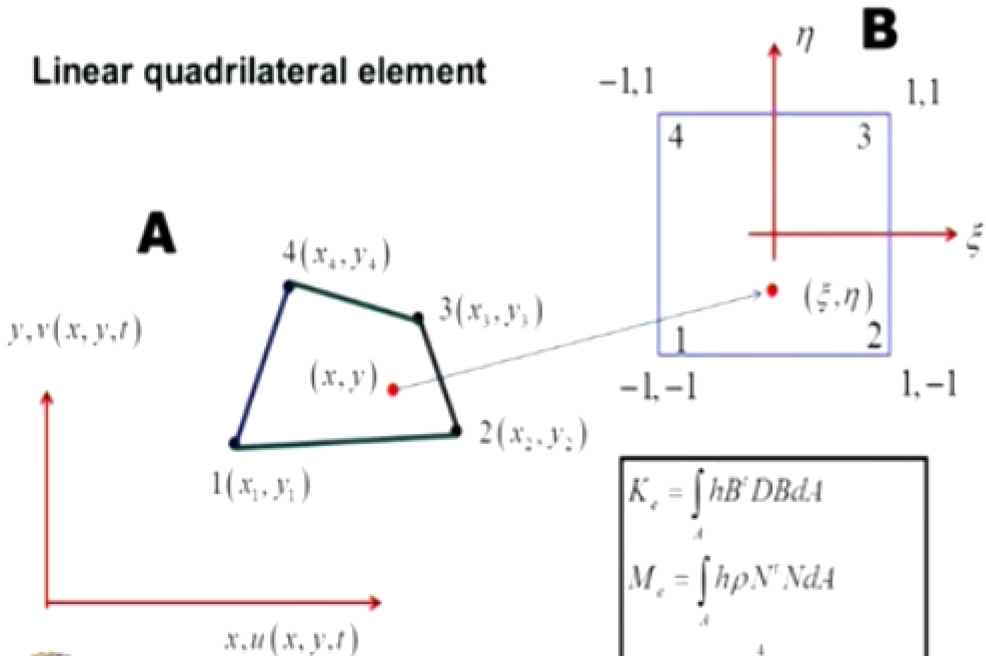
Linear quadrilateral element



$$K_e = \int_A hB^t DB dA$$

$$M_e = \int_A h\rho N^t N dA$$

$$x(\xi,\eta) = \sum_{j=1}^{4} N_j(\xi,\eta) x_j$$

$$y(\xi,\eta) = \sum_{j=1}^{4} N_j(\xi,\eta) y_j$$

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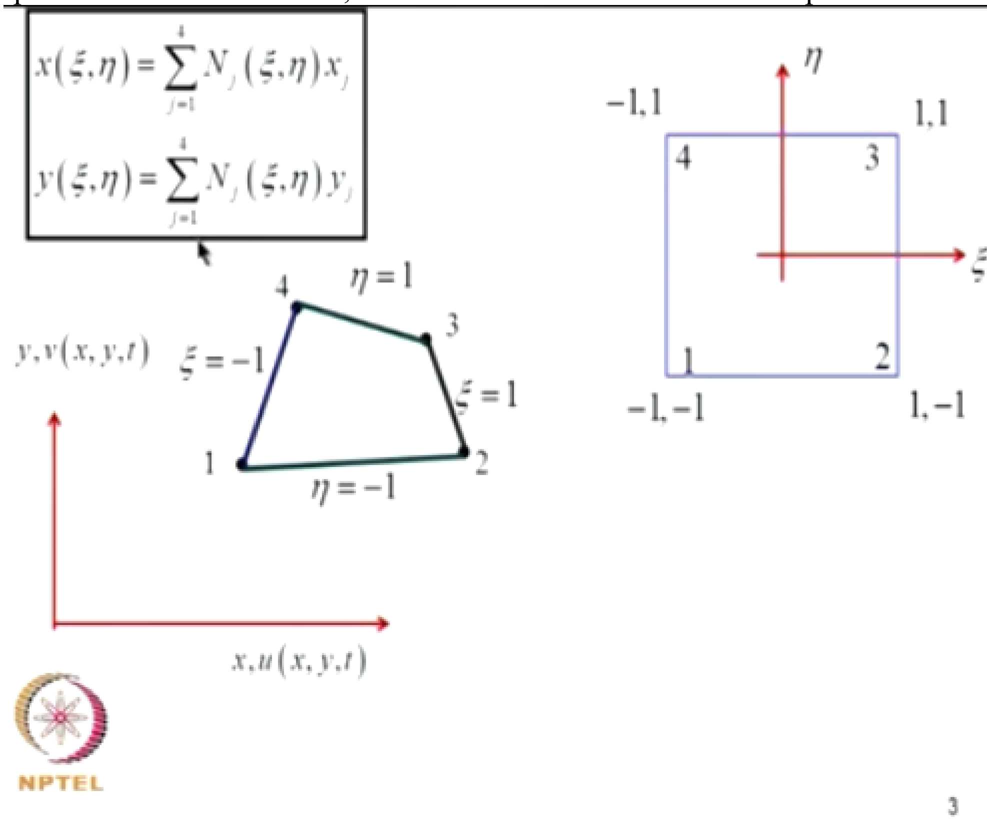
So to quickly recall one of the element that we developed was a linear quadrilateral element having geometry as shown here, and we mapped this to a unit square and this mapping was essentially done to evaluate these integrals, and in doing so we represented the coordinates here X and Y, in terms of this new coordinates XI and eta using the same trial function that were used to represent the field variables, so this formulation is known as isoparametric formulation.



So this is how we did and this was a representation, so this we showed that the edges of this master element correspond to edges of these quadrilateral elements, and we discussed how to carry out the evaluation of stiffness and mass matrices with these representations.

- Isoparametric formulation

$$x(\xi,\eta)=\sum_{j=1}^{4}N_j(\xi,\eta)x_j \;\&\; y(\xi,\eta)=\sum_{j=1}^{4}N_j(\xi,\eta)y_j$$

$$u(\xi,\eta,t)=\sum_{j=1}^{4}N_j(\xi,\eta)u_j(t) \;\&\; v(\xi,\eta,t)=\sum_{j=1}^{4}N_j(\xi,\eta)v_j(t)$$

- $M_e=\int_{A_e}\rho h[N]^t[N]dA=\int_{-1}^{1}\int_{-1}^{1}\rho h[N]^t[N]|J|d\xi d\eta$
- $K_e=\int_{A}hB^tDBdA=\int_{-1}^{1}\int_{-1}^{1}hB^tDB|J|d\xi d\eta$
- Use Gauss quadrature to evaluate these integrals.
- A polynomial of order p is integrated exactly by employing n=smallest integer greater than $0.5(p+1)$.
- Choice of order of integration needs to be made carefully.

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Now in doing so what we observed was for the kind of geometries that we considered this, the evaluation of this mass and stiffness, elements of mass and stiffness matrices required evaluation of quadratures as shown here, and we propose that these integrals be evaluated using Gauss quadrature, and one of the issue in implementing this is to decide upon the order of Gauss quadrature, so the rule that was mentioned was that a polynomial of order P is integrated exactly by employing n smallest integer greater than 0.5(p+1), so I outlined the rationale behind making this decision, we showed that the points where we need to evaluate this integrands coincide are actually zeros of Legendre polynomials and that formulation was discussed in the previous lecture.

So the choice of order of integration needs to be made carefully, a wrong choice can immediately lead to poor estimates of elements of structural matrices. Now two references that

References

- M Petyt, 1998, Introduction to finite element vibration analysis, Cambridge University Press, Cambridge
- S S Rao, 2011, The finite element method in engineering, 5th Edition, Elsevier, Amsterdam.

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I have been using in these discussions is books by Petyt and SS Rao, so the details are mentioned here.

Discussion & miscellaneous remarks

Convergence and choice of order of interpolation polynomial.

- What happens if we reduce the element size successively? Does the FE solution converge?

Requirements to be satisfied by the interpolation functions

(A) The displacement field must be continuous within the element domains. This is automatically satisfied if since we are using polynomials as interpolation functions.

(B) Consider the Lagrangian $L = T - V$. This would be a function of the field variables and its spatial derivatives. Let n=highest order of partial derivative of field variable that appears in L. All the uniform states of the field variable and its derivatives up to order n must be correctly represented in the limit of element size going to zero.

(C) The displacement field and its derivatives up to order $n-1$, must be continuous at the element boundaries.

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Now before we take up the problem of 3-dimensional elasticity, a continuum elements we can raise few miscellaneous points and have some discussions. Now the first point I would like to discuss is convergence and choice of order of interpolation polynomial, so the question that we can ask is what happens if you reduce the element size successively in a given finite element model for a structure, does the finite element solution converge, so under what condition it happens? So there are certain requirements that needs to be satisfied by the interpolation functions to guarantee that this happens, the first is the displacement field must be continuous within the element domains, this is actually automatically satisfied since we are using polynomials as interpolation function in terms of nodal values of the field variables, so within an element continuity is guaranteed, so this condition A is not a problem.

Now condition B will consider the Lagrangian T – V, all these conditions I am discussing with reference to structural mechanics problems, now the Lagrangian would be a function of the field variables and its spatial derivatives, and also time derivatives but we are not discretizing in time at this stage, so we need not discuss that aspect. Now if N is the highest order of partial derivative of field variable that appears in L, let us define N as that, then all the uniform states of the field variable and it's derivatives up to order N must be correctly represented in the limit of element size going to 0, this is one of the requirement, I'll explain what it means. Next condition is the displacement field and its derivatives up to order N - 1 must be continuous at the element boundaries.

(B) $\Rightarrow$

- If all nodal displacements are identical, the field variable must be constant within the element, that is the element must permit rigid body state.
- Requirement on derivative $\Rightarrow$ requirement that the element must permit constant strain state.

Axially deforming element $V = \frac{1}{2}\int_0^L AE\left(\frac{\partial u}{\partial x}\right)^2 dx$; Field variable: $u(x,t)$

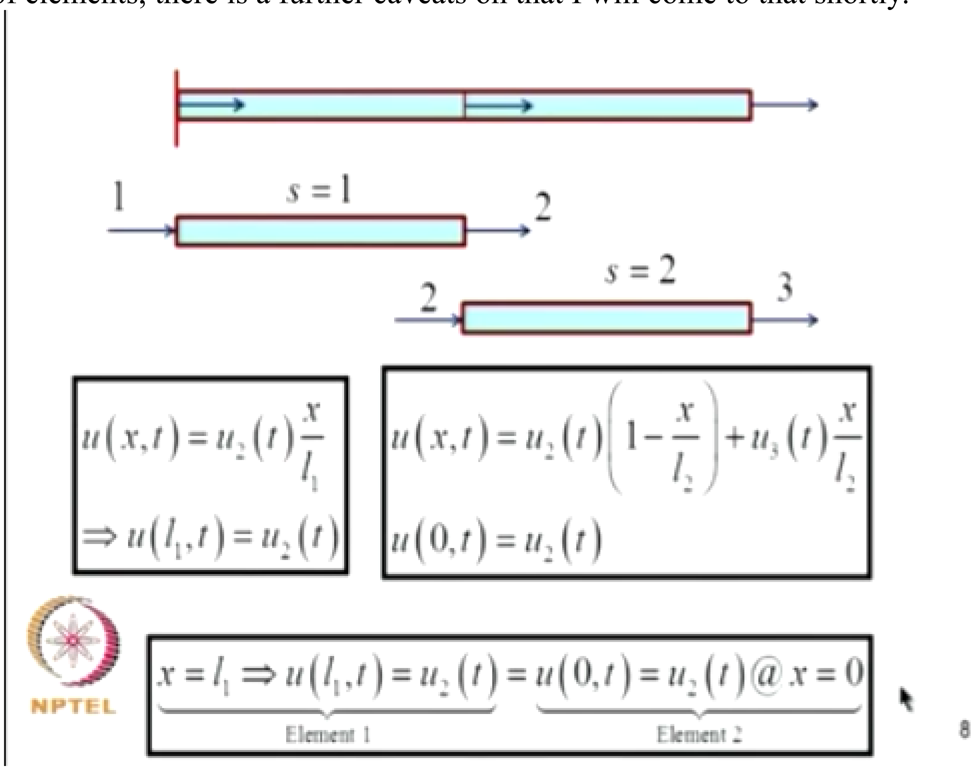
Highest order of derivative: 1; Interpolation used: $u(x,t) = u_1(t)\left(1-\frac{x}{l}\right)+u_2(t)\frac{x}{l}$

$u(x,t)$ = constant $= u_0(t) \Rightarrow u_1(t) = u_2(t) = u_0 \Rightarrow u_0 = u_0\left(1-\frac{x}{l}\right)+u_0\frac{x}{l} = u_0$ (OK)

$u'(x,t) = \frac{u_2(t)-u_1(t)}{l}$ = constant $\Rightarrow$ the requirement on 1st derivative is satisfied.

Now let's consider condition B, now what is condition B? If N is the highest order of partial derivative of the field variable that appears in our Lagrangian, all the uniform states of the field variable and its derivatives up to order N must be correctly represented in the limit of element size going to 0, so now if all nodal displacements are identical the field variable must be constant within the element, so that is the element must permit rigid body state. Then requirement on derivative actually translates into requirement that the element must permit constant strain states, see for example we can explain with respect to a 2 noded axially deforming rod with 2 degrees of freedom, U1, U2, we know the expression for strain energy is given by this, the field variable is U.

Now what is the highest order of derivative appearing here? Is 1, the interpolation used is U1, 1 – X/L + U2 X/L. Now if U(x,t) is constant that means U1 = U2, I want that U should be constant, so if I put that in the assume displacement form I get U(x,t) is U naught, actually U(x,t) turns out to be U naught which is what we are checking. Now similarly if I find the first derivative it is U2 - U1/L which is a constant, so that second condition is also satisfied, so when we use this type of element we can expect convergence of the FE solution as we increase number of elements, there is a further caveats on that I will come to that shortly.



Now this also we have seen which is now, this is a condition C, the displacement field must be and it's derivative up to order N - 1 must be continuous at the element boundaries, this we have seen, suppose there are 2 elements 1 and 2 with the degrees of freedom as named here, U(x,t) for the first element is given by, suppose this end is fixed it is U2 into X/L1, so if I now evaluate the field variable at L1 it is U2(t). Now similarly U(x,t) here is given by U2 into 1 – X/L2 + U3(t) X/L2. Now at X = 0 for this element U(0,t) is U2(t) as you can see here, so the field variable is continuous.

Now that means we have to look at continuity of field variable and it’s derivative up to N – 1,.n here is 1, therefore I should only look at continuity of the field variables, indeed this is satisfied in this element.

Remarks

- Elements which satisfy conditions A and C are called compatible or conforming elements.
- Elements which satisfy B are called complete elements.
- The field variable is said to possess C^r – continuity if its r^{th} derivative is continuous.
- Completeness requirement $\Rightarrow$ field variable has C^n – continuity within the element.
- Compatibility requirement $\Rightarrow$ field variable has C^{n-1} – continuity across element interface.

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Now elements which satisfy conditions A and C are called compatible or conforming elements, elements which satisfy B are called complete elements, the field variable is said to possess CR continuity, if it is R-th derivative is continuous. The completeness requirement implies that field variable has CN continuity within the element, and a compatibility requirement implies that the field variable has CN-1 continuity across element interface, so these are the requirements that we need to satisfy.

If the requirments A,B and C are satisfied, the FE approximation converges to the correct solution if the FE mesh is refined (that is if we use inreasing number of elements with smaller dimensions).

Note:

While the mesh is refined, the form of the interpolation funciton must remain unchanged and the mesh refinement must be such that the mesh with larger number of elements contains the mesh with with smaller number of elements.

Also, the mesh refinement must ensure that all points in the stucture are within an element.

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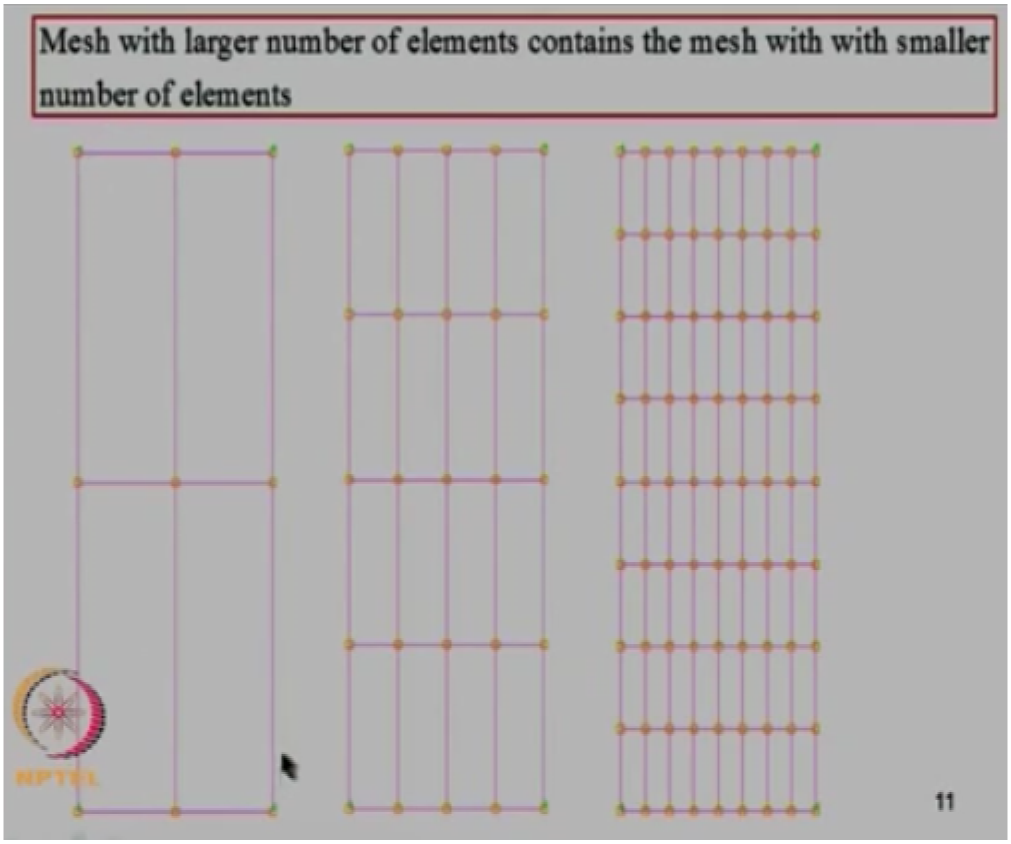
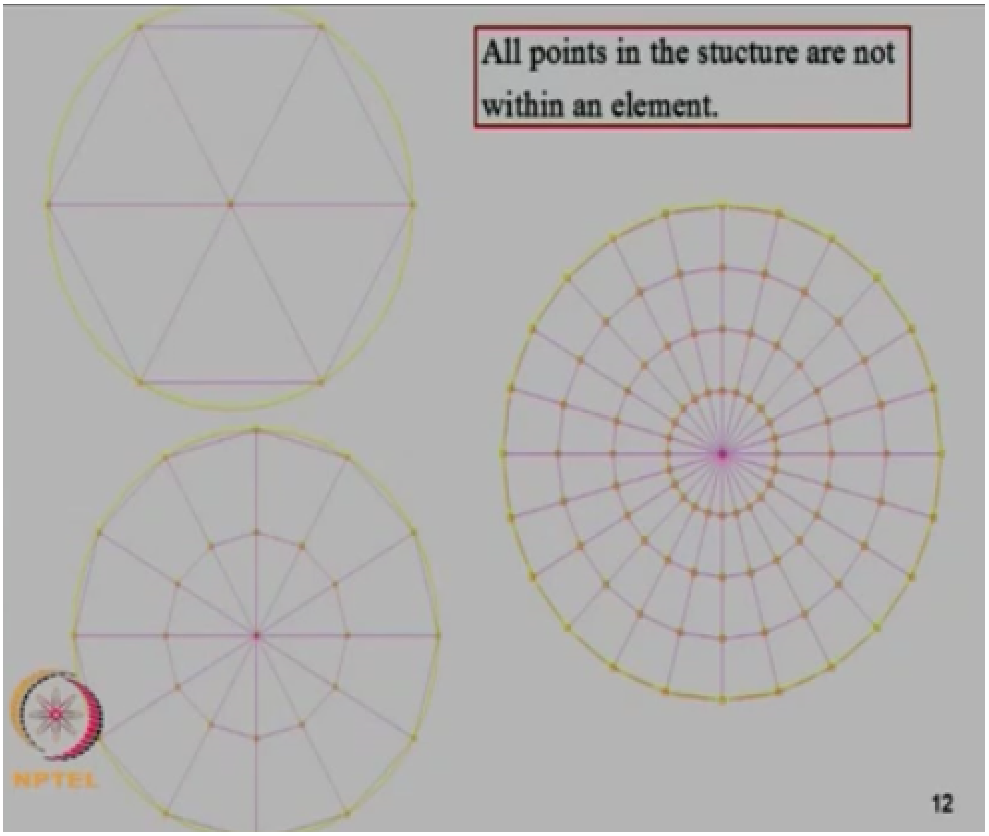
If the requirements A, B and C are satisfied, the FE approximation converges to the correct solution if the FE mesh is refined, that is if we use increasing number of elements with smaller dimensions following certain requirements, so what is this? When we are refining the mesh the form of the interpolation function must remain unchanged, you cannot change the form of the interpolation function and the mesh refinement must be such that the mesh with larger number of elements contains the mesh with smaller number of elements, so also the mesh refinement must ensure that all points in the structure are within an element, so what all this means?



Suppose you use for a rectangular domain a mesh with 4 elements as shown here, so when you make a mesh with 8 elements, this mesh should contain these 4 nodes okay, so these nodes must be there, that means I have to partition this element, okay, so in the next refinement I should partition each of these elements, okay, suppose this is now 2/2 mesh, if I use 3/3 then this mesh won't be a subset of this mesh, so when we talk about convergence we cannot talk of convergence from that point of.



Similarly all points on the structure you know what it means by saying that they have to be all, all the points on the structure must be within an element, suppose you've circular domain and you are using straight edged elements to discretize that, clearly this portion is outside the, say if you consider this element with this discretization, this point is within the structure but not within the any of the element, so this type of discretization we cannot expect convergence, so as I refine this still there will be parts of the structure which don't get into our finite element, the domain that the finite elements cover. So the remedy to this would be to use a curved element.

Factors contributing to the development of an accurate FE model

- Accuracy with which the structure geometry is represented
- Choice of polynomials used for interpolation
- Distribution of elements and nodes
- Details of integration used in time marching

How to refine the FE model to improve accuracy?

- Reduce the element size (*h*-refinement)
- Increase the order of polynomial (*p*-refinement)
- Locate node points differently in a fixed element topology(*r*-refinement)
- Alter the mesh having differing element distributions
- Improvments to the time integration schemes

Alternatives invovling a combination of the above strategies.

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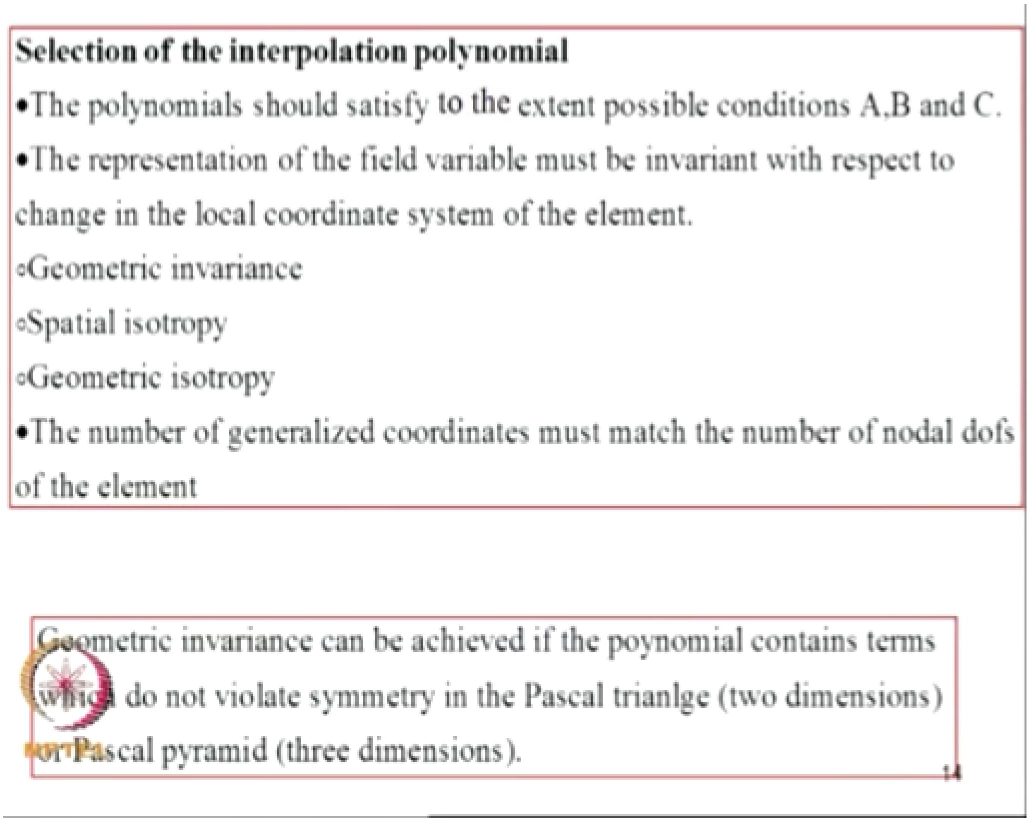
Now what are the factors which contribute to the development of an accurate finite element model? So we can say that accuracy with which the structure geometry is represented is one of the issues, for example in that circle example that I showed this type of issue is quite evident, then choice of polynomial used for interpolation, then distribution of elements and nodes for the same degree of freedom there can be 2 alternative placement of nodes and different shapes of finite elements, so one mesh may be superior to the other, then details of integration used in time marching, and also one could also include how you evaluate stiffness and mass matrices using Gauss quadrature.

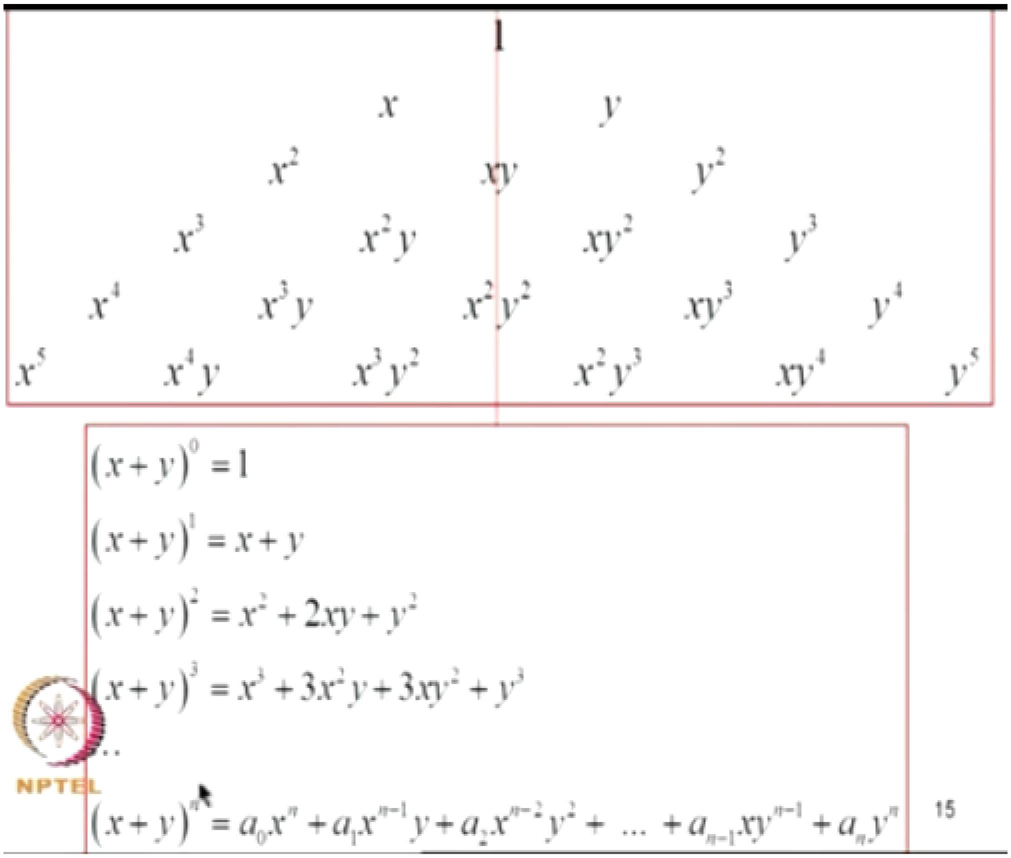
Now given that the accuracy of the FE model depends on these features, if we now ask the question how to refine the FE model to improve accuracy? Here you can reduce the element size that means the same domain will be covered with more elements, and safe for the model will have higher degrees of freedom, or retain the degrees of elements but increase the order of polynomial, instead of first-order interpolation use a higher order interpolation, so this first refinement is known as H refinement, where H refers to the size of the element, the second one is a P refinement where P refers to the order of the polynomial used. Next alter possibility locate node point differently in a fixed element topology, so this is known as R refinement, the other thing is alter the mesh having differing element distributions.

Next improvements to the time integration schemes if you are doing dynamics problem that also, you know, contributes to the accuracy that you can achieve, so alternatives involving a combination of all these strategies also can be thought of, so what are the issues in selection of the interpolation polynomial? The polynomial should satisfy to the extent possible conditions



A, B and C that is one of the requirement, the representation of the field variable must be invariant with respect to change in the local coordinate system of the element, suppose in a local coordinate system you have X, Y, Z coordinate by renaming X as Y and others similarly renaming other axis the behavior of the element should not change, okay, so then this is known as geometric invariance of spatial isotropy or geometric isotropy.

Next the number of generalized coordinates must match the number of nodal degrees of freedom of the element, this is you know essential. Now how to achieve geometric invariance? The geometric invariance can be achieved if the polynomial contains terms which do not violate symmetry in the Pascal triangle, which I will show now in 2 dimension or a Pascal pyramid in 3



dimension, what I mean is if you expand X + Y to the power of N in a binomial expansion you can arrange these terms as shown here, the first term is a constant it is here, the second one we will have X and Y, X square, XY, Y square, X cube, X square Y, for example X + Y to the power of 0 is 1, X + Y to the power of 1 is X + Y, X + Y to the power of 2 is X square + 2XY + Y square, so this is X square, XY, Y square and so on and so forth, so the last term will be having this distribution. Now whenever you are choosing the interpolation polynomial we have

Triangle element: $u(x,y,t) = \alpha_1 + \alpha_2 x + \alpha_3 y$

Rectangular element: $u(x,y,t) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$

$$\left.\begin{aligned} u(x,y,t) &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 \\ u(x,y,t) &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 y^2 \end{aligned}\right\}$$

Not appropriate. Interchanging of x and y would change the representation

Four noded element with 3 dofs/node

$$\begin{aligned} u(x,y,t) = \alpha_1 & \\ &+ \alpha_2 x + \alpha_3 y \\ &+ \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 \\ &+ \alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3 \\ &+ \alpha_{11} x^3 y + \alpha_{12} xy^3 \end{aligned}$$

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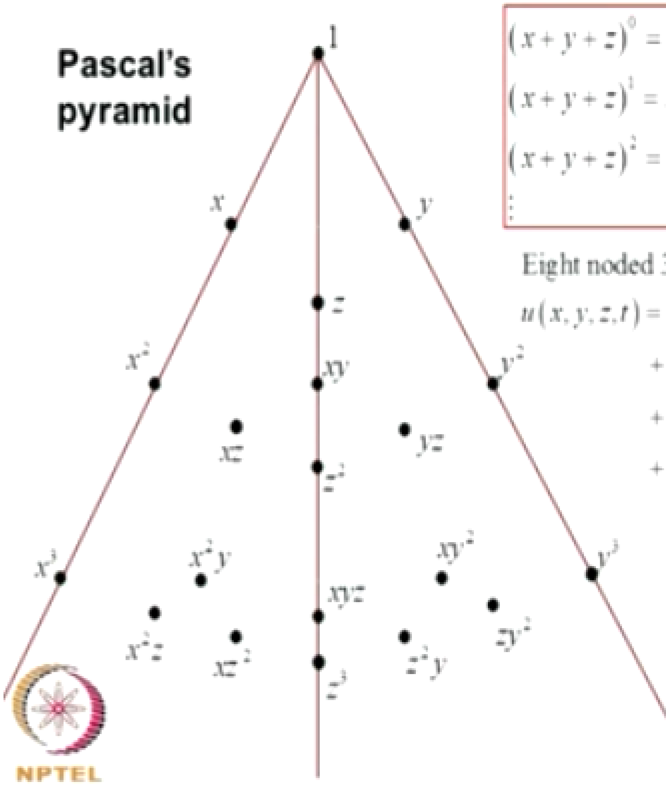
to keep in mind how to do that, for example for a triangle element we had 3 degrees of freedom, I mean 3 nodes and each node there are 2 degrees of freedom, and U was represented alpha 1 + alpha 2X + alpha 3Y, so we needed there are 3 degrees of, 3 nodal displacement values, therefore I should use a 3 term expansion for U, so that would be 1, X and Y, okay, that's what it means, a rectangular element how do you select? I need 4 terms 1, X, Y, and when I come here I can select any one of this in principle, if I select X square it won't be all right because if I rename X and Y axis, this will be Y square, so that's not right, so what we do is we take the term closest to the axis of symmetry which is XY, so this is alpha 1 + alpha 2X + alpha 3Y + alpha 4XY, so just to give this a emphasize that if you were to take this or this, that is either you retain this term or retain this term as the fourth you know, for the fourth term that you need use this as a candidate or use this as a candidate you will get these two, this is not appropriate because interchanging of X and Y would change the representation.

Now just to again to give an example suppose you have 4 noded element with 3 degrees of freedom per node, now how do we, you need 12 terms to represent that field variable, so how do you select 12 terms? 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, up to this there is no problem, but when it comes to this axis, which one will you select? So you should maintain symmetry therefore you should take X cubed Y, and XY cube, you could as well for geometric invariance you could have selected X square Y square but the next time we will not be able to select symmetrically, and also you could have selected X to the power of 4, and Y to the power of 4, but as a rule we select terms which are closest to the axis of symmetry, and therefore we take this term, so we will come to this, this will be using when we discuss plate bending, we will see that.



$$(x+y+z)^0 = 1$$
$$(x+y+z)^1 = x+y+z$$
$$(x+y+z)^2 = x^2+2xy+2yz+2xz+y^2+z^2$$
$$\vdots$$

Eight noded 3d element with 2 dofs per node

$$\begin{aligned} u(x,y,z,t) &= \alpha_1(t) \\ &+ \alpha_2(t)x + \alpha_3(t)y + \alpha_4(t)z \\ &+ \alpha_5(t)xy + \alpha_6(t)yz + \alpha_7(t)xz \\ &+ \alpha_8(t)xyz \end{aligned}$$

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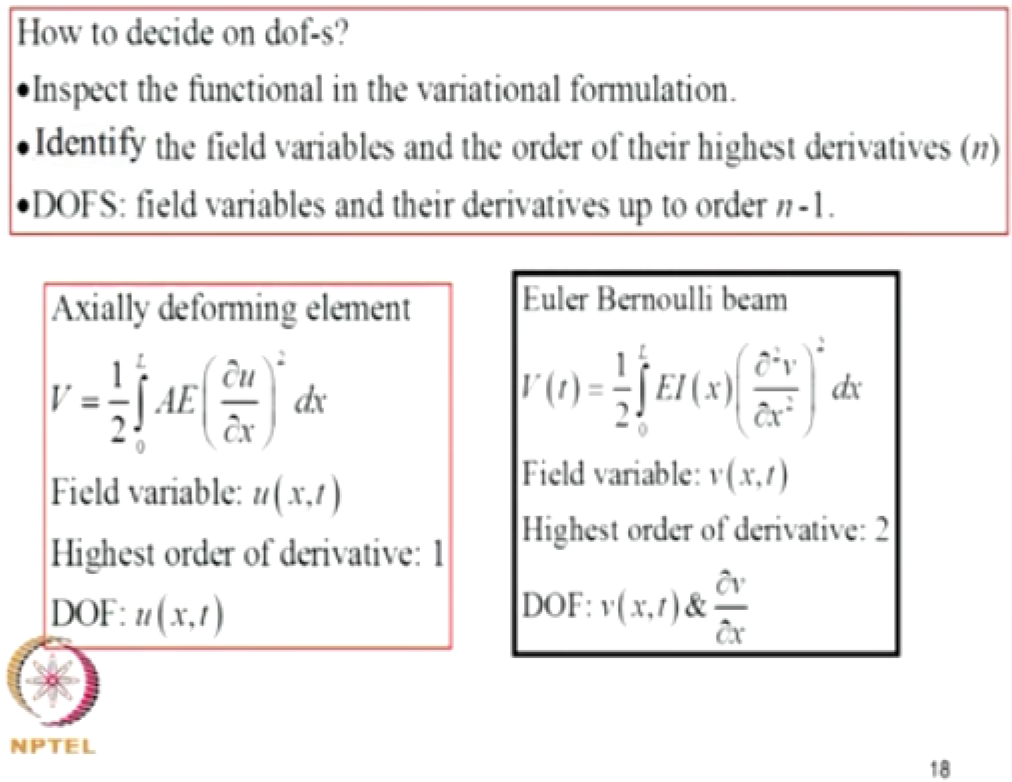
How about in 3 dimensions? So again we consider X + Y + Z to the power of 0 is 1, so that is 1, so on this line I have X, on this I have Z, on this axis I have Y, X + Y + Z to the power of 1 is X + Y + Z, so that is X, Y, Z, square X square + 2XY + 2YZ etcetera that is this, X square, Y square, Z square, 2XY, 2YZ, 2XZ, similarly you have cubic terms and so on and so forth.



Now just the another point which I am sure we have come across several times I just want to articulate that in clear terms, how do you decide on degrees of freedom, okay, so the rule is inspect the functional in the variational formulation identify the field variables and the order of the highest derivatives, we call it as N, so the degrees of freedom will be the field variables and the derivatives up to order N – 1, see for example in axially deforming element the functional is, in the variational formulation will be AE dou U/dou X whole square, so which is a field variable? U, and what is the highest order? N, so we need field variables and their derivatives up to order N – 1, so N is 1 here it is 0, so we need only U, so U is a degree of freedom, whereas if you consider beam element, V is the field variable highest derivative is 2, so I need the field variable and its derivative dou V/dou X, V and dou V/dou X so they are the degrees of freedom.

So as we consider more complicated problems it won't be immediately clear how to do this, that is why I thought I must emphasize this fact at this step. So similarly just to complete the

Plane stress element

$$\sigma(x,y)=\{\sigma_{xx}(x,y)\quad \sigma_{yy}(x,y)\quad \sigma_{xy}(x,y)\}^t;\ \varepsilon(x,y)=\{\varepsilon_{xx}(x,y)\quad \varepsilon_{yy}(x,y)\quad \varepsilon_{xy}(x,y)\}^t$$

$$\sigma(x,y)=D\varepsilon(x,y)$$

$$\varepsilon(x,y)=\begin{bmatrix}\frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}\begin{Bmatrix}u(x,y)\\ v(x,y)\end{Bmatrix}$$

$$V=\frac{1}{2}\int_{V_0}\sigma^t\varepsilon dV_0=\frac{2h}{2}\int_A\sigma^t\varepsilon dA=\frac{2h}{2}\int_A\varepsilon^t D\varepsilon dA$$

Field variables: $u(x,y,t)\,\&\,v(x,y,t)$

Highest order of derivative: 1

DOFs: $u(x,y,t)\,\&\,v(x,y,t)$

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discussion in plane stress element this is the stress field, this is a strain field, and this is how the stress and strain were related, so when we consider the functional we had epsilon transpose D epsilon as the, in the variational formulation this was the functional, now here therefore if you examine these relations carefully we have the field variables are U and V, what is the highest order derivative? You will see here it will be dou U/dou X, and dou U/dou Y and things like that, so highest order derivative is 1 so the nodal degrees of freedom therefore will be U and V, okay. Now equipped with this now we will start discussing 3D solid elements, so quickly we

3D solid elements

$$\sigma = \{\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \sigma_{xy} \quad \sigma_{xz} \quad \sigma_{yz}\}^t$$

$$\varepsilon = \{\varepsilon_{xx} \quad \varepsilon_{yy} \quad \varepsilon_{zz} \quad 2\varepsilon_{xy} \quad 2\varepsilon_{xz} \quad 2\varepsilon_{yz}\}^t$$

$$\sigma = D\varepsilon$$

$$D = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{\nu} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{\nu} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{\nu} \end{bmatrix}$$

$$V = \frac{1}{2}\int_{V_0} \sigma^t \varepsilon dV_0 = \frac{1}{2}\int_{V_0} \varepsilon^t D\varepsilon dV_0 \ \& \ T = \frac{1}{2}\int_{V_0} \rho\left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right) dV_0$$

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will recall the various variables, this is the state of stress at a point it can be expressed either as 3 x 3 matrix or a 6 + 1 vector, this is state of strain at a point there are 6 variables again this is a, can be represented as a 3 x 3 tensor or a 6 cross 1 vector, and we prefer the 6 cross 1 representation in finite element development, so the stress and strain are related through the constitutive law I am assuming material is elastic, linearly elastic and isotropic so this is the constitutive law for that material behavior model.

Strain energy we have seen it is integral over volume element sigma transpose epsilon DV naught multiplied by 1/2, so for sigma transpose I will write epsilon transpose D transpose, now D is a symmetric matrix, so D transpose is D, I get epsilon transpose D epsilon. Kinetic energy, rho into DV naught is the mass of an infinitesimal element with dimension DX, DY, DZ and the kinetic energy of that in X Direction is 1/2 mass into U dot square, in Y direction 1/2 mass into V dot square, Z direction 1/2 mass into W dot square and you add up this is the kinetic energy.

$$V = \frac{1}{2}\int_{V_0} \sigma^t \varepsilon dV_0 = \frac{1}{2}\int_{V_0} \varepsilon^t D\varepsilon dV_0 \ \& \ T = \frac{1}{2}\int_{V_0} \rho\left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right) dV_0$$

$$\varepsilon = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

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Now we will consider the expression for strain energy, this epsilon we really want to express this now in terms of displacements, so we use strain displacement relations, so strain is given in terms of displacement when this matrix operates on this displacement we get the strain, and

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [N] \begin{Bmatrix} u_e \\ v_e \\ w_e \end{Bmatrix} = Nu_e \Rightarrow \varepsilon = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} Nu_e = Bu_e$$

$$\Rightarrow V = \frac{1}{2}\int_{V_0} u_e^t B^t D B u_e dV_0 \ \& \ T = \frac{1}{2}\int_{V_0} \rho \dot{u}_e^t N^t N \dot{u}_e dV_0$$

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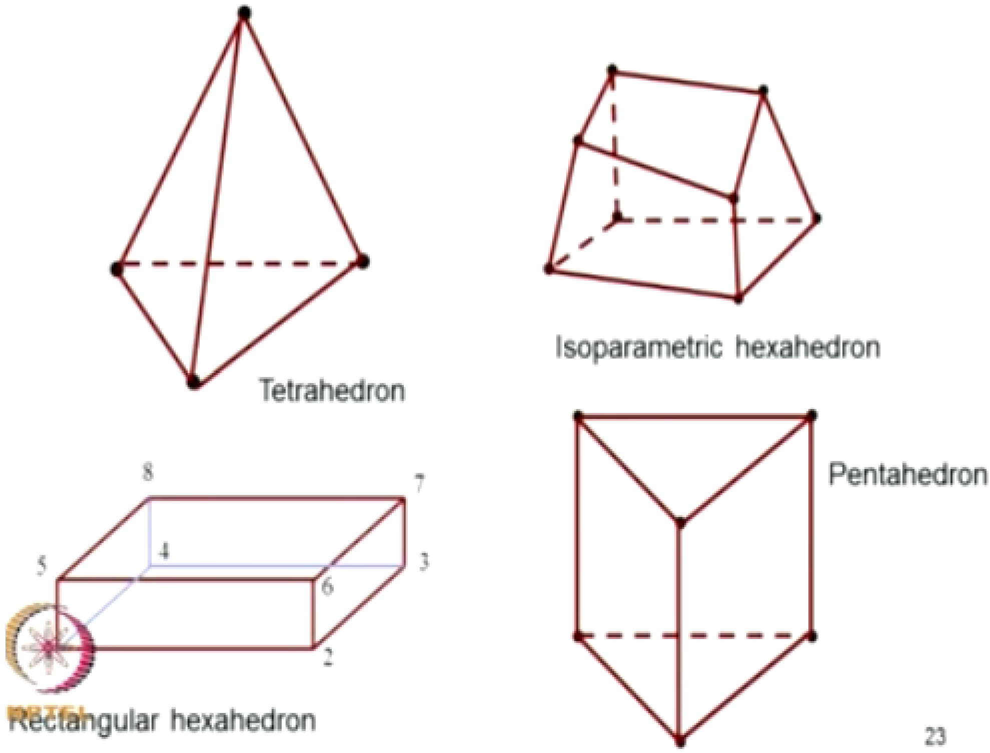
using that I represent U, V, W now in terms of the shape functions and nodal displacements which I rated as NUE and where this epsilon, given this the epsilon will now be this matrix of derivatives into N into UE, because this is N into U this is UE, so this operator acting on N is known as B matrix and therefore the strain energy becomes in terms of displacement field it is given by this expression, and kinetic energy is given by this, this we have seen but just I am reiterating so that we recall quickly what we need to use now immediately.



Now in considering problems of 3 dimensional continuum various shapes become possible, so whereas in line elements there was no such a dilemma it was everything was simple, in plane problems we had quadrilaterals, you know rectangles, triangles, quadrilaterals, isoparametric, you know curve quadrilaterals with curved edges and so on and so forth, similar issues will come up in solid elements now so we can discretize a 3 dimensional domain using tetrahedrons are using rectangular hexahedron or pentahedrons or in a more general situation isoparametric hexahedron, so what we will do is we will try to develop the logic for developing the structural matrices for at least for some of these elements.

Tetrahedron element

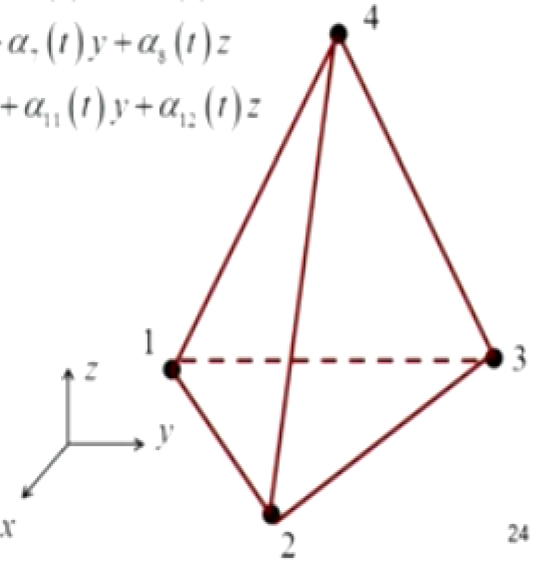
4 noded element with 3 dof-s per node

Dofs=12

$$u(x,y,z,t) = \alpha_1(t) + \alpha_2(t)x + \alpha_3(t)y + \alpha_4(t)z$$

$$v(x,y,z,t) = \alpha_5(t) + \alpha_6(t)x + \alpha_7(t)y + \alpha_8(t)z$$

$$w(x,y,z,t) = \alpha_9(t) + \alpha_{10}(t)x + \alpha_{11}(t)y + \alpha_{12}(t)z$$



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Now let us start with considering tetrahedron element, so tetrahedron has 4 nodes, this is a Cartesian coordinate system 1, 2, 3, 4 are the 4 nodes and each node has 3 degrees of freedom U, V and W, so this is 4 noded element with 3 degrees of freedom per node and hence we have 12 degrees of freedom for the element. Now the field variables are U, V, W which vary with respect to X, Y, Z and T, so what we do is we represent the field variables in terms of, we need 4 terms so we have to go to now the Pascal's pyramid so the 4 terms are 1, X, Y and Z, so that is what we are doing, I have alpha 1(t), alpha 2(t)into X, alpha 3 into Y, alpha 4 into Z, so similarly I have for V and W similar representations, this is alpha 1 to alpha 12 are now the generalized coordinates, now this have to be selected by knowing the value of these field variables at the nodes, so node 1 I will have U1, V1, W1, here U2, V2, W2 and so on and so

At (x_1, y_1, z_1) $u(x,y,z,t) = u_1(t), v(x,y,z,t) = v_1(t), w(x,y,z,t) = w_1(t)$

At (x_2, y_2, z_2) $u(x,y,z,t) = u_2(t), v(x,y,z,t) = v_2(t), w(x,y,z,t) = w_2(t)$

At (x_3, y_3, z_3) $u(x,y,z,t) = u_3(t), v(x,y,z,t) = v_3(t), w(x,y,z,t) = w_3(t)$

$\Rightarrow$

$$u(x,y,z,t) = \sum_{i=1}^{4} u_i(t) N_i(x,y,z);$$

$$v(x,y,z,t) = \sum_{i=1}^{4} v_i(t) N_i(x,y,z)$$

$$w(x,y,z,t) = \sum_{i=1}^{4} w_i(t) N_i(x,y,z)$$

forth. So at X1, Y1, Z1, I have U is U1, V is V1, W is W1, and so on and so forth. So now I will use these conditions and I will be able to evaluate those 12 generalized coordinates in terms of nodal displacements, this we have done for a triangle element, so I get the interpolation, the formula for interpolating the field variable within an element in terms of nodal displacements and the trial functions as shown here.

$\Rightarrow$

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} = \frac{1}{6V_0} \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix}; V_0 = \frac{1}{6} \begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix}$$

$$a_1 = \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}; b_1 = -\begin{vmatrix} 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix}; c_1 = \begin{vmatrix} 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{vmatrix}; d_1 = \begin{vmatrix} 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{vmatrix}$$

Obtain other constants by cyclic interchange of subscripts in the order 1,2,3,4. Sign of a_3 will be same as a_1 and both a_2 and a_4 will have sign opposite to that of a_1. Similar rules apply to the coefficients $b, c,$ and d.

NPTEL

Now in terms of the nodal coordinates we can derive the shape functions, the formulary is given here V naught is the volume of the element and the N1 for example is a constant A1 + B1 X +

C1Y + D1Z, so it is a linear first-order polynomial in 3 variables. So similarly this V naught itself is a volume of the element that will turn out to be in terms of the coordinates of the nodes, and there are some constant A1, B1, C1, they are all explained here. Actually there is A1, B1, C1, D1, they have been described here and there is a principal for evaluating the other elements, some rules are given here that can be easily followed. So we have now got a representation like

At (x_1, y_1, z_1) $u(x,y,z,t) = u_1(t), v(x,y,z,t) = v_1(t), w(x,y,z,t) = w_1(t)$

At (x_2, y_2, z_2) $u(x,y,z,t) = u_2(t), v(x,y,z,t) = v_2(t), w(x,y,z,t) = w_2(t)$

At (x_3, y_3, z_3) $u(x,y,z,t) = u_3(t), v(x,y,z,t) = v_3(t), w(x,y,z,t) = w_3(t)$

$\Rightarrow$

$$u(x,y,z,t) = \sum_{i=1}^{4} u_i(t) N_i(x,y,z);$$

$$v(x,y,z,t) = \sum_{i=1}^{4} v_i(t) N_i(x,y,z)$$

$$w(x,y,z,t) = \sum_{i=1}^{4} w_i(t) N_i(x,y,z)$$

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this where this N1, N2, N3, N4 are linear functions of X, Y, Z and there is a constant term as well.

$$\begin{Bmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ w(x,y,z,t) \end{Bmatrix} = [N]\{u\}_e$$

$$\{u\}_e^T = \begin{bmatrix} u_1 & v_1 & w_1 & \cdots & u_4 & v_4 & w_4 \end{bmatrix}$$

$$[N] = \begin{bmatrix} N_1 & 0 & 0 & \cdots & N_4 & 0 & 0 \\ 0 & N_1 & 0 & \cdots & 0 & N_4 & 0 \\ 0 & 0 & N_1 & \cdots & 0 & 0 & N_4 \end{bmatrix}$$

$$M_e = \int_{V_e} [N]^T [N] \, dV_e$$

This integral can be evaluated exactly.

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So now this is a representation U, V, W is N into UE, UE consists of 12 variables now, U1, V1, W1 up to U4, V4, W4, so N is this 3 into 12 matrix, and ME is our N transpose N matrix, we can evaluate this exactly, so you have to decide upon the order of the terms, N is linear, N transpose N will be quadratic and you can use the appropriate order of integration and this can be evaluated exactly, or you can actually carry out the integration in closed form, there is no

$$M_e = \frac{\rho V_0}{20}\begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}$$

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problem, so this is the mass matrix, this is symmetric 12x12. Now strain energy leading to the

$$\varepsilon = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} [N]\{u\}_e = [B]\{u\}_e$$

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evaluation of stiffness matrix this is epsilon into U, this operator matrix into U, and this is B, and now B is this operator acting on this matrix, so we can evaluate B here, in this case it

$$B = \frac{1}{6V_0}\begin{bmatrix} b_1 & 0 & 0 & b_2 & 0 & 0 & b_3 & 0 & 0 & b_4 & 0 & 0 \\ 0 & c_1 & 0 & 0 & c_2 & 0 & 0 & c_3 & 0 & 0 & c_4 & 0 \\ 0 & 0 & d_1 & 0 & 0 & d_2 & 0 & 0 & d_3 & 0 & 0 & d_4 \\ c_1 & b_1 & 0 & c_2 & b_2 & 0 & c_3 & b_3 & 0 & c_4 & b_4 & 0 \\ 0 & d_1 & c_1 & 0 & d_2 & c_2 & 0 & d_3 & c_3 & 0 & d_4 & c_4 \\ d_1 & 0 & b_1 & d_2 & 0 & b_2 & d_3 & 0 & b_3 & d_4 & 0 & b_4 \end{bmatrix}$$

Note: B is independent of $x, y,$ and z.

$$K_e = \int_{V_e} B^t DB dV_e = V_0 B^t DB$$

$$\left.\begin{aligned} \varepsilon &= Bu_e \\ \sigma &= DBu_e \end{aligned}\right\} \text{constant over the element}$$

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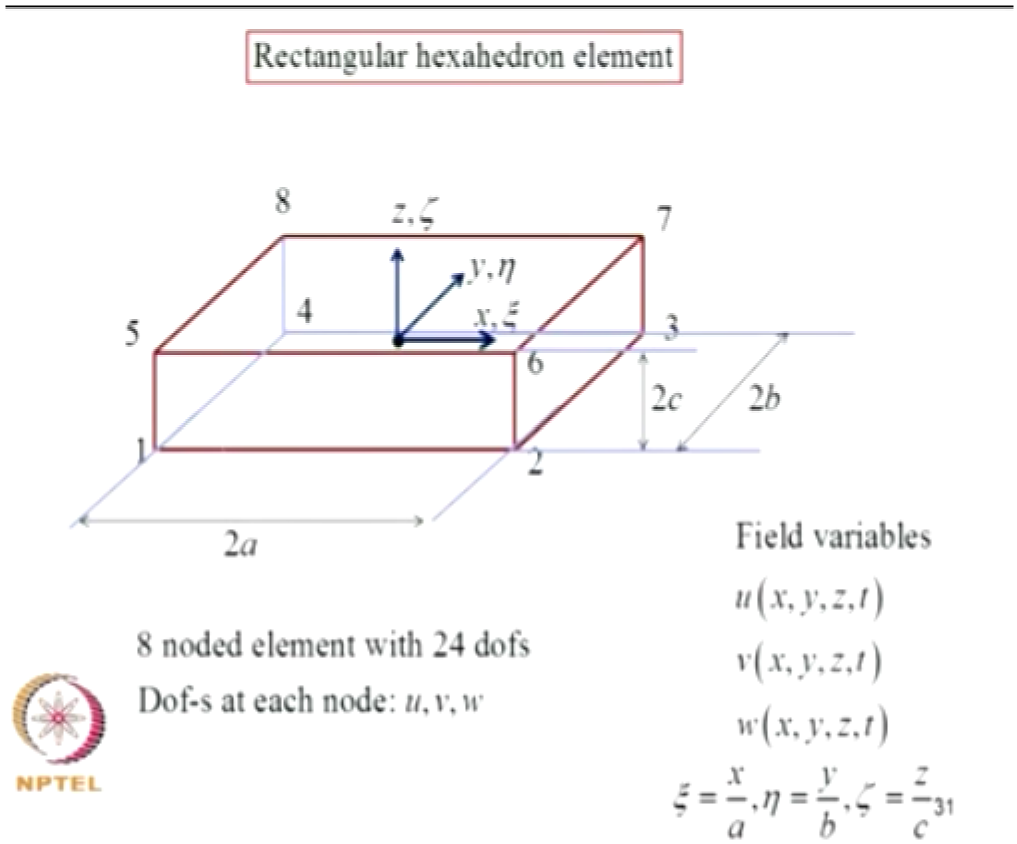
becomes a constant you recall what happened for a triangular element in plane problems a similar thing happens here, this is a constant, so then K, this KE can thus be evaluated exactly there is no need for quadrature because this is simply a constant term, so once we find all these we know the strain which is constant over the volume, and stress is DB into UE which is also constant over the volume that we have considered.

So a tetrahedral element, formulation of tetrahedral element is straightforward it follows the same steps that we use for formulating a triangle elements.



Now how about a rectangular hexahedron element? So this element has 8 nodes 1, 2, 3, 4, 5, 6, 7, 8, so it is 8 noded element with 3 degrees of freedom per node, and therefore it has 24 degrees of freedom, the field variables are U, V, W which are functions of X, Y, Z, and T, so at each node there will be 3 field variables, so there are 3 degrees of freedom therefore it is a 8 noded element with 3 degrees of freedom per node, therefore 24 degrees of freedom element so all the structural matrices will be 24x24. Now we introduce the coordinate transformation XI is X/A, eta is Y/B and zeta is Z/C so that this gets mapped to a cube of dimensions 2.

Each of the variables $u(x,y,z,t), v(x,y,z,t), \& w(x,y,z,t)$ needs to be represented by a polynomial with 8 terms.

Terms: $1, x, y, z, xy, xz, yz, xyz$

$\Rightarrow$ Ensures geometric invariance

$\Rightarrow$

$$u(x,y,z,t) = \alpha_1(t) + \alpha_2(t)x + \alpha_3(t)y + \alpha_4(t)z + \alpha_5(t)xy + \alpha_6(t)yz + \alpha_7(t)xz + \alpha_8(t)xyz$$

Select $\alpha_i(t), i = 1, 2, \cdots, 8$ so that the value of $u(x,y,z,t)$ matches with its respective nodal values at the 8 nodes.

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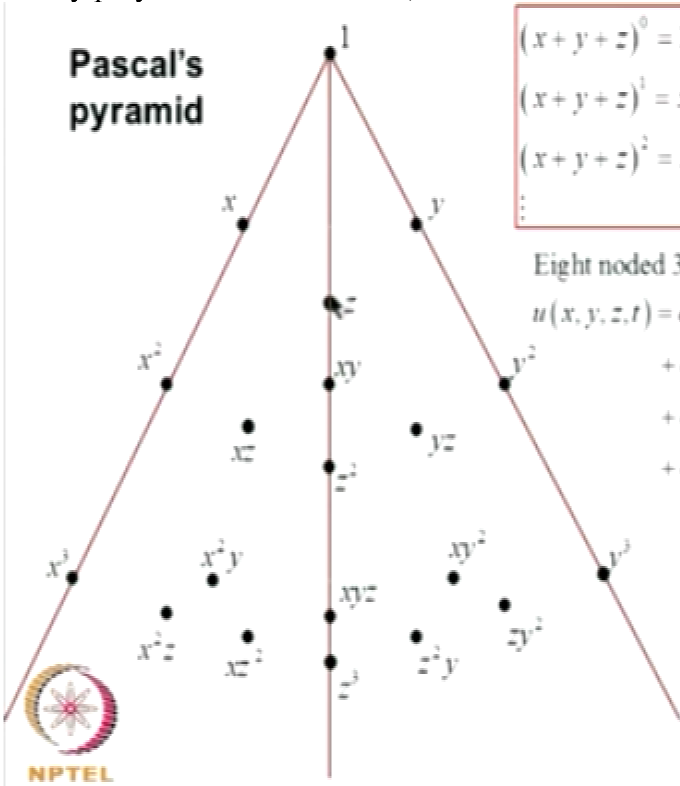
Now each of these field variables now need to be, there are 8 nodes, therefore each field variable needs to be represented in terms of 8 terms, okay, so each of these variables need to be represented by polynomial with 8 terms, how do we select that? So we go back to the Pascal's



$(x+y+z)^0 = 1$

$(x+y+z)^1 = x+y+z$

$(x+y+z)^2 = x^2 + 2xy + 2yz + 2xz + y^2 + z^2$

$\vdots$

Eight noded 3d element with 2 dofs per node

$$u(x,y,z,t) = \alpha_1(t) + \alpha_2(t)x + \alpha_3(t)y + \alpha_4(t)z + \alpha_5(t)xy + \alpha_6(t)yz + \alpha_7(t)xz + \alpha_8(t)xyz$$

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pyramid, we have to take 8 terms 1, X, Y, Z, 4 are over, X square, so the term that we are taking is 1, X, Y, Z, XY, XZ, YZ, and XYZ, this is what we will select. We are again taking terms closest to the central axis and symmetry is a requirement, okay, so X square, Y square, Z square, we are not taking, we are taking XY, XZ, YZ, subsequently we go to the third order term, and we don't take X cube or Y cube or Z cube instead we take XYZ, so this ensures geometric invariance, so I get this representation. Now these alphas are the generalized coordinates which need to be selected so that the values of U matches with its respective nodal values at the eighth notes, so assuming that we have done that I get the representation for the

$$u(\xi,\eta,\zeta,t)=\sum_{j=1}^{8}N_j(\xi,\eta,\zeta)u_j(t)$$

$$v(\xi,\eta,\zeta,t)=\sum_{j=1}^{8}N_j(\xi,\eta,\zeta)v_j(t)$$

$$w(\xi,\eta,\zeta,t)=\sum_{j=1}^{8}N_j(\xi,\eta,\zeta)w_j(t)$$

$$N_j(\xi,\eta,\zeta)=\frac{1}{8}(1+\xi_j\xi)(1+\eta_j\eta)(1+\zeta_j\zeta),\ j=1,2,\cdots,8$$

Remark

Continuity of field variables across element boundaries is ensured. (Why?)

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}=[N]\{u\}_e$$

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field variables, now we have transformed now to XI, eta, zeta coordinate system I get this representation, and we can show that in XI, eta, zeta coordinate these trial functions are given by this, this is again similar to what we did for linear rectangular plane stress element.
Now how do you check for continuity of field variables across the element boundaries? I claim that it is insured and I leave it as an exercise for you to verify, so you need to consider 2 neighboring elements and a point lying on the interface and argue out why the field variables are continuous across the interface. So the representation is therefore now U, V, W is N into UE, that means I have combined all this where N is this matrix 3x24, and this is the 24 cross 1

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [N]\{u\}_e = \begin{bmatrix} N_1 & 0 & 0 & \cdots & N_8 & 0 & 0 \\ 0 & N_1 & 0 & \cdots & 0 & N_8 & 0 \\ 0 & 0 & N_1 & \cdots & 0 & 0 & N_8 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w \\ \vdots \\ u_8 \\ v_8 \\ w_8 \end{Bmatrix}$$

$$T = \frac{1}{2}\int_{V_0} \rho \dot{u}_e^t N^t N \dot{u}_e dV_0 = \frac{1}{2}\dot{u}_e^t \left[\int_{V_0} \rho N^t N dV_0 \right] \dot{u}_e = \frac{1}{2}\dot{u}_e^t [m]_e \dot{u}_e$$

$$[m]_{eij} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \rho abc N_i(\xi,\eta,\zeta) N_j(\xi,\eta,\zeta) d\xi d\eta d\zeta$$

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nodal degrees of freedom vector. So we go back to the expression for kinetic energy, so I get the mass matrix to be given by this where an IJ-th element is given by this integral. Now since this is you know cubic function, you know you will have the nonlinear term that will be present here will be products of XI eta zeta second and third-order products so you can evaluate this integral easily, so if this is done in this case it is possible again to evaluate it in closed form I

$$[m]_{ey} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \rho abc N_i(\xi,\eta,\zeta) N_j(\xi,\eta,\zeta)\, d\xi d\eta d\zeta$$

$$= \frac{\rho abc}{64}\int_{-1}^{1}\int_{-1}^{1}(1+\xi_i\xi)(1+\eta_i\eta)(1+\zeta_i\zeta)(1+\xi_j\xi)(1+\eta_j\eta)(1+\zeta_j\zeta)\, d\xi d\eta d\zeta$$

$$= \frac{\rho abc}{64}\left[\int_{-1}^{1}(1+\xi_i\xi)(1+\xi_j\xi)\, d\xi\right]\left[\int_{-1}^{1}(1+\eta_i\eta)(1+\eta_j\eta)\, d\eta\right]$$

$$\left[\int_{-1}^{1}(1+\zeta_i\zeta)(1+\zeta_j\zeta)\, d\zeta\right]$$

$$= \frac{\rho abc}{8}\left(1+\frac{1}{3}\xi_i\xi_j\right)\left(1+\frac{1}{3}\eta_i\eta_j\right)\left(1+\frac{1}{3}\zeta_i\zeta_j\right)$$

$$m = \begin{bmatrix} 2m & m \\ m & 2m \end{bmatrix}$$

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get the element mass matrix in this form which is shown here, where this M itself is a huge

$$m = \frac{\rho abc}{27}\begin{bmatrix}
4 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\
0 & 4 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\
0 & 0 & 4 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 \\
2 & 0 & 0 & 4 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 2 & 0 & 0 & 4 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\
0 & 0 & 2 & 0 & 0 & 4 & 0 & 0 & 2 & 0 & 0 & 1 \\
1 & 0 & 0 & 2 & 0 & 0 & 4 & 0 & 0 & 2 & 0 & 0 \\
0 & 1 & 0 & 0 & 2 & 0 & 0 & 4 & 0 & 0 & 2 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 4 & 0 & 0 & 2 \\
2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 4 & 0 & 2 \\
0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 4 & 0 \\
0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 4
\end{bmatrix}$$

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matrix of this kind, so this is 12x12 matrix, the mass matrix will be 24x24 matrix, okay so the partitioning is in terms of 12 by 12 square matrices.

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [N]\{u\}_e \ \& \ \varepsilon = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} [N]\{u\}_e = B\{u\}_e$$

$$V = \frac{1}{2}\int_{V_0} \sigma^t \varepsilon dV_0 = \frac{1}{2}\int_{V_0} \varepsilon^t D\varepsilon dV_0 = \frac{1}{2}\int_{V_0} \{u\}_e^t B^t DB\{u\}_e dV_0$$

$$= \frac{1}{2}\{u\}_e^t \left[\int_{V_0} B^t DBdV_0\right]\{u\}_e = \frac{1}{2}\{u\}_e^t K_e \{u\}_e$$

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Now how about strain energy? So I have displacement is N into UE, and strain is given by this and this is our expression for displacements in terms of nodal, strains in terms of nodal displacement values, and this B matrix is given by this into this, this operation on N matrix, so as before I get K matrix to be given by, KE to be given by integral over the volume B tranpsoe DBDV naught, so how do we get these terms now? So this B matrix I can you know carry out

$$V = \frac{1}{2}\{u\}_e^T K_e \{u\}_e$$

$$K_e = \int_{V_0} B^t D B dV_0 = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} abc B^t D B d\xi d\eta d\zeta$$

$$B = \begin{bmatrix} B_1 & \cdots & B_8 \end{bmatrix} \text{ with } B_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial z} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 \\ \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_i}{\partial x} \\ 0 & \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial z} \end{bmatrix}$$

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that differentiation operation and take it through the trial functions I get this, and if we spend some you know some effort we can evaluate all the gradients that appears here, so we can

$$\frac{\partial N_i}{\partial x} = \frac{1}{a}\frac{\partial N_i}{\partial \xi} = \frac{\xi_i}{8a}(1+\eta_i\eta)(1+\zeta_i\zeta)$$

$$\frac{\partial N_i}{\partial y} = \frac{1}{b}\frac{\partial N_i}{\partial \eta} = \frac{\eta_i}{8b}(1+\xi_i\xi)(1+\zeta_i\zeta)$$

$$\frac{\partial N_i}{\partial z} = \frac{1}{c}\frac{\partial N_i}{\partial \zeta} = \frac{\zeta_i}{8c}(1+\xi_i\xi)(1+\zeta_i\zeta)$$

$\Rightarrow K_e = \int_{V_0} B^t D B dV_0$ can be evaluated. ?

Simpler alternative: evaluate $K_e = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} abc B^t D B d\xi d\eta d\zeta$

using Gaussian quadrature.

$$I = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta,\zeta) d\xi d\eta d\zeta \approx \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} w_i w_j w_k f(\xi_i,\eta_j,\zeta_k)$$

Use $2\times2\times2$ quadrature.

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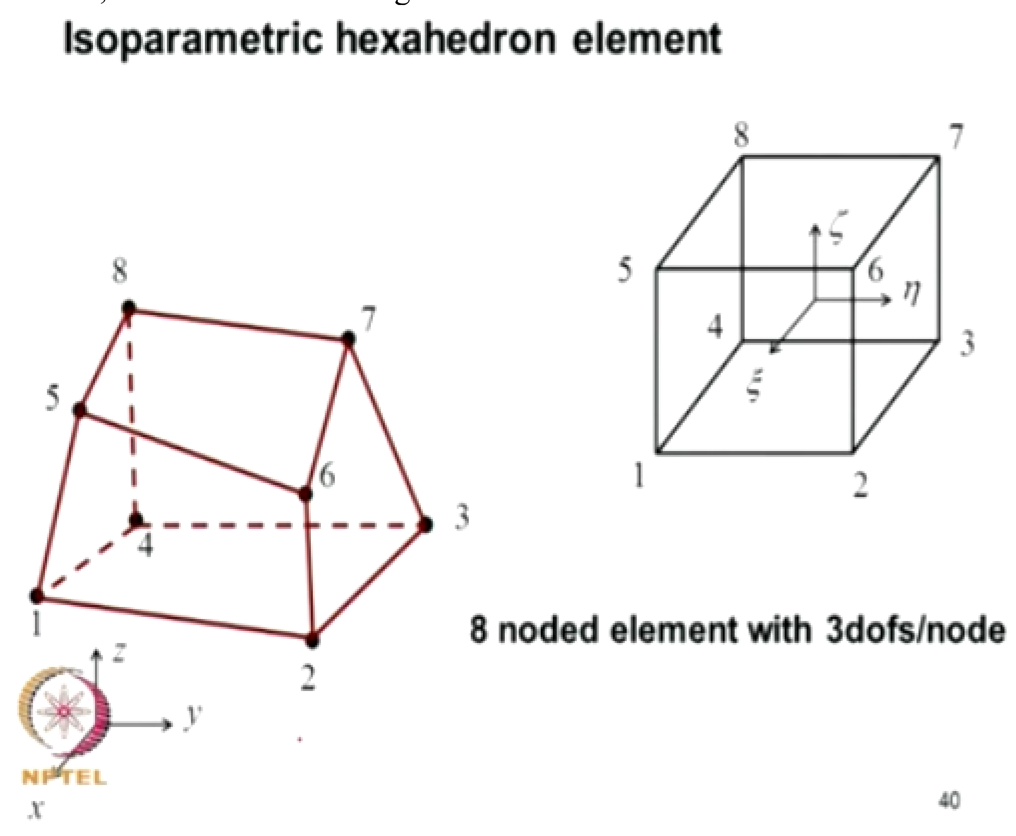
evaluate KE element as shown here, or we can use Gauss quadrature to evaluate this and you can argue you out that a 2 x 2 x 2 quadrature would you know complete the, you know, the quadrature here, and we would offer a good solution here.



Now the similar logic can be used now, how about an isoparametric hexahedron element? So here I consider this as my element, so it has 8 nodes 1, 2, 3, 4, 5, 6, 7, 8, so the phases are, you know, not orthogonal to each other and they are not rectangles and so on and so forth, this I will map through a transformation to a unit cube of dimensions 2, lateral dimensions 2, so then we will carry out the integrations needed to implement evaluation of KE and ME in this coordinate system, so the transformation that we are looking for is X is represented in terms of the same

Coordinates

$$x = \sum_{i=1}^{8} N_i(\xi,\eta,\zeta)\,x_i;\; y = \sum_{i=1}^{8} N_i(\xi,\eta,\zeta)\,y_i;\; z = \sum_{i=1}^{8} N_i(\xi,\eta,\zeta)\,z_i$$

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{8} N_i(\xi,\eta,\zeta)\,x_i \\ \sum_{i=1}^{8} N_i(\xi,\eta,\zeta)\,y_i \\ \sum_{i=1}^{8} N_i(\xi,\eta,\zeta)\,z_i \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & \cdots & N_1 & 0 & 0 \\ 0 & N_1 & 0 & \cdots & 0 & N_1 & 0 \\ 0 & 0 & N_1 & \cdots & 0 & 0 & N_1 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \\ \vdots \\ x_8 \\ y_8 \\ z_8 \end{Bmatrix}$$

$$N_j(\xi,\eta,\zeta) = \frac{1}{8}(1+\xi_j\xi)(1+\eta_j\eta)(1+\zeta_j\zeta),\; j = 1,2,\cdots,8$$

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trial function that eventually I will use for representing displacement fields, field variables this is Y, this is Z, in terms of nodal values, so I can assemble them in a matrix form I can write this matrix into this vector of nodal coordinates. In XI eta zeta plane I know that this is the form of the trial functions, so I have now the representation that I need for the nodal, the XY, the transformation from X to XI eta zeta plane.

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [N]\{u\}_e \ \& \ \varepsilon = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} [N]\{u\}_e = B\{u\}_e$$

$$B = [B_1 \quad B_2 \quad \cdots \quad B_8]$$

$$\underbrace{B_i}_{6\times 3} = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial z} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 \\ \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_i}{\partial x} \\ 0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial y} \end{bmatrix}$$

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Now how about displacement field? The same representation is used, and the strain following this representation I get B into UE, where B is given by B1, B2, B8, where each of these matrices have this form, where I is 1 to 8, okay, so we have these matrices now. Now here I

$$\begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \zeta} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \xi} + \frac{\partial N_i}{\partial z}\frac{\partial z}{\partial \xi} \\ \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \eta} + \frac{\partial N_i}{\partial z}\frac{\partial z}{\partial \eta} \\ \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \zeta} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \zeta} + \frac{\partial N_i}{\partial z}\frac{\partial z}{\partial \zeta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{Bmatrix}$$

$$\underbrace{[J]}_{3\times 3} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} y_i & \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} z_i \\ \sum_{i=1}^{8} \frac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{8} \frac{\partial N_i}{\partial \eta} y_i & \sum_{i=1}^{8} \frac{\partial N_i}{\partial \eta} z_i \\ \sum_{i=1}^{8} \frac{\partial N_i}{\partial \zeta} x_i & \sum_{i=1}^{8} \frac{\partial N_i}{\partial \zeta} y_i & \sum_{i=1}^{8} \frac{\partial N_i}{\partial \zeta} z_i \end{bmatrix}$$

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have this term dou NI/dou X, dou NI/dou Y, dou NI/dou Z etcetera, so that I need to evaluate so to do that we can start by finding a derivative of dou NI/dou XI and NI with respect to eta, NI with respect to zeta, these are straightforward application of rules of differentiation, so NI/XI is NI/X into dou X/dou sai + dou NI/dou Y into dou Y/dou XI etcetera, so you do for all the variables I get this and this itself I can write it in a matrix form as shown here, and this matrix is a Jacobian matrix of the transformation, so I will get J into this vector of gradients, this is J, J itself I can write in terms of you know X, I will now use this representation therefore I will be able to write elements of J in this form. So J is this now since I have I know the form of this

$$\underbrace{[J]}_{3\times 3}=\begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta}\\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta}\end{bmatrix}=\begin{bmatrix}\sum_{i=1}^{8}\frac{\partial N_i}{\partial \xi}x_i & \sum_{i=1}^{8}\frac{\partial N_i}{\partial \xi}y_i & \sum_{i=1}^{8}\frac{\partial N_i}{\partial \xi}z_i\\ \sum_{i=1}^{8}\frac{\partial N_i}{\partial \eta}x_i & \sum_{i=1}^{8}\frac{\partial N_i}{\partial \eta}y_i & \sum_{i=1}^{8}\frac{\partial N_i}{\partial \eta}z_i\\ \sum_{i=1}^{8}\frac{\partial N_i}{\partial \zeta}x_i & \sum_{i=1}^{8}\frac{\partial N_i}{\partial \zeta}y_i & \sum_{i=1}^{8}\frac{\partial N_i}{\partial \zeta}z_i\end{bmatrix}$$

$$N_i(\xi,\eta,\zeta)=\frac{1}{8}(1+\xi_i\xi)(1+\eta_i\eta)(1+\zeta_i\zeta),\ i=1,2,\cdots,8$$

$$\frac{\partial N_i}{\partial \xi}=\frac{1}{8}\xi_i(1+\eta_i\eta)(1+\zeta_i\zeta),\ i=1,2,\cdots,8$$

$$\frac{\partial N_i}{\partial \eta}=\frac{1}{8}\eta_i(1+\xi_i\xi)(1+\zeta_i\zeta),\ i=1,2,\cdots,8$$

$$\frac{\partial N_i}{\partial \zeta}=\frac{1}{8}\zeta_i(1+\eta_i\eta)(1+\eta_i\eta),\ i=1,2,\cdots,8$$

$$\begin{Bmatrix}\frac{\partial N_i}{\partial x}\\ \frac{\partial N_i}{\partial y}\\ \frac{\partial N_i}{\partial z}\end{Bmatrix}=[J]^{-1}\begin{Bmatrix}\frac{\partial N_i}{\partial \xi}\\ \frac{\partial N_i}{\partial \eta}\\ \frac{\partial N_i}{\partial \zeta}\end{Bmatrix}$$

- Elements of J are tri-quadratic functions

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interpolation function I can carry out these integrations, so the differentiation with respect to XI will give me this, with respect to eta will give me this, zeta will give me this, so that means I can evaluate elements of J matrix. See why I am doing all this is I need to find out which order terms will be present, right so we discover that elements of J are tri-quadratic functions.

Now this function that I am basically looking for which one, this is now given by J inverse into

$$K_e = \int_{V_0} B^t DB dV_0 = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} B^t DB |J| d\xi d\eta d\zeta$$

Elements of this matrix ratios of triquadratic functions of $\xi, \eta, \& \zeta$.

These elements cannot be evaluated exactly.

Use $2 \times 2 \times 2$ Gauss quadrature to get

$$K_{ers} \approx \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} w_i w_j w_k \left[B^t DB |J|\right]_{rs} \left(\xi_i, \eta_j, \zeta_k\right)$$

Note :

$$V_e = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} |J| d\xi d\eta d\zeta$$

The integrand is a triquadratic function and hence the integral can be evaluated exactly using $2 \times 2 \times 2$ Gauss quadrature.

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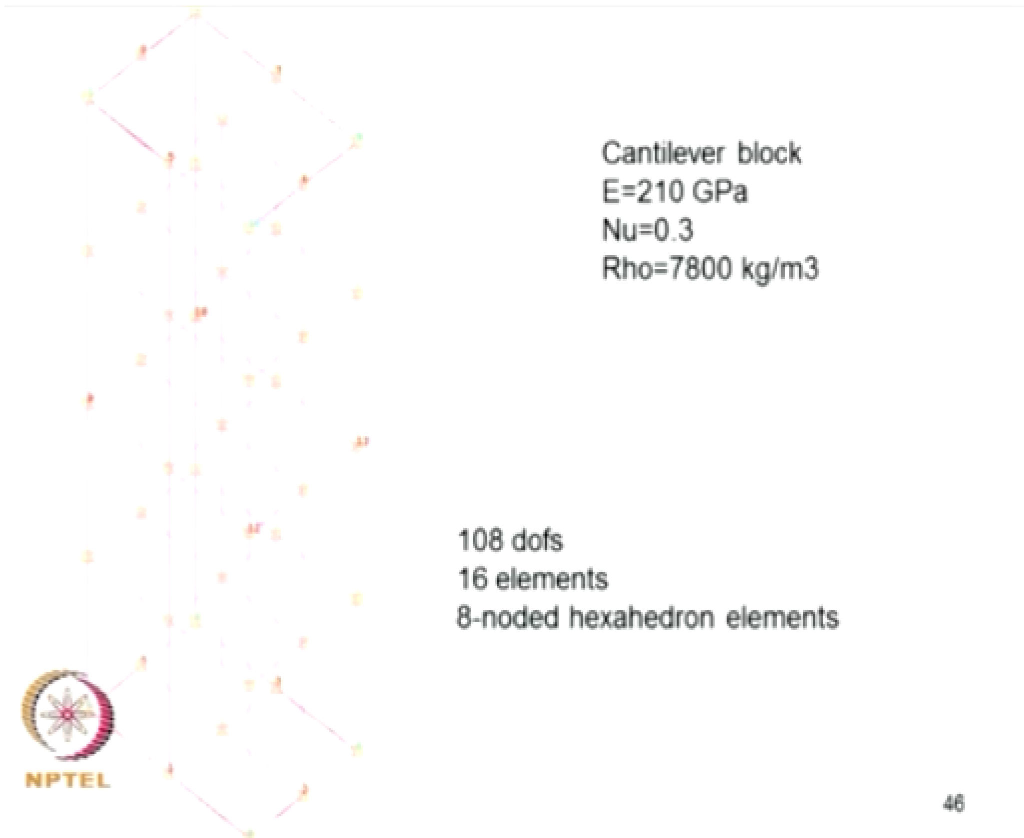
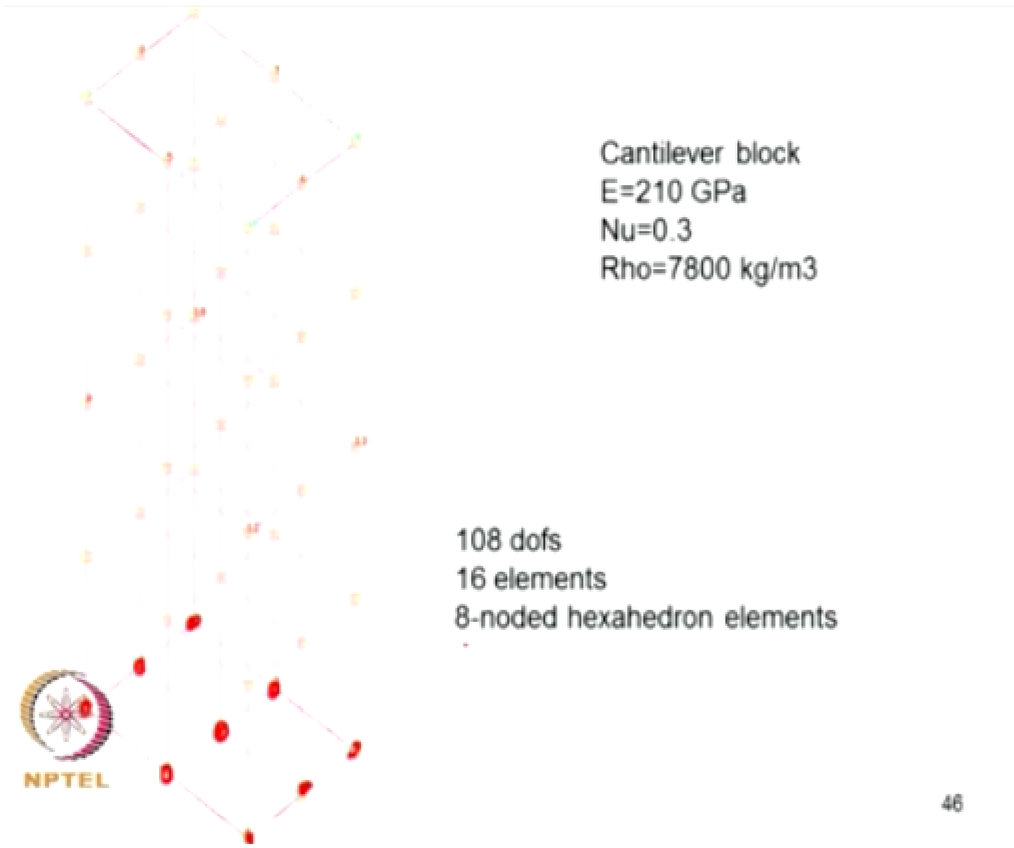
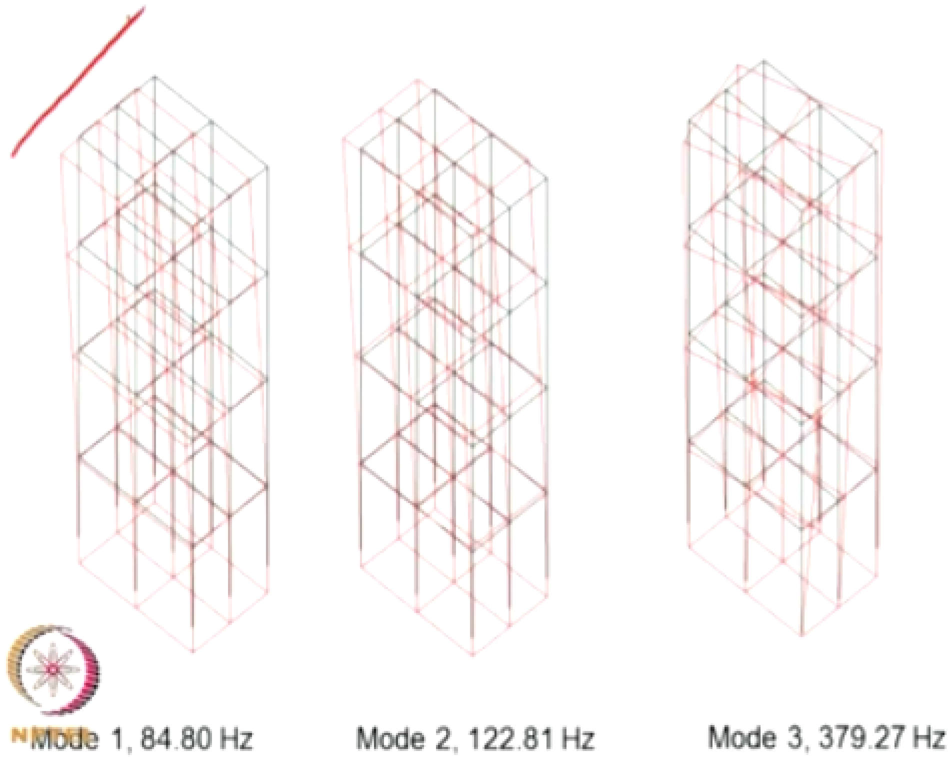
this, now KE is thus given by this integral over volume element now becomes B transpose DB determinant of J into this, now therefore if you can now see this the elements of this matrix are ratios of tri-quadratic functions of XI eta and zeta, so the Gauss quadrature will not evaluate this integral exactly, so you will have to have some judgment on this and what is recommended is use a 2 x 2 x 2 by 2 Gauss quadrature and accordingly we get this, this is not an exact evaluation, the mass matrix may get evaluated exactly but not the stiffness matrix, and also we'll be needing the volume in this calculation the volume itself can be evaluated again using Gauss quadrature it can be done exactly but since we are using Gauss quadrature this can be done, and since the integrand here will be a polynomial this can be evaluated exactly by using 2 x 2 x 2 Gauss quadrature.



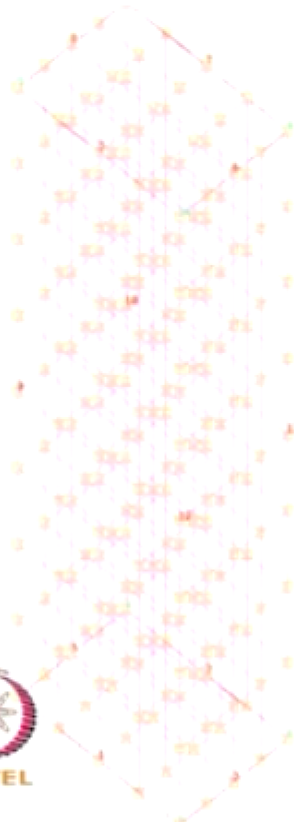
So now with some effort we have formulated the elements, so now we can illustrate with some simple examples suppose I consider a cantilever block which is fixed at the bottom, at these



nodes it is fixed, so in the first model that we have used I have 16 elements and 108 degrees of freedom, and we have used by 8 noded hexahedron elements, so these are the parameters of the structure, simple structure and let us try to find first few natural frequencies and mode shapes, so you can easily see that first mode will be a bending mode in this direction, the second mode



is likely to be this, and depending on the geometry the third mode could be a torsional, so indeed that happens the first mode is bending along this direction, and the second one is bending along this direction, and third mode is torsional mode, and the frequencies we get are shown here 84 hertz, 122 hertz, 379 hertz. Now what I will do is I will refine the mesh now,



DOFs=600
128 elements
8 noded hexahedron elements

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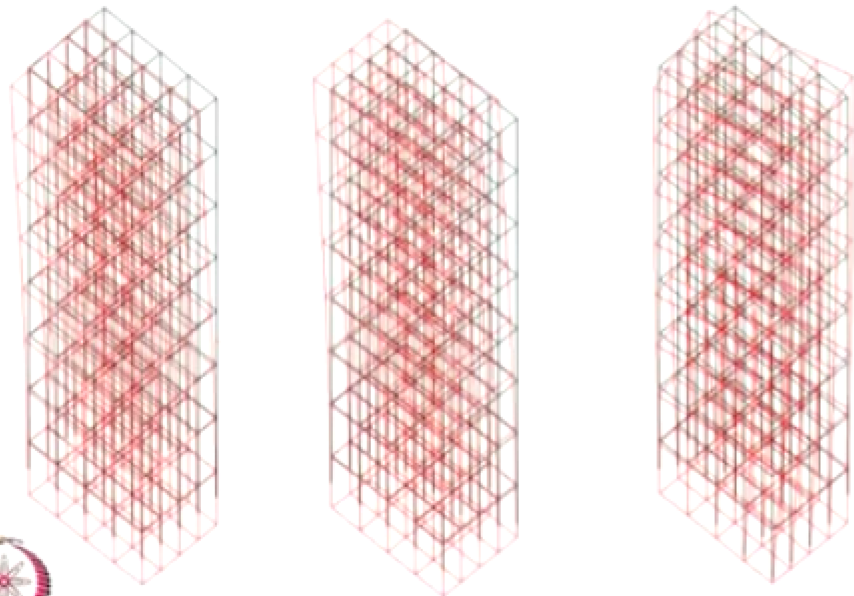
again I am using the same, the refinement is such that within an element I am creating more elements, okay so that leads to a model with 600 degrees of freedom and 128 elements, and again I am using 8 noded hexahedron element that means I am keeping the interpolation function the same, and the mesh in the previous edition of the model is a subset of this.



Mode 1, 83.33 Hz Mode 2, 120.43 Hz Mode 3, 357.38 Hz

So here these are the modes, again first mode is bending here, this and twisting, the frequencies of course are now different numbers so I am getting now is 83.33, this is 84.8, 122.81 379, so this if we follow all these rules that I mentioned this convergence will be from the above, okay.

2D approximations

Object is so thin that stresses across the thickness are neglected (plane stress)

Object so thick that displacements across the thickness are neglected (plane strain)

Objects possessing rotational symmetry about an axis and loaded and supported in an axisymmetric manner.

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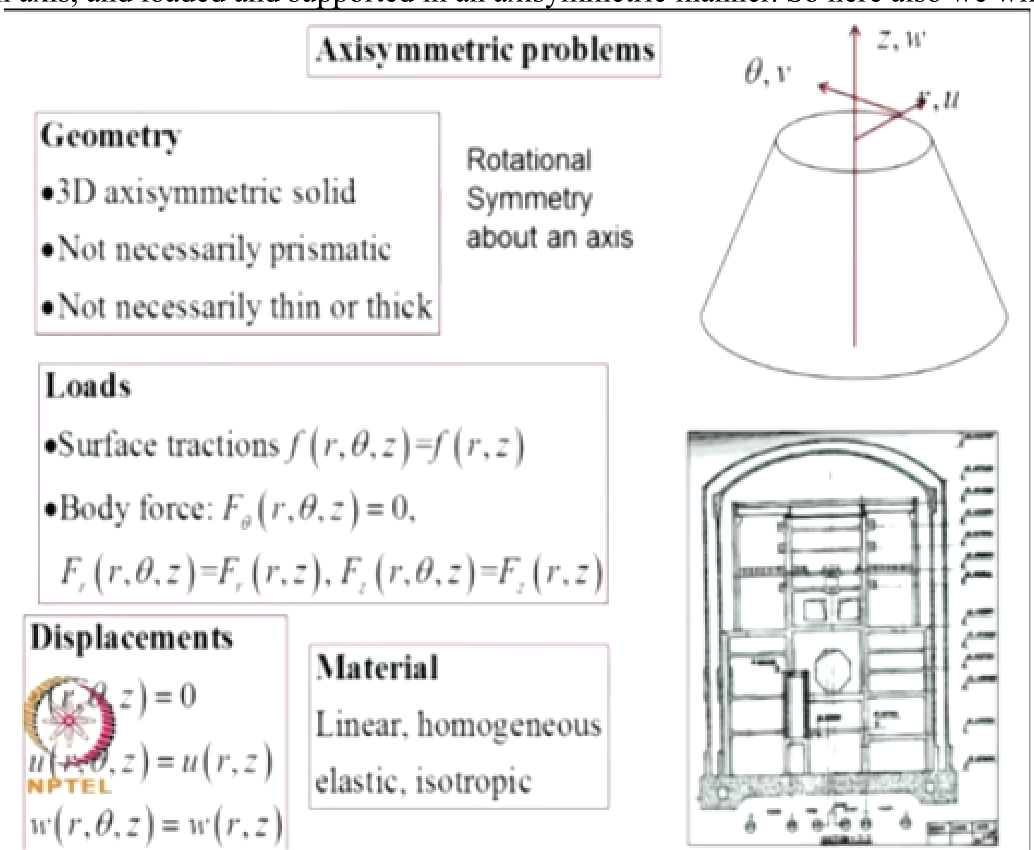
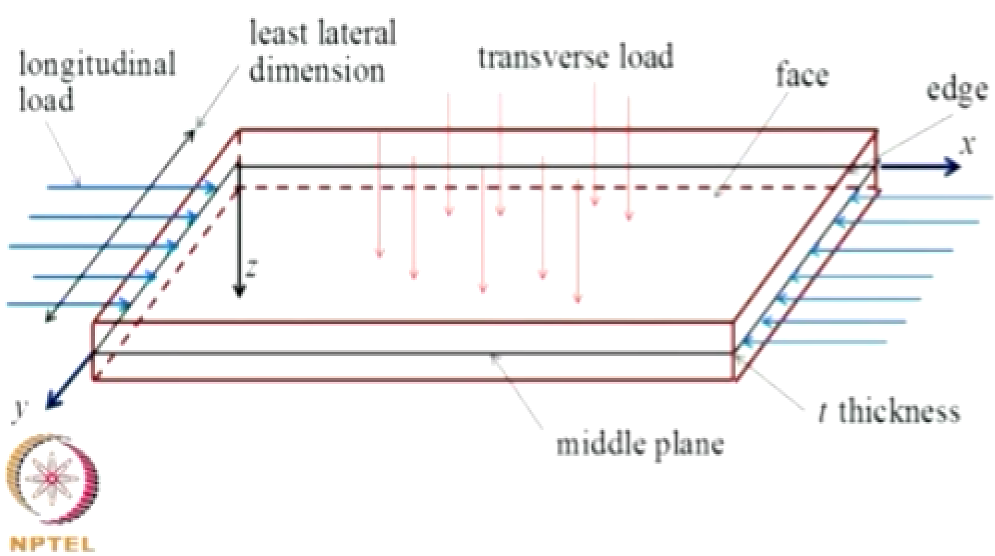
Now in the next part of the discussions, we will consider another 2D approximation to problems of solid continuum, if you recall now we had a 2 dimensional approximations already, in the plane stress model the object was so thin that the stresses across the thickness were neglected we got the plane stress model. In the next plane strain model the object was so thick that the displacements across the thickness were neglected and we got a plane strain model. Now in the next part of our discussion we consider objects processing rotational symmetry about an axis, and loaded and supported in an axisymmetric manner. So here also we will get a



2-dimensional approximation, so how does it work? Now this is an object with rotational symmetry so you take a curve and an axis and rotate about this, so you take this line and this axis and rotate about Z axis and you get this frustum of the cone, okay so this is an example of an object with rotational symmetry about an axis, such structures are found extensively in many application, for example this is a cross-section of a nuclear reactor vessel and the outer shell that you can see here has this type of property, it is rotationally symmetric about the Z axis here there will be many internals that may not have this type of rotational symmetry, but still the outer dome for example has this property. So if we are analyzing this type of properties it is useful to take advantage of this simplification in formulating the problem, the dimensions of the problem, the numerical, the dimension of the equation that we need to solve reduces.

Now what are the characteristics of this type of objects? Geometry, there is 3 dimensional axis symmetry, axis matrix solid, it is not necessarily prismatic, and not necessarily thin or thick, by that what I mean instead of this curve I can rotate this curve about Z axis, I get different geometries okay, so the cross sectional properties will not be the same at different values of Z. How about the loads? Now we are going to use a cylindrical polar coordinate system Z axis is this, and this is a radial axis, and this is the angle, theta axis, so we are going to use actually, suppose this origin, this is theta, this is R, and this is Z, that is the axis that we are going to use, now with reference to that axis the surface tractions are independent of theta, that means they are constant with respect to theta, the body forces in the theta direction are 0, and the surface tractions in the R and Z direction are independent of theta, that means they are uniform for all values of theta, like a cylindrical vessel like this, internally pressurized for example, suppose it is closed at the bottom and top and it is internally pressurized will have this type of model for surfaced tractions. Then using the property of the symmetry we will postulate that, there are 3 displacement fields, U along R, theta is V, and W is along Z, because of the symmetry and of loading boundary conditions and surface tractions and body forces it emerges at V is 0, then U is independent of theta, and W is independent of theta. Now the material that we are using is linear, homogenous, elastic and isotropic, so what we will do in the next class is we will develop a finite element model for problems that satisfy these requirements, so that would offer as another 2-dimensional approximation to a more complicated behavior.

Plate bending element



Following that we will consider behavior of, again thin elements of a different kind, we have now considered in plane stress models prismatic objects which carry loads in their own plane. Now we want to consider what are known as plate bending problems? So suppose you have a folded plate structure, and it is loaded then this plate for example there will be both transverse loads and in plane loads, okay, this could be a wall or a shear you know panel in a structure also, so we consider objects like this whose thickness is constant, right initially they are flat in addition to the in-plane loads we now consider transverse loads like a slab in a building deforming under its own weight, so that is the plate action.

Now a problem like this has 2 components, one is what is the membrane action where we consider the behavior of the structure only under inline in plane loads, and the bending action which is behavior of this type of structures under transverse loads, the bending action is known as plate action, and the response due to inline loads or in plane loads is known as membrane action. The analysis of membrane action can be carried out using plane stress models, so we need to next analyze the behavior under transverse loads, this requires generalization of beam, actually is a generalization of a grid if you recall we analyze the problems, grid structures consisting of beam elements and it was pointed out that bending in this member is twisting in this member, so a plate can be considered as a continuous analog of a grid where bending in one direction causes twisting in the other direction, so the stress resultants will consist of apart from bending moment and shear forces there will be twisting moment, so what we will do in the subsequent class is we will consider these two problems, problems of axis symmetry and problems of plate bending and this will develop based on 2-dimensional theory of elasticity and this we will develop by developing theory based on Kirchhoff–Love assumptions and Mindlin assumptions.

So in Kirchhoff–Love assumption the thickness of this member is taken to be small and certain assumptions on how a line element like this behaves will be made, whereas in midline theory which is generalization of Timoshenko beam theory, the beam can be thick and we will include shear deformation and also while computing inertia we will compute, include effects of rotary inertia. So these 2 problems will consider in the next class, and will conclude the present lecture at this stage.

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