Indian Institute of Science
Bangalore
NP-TEL
National Programme on
Technology Enhanced Learning

Course Title
Finite element method for structural dynamic
And stability analyses
Lecture – 20
Plane stress models
(continued)

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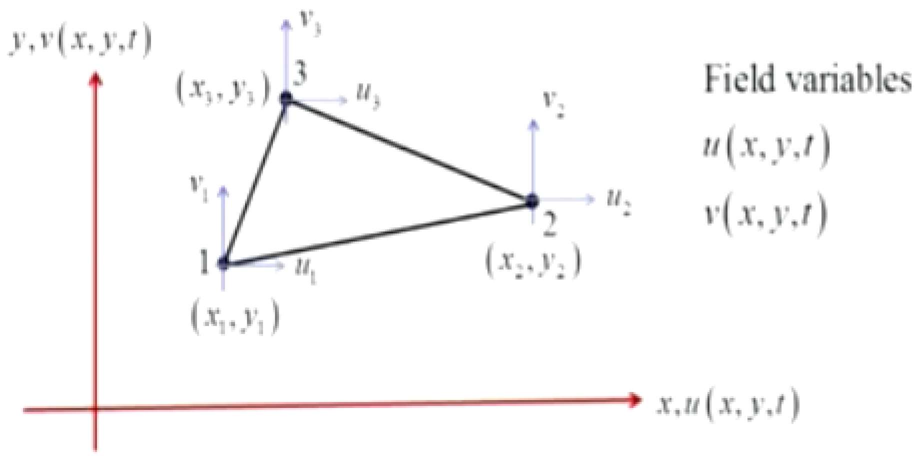
We have been discussing analysis of 2-dimensional continuum, so we will continue with that.

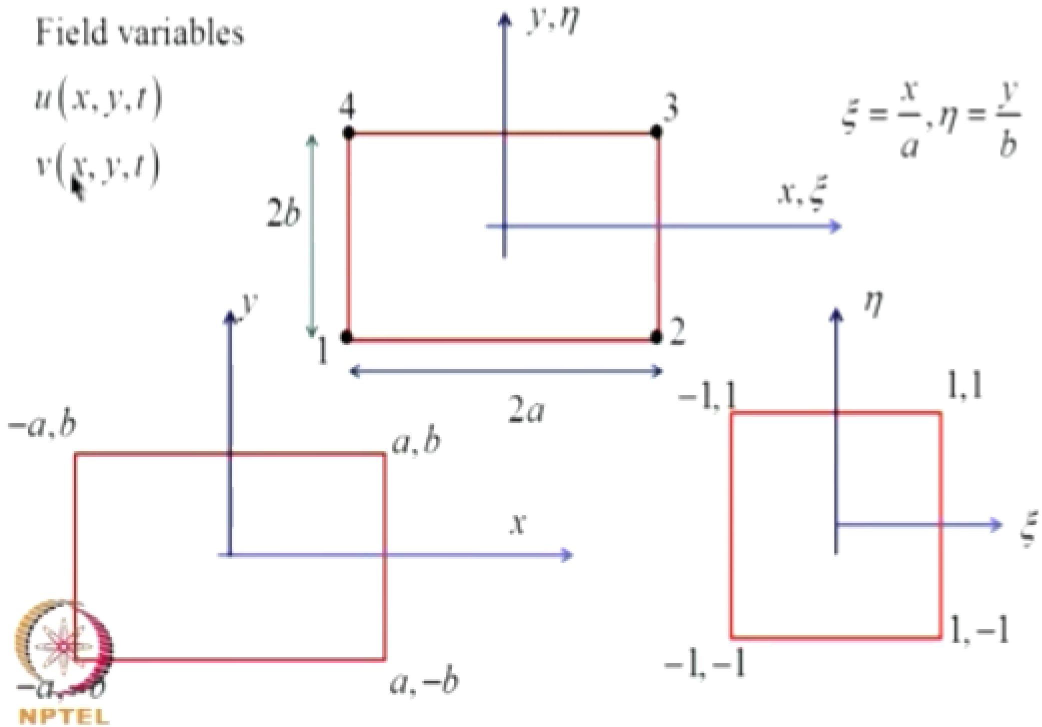
Linear triangular plane stress element



3 noded element with 6 dofs

So what we did in the previous lecture was we considered a triangular element, so this element has 3 nodes 1, 2, 3 and at each node there are 2 degrees of freedom, so this is a 3 noded element with 6 degrees of freedom, so the field variables are U and V which are functions of X, Y and T and we interpolate the field variable in terms of the nodal values, that means U is interpolated in terms of U1, U2, U3, and V is interpolated in terms of V1, V2, V3. And the interpolation functions are derived, so that for example the interpolation function N1 at this node will have value 1, and at these 2 nodes it will have value 0 and in this particular case we use linear interpolation functions.

Linear rectangular plane stress element



4 noded element with 8 dofs

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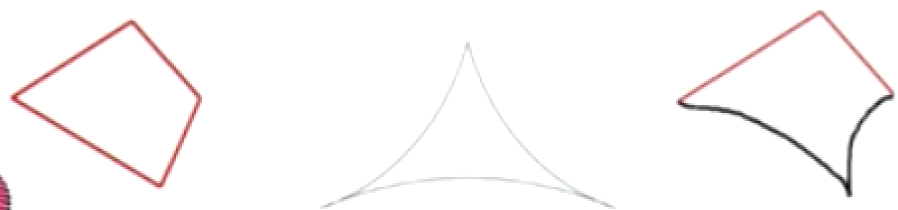
We also discussed rectangular plane stress element, again the field variables here are U and V, this element has 4 nodes 1, 2, 3, 4, and at each node there are 2 degrees of freedom, therefore it is 4 noded element with 8 degrees of freedom. Now to facilitate the development of the element

$$K_e = \int_A hB^t DB dA$$

$$M_e = \int_A h\rho N^t N dA$$

We are able to evaluate these integrals in closed form.

This may not be posisble in general situations.



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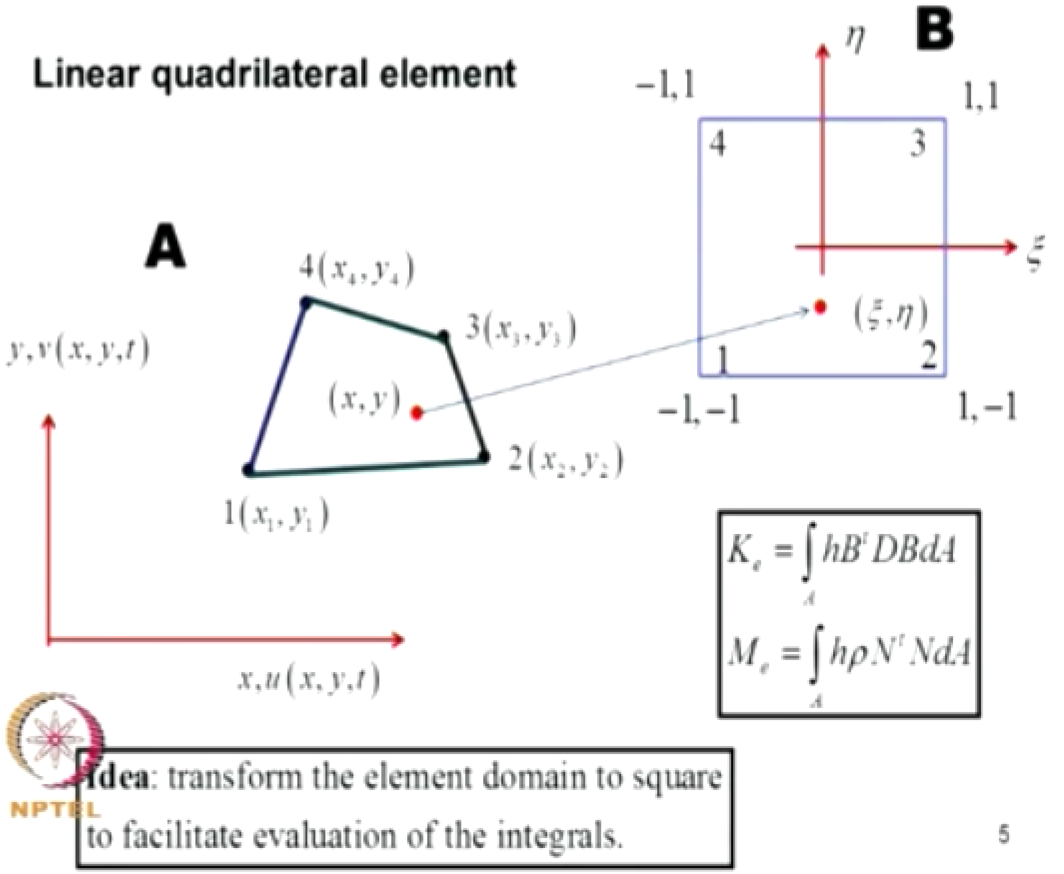
we made a coordinate transformation, so if you recall the elements of stiffness and mass matrices for the element have to be evaluated by carrying out these integrations, this is stiffness, and this is stiffness matrix, this is the mass matrix, so to facilitate this integration we transform this rectangular region to a square region so that integration can be done from - 1 to + 1. Now in the examples that we have considered so far it has been possible for us to evaluate these integrals in closed form, but this may not be always possible, for example if you have a quadrilateral element or a triangular element with curved edges or quadrilateral element with curved edges like this, evaluation of these integrals will not be straightforward, so it becomes necessary to deal with two complexities, one is a complexity associated with the geometry, and the complexity associated with evaluation of these integrals, so these two issues we will try to take a look in today's class.



So we will consider to start with a linear quadrilateral element, so this has 4 nodes, this is a coordinate system U and V, X and Y and even V are the displacement fields, U is along X, and V is along Y, and this is the quadrilateral element with nodes 1, 2, 3, 4, and the nodal coordinates are X1 Y1, X2 Y2, X3 Y3, and X4 Y4, now we need to now approximate the field variable U and V, variables U and V within this domain, now what we do is we introduce the transformation so that a point in XY here gets mapped to a new set of coordinates XI and eta that we have to introduce now what these are, and this coordinate transformation is implemented exclusively for the purpose of evaluating these integrals, we will see what transpires when we implement this transformation. The main idea is to you know introduce this transformation to facilitate evaluation of the integrals, the stiffness matrix is given by this, and

$$K_e = \int_A hB^t DBdA; M_e = \int_A h\rho N^t NdA$$

$$x = x(\xi,\eta)$$

$$y = y(\xi,\eta)$$

Following the way we represent the field variables, we write

$$x(\xi,\eta) = \sum_{j=1}^{4} N_j(\xi,\eta)x_j$$

$$y(\xi,\eta) = \sum_{j=1}^{4} N_j(\xi,\eta)y_j$$

Recall

$$N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta); N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta); N_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta)$$

$$N_j(\xi,\eta) = \frac{1}{4}(1+\xi_j\xi)(1+\eta_j\eta); j = 1,2,3,4$$

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the mass matrix is given by this. Now the transformation that we are doing is from X, Y is taken to sai eta, therefore we can consider the transformation in the mathematical form as X is a function of XI and eta, Y is a function of X and eta.

Now following the way we represent the field variables, we write now for this X and Y, this expression where, what we are doing is X(sai, eta) is being evaluated in terms of its nodal values and interpolation functions. Similarly Y is interpolated in the similar manner, so what we are going to do is we are going to use the same interpolation function that we use for representing the field variables, to represent the geometry also, that is geometry of this transformation. Now if you recall the N1, N2, N3, N4 for this configuration we have arrived, we have shown that this is, that is of this form and compactly for all the 4 indices, 1, 2, 3, 4 we have obtained this in the previous lecture, so we'll use this now here.

Consider the line connecting (x_2, y_2) to (x_3, y_3).

Upon making the tranformation $x = x(\xi,\eta)$ & $y = y(\xi,\eta)$

how does this line gets mapped?

Consider the line $\xi = 1$ connecting $(1,-1)$ & $(1,1)$.

$(\xi,\eta) = (1,-1) \Rightarrow$

$$x(\xi,\eta) = x_1 N_1(1,-1) + x_2 N_2(1,-1) + x_3 N_3(1,-1) + x_4 N_4(1,-1) = x_2$$

$$y(\xi,\eta) = y_1 N_1(1,-1) + y_2 N_2(1,-1) + y_3 N_3(1,-1) + y_4 N_4(1,-1) = y_2$$

$(\xi,\eta) = (1,1) \Rightarrow$

$$x(\xi,\eta) = x_1 N_1(1,1) + x_2 N_2(1,1) + x_3 N_3(1,1) + x_4 N_4(1,1) = x_3$$

$$y(\xi,\eta) = y_1 N_1(1,1) + y_2 N_2(1,1) + y_3 N_3(1,1) + y_4 N_4(1,1) = y_3$$

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Now let us consider the line 2, 3, and this line 2, 3, now we want to investigate what is the relationship between the two? So let us consider the line connecting X2, Y2 to X3, Y3, upon making the transformation X = sai, eta and Y = Y(sai, eta), how does this line gets mapped, or alternatively we can ask if this is a transformation we are using what happens to this line 2, 3 in this configuration, whichever way you look at we will find the answer to the required question. Now let us therefore consider the line XI = 1 connecting 1 - 1 and 1, 1 that means XI = 1 this line.

Now XI, at the one of the vertices the coordinates are 1, -1 so what happens to X there? X is X1 into N1 + X2 into N2, X3, X4, now how does these shape functions behave? Except this N2 all other shape functions will be 0, so this becomes X2. Similarly you can write the value of Y (XI, eta) that means where does this vertex 1, - 1 in this coordinate system go, in the original coordinate system, so X has gone to, I mean this, okay let us see, X2 we have found out, similarly Y will go to Y2 that means if I am considering 1 -1 that means the second point here it is going to 2 here. Similarly the other vertex 1, 1 you can verify that this goes to X3, Y3, now between these 2 point does the variation remain linear or not, that is the next question we have to see.

Consider the line $\xi=1$

$$x(\xi,\eta)=x_1N_1(1,\eta)+x_2N_2(1,\eta)+x_3N_3(1,\eta)+x_4N_4(1,\eta)$$

$$=x_2N_2(1,\eta)+x_3N_3(1,\eta)=\frac{1}{2}(1-\eta)x_2+\frac{1}{2}(1+\eta)x_3$$

$$y(\xi,\eta)=\frac{1}{2}(1-\eta)y_2+\frac{1}{2}(1+\eta)y_3$$

$$x=\frac{1}{2}(x_2+x_3)+\frac{1}{2}(1+\eta)(x_3-x_2)$$

$$y=\frac{1}{2}(y_2+y_3)+\frac{1}{2}(1+\eta)(y_3-y_2)$$

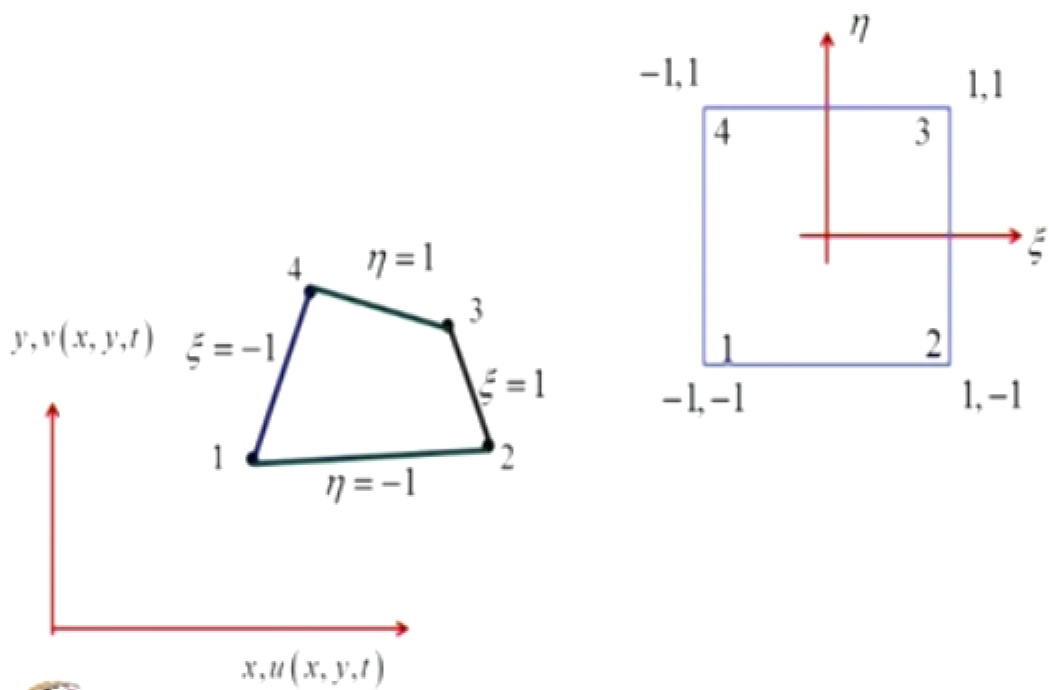
Eliminating η, we get $y-\frac{(y_2+y_3)}{2}=\left\{x-\frac{(x_2+x_3)}{2}\right\}\frac{(y_3-y_2)}{(x_3-x_2)}$

This is the equation of straight line passing through (x_2,y_2) & (x_3,y_3)

$\Rightarrow$ Line 2-3 in figure A is transformed to line 2-3 in B

The quadrilateral in A is transformed to the square in B. 8

Now consider the line XI = 1, which is that line? This line, XI = 1, now X (sai, eta) is X1, N1 eta and wherever XI is 1, XI is there I will write it as 1, so I get this expression, okay, I am simply putting for XI 1, then I will use the properties of these functions and I will be able to show that this is given by this expression, it’s a linear function of eta. Similarly the Y coordinate also get mapped to another linear function of eta and I get X and Y to be this. Now I can eliminate eta from this and get the equation in XY plane what it is, so if you do that I get Y minus this equal to this minus this, so what I am doing is this I am taking to the left side, this to the left side, and dividing this 1 + eta gets cancelled this is what I get. Now this is actually the equation of straight line passing through X2, Y2 and X3, Y3, so we conclude that line 2, 3 in figure A is transformed to line 2, 3 in figure B, that means 2, 3 is mapped to 2, 3 here, so we



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can summarize this, this is the line 2, 3 which comes here sai = 1, sai = 1 is this line that comes here, eta = 1 is this line it comes here, sai = -1 is this line it comes here, and 1, 2 is eta = -1 that comes here, so that means that this quadrilateral is getting mapped to this through the transformation that we are proposing, so these transformations of coordinates that is X (sai, eta)

Remark

The transformations on coordinates

$$x(\xi,\eta)=\sum_{j=1}^{4}N_j(\xi,\eta)x_j \ \& \ y(\xi,\eta)=\sum_{j=1}^{4}N_j(\xi,\eta)y_j$$

are similar to the transformations

$$u(\xi,\eta,t)=\sum_{j=1}^{4}N_j(\xi,\eta)u_j(t) \ \& \ v(\xi,\eta,t)=\sum_{j=1}^{4}N_j(\xi,\eta)v_j(t)$$

on the field variables.

That is the same interpolation functions are used to represent the geometry of the element and the displacement field variables.

The resulting element formulation is called isoparametric formulation.

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Next, let us, consider the evaluation of

$$M_e=\int_{A_e}\rho h[N]^T[N]dA \ \& \ K_e=\int_{A_e}h[B]^T[D][B]dA$$

by transforming (x,y) to (ξ,η) coordinates.

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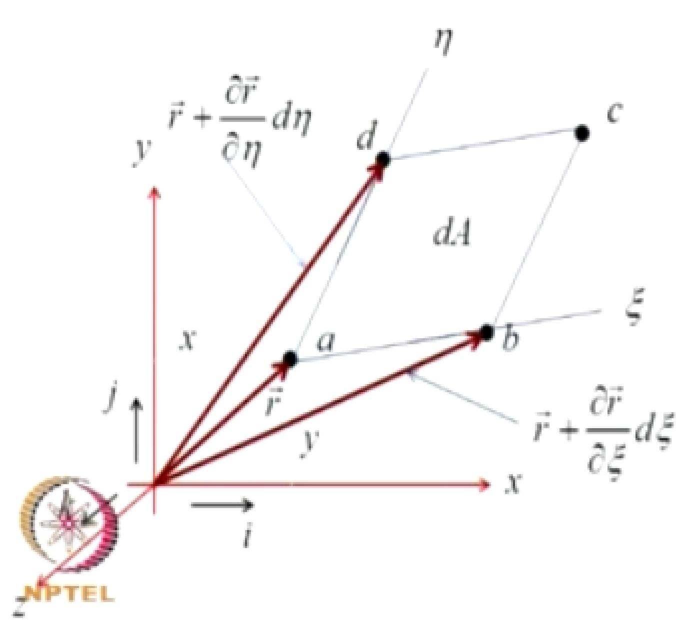
in terms of N, and a nodal values of X, and nodal values of Y for Y, this is quite similar to the transformation on field variable, this is what I have been pointing out, this is a field variable UJ(t) are the nodal values which are their degrees of freedom in our finite element model, that is what we conclude from this is the same interpolation functions these NJ's are being used to represent both the field variables as well as a geometric variables. The displacement field and the geometry of the element are getting represented through the same set of interpolation functions, resulting element formulation is called isoparametric formulation, okay, fine.

So what we have achieved is we have mapped a quadrilateral to a square, so when we are performing integration instead of performing over a quadrilateral which can be quite you know unwieldy, we have now the need to situation of being able to integrate or a square region, but what about the integrand, how does that behave? So let us consider the evaluation of the mass and stiffness matrices by transforming X, Y to sai, eta coordinates, and this transformation will

Digress - 1

$$I = \iint_A f(x,y)\,dxdy = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,|J|\,d\xi d\eta$$



$$\vec{r} = ix + jy$$

$$\frac{\partial \vec{r}}{\partial \xi} = i\frac{\partial x}{\partial \xi} + j\frac{\partial y}{\partial \xi}$$

$$\frac{\partial \vec{r}}{\partial \eta} = i\frac{\partial x}{\partial \eta} + j\frac{\partial y}{\partial \eta}$$

$$ab = \frac{\partial \vec{r}}{\partial \xi} d\xi$$

$$ad = \frac{\partial \vec{r}}{\partial \eta} d\eta$$

$$dA = \left(\frac{\partial \vec{r}}{\partial \xi} d\xi \times \frac{\partial \vec{r}}{\partial \eta} d\eta\right) \bullet k$$

now implement, so we need to digress a bit now to understand what the problem is, so first is we will consider a function F(x,y) in Cartesian coordinates, and using this transformation XI eta, if we map X and Y to X and eta, what is the rule of transformation? See this is the rule as you may be knowing but how does it really originate, to be able to do that let us consider a point, this point with a position vector R in XY plane, so that is R is IX + JY. Now sai and X are functions, sai and eta are functions of X and Y, so now I want, that is this, this is a sai, and this is a eta, some coordinates, okay, so now I want to deal with this situation so what is dou R/dou sai? This I dou X/dou sai + J dou Y/dou sai, where I, J, K are unit vectors along X, Y and Z respectively.

Similarly dou R/dou eta is I dou X/dou eta + J dou Y/dou eta. A, B the line A, B if you see is, what is this? so this plus this is equal to this, this is a vector sum, so if you are looking at A, B it is simply the difference between the two which is this, dou R/dou sai to this I, similarly if you look A, D it is the vector difference of this and this, position vector of D, and position vector of A and that AD is this. Now the area element DA is given by the triple product of these two position vectors and the K along this, this gives the area this is a result from algebra so you

$$dA = \left(\frac{\partial \vec{r}}{\partial \xi} d\xi \times \frac{\partial \vec{r}}{\partial \eta} d\eta \right) \bullet k$$

$$= \left(\frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi} \right) d\xi d\eta$$

$$= \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix} d\xi d\eta = |J| d\xi d\eta$$

$$\Rightarrow I = \iint_A f(x,y) dxdy = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)|J| d\xi d\eta$$

NPTEL

have, you can recall that, and if we do that and now substitute for dou R/ dou sai and dou R/ dou eta, the quantities that we have just now derived and this itself can be expressed as a determinant, see we have shown just here dou R/dou sai and dou R/dou eta and this is what I am substituting here, and implement this cross product, you should know the rule of cross product and the dot product, so if you do that you get this expression.

Now this quantity inside this parenthesis can be written as a determinant, and the matrix associated with this is known as Jacobian matrix, and this is a determinant of the Jacobian. So DA is therefore Jacobian of the transformation into D sai D eta, therefore I is integral F(x,y) DXDY A becomes now - 1 to + 1 this. Now that is first part of our digression, the second is the

Digress - 2

Numerical integration

Consider the one dimensional integral $\hat{I} = \int_a^b f(x)\,dx$

First let us transform x to a new variable u so that the limits of integration are from $-\frac{1}{2}$ to $\frac{1}{2}$.

$$u = \alpha x + \beta \Rightarrow \begin{matrix} -\frac{1}{2} = \alpha a + \beta \\ \frac{1}{2} = \alpha b + \beta \end{matrix} \Rightarrow x = (b-a)u + \frac{a+b}{2}$$

$$\Rightarrow \hat{I} = \int_{-\frac{1}{2}}^{\frac{1}{2}} f\left((b-a)u + \frac{a+b}{2}\right)(b-a)\,du = (b-a)I \text{ with } I = \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(u)\,du$$

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question of numerical integration so you may be familiar with the Trapezoidal rule, Simpson rule, and Newton Cotes formula etcetera, but we will consider what are known as gauss quadrature rules, and I will explain what the rationale of those rules are.

So let us consider for sake of discussion and evaluation of an integral I cap, A to B F(x) DX, first let us transform X into to a new variable U so that the limits of integration are from – 1/2 to + 1/2, so what I will do, I will substitute U as alpha X + beta, alpha and beta are the 2 parameters which I don't know, so when we are at the lower limits I want this to be – 1/2, so I want U to be – 1/2 when X is A, and U to be + 1/2 when X is B, so that helps us to solve for alpha and beta, and the transformation I am looking for is given by this. So if you substitute that into this equation I will get – 1/2 to + 1/2, F of this, U now (B – A) DU, so this is equal to B –A into I, where I is this integral, so I call this F(b-a) u + A+B/2 as phi(u) DU, so we will focus our attention on discussing how to evaluate this integral. So now what I wish to do is I have this

$$I = \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(u)\,du \approx \sum_{i=1}^{n} R_i \phi(u_i)$$

How to select n, R_i, and $u_i, i = 1, 2, \cdots, n$?

Keeping $(u_{i-1} - u_i)$ as constant for all $i = 1, 2, \cdots, n$ may not lead to the best solution.

Assuming that $\phi(u)$ is continuous between $-\frac{1}{2}$ and $\frac{1}{2}$, we write

$$\phi(u) = a_0 + a_1 u + a_2 u^2 + \cdots + a_m u^m + \cdots$$

$$\Rightarrow \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(u)\,du = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{m=0}^{\infty} a_m u^m\,du = \sum_{m=0}^{\infty} a_m \left(\frac{u^{m-1}}{m+1}\right)_{-0.5}^{0.5}$$

$$= a_0 + \frac{1}{12}a_2 + \frac{1}{80}a_4 + \frac{1}{448}a_6 + \frac{1}{2304}a_8 + \cdots$$

function phi(u) and this is my U axis from – 1/2 to + 1/2, I want to evaluate this function at some sets of points, U1, U2 some UN, and I want to attach a weight R1 to phi(U1) R2 to phi(U2) and be able to evaluate this integral, so I am looking for evaluation of I by using this representation, so the effort here will be spent in evaluating phi(U), so any integration method that we use, we should get the ideal integration scheme is get a good approximation for I with minimum number of evaluation of phi(U), or in other words if you evaluate say phi at say 10 points which is the best way that I can combine them, right for any other combination you know you can use trapezoidal rule and things like that, then you can evaluate R, but then you are constraining yourself to place UI in a particular manner, so the weighting factor also can be optimized.

Now the question is therefore how to select N, that means how many number of points, how to select these weights, and where to place this point where I am going to evaluate these functions so that I get a good approximation to I. Keeping this UI + 1 – UI as a constant, that means you sample it at a constant step size that may not be the ideal situation, because that depends on how this function varies, okay, it may not lead to the best solution always. Now assuming that phi(U) is continuous between – 1/2 and 1/2, we write now a power series expansion convergent, power series expansion in this form, so in this form. Now this – 1/2 to + 1/2, phi(U) DU, now for phi(U) I substitute this summation, assume summation which is this, 0 to infinity AM, UM, DU and upon performing integral I get this, so I can evaluate these terms, and in terms of the unknowns A naught, A1, A2 etcetera you can get an expansion like this for the integral.

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(u)\,du = a_0 + \frac{1}{12}a_2 + \frac{1}{80}a_4 + \frac{1}{448}a_6 + \frac{1}{2304}a_8 + \cdots$$

$$\phi(u) = \sum_{m=0}^{\infty} a_m u^m \Rightarrow \phi(u_i) = \sum_{m=0}^{\infty} a_m u_i^m$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(u)\,du \approx \sum_{i=1}^{n} R_i \phi(u_i) = \sum_{i=1}^{n} R_i \left(\sum_{m=0}^{\infty} a_m u_i^m \right) = \sum_{m=0}^{\infty} a_m \left(\sum_{i=1}^{n} R_i u_i^m \right)$$

$$= R_1\left(a_0 + a_1u_1 + a_2u_1^2 + \cdots + a_m u_1^m + \cdots\right)$$

$$+R_2\left(a_0 + a_1u_2 + a_2u_2^2 + \cdots + a_m u_2^m + \cdots\right) + \cdots$$

$$+R_n\left(a_0 + a_1u_n + a_2u_n^2 + \cdots + a_m u_n^m + \cdots\right) + \cdots$$

$$= a_0\left(R_1 + R_2 + \cdots + R_n\right) + a_1\left(R_1u_1 + R_2u_2 + \cdots + R_nu_n\right) + \cdots$$

$$+a_m\left(R_1u_1^m + R_2u_2^m + \cdots + R_nu_n^m\right) + \cdots$$

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Now, so this is expansion we have got, now we have this power series representation for phi(U), now if I evaluate phi(U) at UI this will be the representation, okay. Now therefore now I will substitute this into the representation that we are using, so phi(U) DU is approximately this, this is what is our proposition, and for phi(UI) I will write this expansion, this is what I get, so I get now a representation I will interchange the order of summation and I will get a summation integral, I mean representation like this, so in long hand if you expand this R1 into this infinite number of terms, R2 into infinite number of terms so on and so forth.

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(u)\,du = a_0 + \frac{1}{12}a_2 + \frac{1}{80}a_4 + \frac{1}{448}a_6 + \frac{1}{2304}a_8 + \cdots$$

$$a_0(R_1 + R_2 + \cdots + R_n) + a_1(R_1u_1 + R_2u_2 + \cdots + R_nu_n) + \cdots$$

$$+a_m(R_1u_1^m + R_2u_2^m + \cdots + R_nu_n^m) + \cdots$$

$$\Rightarrow$$

$$\left.\begin{array}{l}(R_1 + R_2 + \cdots + R_n) = 1 \\ (R_1u_1 + R_2u_2 + \cdots + R_nu_n) = 0 \\ (R_1u_1^2 + R_2u_2^2 + \cdots + R_nu_n^2) = \frac{1}{12} \\ (R_1u_1^3 + R_2u_2^3 + \cdots + R_nu_n^3) = 0 \\ (R_1u_1^4 + R_2u_2^4 + \cdots + R_nu^4) = \frac{1}{80}\end{array}\right\} \text{If we consider } 2n \text{ equations, there would be } 2n \text{ unknowns}$$

$$\vdots$$

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Now this, I know right hand side what it is, and the left hand side I have, I am rearranging the terms now A naught I will collect, coefficient of A naught are collected that is R1 + R2 + RN, A1 is this, AM is this, and this continues. Now I will now compare the coefficients of A naught, A1, A2, A3 on both sides I will get a set of equations now, for this to be equal coefficient of A naught here is 1, and coefficient of A naught here is this, this should be satisfied. And similarly coefficient of, this is A naught, A1 is 0 here, so that must be 0, so by writing this we can write as many equations as you want, suppose if you consider 2N equation, there will be 2N unknowns, what are the 2N unknowns? R1, R2, RN first set, then U1, U2, UN, so that constitute to 2N unknowns, now you select these 2N equations and solve for those 2N unknowns and that gives you the representation that you are looking for.

The equations are nonlinear in nature and the question on how to solve them still remains.

If $\phi(u)$ is a polynomial of degree not higher than $2n-1$, then $u_1, u_2, \cdots, u_n$ are the zeros of Legendre polynomials $P_n(u)$.

That is, $\frac{d^n}{du^n}\left[u^2-\left(\frac{1}{2}\right)^2\right]^n = 0$

The roots are real. Once these roots are determined, $(R_i)_{i=1}^n$ can be determined by solving a set of algebraic equations.

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But how do you solve for that? This is a set of nonlinear equations, you can see quadratic terms here so there are nonlinear equations, but there is a nice result which shows that if phi (U) is a polynomial of degree not higher than 2N – 1, then this U1, U2, UN which are the points where I want to evaluate the function according to the prescription that I just outlined would turn out to be the zeros of Legendre polynomials PN(U), that means these U's will satisfy this equation, so the roots, and these roots are real, so once these roots are determined if I know U1, U2, UN then there remaining N unknowns which are R1, R2, RN can be solved by solving a linear problem, okay, so I can find this constants as I wish. So if you are dealing with a polynomial then if you retain adequate number of terms you will be evaluating the integral exactly, there is no approximation at all, okay. Now for example N = 3, this is a rule according to which we

Example: $n = 3$

That is, $$\frac{d^3}{du^3}\left[u^2 - \left(\frac{1}{2}\right)^2\right]^3 = 0$$

$$\Rightarrow \frac{d^3}{du^3}\left[u^6 - \left(\frac{1}{2}\right)^6 - 3u^4\left(\frac{1}{2}\right)^2 + 3u^2\left(\frac{1}{2}\right)^4\right] = 0$$

$$u\left(20u^2 - 3\right) = 0$$

$$u = 0, \pm\frac{1}{2}\sqrt{\frac{3}{5}}$$

$$\Rightarrow u_1 = -\frac{1}{2}\sqrt{\frac{3}{5}}, u_2 = 0, u_3 = \frac{1}{2}\sqrt{\frac{3}{5}}$$

$$\Rightarrow R_1 = \frac{5}{18}, R_2 = \frac{4}{9}, R_3 = \frac{5}{18}$$

want to evaluate, so if you expand this now for N = 3 and carry out this differentiation I get this equation, so it turns out that the roots of the equations are U = 0 and +-1/2 square root 3/5, so the first point I have to select is this, second one is 0, third point is this, with these weights if I add and evaluate the integral so 2N - 1 means 4, 6 - 1 a fifth order polynomial will be exactly valuated by these weights, okay because a function is polynomial.

$$I = \int_{-1}^{1} f(u)\,du = \sum_{i=1}^{n} R_i f(u_i)$$

n	$\pm u_i$	R_i
1	0	2
2	$1/\sqrt{3}$	1
3	0	$8/9$
	$\sqrt{0.6}$	$5/9$
4	$\left[\frac{3 \pm \sqrt{4.8}}{7}\right]^{\frac{1}{2}}$	$\left[\frac{1}{2} \mp \frac{\sqrt{30}}{36}\right]$

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Now I have done a few calculations here for different values of N the roots, these are the roots of the Legendre polynomial, this is for N = 1, 2, 3, 4, and these are the weights, okay, so if you are implementing this you can use this information to do this. Now we can see how this formula

A polynomial of order p is integrated exactly by employing n=smallest integer greater than $0.5(p+1)$

Integral	Exact	n=1	n=2	n=3	n=4
$\int_{-1}^{1} dx$	2	2	2	2	2
$\int_{-1}^{1} x dx$	0	0	0	0	0
$\int_{-1}^{1} x^2 dx$	2/3	0	0.6667	0.6667	0.6667
$\int_{-1}^{1} x^3 dx$	0	0	0	0	0
$\int_{-1}^{1} x^4 dx$	0.4	0	0.2222	0.4000	0.4000
$\int_{-1}^{1} x^5 dx$	0	0	0	0	0
$\int_{-1}^{1} x^6 dx$	2/7=0.2857	0	0.0741	0.2400	0.2857

perform, now if I want to evaluate integral -1 to +1, the answer is 2, so if I take one term it gives the exact answer, so the rule is a polynomial of order P is integrated exactly by employing N number of terms which is smallest integer greater than 0.5 P+1. So now X, so 1, so P + 1 is 2, you need at least one term, if you use one term you will get exact solution but this is of course 0, because it is a odd function under symmetric limits, there is no problem, but you come here P = 2, so 2 + 1 is 3, 1.5 I need at least 2 terms before I can get the value of the integral correctly, so if you see here if I take only one point I get answer as 0 which is wrong, if I get 2 points this is an exact answer X cube by, whatever you can quickly do this and we get for N = 2 right answer.

Similarly 3, of course there is no problem, 4 this is 4 + 1, 2.5 I should have at least three terms, so with one time I get 0, two terms I get 0.22 and this is a 0.4. Now 6, 6 + 1 is 7, 3.5 I would need at least 4 terms to get that, so the 0.2857 is the exact answer, 1 does not give a good answer, 2 is this answer, 3 is this, 4 gives the exact answer. So you can quickly verify these things in the information needed to check these tables are here, now we will return to the

Return to

$$M_e = \int_{A_e} \rho h[N]^T [N] dA = \int_{-1}^{1}\int_{-1}^{1} \rho h[N]^T [N]|J| d\xi d\eta$$

We propose to evaluate this integral using the Gauss quadrature.

For this we need to establish the order of the integrand

so as to correctly choose the number of integration points.

We have

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}, x(\xi,\eta) = \sum_{j=1}^{4} N_j(\xi,\eta)x_j \ \& \ y(\xi,\eta) = \sum_{j=1}^{4} N_j(\xi,\eta)y_j$$

$$\Rightarrow J = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

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evaluation of the integrals, this is a mass matrix, and as using the first result I will be able to write in this form, okay, upon transformation that is our first set of results that we got. Now how do we evaluate this integral is a question. Now we propose to evaluate this integral using Gauss quadrature, for that too to do that correctly I should know the order of the function which is this integrand, that means how does N transpose N into determinant of J vary with respect to eta and XI, if I know that then I will be able to select correctly the number of integration points, otherwise I will again as you saw here if your integrand is X to the power of 6, and if you happen to take an N = 3, you will not get a right answer, you have to take N = 4, because you know this rule you will be able to make this choice.

Now we have to now work through the evaluation of this determinant of the Jacobian, so what we know right now Jacobian is given by this, Jacobian of the transformation, X we are writing like this, and Y we are writing like this, so given this I can now substitute for these derivatives in terms of NJ'S and using this expression I will be able to write J as product of 2 matrices, the first matrix is matrix contains derivatives of N1 and N2 with respect to XI and eta, and second matrix contains a coordinates of the nodes, okay, so all these quantities are known, so we can

It can be shown that

$$J=\frac{1}{4}\begin{bmatrix} e_1+e_2\eta & f_1+f_2\eta \\ e_3+e_2\xi & f_3+f_2\xi \end{bmatrix}$$

$$e_1=(-x_1+x_2+x_3-x_4)$$

$$e_2=(x_1+x_2+x_3-x_4)$$

$$e_3=(-x_1-x_2+x_3+x_4)$$

$$f_1=(-y_1+y_2+y_3-y_4)$$

$$f_2=(y_1+y_2+y_3-y_4)$$

$$f_3=(-y_1-y_2+y_3+y_4)$$

$$\Rightarrow \det[J]=\frac{1}{16}(c_1+c_2\xi+c_3\eta)$$

$$c_1=e_1f_3-e_3f_1;\, c_2=e_1f_2-e_2f_1;\, c_3=e_2f_3-e_3f_2$$

$\Rightarrow [N]^t[N]|J|$ is a biquadrtatic or bicubic function.

$\Rightarrow$ Use (2×2) array of integration points.

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go ahead and perform the calculation. If you can, it is quite straightforward to do this, if you do this I get J to be in this form where E1, E2, F1, F2 and F3 are explained here in terms of nodal coordinates. So moment I have this I can evaluate the determinant of J, and it turns out to be this, and interestingly it turns out that the determinant of J in this case is a linear function of X and eta, it is precisely this that I wish to determine then only I will know which order of integration I need to implement. We knew the order of N transpose N, because we know entries in N, and so we know what order that product would be, but we didn't know what was the order of this J is, I mean we didn't even know that it is linear or not first of all, so if we now look into this we see that it is a biquadratic or a bicubic function, depending on which terms you are looking at, so using the rule that we are using a 2x2 array of integration points would be adequate to perform this integration, so mass matrix can be evaluated exactly by using Gauss quadrature with 2x2 Gauss quadrature point mesh.

$$K_e = \int_A hB^t DBdA = \int_{-1}^{1}\int_{-1}^{1} hB^t DB|J|d\xi d\eta$$

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \eta} = \frac{\partial x}{\partial \eta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial}{\partial y}$$

$$\Rightarrow \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix} = [J]\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = [J]^{-1}\begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix}$$

How about stiffness matrix? So I need here B transpose DB into the Jacobian, so again I need to perform some calculations here so this D matrix has to operate now, so I need this information dou/dou sai is dou X/dou sai into dou/dou X + dou Y/dou sai into dou/dou Y etcetera, and this if you put in the matrix form I have this vector of gradient dou/dou sai and dou/dou eta and this is a Jacobian matrix this, so I can either write this or the inverse of this I can express dou/ dou X and dou/dou Y in terms of dou/dou sai and dou/ dou eta through J inverse. Now B, recall B

$$B=\begin{bmatrix}\frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}\begin{bmatrix}N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0\\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4\end{bmatrix}$$

$$=\begin{bmatrix}\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x}\\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y}\end{bmatrix}=[J]^{-1}\begin{bmatrix}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi}\\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta}\end{bmatrix}$$

$$\left(\frac{1}{4}\right)^{-1}\begin{bmatrix}e_1+e_2\eta & f_1+f_2\eta\\ e_3+e_2\xi & f_3+f_2\xi\end{bmatrix}^{-1}\begin{bmatrix}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi}\\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta}\end{bmatrix}$$

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was this for this problem so we get after multiplication I get this, and it turns out that the quantity that we are looking for is given in this form, okay, so elements of B transpose DB into

$\Rightarrow$ Elements of $B^t DB|J|$ are ratios of biquadratic functions and linear functions.

$K_e = \int_{-1}^{1}\int_{-1}^{1} hB^t DB|J| d\xi d\eta$ cannot be evaluated exactly by using Gaussian quadrature.

Recommendation based on experience: use (2×2) Gauss quadrature.

More on this later.

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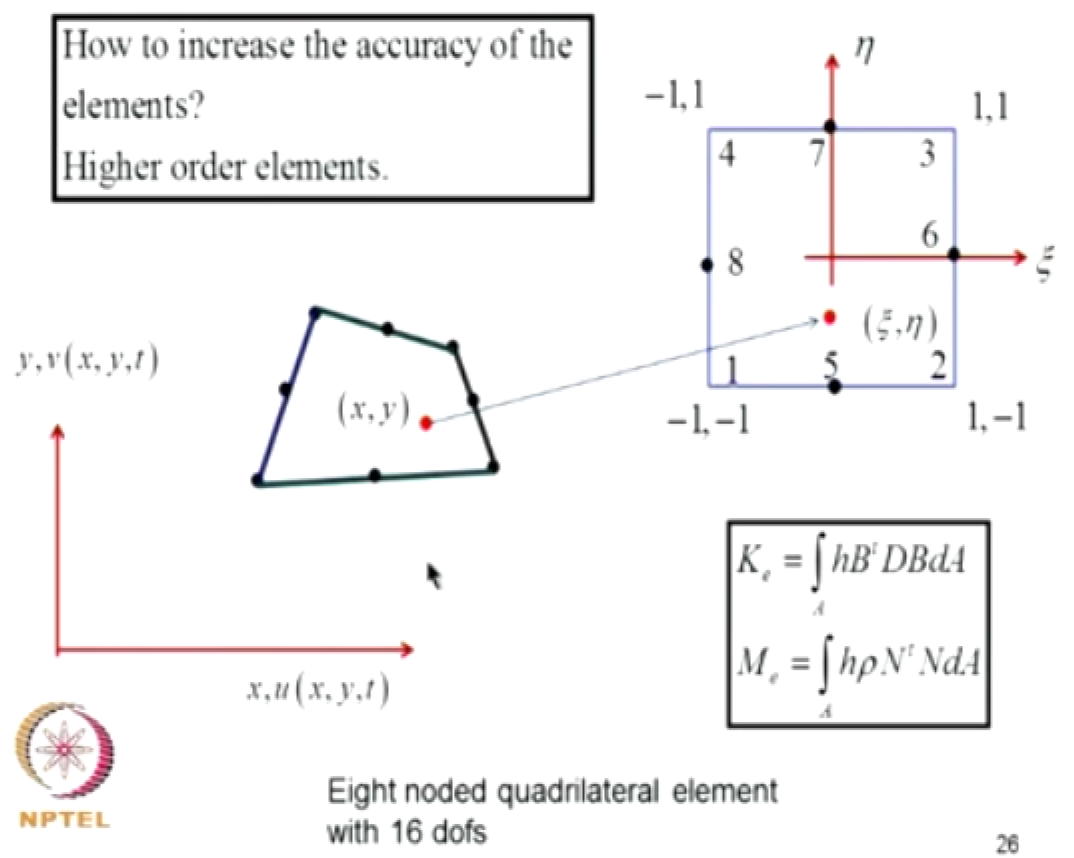
determinant of J are ratios or biquadratic functions and linear functions, see there is a inverse here, so therefore if you look at individual terms they won't be polynomials, there will be ratios of polynomials, okay, it turns out they are ratios of biquadratic functions and linear functions, so this would mean Gauss quadrature won't be able to evaluate elements of KE in exact manner, so if you use Gauss quadrature to evaluate this we are introducing certain approximation, but Gauss quadrature is so convenient we will make some discussion on this subsequently, but right now the recommendation based on experience is used a 2x2 Gauss quadrature, we will come to this slightly later once again, okay, so summary is mass matrix gets evaluated exactly, and stiffness matrix there will be an approximation.



Now this is, I have used now a 4 noded quadratic element, now the question is if you are using discretization, if you are modeling any structural behavior how do you improve upon the accuracy? One way is stick to 4 noded quadrilateral elements and use more of them, that means refine the mesh introduce more number of elements and do that, the other way is use a higher order polynomial to interpolate the field variables, if you want to do that it amounts to increasing the number of nodes in an element, so instead of having 4 noded element we can have 8 noded element okay, so then when I interpolate the interpolation functions will be of higher order polynomials, there won't be what we saw for a 4 noded element, because now there are additional nodes that come into our formulation. Suppose I take this, as before I will retain these 3 nodes and introduce additional nodes at the midpoint of each of these elements. Again we will transform this to a square region and evaluate the elements of stiffness and mass matrices using Gauss quadrature, so this is a 8 noded quadrilateral element with 16 degrees of freedom.

Eight noded rectangular element

$$u(\xi,\eta,t)=\sum_{j=1}^{8}N_j(\xi,\eta)u_j(t) \;\&\; v(\xi,\eta,t)=\sum_{j=1}^{8}N_j(\xi,\eta)v_j(t)$$

Nodes 1 to 4: $N_j(\xi,\eta)=\frac{1}{4}(1+\xi_j\xi)(1+\eta_j\eta)(\xi_j\xi+\eta_j\eta-1)$

Nodes 5 and 7: $N_j(\xi,\eta)=\frac{1}{2}(1-\xi^2)(1+\eta_j\eta)$

Nodes 6 and 8: $N_j(\xi,\eta)=\frac{1}{2}(1-\eta^2)(1+\xi_j\xi)$

Use (3×3) Gauss integration points.

With (3×3) mesh, mass matrix is evaluated exactly.

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Now the field variable now needs to be represented in terms of the nodal values and there are 8 nodal values for U, so if you go back here for U1, U2, U3, U4, U5, U6, U7, U8, so I have to utilize all of that to interpolate U(x,y,t) you know to facilitate the development of this element. Now similarly V is represented like this, now using the logic that we already discussed we need not have to go into all the details every time, we get these interpolation functions which have this Kronecker delta property, for nodes 1 to 4 I get, this is the interpolation function, for 5 and 7 these are the interpolation functions, and 6 and 8 we get this. Now I would not like to get into all the details of the formulation of this integrands, but if you indeed perform all that you will be able to notice that you need to use 3x3 Gauss integration points, with a 3x3 mesh mass matrix is evaluated exactly, but the stiffness matrix continues to be evaluated approximately.

Eight noded quadrilateral element

$$u(\xi,\eta,t)=\sum_{j=1}^{8}N_j(\xi,\eta)u_j(t)\,\&\,v(\xi,\eta,t)=\sum_{j=1}^{8}N_j(\xi,\eta)v_j(t)$$

Nodes 1 to 4: $N_j(\xi,\eta)=\frac{1}{4}(1+\xi_j\xi)(1+\eta_j\eta)(\xi_j\xi+\eta_j\eta-1)$

Nodes 5 and 7: $N_j(\xi,\eta)=\frac{1}{2}(1-\xi^2)(1+\eta_j\eta)$

Nodes 6 and 8: $N_j(\xi,\eta)=\frac{1}{2}(1-\eta^2)(1+\xi_j\xi)$

$$x=\sum_{j=1}^{8}N_j(\xi,\eta)x_j$$

$$y=\sum_{j=1}^{8}N_j(\xi,\eta)y_j$$

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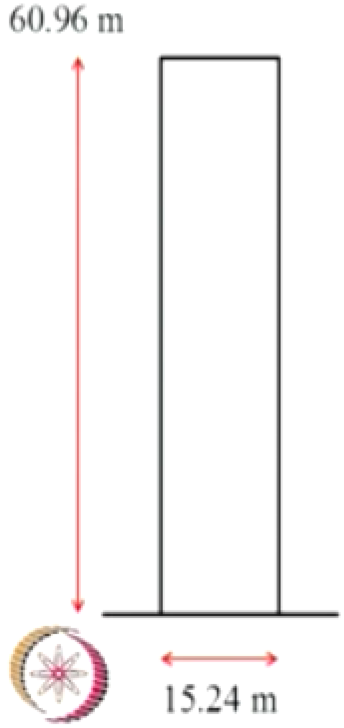
Use (4×4) Gauss integration points.

With (4×4) mesh, mass matrix is evaluated exactly.

Now how about, this is a rectangular element, this discussion is not for this element, this discussion is for a rectangular element, for a quadrilateral element the same formulary works except now that the X and Y also need to be interpolated using the same shape functions, okay, so in this case it turns out, we need to use 4x4 Gauss integration points in evaluating stiffness and mass matrices, and if you use 4x4 mesh, the mass matrix is evaluated exactly, so the mesh size that you need to select should be such that the mass matrix, elements of mass matrix are evaluated exactly, see mass matrix elements turn out to be polynomials, because N transpose N remains as polynomial, whereas here because of the Jacobian also of course would play a role but here because of this you know various operations involved here the K matrix there will be difficulties as we saw while before.



$E = 34.474 \times 10^9$ N/m^2

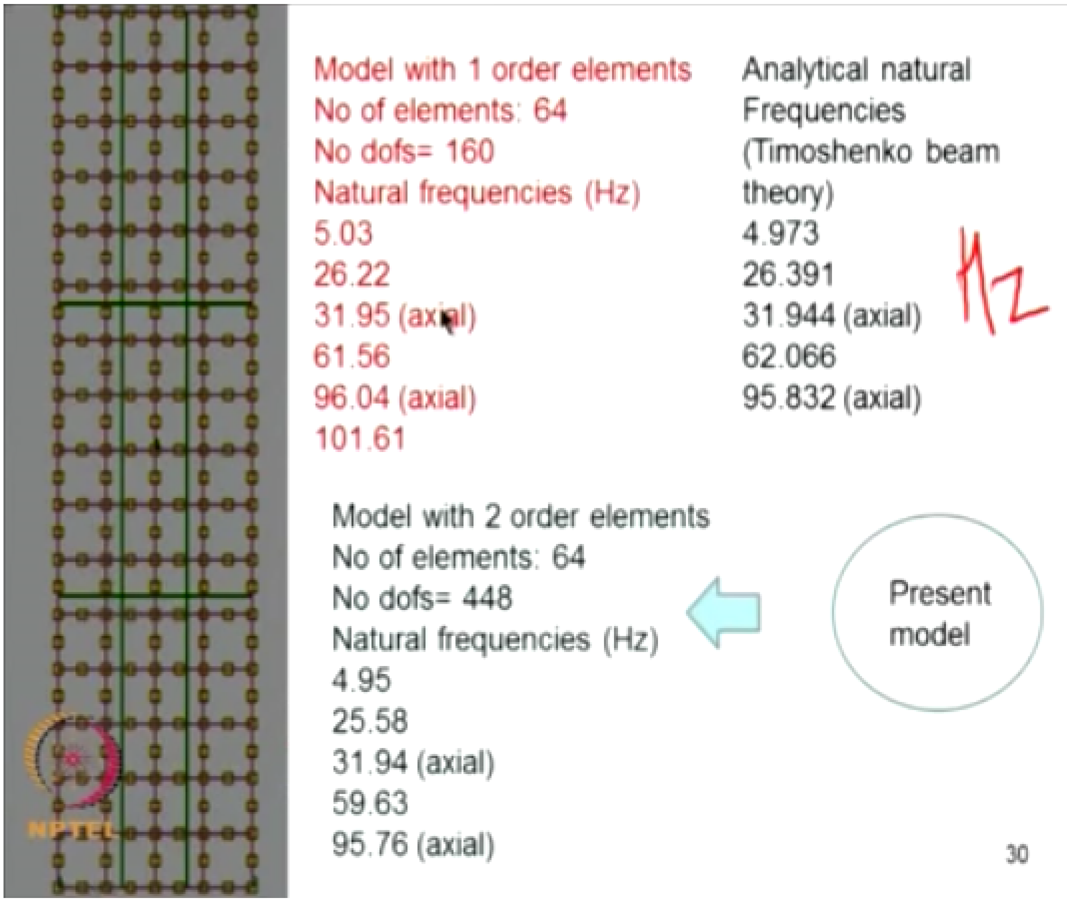
$\nu = 0.11$

$\rho = 568.7$ kg/m^3

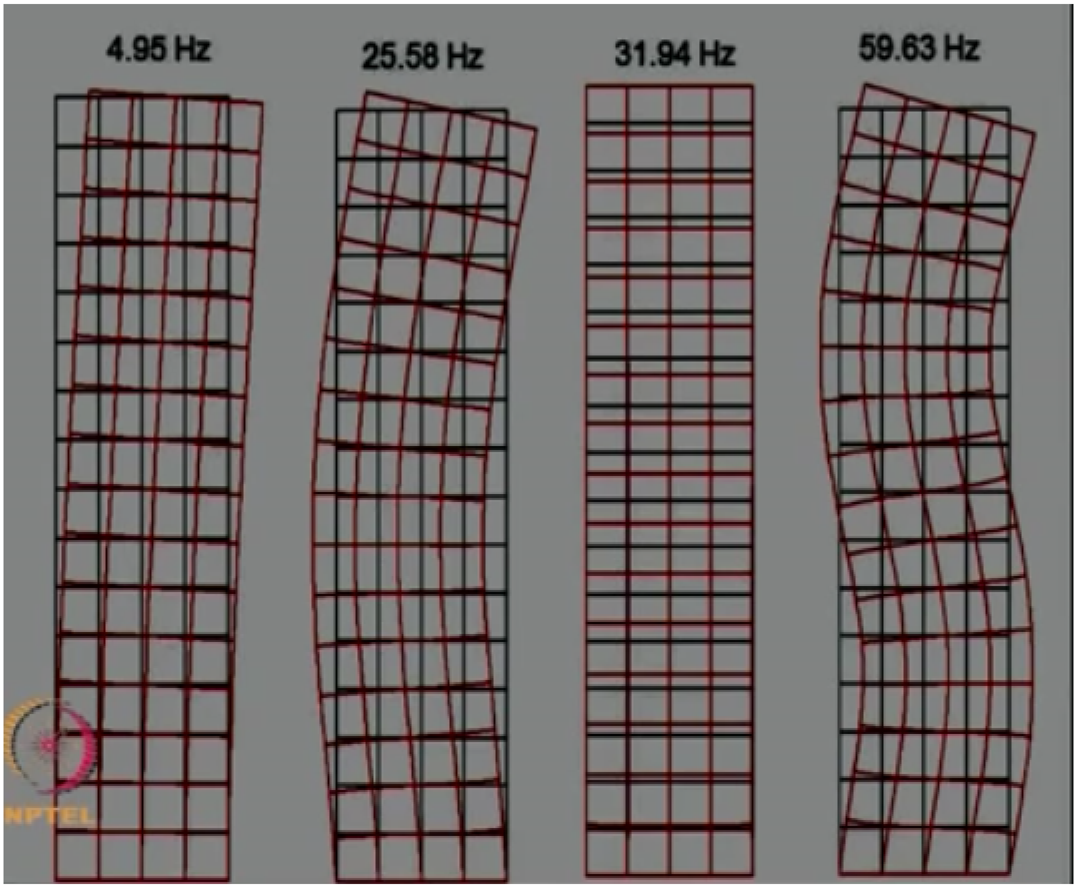
$h = 0.2286$ m

Determine the first five natural frequencies and normal modes.

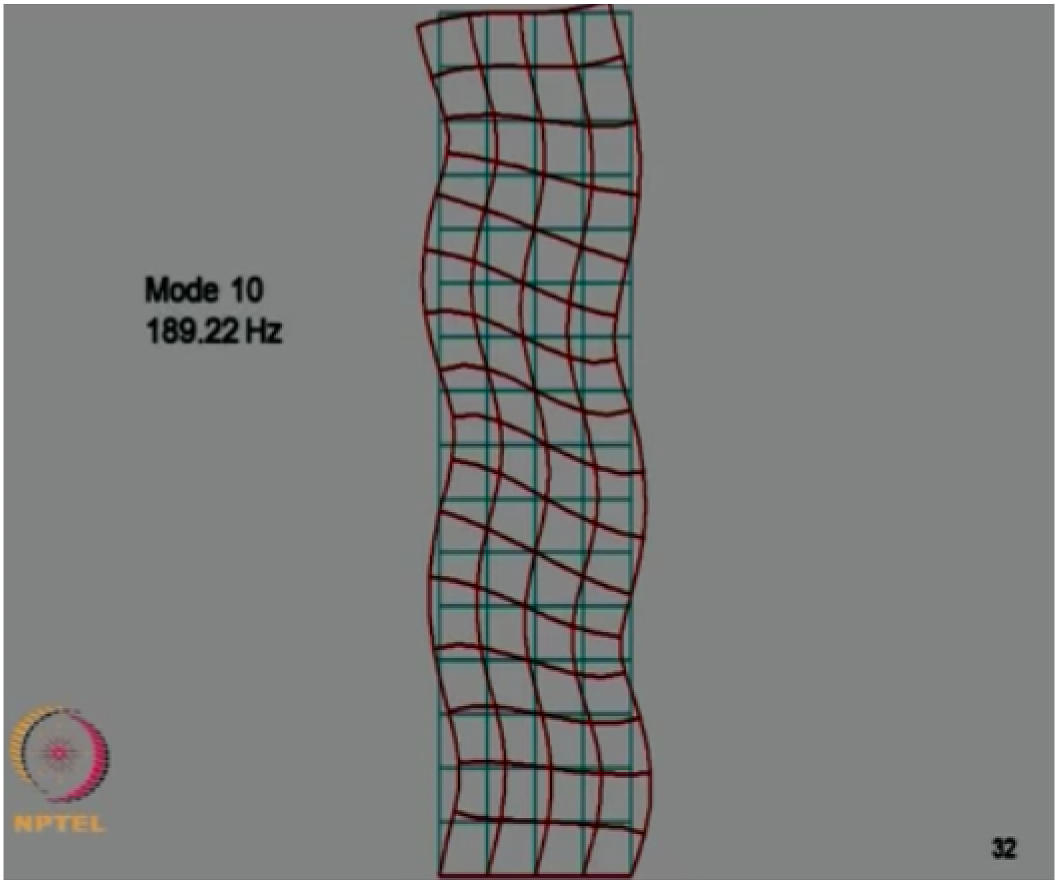
Now let us return to the shear wall and earth dam example that we considered in the previous lecture, so the problem was there is a shear wall of thickness 0.2286 meters, and these are the dimensions and the problem was to find first 5 natural frequencies and normal modes. Now



analytical estimates of natural frequencies using Timoshenko beam theory were obtained, if you pointed out these were the numbers, these are frequencies in hertz. Now we discuss this model in the previous lecture, we use 64 elements with first order elements, that means we used 4 noded quadrilateral elements, actually the rectangular elements here and the degrees of freedom that the model had was 160, now what I have done is we have retained the same mesh and same element configuration, but now the elements are modified to be second order elements, that means they are now 8 noded quadrilateral elements, so the degree of freedom increases to now 448 and the natural frequencies from 5.03 had come to 4.95 and these numbers change, and you can see here they are reasonable, if this is believed to be correct because an analytical solution and the structural geometry satisfies the requirements for the application of Timoshenko beam theory, so consequently we can trust this result, and this analytically derived so we see that the first natural frequency is much better approximated than this, similarly the other you know frequencies also get represented better.

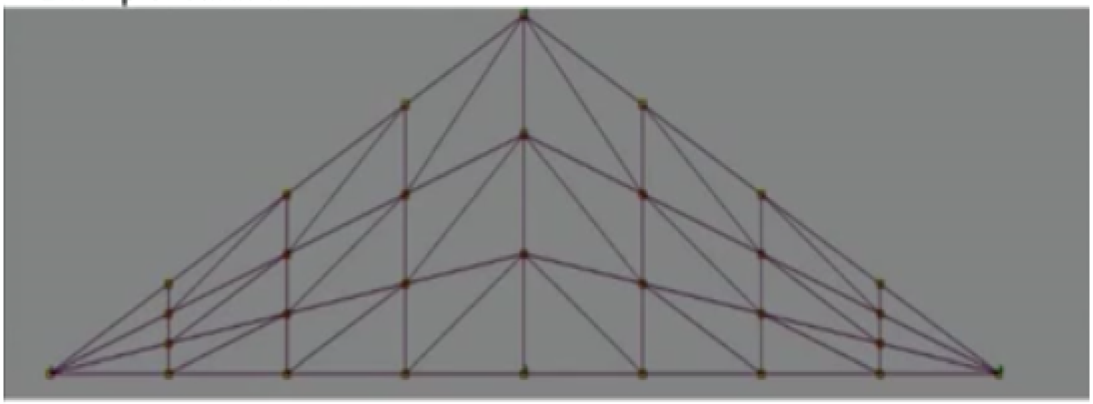


So these are the mode shapes, first 4 mode shapes are shown here so you can see that now the mode shapes are smoother, because they are represented with greater detail within an element. This is the tenth mode, higher modes become, a representation of higher modes would not be so



smooth as it is seen here, if we had used the first order element this mode would not be as smooth as this seen here. Now so this earth dam problem, it was a triangular wedge model

Example: earth dam



E=5.605E08 N/m2
Nu=0.45
Rho=2082 kg/m3

42 elements 42 dofs Nat frequencies in Hz	Shear beam model Nat freqs Hz
1.25	1.227
2.67	1.993
3.70	2.324
4.05	3.073
4.10	

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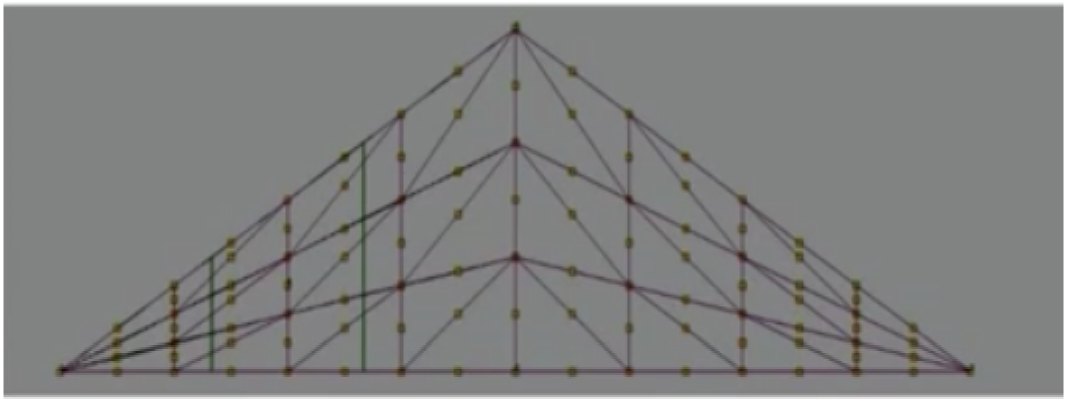
using plane strain approximation, and these were the properties persons ratio 0.45, this is a density, and this is the Young's modulus, with triangular elements with 42 elements and 42 degrees of freedom we got these natural frequencies, and if you use a shear beam model and this is analytical estimate of the shear beam a natural frequency, this is available in terms of bessel functions, then exact solution to this is available, so based on that these numbers have been computed. Now this we have done in the last class, now what I have done now is this

Model with 2nd order triangular elements



42 elements
168 dofs
Nat frequencies in Hz
1.24
2.03
2.39
3.05
3.40

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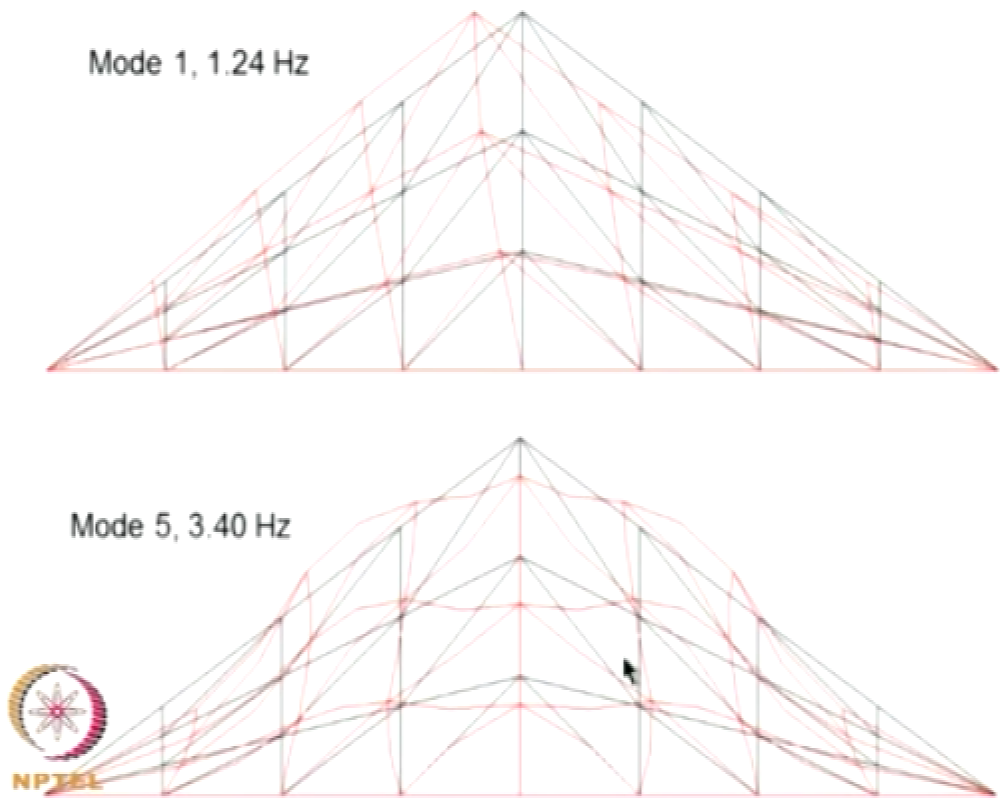
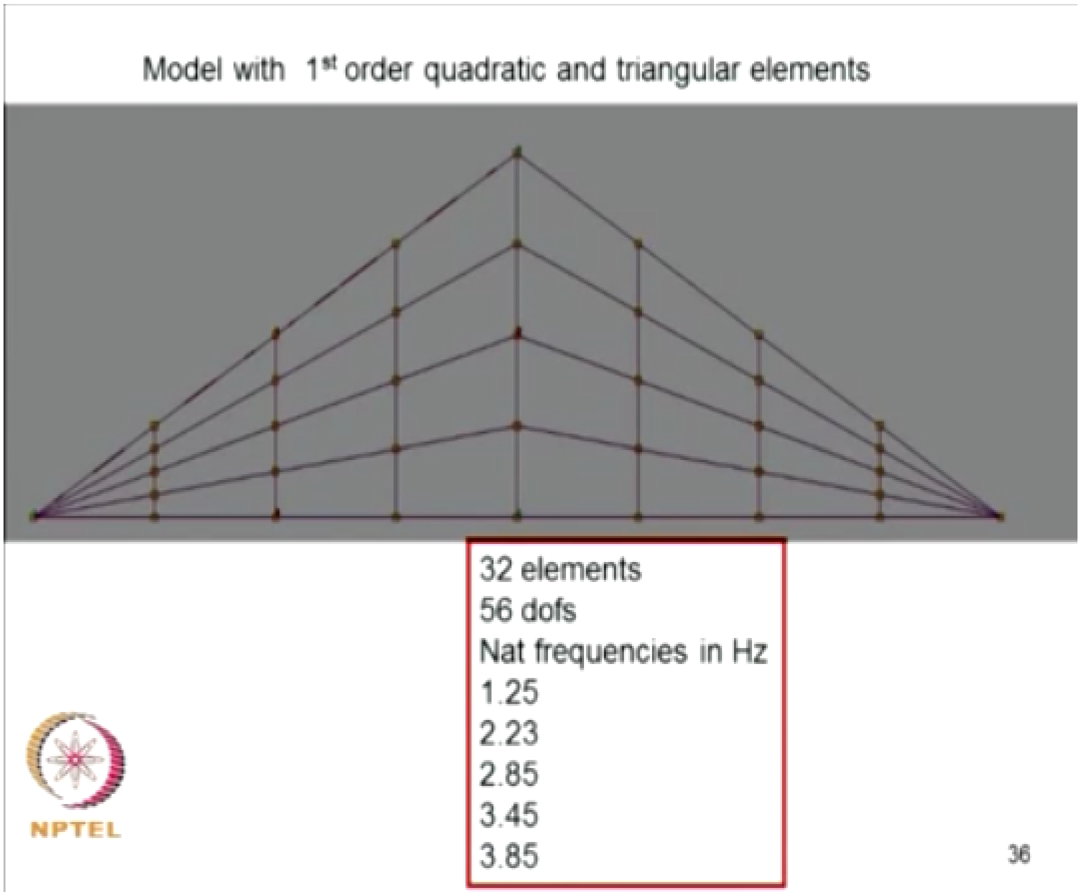
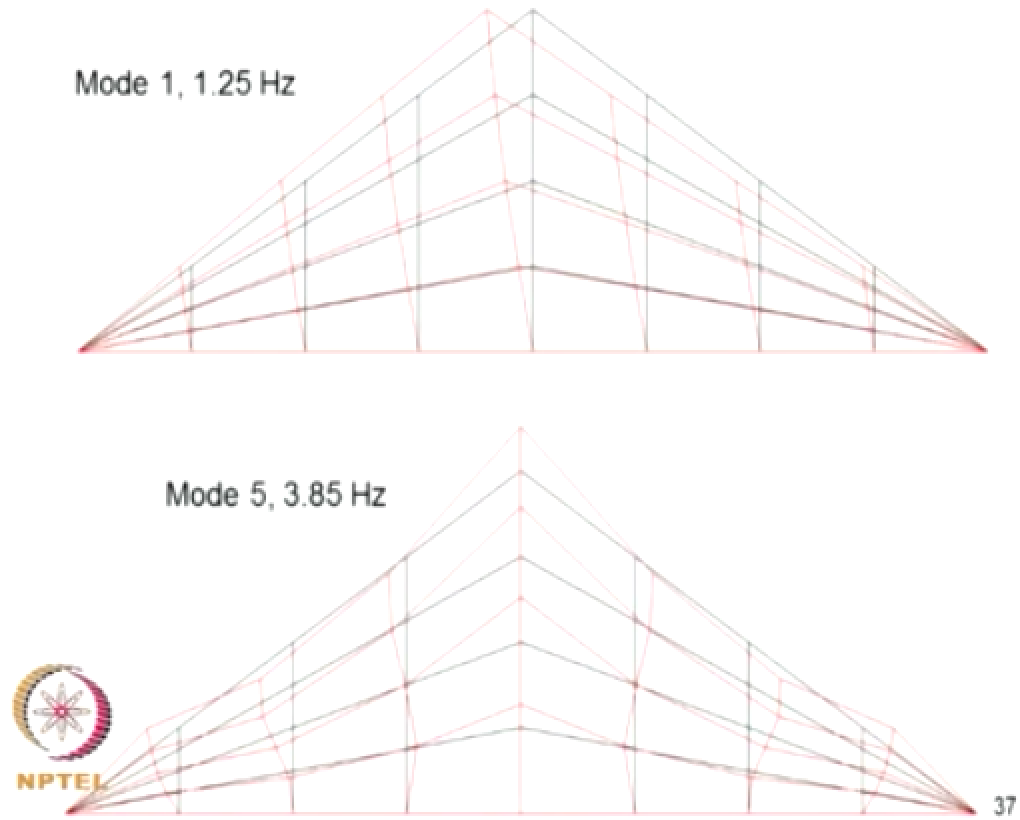
analysis is with first order triangular elements, now this analysis is the second order triangular although I didn't discuss the formulation of second order quadrilateral element the logic is conceptually quite similar to what we briefly mentioned for quadrilateral elements, so if I use this same number of elements but the order has increased so degrees of freedom will increase, so the frequency is now change, and the first frequency is if we have to believe this, this is 1.25, this is 1.227 and this seems to be moving towards that, but mind you this is also an approximation to this behavior of this wedge, so the underlying assumptions of shear beam behavior must be satisfied by this structure, so it is difficult to you know pass a clear judgment on which of these two are exact.



Now these are the mode shapes from a higher order element, this is the first mode and fifth mode, this matches with what we did with coarser element, I mean first order element. Now in

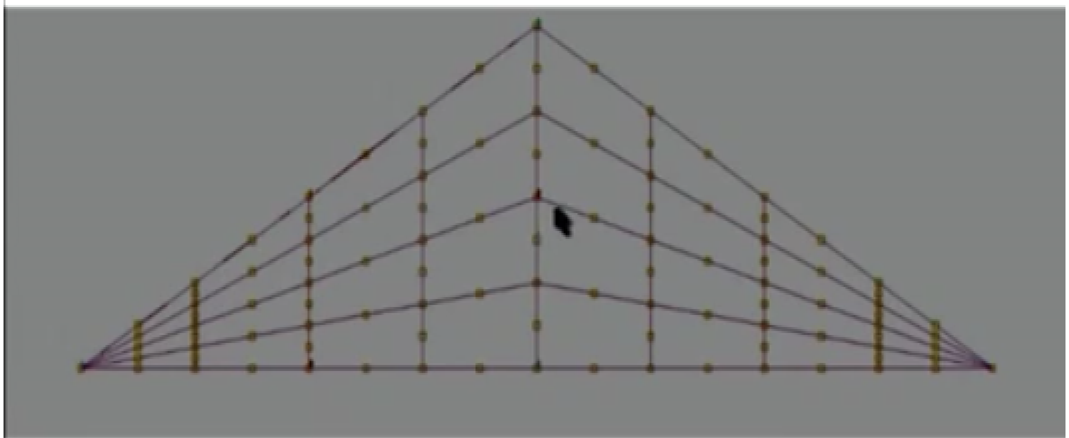


this illustration I am using a combination of triangle and quadrilateral elements, this is quadrilateral element, this here we are using triangular elements so this is again first order elements, so there are 32 elements, 56 degrees of freedom and we get natural frequencies like this, so this is just for you know illustration, and these are the mode shapes that we obtain from



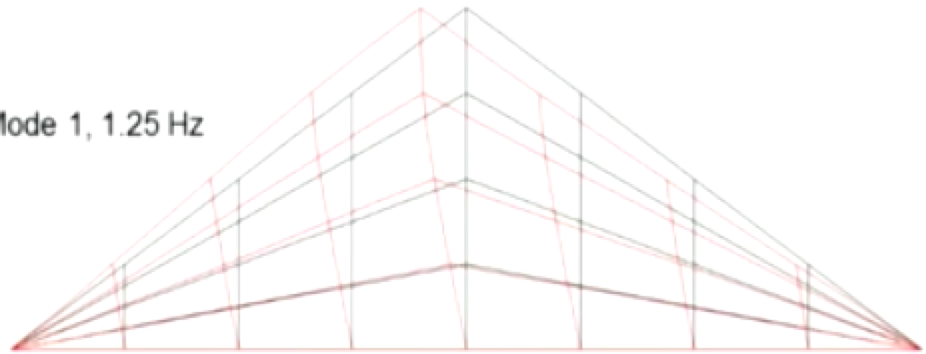
this model.

Model with 2^{nd} order quadratic and triangular elements

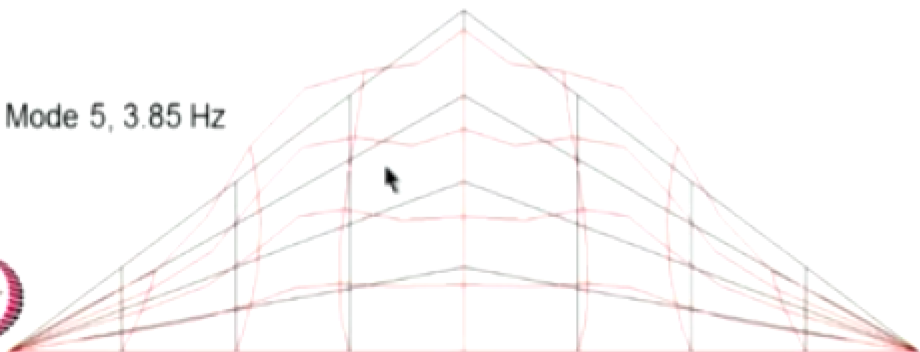


32 elements
210 dofs
Nat frequencies in Hz
1.25
2.23
2.85
3.45
3.85

Now what I do, I will repeat this exercise using the same mesh instead of first order element, I will use second order element, so this quadrilateral element here has 8 nodes and 8 degrees of freedom, the triangular elements here will have 6 nodes and 12 degrees of freedom, so number of elements remain the same, there are 210 degrees of freedom and I get different sets of estimates for natural frequency, so these are the mode shapes for the two, first and the fifth mode.



Mode 1, 1.25 Hz



Mode 5, 3.85 Hz

Equivalent nodal forces

$$\delta W_e = \int_S \left(p_x \delta u + p_y \delta v \right) dS$$

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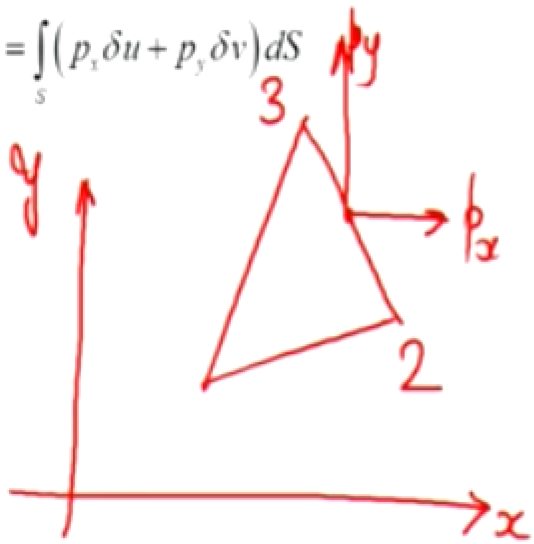
Now so far what we have discussed is basically free vibration problems, now if we have to consider forced vibration response, we need to worry about how the equivalent forces are computed, so let me just, basic rule for that is this okay, so to explain this what we will do is we will consider a suppose there is a edge 2, 3 in a triangular element and the loads will be in the plane of the element, this is a plane stress problem, suppose at any point here there is a load PX

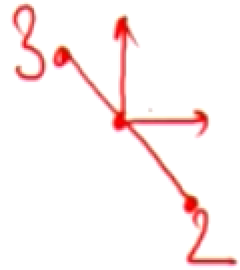
Equivalent nodal forces

$$\delta W_e = \int_S \left(p_x \delta u + p_y \delta v \right) dS$$



and load sorry, this load PY, now when we formulate the finite element model we also need to assemble the nodal forces, first of all we should evaluate the equivalent nodal forces, so what we do is we have this over the surface of the element delta U, delta V transpose PX, PY into DS, now we also have the approximation UV which is N into UE. So now we write delta WE for the nodal, in terms of the equivalent nodal forces these are the forces that are acting on the surface I want to represent them in terms of equivalent nodal forces, so we demand that equivalence of these two will give me the expression FE equal to, see I have to consider the edge 2, 3, I am assuming if you recall we are considering force on the edge 2, 3, so this has to

$$\delta W_e = \int_s \begin{bmatrix}\delta u\\ \delta v\end{bmatrix}^t \begin{bmatrix}p_x\\ p_y\end{bmatrix} ds \qquad \begin{pmatrix}u\\ v\end{pmatrix} = [N]u_e$$



$$\delta W_e = \{\delta u\}^t \{f\}_e$$

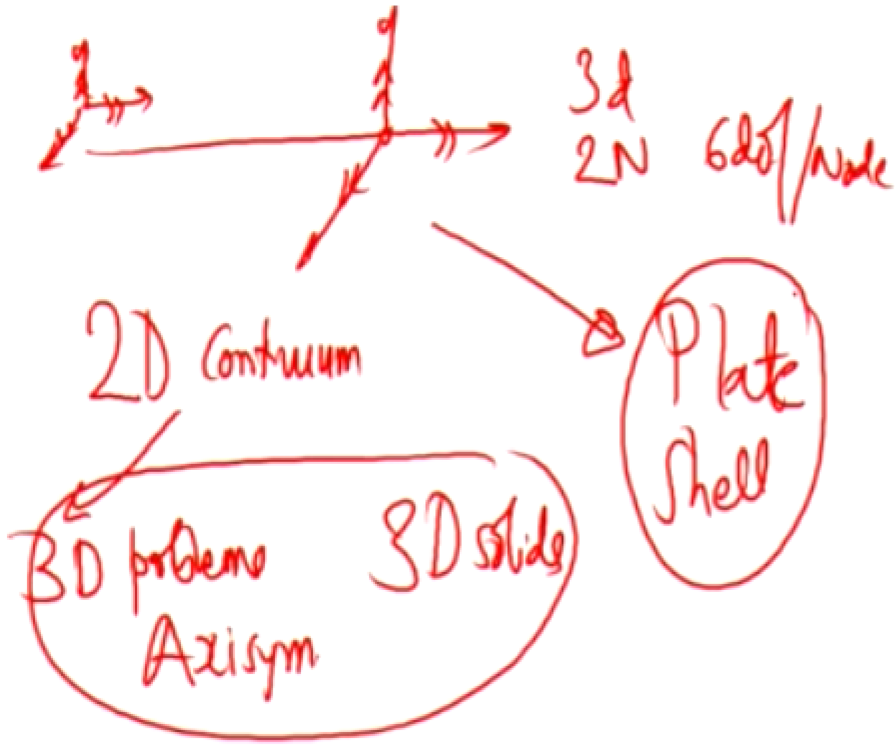
$$= \{\delta u\}_e^t \{$$

$$f_e = \int_s [N]^t_{2-3} \begin{bmatrix}p_x\\ p_y\end{bmatrix} ds$$

$$f_e = \frac{1}{2} l_{2-3}(0\ 0\ p_x\ p_y\ p_x\ p_y)^t$$

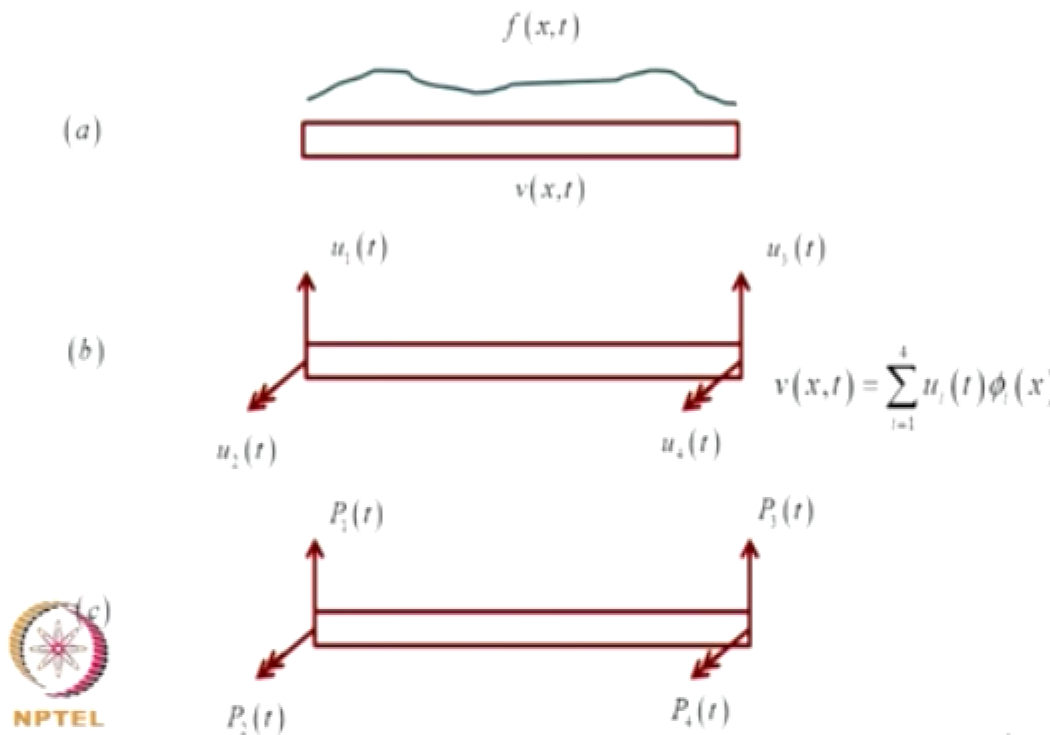
be evaluated along that edge, so if I do this we will get a FE for the example that we have considered as 1/2 length of 2, 3 into 0, 0, 0, PX, PY, PX PY transpose, so what we are getting is at the two nodes 1/2 of the load comes here, 1/2 of the load goes there as per this formulation, so this is what the equivalent nodal forces logic for computing equivalent nodal force, this can be done for quadrilateral elements and rectangular and quadrilateral elements as well.

So at some point we will illustrate this with some examples, now in the subsequent lectures what we will do now, so far what we have done is we have dealt with a 3D beam element with 2 nodes and 6 DOF for node. Now we have now considered 2D continuum, we have derived



different elements, now the generalization for further analysis now there are 3 things that we can do, this beam theory will get extended to plate and shell, this 2D continuum we can consider now 3 dimensional problems that means when conditions of plane stress and plane strain are not satisfied here another 2D model becomes possible that is axisymmetric solids, and then 3D solids, so what in the remaining next few lectures what we will do is we will develop the structural matrices for these problems, and these problems, and at the end of it I will, at the end of these developments I will make a few remarks on choice of shape functions, and their role on how the answers converge.

Generalized forces



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Now we will consider the question of how to compute equivalent nodal forces, we will quickly recall what we did for a beam element so this is a beam element say 2-dimensional beam element deforming in its own plane, this is the distributed load F(x,t) so a discretization of this element we had 2 nodes and 2 degrees of freedom per node U1, U2, U3, U4 are the 4 degrees of freedom for the element. The problem here is see the displacement field is represented in terms of nodal displacement and these interpolation functions, so how do we represent this distributed load in terms of equivalent nodal forces, so how to find P1, P2, P3, P4 in a way that these form, there is an equivalence between this load and this set of loads, so what we considered was a

Let $\delta v(x,t)$ = a virtual displacement. We can write

$$\delta v(x,t)=\sum_{i=1}^{4}\delta u_i(t)\phi_i(x)$$

$$\Rightarrow \sum_{i=1}^{4}P_i(t)\delta u_i(t)=\int_0^l f(x,t)\delta v(x,t)\,dx$$

$$=\int_0^l f(x,t)\left\{\sum_{i=1}^{4}\delta u_i(t)\phi_i(x)\right\}dx$$

$$\Rightarrow \sum_{i=1}^{4}\left\{P_i(t)-\int_0^l f(x,t)\phi_i(x)\,dx\right\}\delta u_i(t)=0$$

Since $\delta u_i(t), i=1,2,3,4$ are arbitrary

$$\Rightarrow P_i(t)=\int_0^l f(x,t)\phi_i(x)\,dx, i=1,2,3,4$$

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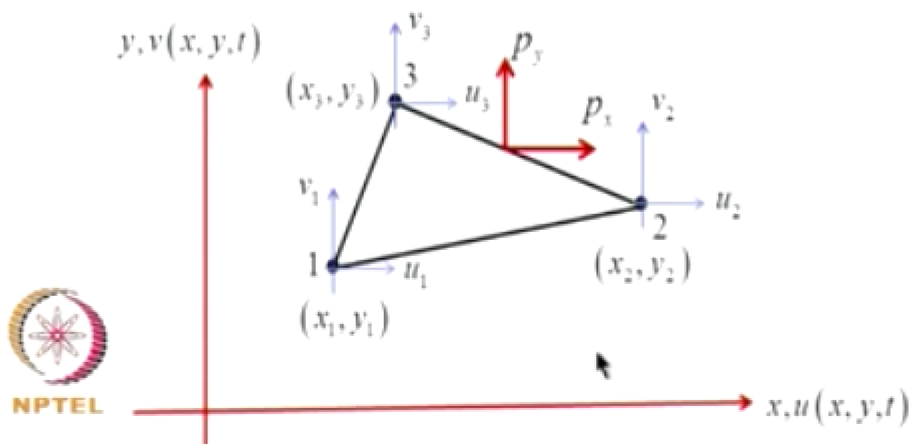
virtual displacement and that we represented in terms of virtual nodal displacements multiplied by the interpolation functions, and the work done was equated, so the work done by virtual displacements on this set of forces and the virtual displacement on this, by this set of forces are equated and by manipulating the terms we determined the nodal forces to be given by these expressions.

Equivalent nodal forces

$$\delta W_e = \int_S \left(p_x \delta u + p_y \delta v \right) dS$$

$$\begin{Bmatrix} u(x,y,t) \\ v(x,y,t) \end{Bmatrix} = \left[N(x,y) \right] \{u(t)\}_e$$



3

So we are going to do something similar for even 2 dimensional elements, so suppose PX and PY are the distributed loads along the edge 2, 3. Now I want to find out what is the equivalent nodal forces because of these forces, so PX and PY are distributed along this edge 2, 3, so the virtual work is given by this expression, and the displacement field where interpolating in terms of nodal degrees of freedom as shown here. Now along this edge so this virtual work can be

$$\delta W_e = \int_S \left(p_x \delta u + p_y \delta v \right) dS = \int_S \begin{bmatrix} \delta u \\ \delta v \end{bmatrix}^t \begin{bmatrix} p_x \\ p_y \end{bmatrix} dS$$

We have

$$\begin{Bmatrix} u(x,y,t) \\ v(x,y,t) \end{Bmatrix} = \left[N(x,y) \right] \left\{ u(t) \right\}_e$$

$$\delta W_e = \int_S \begin{bmatrix} \delta u \\ \delta v \end{bmatrix}^t \begin{bmatrix} p_x \\ p_y \end{bmatrix} dS = \int_S \left\{ \delta u(t) \right\}_e^t \left[N(x,y) \right] \begin{bmatrix} p_x \\ p_y \end{bmatrix} dS$$

$$= \left\{ \delta u(t) \right\}_e^t \left[\int_S \left[N(x,y) \right] \begin{bmatrix} p_x \\ p_y \end{bmatrix} dS \right]$$

Note : $\left[N(x,y) \right]$ here needs to be evaluated along the edge 2-3

$$\delta W_e = \left\{ \delta u(t) \right\}_e^t \left\{ f \right\}_e$$

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written in terms of virtual displacement and applied forces as shown here, and this is a representation, so the virtual work which is this is also equal to virtual nodal displacements into these forces, and after we manipulate this we get this expression and we should note that this trial functions need to, here we need to be evaluate this along the edge 2, 3. Now this is the

$$\delta W_e = \left\{ \delta u(t) \right\}_e^t \left\{ f \right\}_e$$

$$\left\{ f \right\}_e = \frac{1}{2} l_{2-3} \begin{Bmatrix} 0 \\ 0 \\ p_x \\ p_y \\ p_x \\ p_y \end{Bmatrix} ; l_{2-3} = \text{length of edge 2-3}$$

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virtual nodal displacements into the nodal forces and by equating these two we get the, by equating this and this we get the expression for equivalent nodal forces which is shown here. Now here we have assumed that PX and PY are constant along the edge, otherwise there will be a quadrature that has to be implemented, the quantity L, 2, 3 represent length of the edge 2, 3 this length, so if you know the nodal coordinates you can derive that length in a straightforward manner.

Remarks

- $I = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi\,d\eta \approx \int_{-1}^{1}\left\{\sum_{i=1}^{n} w_i^{\xi} f(\xi_i,\eta)\right\}d\eta \approx \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} w_i^{\xi} w_j^{\eta} f(\xi_i,\eta_j)$
- Body force

Consider the four noded quadrilateral element.

Let the body force $f = \begin{bmatrix} f_x & f_y \end{bmatrix}^T$ be constant within the element.

It can be shown that

$$f^e = h\left[\int_{-1}^{1}\int_{-1}^{1} N^T |J|\,d\xi\,d\eta\right]\begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \qquad \text{(Show this)}$$

The integral in the above equation can again be evaluated using the Gauss quadrature.

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Now we can make some remarks, first is the quadrature rule that we discussed the was in the context of a scalar integral, the same formulation can also be extended to 2 dimensional integrals and 1 gets quadrature rule of this form. Now in this calculation of equivalent nodal forces we considered surface tractions, but if there are equivalent there are body forces within the element we can denote them by FX FY transpose, suppose if that body force is constant within the element how we can find the equivalent nodal forces that again we have to use the same formulation, I leave this as an exercise this will be the equivalent nodal forces and here again we need to evaluate this integral and we need to examine the order of terms in the integrand and decide upon the quadrature rule, but this needs to be evaluated again using Gauss quadrature.

Remarks (continued)

- Consider the four noded linear rectangular element. Consider the displacements given by

$$u = \sum_{j=1}^{4} N_j u_j + \alpha_1 (1-\xi^2) + \alpha_2 (1-\eta^2) \ \& \ v = \sum_{j=1}^{4} N_j v_j + \alpha_3 (1-\xi^2) + \alpha_4 (1-\eta^2)$$

where $N_j(\xi,\eta) = \frac{1}{4}(1+\xi_j\xi)(1+\eta_j\eta);\ j = 1,2,3,4$

$\alpha_i, i = 1,2,3,4$ here are not nodal displacements but need to be interpreted as additional generalized coordinates. Note that $(1-\xi^2)\ \&\ (1-\eta^2)$ are zero at the four vertices.

Excercise

- Obtain the 12×12 stiffness matrix
- Eliminate $\alpha_i, i = 1,2,3,4$ in terms of $(u_i, v_i), i = 1,2,3,4$ by using static condensation and hence deduce the 8×8 stiffness matrix.
- Are the dispacement fields continuous across element boundaries?

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Now a small exercise, so let's consider a 4 noded linear rectangular element, now we consider the displacements, in the discussion so far we consider this part, now to this I will now add two more terms alpha 1, 1 - XI square, alpha 2 1 - eta square, and similarly for V I will add this, now you can see here that at the nodes which are XI = + - 1, and eta = +- 1, these terms are 0, so this is like a bubble function so they don't affect the behavior of U near the nodes, at the nodes, and V is given by this similarly, so now again these trial functions are same as what we have used for a linear rectangular element, now these alpha 1, alpha 2, and alpha 3, alpha 4 are not nodal displacements now, they are generalized coordinates, so there will be now 12 degrees of freedom for the element, 4U nodal displacement U1, U2, U3, U4 and 2 generalized coordinates alpha 1, alpha 2, similarly this is 6 for U, similar 6 quantities for V.

Now the exercise that is being suggested is use the energy expressions and obtain the 12x12 stiffness matrix, and eliminate this alpha 1, 2, 3, 4 in terms of the nodal degrees of freedom using static condensation, and obtain a 8x8 stiffness matrix, and also examine are the displacement fields continuous across the element boundaries, so this exercise can be carried out, so in using this element you may notice that we can continue to use the consistent mass matrix that was derived using only this approximation, the kinetic energy can be characterized using only this, and whereas the displacement fields will be, strain energy will be computed by an alternative representation.

3D solid elements

$$\sigma = \{\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \sigma_{xy} \quad \sigma_{xz} \quad \sigma_{yz}\}^t$$

$$\varepsilon = \{\varepsilon_{xx} \quad \varepsilon_{yy} \quad \varepsilon_{zz} \quad 2\varepsilon_{xy} \quad 2\varepsilon_{xz} \quad 2\varepsilon_{yz}\}^t$$

$$\sigma = D\varepsilon$$

$$D = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{\nu} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{\nu} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{\nu} \end{bmatrix}$$

$$V = \frac{1}{2}\int_{V_0} \sigma^t \varepsilon dV_0 = \frac{1}{2}\int_{V_0} \varepsilon^t D\varepsilon dV_0 \;\&\; T = \frac{1}{2}\int_{V_0} \rho\left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right) dV_0$$

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Now in the subsequent lectures now, we will now consider problems of 3-dimensional elasticity, we have briefly discussed this earlier so will quickly recall, so stress, state of stress is now described in terms of 6 stress components either it can be arranged as a 10 matrix or a vector like this, these are the strain components, and for an isotropic linear elastic material the stress and strain are related through this relation, and this is the matrix D and we have the expression for strain energy and kinetic energy. This is sigma transpose epsilon DV naught is the expression integral of that over volume element into 1/2 is strain energy and this is a kinetic energy.

$$V = \frac{1}{2}\int_{V_0} \sigma^t \varepsilon dV_0 = \frac{1}{2}\int_{V_0} \varepsilon^t D\varepsilon dV_0 \; \& \; T = \frac{1}{2}\int_{V_0} \rho\left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right) dV_0$$

$$\varepsilon = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

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Now to compute strain energy we can now, what we are doing is where expressing sigma in terms of strain, and the strain subsequently we will express in terms of displacement as we have been doing, this is a strain displacement relation for small deformations, and substituting that

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [N] \begin{Bmatrix} u_e \\ v_e \\ w_e \end{Bmatrix} = Nu_e \Rightarrow \varepsilon = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} Nu_e = Bu_e$$

$$\Rightarrow V = \frac{1}{2}\int_{V_0} u_e^t B^t DBu_e dV_0 \;\&\; T = \frac{1}{2}\int_{V_0} \rho \dot{u}_e^t N^t N\dot{u}_e dV_0$$

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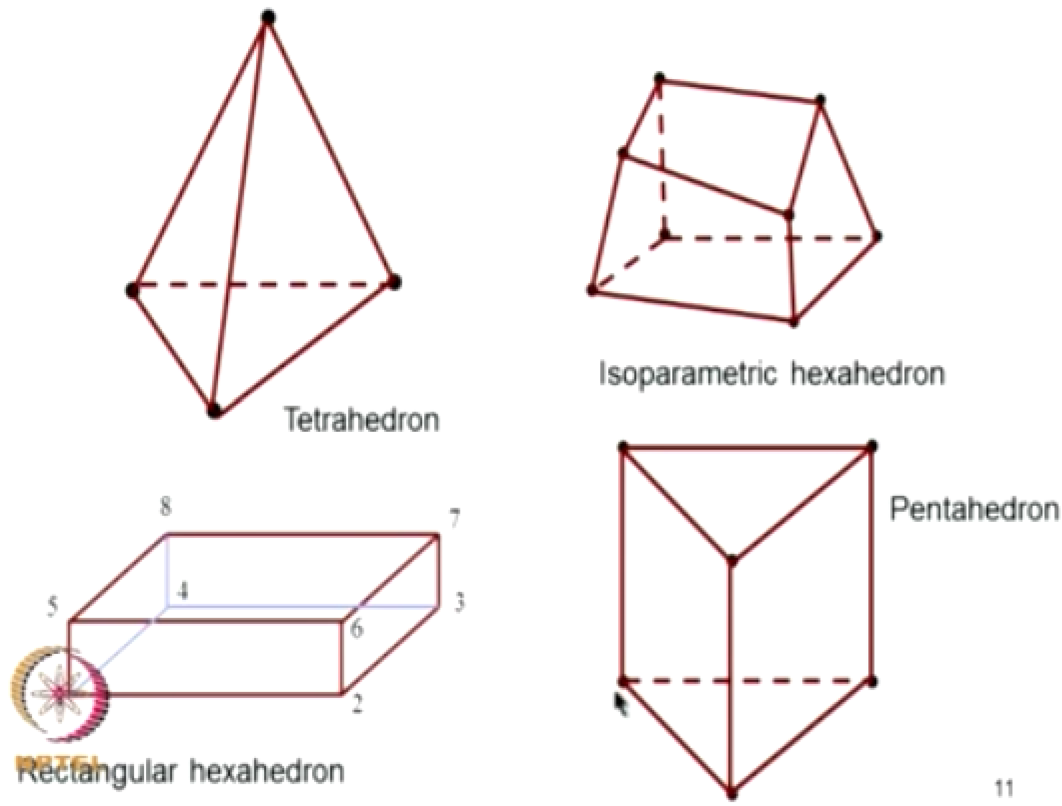
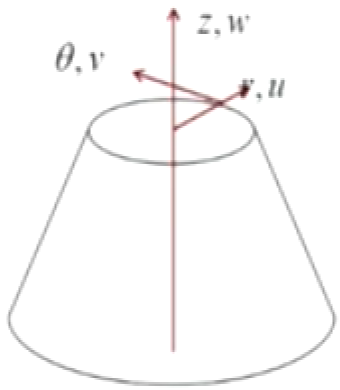
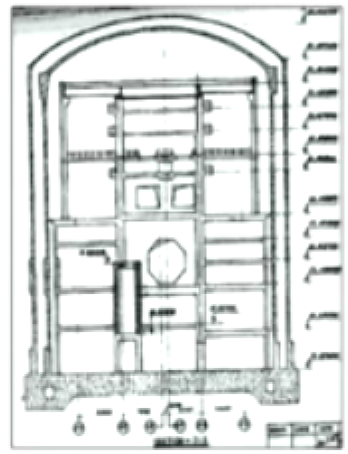
we will get displacements as N into UE, where UE are the nodal displacements and consequently epsilon is obtained in this form, where B is equal to this matrix into N, so given that we will have the expression for strain energy and kinetic energy in terms of assumed displacement fields as given here, and we need to now carry out this integration over the volume elements. Now the volume elements in a 3-dimensional solid becomes more diverse we



can have a tetrahedron or a rectangular hexahedron or pentahedron or an isoparametric hexahedron, and we can also have isoparametric questions of tetrahedral and pentahedron, and this can be first order elements or higher order elements, so there is a great diversity of elements so we can examine some of them as we go along.

Axisymmetric problems





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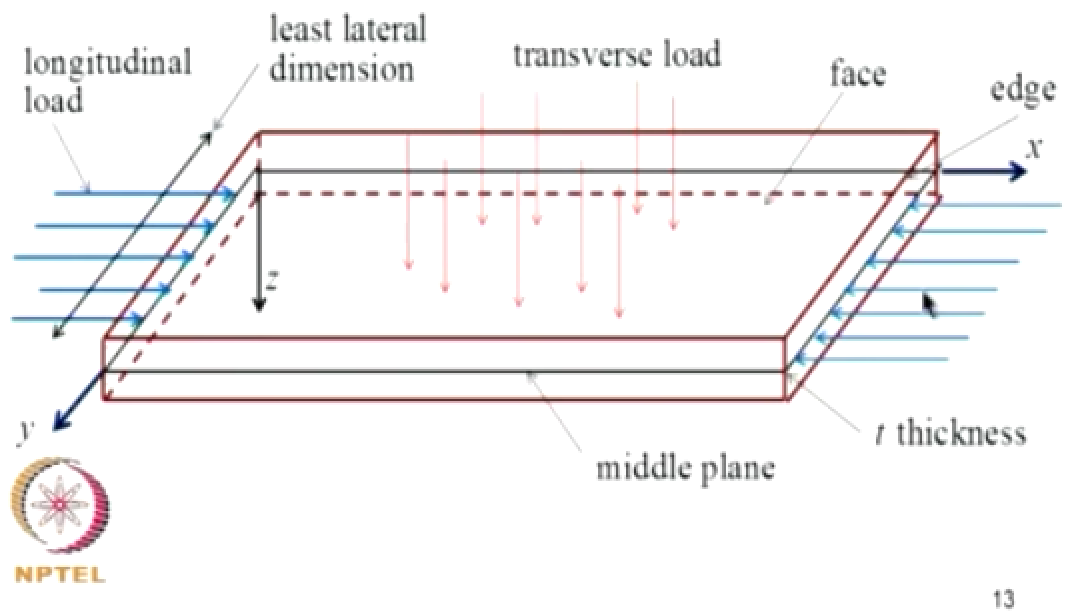
Subsequently in 3 dimensional problems we can also consider solids of revolution, these are called axisymmetric problems, we will be doing that and we will develop at least one element

Plate bending element



for this type of applications and this would be followed by discussion on a plate bending element, so this is a thin say lamina which carries now transverse loads in addition to the inline loads, so the action of this transverse loads on this lamina is to, the plate is to induce bending and about X and Y axis and we need to formulate the correct expressions for strain energy and kinetic energy and develop the element, so we will take up these exercises in the following lectures and maybe some examples on force response analysis. So with this we will conclude this today's lecture.

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