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Course Title

Finite element method for structural dynamic And stability analyses

Lecture – 17

Model reduction schemes.

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Finite element method for structural dynamic and stability analyses

Module-6

Model reduction and substructuring schemes

Lecture-17 Model reduction schemes

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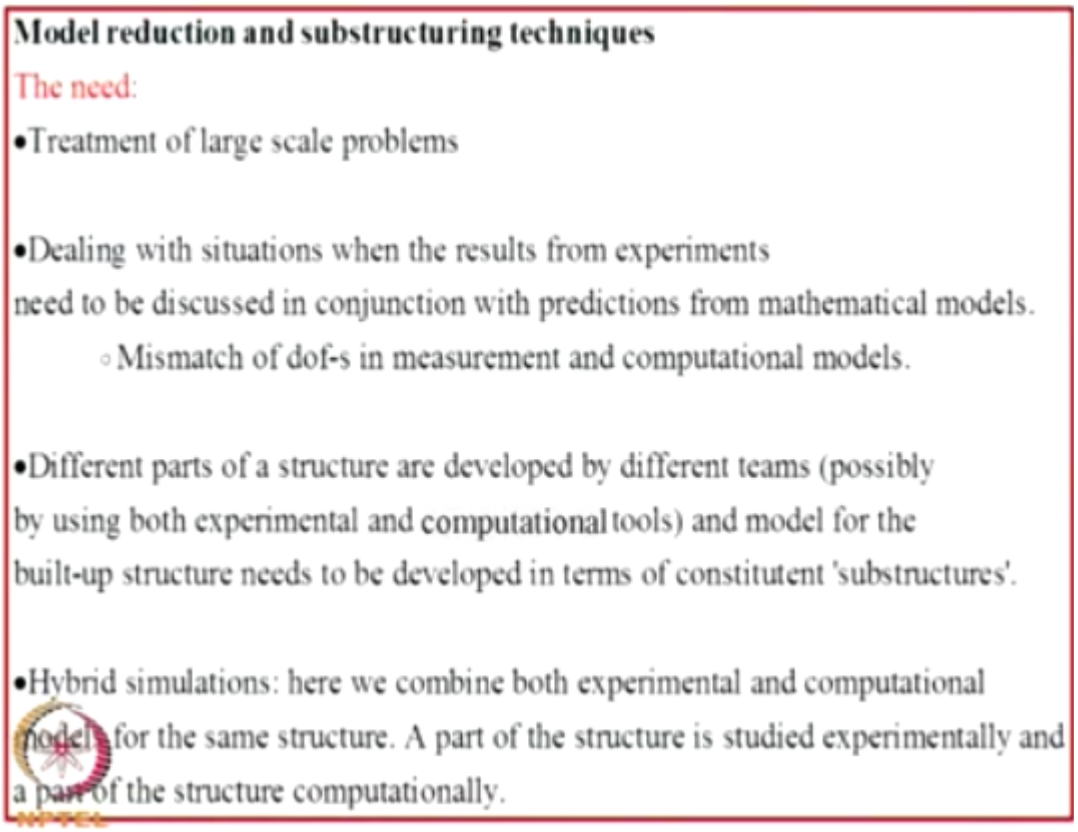
Towards the end of the previous lecture we started talking about issues related to model reduction and sub-structuring schemes. So we'll continue with this discussion now, so we'll



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quickly recall where do we need you know, model reduction and sub-structuring's. Actually, the need for using model reduction and sub-structuring arises in several context, one is for example in treating very large scale problems or in dealing with situations when the results from experiments need to be discussed in congestion with prediction from mathematical models, that means for a given structure we have made both finite element model as well as we have done some experimental investigations and we want to now reconcile the two, and this is an essential step in problems of finite element model updating, and in such situations the question of model reduction arises, because of mismatched between degrees of freedom in measurement and computational models, typically in a computational model the size of the, the number of degrees of freedom can be very large and for every degree of freedom in the computational model we may not have a sensor for instance it may not be feasible to measure degrees of freedom at interior nodes in a 3-dimensional structure or measurement of rotations and so on and so forth. So the number of degrees of freedom that we'll be able to measure in an experimental work will typically be much less than the degrees of freedom in a finite element model.

Another situation is when different parts of a structure are developed by different teams, possibly by using both experimental and computational tools, and based on this we need to construct the model for built up structures, so this typically happens in applications such as space applications and automotive applications, and this could also occur in problems of secondary systems in civil engineering applications. There is a modern testing strategy known as hybrid simulations, where we combine both experimental and computational models for the same structure, so what we do is a part of the structure is studied experimentally and a part of

the structure computationally, and we want to in some way couple these two disparate studies and arrive at certain conclusions on global behavior of the complete structure.

Problem of model reduction

Consider a N dof FE model for a linear system governed by

$$M\ddot{X} + C\dot{X} + KX = F(t); X(0) = X_0 \;\&\; \dot{X}(0) = \dot{X}_0$$

The objective of model reduction is to replace the above N dof system by an equivalent n dof system $(n < N)$

$$M_r\ddot{X}_m + C_r\dot{X}_m + K_r X_m = F_r(t)$$

M_r, C_r, K_r = Reduced $n \times n$ structural matrices.

X_m = $n \times 1$ vector of dof-s which has been retained in the reduced model.

In all the model reduction techniques, the displacement vector $X(t)$ is taken to be of the form

$$X(t) = \begin{Bmatrix} X_m(t) \\ X_s(t) \end{Bmatrix}; \quad \begin{matrix} X_m(t): n \times 1 \text{ master dof-s} \\ X_s(t): (N-n) \times 1 \text{ slave dof-s} \end{matrix}$$

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So let us start with discussion on problem of model reduction, so we will be limiting our attention to linear time-invariant vibrating systems, so a typical finite element model for a linear system will be of this form MX double dot + CX dot + KX = F(t) and certain specified initial conditions, this is an end product of making finite element model as you have seen in previous lectures.

The objective of model reduction is to replace this above this end degree of freedom system by an equivalent lowercase n degree of freedom system where the reduced degree of freedom is much less than the capital N degrees of freedom here, so for the reduced system the equation will be again of the form MR XM double dot + CR XM dot + KR XM is FR(t), the subscript R here refers to reduced model, whereas the subscript M, I will shortly come to that, this is the vector of degrees of freedom which have been retained in the reduced model, so in the original model what we do is the degree of freedom is partitioned into two sets, one set is called master degrees of freedom, the other set is called slave degrees of freedom, so this subscript M here refers to the master degrees of freedom which have been retained in the reduced model. The slave degrees of freedom XS(t) have been eliminated from this model and a reduced model of this type has been arrived at, so the size of the master degree of freedom will be lower case n cross 1, and slave degrees sorry, master degrees of freedom, slave degrees of freedom will be N – N cross 1 vector, XS(t) is a N – N cross 1 vector.

We represent

$$X(t) = \begin{Bmatrix} X_m(t) \\ X_s(t) \end{Bmatrix} = \Psi X_m(t)$$

where Ψ is an $N \times n$ transformation matrix.

$\Rightarrow$

$$M\Psi\ddot{X}_m(t) + C\Psi\dot{X}_m(t) + K\Psi X_m(t) = F(t)$$

$\Rightarrow$

$$\Psi^t M\Psi\ddot{X}_m(t) + \Psi^t C\Psi\dot{X}_m(t) + \Psi^t K\Psi X_m(t) = \Psi^t F(t)$$

$\Rightarrow$

$$M_r\ddot{X}_m + C_r\dot{X}_m + K_r X_m = F_r(t)$$

$M_r = \Psi^t M\Psi$ = Reduced mass matrix; $M_r^t = M_r$

$C_r = \Psi^t C\Psi$ = Reduced damping matrix; $C_r^t = C_r$

$K_r = \Psi^t K\Psi$ = Reduced stiffness matrix; $K_r^t = K_r$

$F_r(t) = \Psi^t F(t)$ = Reduced force vector

Question: how to select Ψ?

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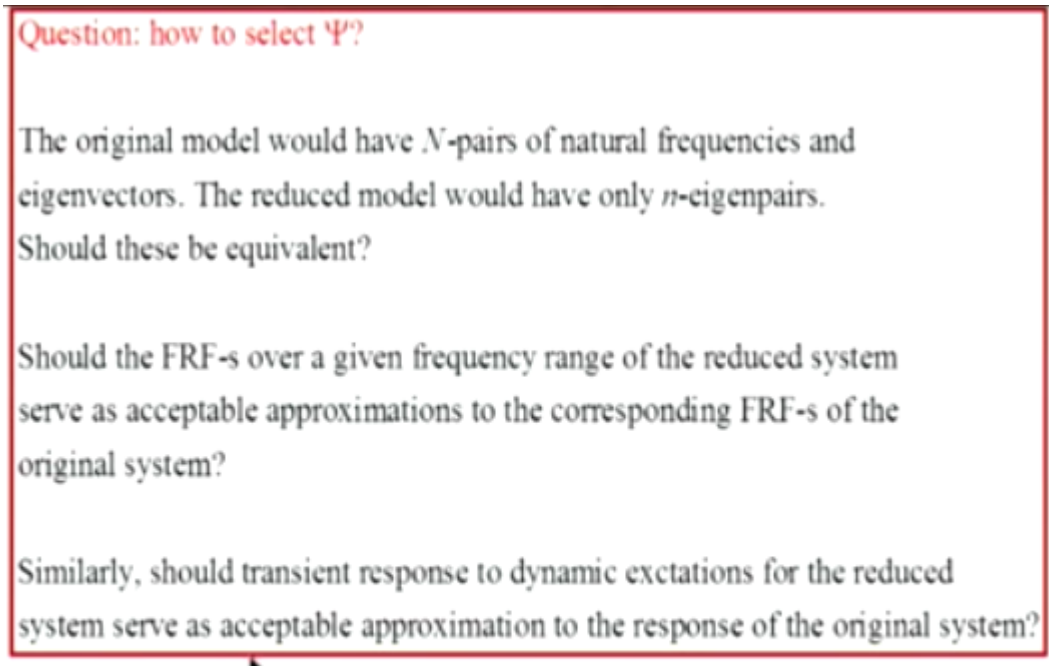
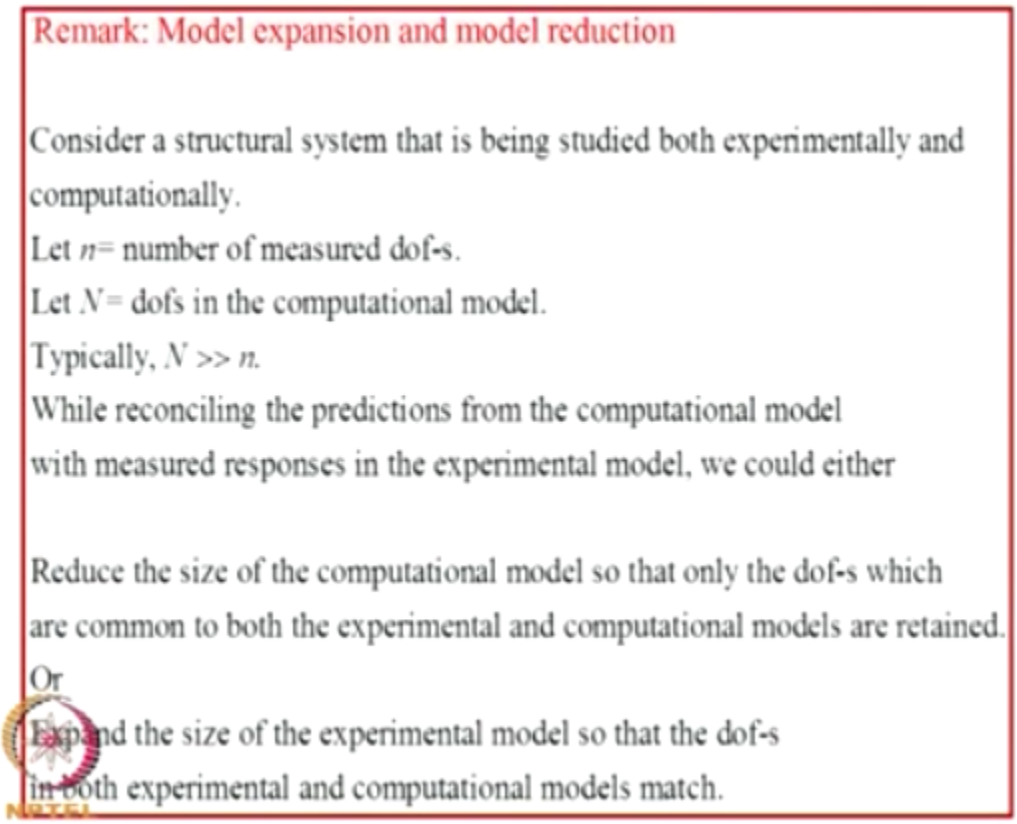
Now in all, there are several methods for model reduction and in all the alternative method there is a generic form to the problem of model reduction, so what we do is X(t) is written as partitioned as already I mentioned as master and slave, and this we take it to the, this X(t) is taken to be related to XM(t) through a transformation matrix capital Sai, so XM is lowercase n cross 1, and capital Sai therefore will be M cross N transformation matrix, so we can substitute this into the governing equation we get the equation at this stage in this form, and if you pre multiply by sai transpose I get equation of this kind, and I call this MR which is sai transpose M sai as a reduced mass matrix, clearly if you take transpose of this it will be a since capital M is symmetric this MR would also be symmetric. Similarly we define reduced damping matrix, and reduced stiffness matrix, we call this quantity FR(t) as sai transpose F(t) as a reduced force vector, so once this is achieved we can analyze this equation using any of the tools that we already developed, but therefore the question now remains how do we select this transformation matrix?



Now there could be different criteria based on which we may like to select this transformation matrix, for example the original model would have capital N number of natural frequencies and mode shapes, on the other hand the reduced model will have only lowercase n Eigen pairs, now suppose I have a 100 degrees of freedom system and I reduce it to a 10 degrees of freedom system, the 100 degrees of freedom system will have 100 natural frequencies and 100/100 modal matrix whereas the reduced model will have 10 natural frequencies and 10/10 model matrix, these 10 natural frequencies of the reduced model should they be equal to any of these 100 natural frequencies of the larger model, that could be one of the criteria, or should the frequency response function over a given frequency range of the reduced system serve as an acceptable approximation to the corresponding FRF's of the original system, so here we are matching response, here we are matching only the natural frequency, so if we match FRF's the issue is related to mode shapes as well as damping models would be allowed for.
Similarly should transient response to dynamic excitation for the reduced system serve as an acceptable approximation to the response of the original system, so we can set forth different objectives depending on which one is of crucial importance to a given situation, we have to suitably design this transformation matrix.



Now before we proceed further we can just as there is a model reduction, there is a counter you know feature that is model expansion, for example consider a structural system that is being studied both experimentally and computationally, so this n is a number of major degrees of freedom, whereas capital N is a degrees of freedom in the computational model, and this typically far exceeds the measured degrees of freedom in the experimental model. So now when we reconcile either I can reduce my computational model to match the number of measured DOF's, so let me go back to again the example of capital N being 100 and lowercase n being 10, so what I can do is from a 100 degree of freedom computational model I can illuminate 90 degrees of freedom and obtain a model with 10 degrees of freedom and the degrees of freedom can be chosen to match what exactly I have measured.

On the other hand we could also expand the measurement model, that means I have 10 degrees of freedom model here I will augment it by additional 90 degrees of freedom, so that augmentation is essentially a transformation, so if I do that then instead of calling it as model reduction, it would become model expansion, because a smaller model is now replaced by a larger model, so the transformation that we discuss can be viewed from both these perspectives. So that is to say reduce the size of the computational model so that only the degrees of freedom which are common to both experimental and computational models are retained, or alternatively expand the size of the experimental model so that the degrees of freedom in both experimental and computational model match.

Three techniques

- Static condensation (Guyan's reduction)
- Dynamic condensation
- System equivalent reduction expansion process

Now we will discuss 3 alternative techniques for model reduction, and the names of these techniques are static condensation method, dynamic condensation, and there is what is known as system equivalent reduction expansion process. So I will just run through the logic of these 3 model reduction schemes and we will discuss the relative merits and demerits.

Guyan's reduction technique

$$X(t) = \begin{Bmatrix} X_m(t) \\ X_s(t) \end{Bmatrix}_{N\times 1} = \underset{N\times n}{\Psi} \underbrace{X_m(t)}_{n\times 1}$$

$$\Rightarrow$$

$$\begin{bmatrix} M_{mm} & M_{ms} \\ M_{sm} & M_{ss} \end{bmatrix} \begin{Bmatrix} \ddot{X}_m \\ \ddot{X}_s \end{Bmatrix} + \begin{bmatrix} C_{mm} & C_{ms} \\ C_{sm} & C_{ss} \end{bmatrix} \begin{Bmatrix} \dot{X}_m \\ \dot{X}_s \end{Bmatrix} + \begin{bmatrix} K_{mm} & K_{ms} \\ K_{sm} & K_{ss} \end{bmatrix} \begin{Bmatrix} X_m \\ X_s \end{Bmatrix} = \begin{Bmatrix} F_m(t) \\ 0 \end{Bmatrix}$$

Assumption

•Slave dofs carry no nodal forces

Idea: Relate slave dofs to the master dofs through relations which are valid under static conditions. That is use the relation

$$\begin{bmatrix} K_{ms} \\ K_{ss} \end{bmatrix} \begin{Bmatrix} X_m \\ X_s \end{Bmatrix} = \begin{Bmatrix} F_m(t) \\ 0 \end{Bmatrix}$$

to eliminate slave dofs in terms of master dofs.

So-called static condensation is also called Guyan's reduction technique, so what we are looking for? We have this global degrees of, the degrees of freedom for the larger system partitioned as master and slaves, and I want to now relate X(t) to the masters through this transformation matrix capital Sai, so this partitioning of states into master and slave induces a partition on the structural matrices and I can write the equation in this form, so here we assume that the slave degrees of freedom do not carry any external force, okay, that's an assumption. Now the idea in static condensation is to relate the master and slave degrees of freedom through a relation which is valid only for under static conditions, that means to establish relationship between XM and XS I will consider the equilibrium equation, this equation I will discard the inertial and damping terms and write only the static equilibrium equation. So using this equation now I will be able to establish a relationship between XM and XS, so if this is what

$$\begin{bmatrix} K_{mm} & K_{ms} \\ K_{sm} & K_{ss} \end{bmatrix} \begin{Bmatrix} X_m \\ X_s \end{Bmatrix} = \begin{Bmatrix} F_m(t) \\ 0 \end{Bmatrix}$$

$$\Rightarrow K_{sm} X_m + K_{ss} X_s = 0$$

$$\Rightarrow X_s = -K_{ss}^{-1} K_{sm} X_m$$

$$\Rightarrow \text{Transformation matrix}$$

$$\Psi = \begin{bmatrix} I \\ -K_{ss}^{-1} K_{sm} \end{bmatrix}$$

$$\Rightarrow \begin{Bmatrix} X_m(t) \\ X_s(t) \end{Bmatrix} = \begin{bmatrix} I \\ -K_{ss}^{-1} K_{sm} \end{bmatrix} \{X_m(t)\}$$

$$T = \frac{1}{2} \dot{X}^t M \dot{X} = \frac{1}{2} \dot{X}_m^t \Psi^t M \Psi \dot{X}_m = \frac{1}{2} \dot{X}_m^t M_r \dot{X}_m$$

$$M_r = \Psi^t M \Psi$$

$$V = \frac{1}{2} X^t K X = \frac{1}{2} X_m^t \Psi^t K \Psi X_m = \frac{1}{2} X_m^t K_r X_m$$

$$K_r = \Psi^t K \Psi$$

$$\Rightarrow M_r \ddot{X}_m + C_r \dot{X}_m + K_r X_m = F_r$$

we are going to accept as a relationship between master and slaves the first of this equation gives KMM XM + KMS XS is FM(t), that we are not considering.

The second equation is of interest to us KSM into XM + KSS into XS is 0, so by rearranging the terms I get XS as - KSS inverse KSM XM, so this is a relation between slave and master. So the transformation matrix therefore can be written as I into, sorry I and - KSSs inverse KSN, so this will be lowercase n by n, this will be capital N - N across N - N square matrix, so this is the sai matrix. Now we can also look at now the expression for kinetic energy and potential energy in the original system, the expression for kinetic energy is 1/2 X dot transpose MX dot, so X I'm writing it as sai into XM, so if I make that substitution for X dot I will write sai XM dot and for X dot transpose it will be XM dot transpose sai transpose. So if I now call this quantity sai transpose M sai as MR, I get the expression for kinetic energy in the reduced model as shown here, so this MR is now the reduced mass matrix.

Similarly the potential energy I can write 1/2 X transpose KX so this again 1/2 sai XM here, and XM transpose, sai transpose here, and I get a reduced stiffness matrix KR which is sai transpose K sai. Now therefore the governing equation, I can now write for the reduced system as MR XM double dot + CR XM dot + KR XM = FR(t), so this equation can now be analyzed, this is the reduced model that we are looking for. So what are the features of this? So we can

$$M_r = \Psi^T M \Psi = \begin{bmatrix} I \\ -K_{ss}^{-1} K_{sm} \end{bmatrix}^T \begin{bmatrix} M_{mm} & M_{ms} \\ M_{sm} & M_{ss} \end{bmatrix} \begin{bmatrix} I \\ -K_{ss}^{-1} K_{sm} \end{bmatrix}$$

$$= M_{mm} - M_{ms} K_{ss}^{-1} K_{sm} - K_{sm}^T K_{ss}^{-1} M_{sm} + K_{sm}^T K_{ss}^{-1} M_{ss} K_{ss}^{-1} K_{sm}$$

$$K_r = \Psi^T K \Psi = \begin{bmatrix} I \\ -K_{ss}^{-1} K_{sm} \end{bmatrix}^T \begin{bmatrix} K_{mm} & K_{ms} \\ K_{sm} & K_{ss} \end{bmatrix} \begin{bmatrix} I \\ -K_{ss}^{-1} K_{sm} \end{bmatrix}$$

$$= K_{mm} - K_{ms} K_{ss}^{-1} K_{sm} - K_{sm}^T K_{ss}^{-1} K_{sm} + K_{sm}^T K_{ss}^{-1} K_{ss} K_{ss}^{-1} K_{sm}$$

$$= K_{mm} - K_{ms} K_{ss}^{-1} K_{sm}$$

work out the details of the MR and KR matrices in a more explicit manner, so sai transpose M sai will give me this, this is sai transpose, this is this, and if I expand this I will get now the reduced mass matrix in this form, you must notice here that the reduced mass matrix is now a function of the stiffness matrices of the original system, this is unusual because the inertial, kinetic energy in the reduced system is now function of stiffness characteristics, okay, that's an artifice induced because of the remodeled reduction that we have done. Similarly the reduced stiffness matrix is sai transpose K sai and this by rearranging the terms I get this as the reduced stiffness matrix.

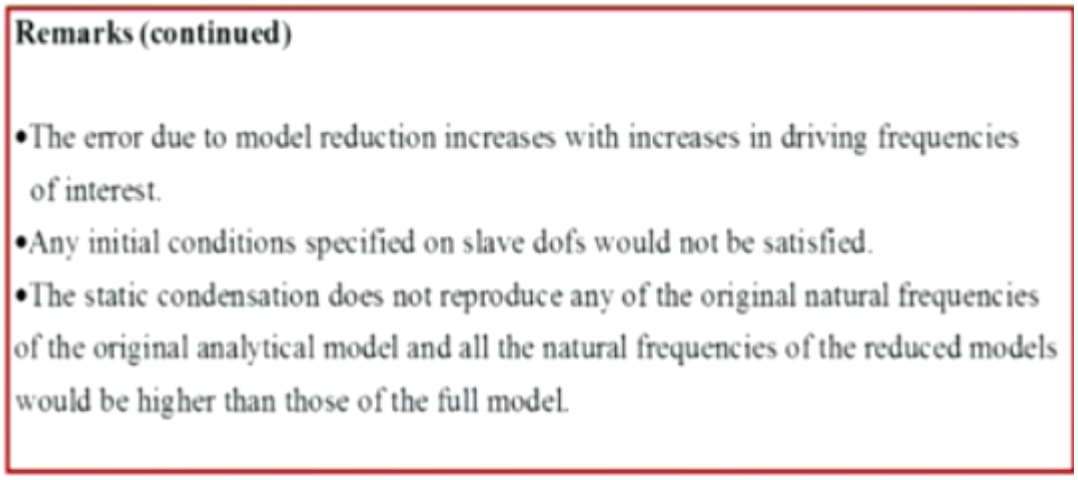
Remarks

- The slave dofs are related to the master dofs through relations that are strictly valid for static situations and hence this method is also known as method of static condensation.
- The partitioning of dofs as being masters and slaves has to be done by the analyst bearing in mind the following points:
  - Slave dofs must contribute little to kinetic energy.
  - Select slave dofs such that the lowest eigenvalue of the equation $K_{ss}\alpha = \lambda M_{ss}\alpha$ has the highest value.
  - Select slave dofs in regions of high stiffness and low mass.
  - Ensure that terms of M_{ss} are small and terms of K_{ss} are large.
  - Those dofs which yield the larger values of the ratio $\frac{K_{ii}}{M_{ii}}$ can be selected as slaves

Okay, now we can make few observations, now the slave degrees of freedom are related to the master degrees of freedom through relations that are strictly valid for static situations, and

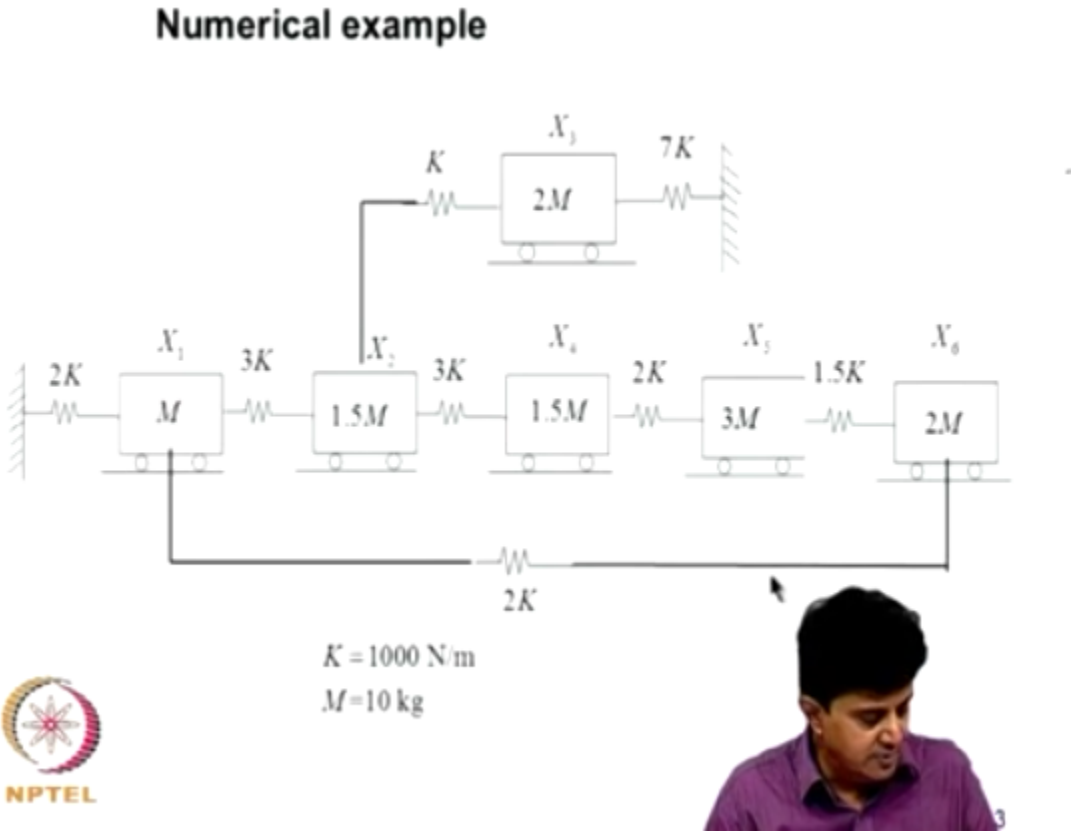
hence this method is known as method of static condensation. The partitioning of degrees of freedom as being masters and slaves has to be done by analyst bearing in mind this assumption, that master and slaves are connected to each other through relations which has strictly valid only under static conditions so what is the consequence of that? The method is likely to perform better if slave degrees of freedom contribute little to the kinetic energy, so in regions of low mass and high stiffness, you should identify the slave degrees of freedom. The select slave degrees of freedom such that the lowest Eigenvalue of the equation KS is alpha = lambda MSS alpha has the highest Eigenvalue, that means between two competing choices for slave degrees of freedom you will get two different KSS and MSS, you perform the Eigenvalue analysis for the two competing choices, and between the two the one which has, the one in which the lowest Eigenvalue is higher is a better choice, so I will illustrate that with an example, this again is a consequence of the basic fact that we are only, they're relating the master and slave only through static relations. So as I was telling select slave degrees of freedom in regions of high stiffness and low mass.

Now we to ensure that terms of MSS are small in, and in terms of KSS are last that is I am reiterating the same statement in a slightly different way, yet another way of saying similar thing is those degrees of freedom which yield the larger value for this ratio can be selected as slaves, so you can alter the all the degrees of freedom based on this ratio and select as many slaves as this you needed by assessing, by comparing this ratio.



Now it's clear that the error due to model reduction increases with increases in driving frequencies of interest, that is because with increase in driving frequencies the kinetic energy goes on increasing and we can't ignore mass, a mass which is small at low frequency will contribute significantly to kinetic energy at a higher frequency therefore the assumption will start breaking down. Now any initial condition specified on slave degrees of freedom would not be satisfied, especially the velocity degrees of freedom and things like, even displacement degrees of freedom because we're even slave is made to you know forcefully related to the master and the initial condition of slave cannot be accommodated in the further modeling work. The static condensation does not reproduce any of the original natural frequencies of the

original analytical model, and all the natural frequencies of the reduced models would be higher than those of the full model.
Now again let me go back to the example of a bigger model with 100 degrees of freedom and a reduced model of 10 degrees of freedom. The reduced model if you compute the 10 natural frequencies for the reduced model, these frequencies need not agree with any of the 100 natural frequencies of the original system, there is no guarantee that is bound to happen if you follow this procedure.



So we can try to understand this through a numerical example, so let's consider a simple example having 6 degrees of freedom with springs as shown here, and we'll assume stiffness is 1000 newton per meter, and mass is 10 kg, so this is simple you know vibrating system, there was the only thing of interest is this X1 and X6 are coupled so the stiffness matrix will have certain you know of diagonal terms to reflect that, so this stiffness matrix, this is a stiffness

$$K = \begin{bmatrix} 7000 & -3000 & 0 & 0 & 0 & -2000 \\ -3000 & 7000 & -1000 & -3000 & 0 & 0 \\ 0 & -1000 & 8000 & 0 & 0 & 0 \\ 0 & -3000 & 0 & 5000 & -2000 & 0 \\ 0 & 0 & 0 & -2000 & 3500 & -1500 \\ -2000 & 0 & 0 & 0 & -1500 & 3500 \end{bmatrix} \text{(N/m)}$$

$$M = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 15 & 0 & 0 & 0 & 0 \\ 0 & 0 & 20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 15 & 0 & 0 \\ 0 & 0 & 0 & 0 & 30 & 0 \\ 0 & 0 & 0 & 0 & 0 & 20 \end{bmatrix} \text{(kg)}$$

$$\Phi = \begin{bmatrix} 0.0649 & 0.0703 & 0.0746 & -0.0050 & 0.1493 & 0.2509 \\ 0.0774 & 0.0023 & 0.1565 & -0.0107 & 0.1258 & -0.1423 \\ 0.0102 & 0.0005 & 0.0369 & -0.2130 & -0.0543 & 0.0143 \\ 0.1040 & -0.0671 & 0.1312 & 0.0723 & -0.1622 & 0.0509 \\ 0.1273 & -0.0850 & -0.0918 & -0.0187 & 0.0335 & -0.0021 \\ 0.1043 & 0.1825 & -0.0439 & 0.0085 & -0.0512 & -0.0345 \end{bmatrix}$$

$$\omega = \begin{bmatrix} 4.6090 & 13.0911 & 13.7122 & 19.9369 & 22.7136 & 29.9605 \end{bmatrix} \text{ rad/s}$$

matrix for the system as you can see there is a term here which reflects coupling between first and this, the sixth mass, this mass and this mass, this is a mass matrix as one could expect, this is a diagonal matrix since we are using lumped mass matrix, modeling, and you can do the Eigenvalue analysis, you can find the, if you perform that the modal matrix comes out to be this and this is the vector of natural frequencies expressed in radian per second, so let us say that this is my larger model.

Static condensation

$$\frac{K(i,i)}{M(i,i)} = \begin{bmatrix} 700.0 & 466.7 & 400.0 & 333.3 & 116.7 & 175.0 \end{bmatrix}$$

Case - 1 Master dofs : 1, 2, and 3; slave dofs : 4, 5, and 6

Eigenvalues associated with $K_{ii}\alpha = \lambda\alpha$

$$\lambda = \begin{bmatrix} 53.89 & 200.00 & 371.11 \end{bmatrix} \text{ (rad/s)}^2$$

$$\Psi = \begin{bmatrix} I \\ -K_{ii}^{-1}K_{im} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.1667 & 0.8333 & 0 \\ 0.4167 & 0.5833 & 0 \\ 0.7500 & 0.2500 & 0 \end{bmatrix}$$

$$\Phi_r = \begin{bmatrix} 0.0944 & 0.0720 & 0.1759 \\ 0.1134 & -0.0416 & -0.1350 \\ 0.0153 & -0.2092 & 0.0774 \end{bmatrix}$$

$$\omega_r = \begin{Bmatrix} 5.5107 & 19.7498 & 22.0729 \end{Bmatrix} \text{ rad/s}$$

Now I want to achieve model reduction, okay by using static condensation, so to begin with I can compute this ratio K(I,I)/M(I,I), now the ratios are shown here so let us consider for purpose of illustration, two alternative choices for master and slave degrees of freedom, in each case we will have, we will try to reduce the model to a 3 degree of freedom system, so in the first case what I will take the master degrees as 1, 2, 3 and slave degrees as 4, 5, 6, now is it a good choice, is it a deliberately bad choice because we want this ratio to be large for slaves, but I'm forcing it do you know what should be ideally slave degrees of freedom, I am making them as masters just to emphasize what would happen. Then I will perform this Eigenvalue analysis KSS alpha = lambda into alpha, and these are the Eigenvalues.

Now the sai matrix turns out to be this and I can get the reduced mass and stiffness matrices and get the Eigenvalue, Eigenvector matrix for the reduced system, and the 3 natural frequencies. Now the 3 natural frequencies we are getting as 5.5, 19.7, and 22.07, so if you go here and see this is 4.6, 13, 13, 19, 22, and 29, so we don't seem to be, we seem to be getting 2 frequencies in this, for these 2 frequencies seem to be giving reasonable answers but the first 3 modes are not captured well.

Static condensation

$$\frac{K(i,i)}{M(i,i)} = \begin{bmatrix} 700.0 & 466.7 & 400.0 & 333.3 & 116.7 & 175.0 \end{bmatrix}$$

Case - 2 Master dofs : 4, 5, and 6; slave dofs : 1, 2, and 3

Eigenvalues associated with $K_{ii}\alpha = \lambda\alpha$

$$\lambda = \begin{bmatrix} 290.01 & 419.91 & 856.75 \end{bmatrix} \ (\text{rad/s})^2$$

$$\Phi_r = \begin{bmatrix} 0.1048 & -0.0785 & -0.1831 \\ 0.1286 & -0.0750 & 0.1057 \\ 0.1053 & 0.1888 & 0.0058 \end{bmatrix}$$

$$\omega_r = \begin{bmatrix} 4.6297 & 13.1467 & 15.1456 \end{bmatrix} \text{rad/s}$$

Case 2 offers better model

Now let us know switch the options and I will make now 4, 5, 6 as masters and 1, 2, 3 as slaves, this is what our recommendation you know suggests, that this is what we should be doing if you are interested in producing a 3 degree of freedom model. Now let's again do this Eigenvalue analysis and I get 290, 419 and 91, now the point I was making was the lowest Eigenvalue here is 53.89, the lowest Eigenvalue here is 290 so between the two model the one which has higher lowest Eigenvalue is the second model, because it is 290, this 290 is much larger than 53 so we could expect that this will perform better. And in static condensation the reduced model should typically represent the behavior of the system in low frequencies well, okay. Now the reduced model looks like this and I get the 3 natural frequencies to be this, so we see here 4.6, 13.1, 15.14 so relatively speaking they seems to be you know a better, you know reduction has been you know obtained through this choice of master and slaves. So at this stage we should notice that in the reduced model the frequencies that I obtain do not match with any of the natural frequencies of the global model.

Now the next question I should ask is suppose I demand, okay some frequency should match, okay how to achieve that? So that takes us to the discussion on what is known as dynamic

Dynamic condensation technique

$$M\ddot{X} + KX = F\exp(i\omega t)$$

$$\Rightarrow \left[-\omega^2 M + K\right]\{X\} = F$$

$$D(\omega) = -\omega^2 M + K \quad \text{[Dynamic stiffness matrix]}$$

$$\Rightarrow DX = F$$

$$X = \begin{Bmatrix} X_m \\ X_s \end{Bmatrix}_{N\times 1} = \underset{N\times n}{\Psi}\ \underset{n\times 1}{X_m}$$

$$\Rightarrow$$

$$\begin{bmatrix} D_{mm} & D_{ms} \\ D_{sm} & D_{ss} \end{bmatrix} \begin{Bmatrix} X_m \\ X_s \end{Bmatrix} = \begin{Bmatrix} F_m \\ 0 \end{Bmatrix}$$

$$\Rightarrow X_s = -D_{ss}^{-1} D_{sm} X_m$$

$$\Rightarrow \Psi = \begin{bmatrix} I \\ -\left[K_{ss} - \omega^2 M_{ss}\right]^{-1}\left[K_{sm} - \omega^2 M_{sm}\right] \end{bmatrix}$$

condensation technique, so let's consider for sake of discussion a harmonically driven undamped system and this is our equation, and this we call it as dynamic stiffness matrix, this is - omega square M + K this we encountered earlier, so the equilibrium equation in the frequency domain is DX into F. So now what I do is I will again partition X into XM and XS and related to, X is related to XM through this matrix. Now this partitioning induces this partitioning of the dynamic stiffness matrix also as shown here. Now what I do here is, I will again assume the slave degrees of freedom are not driven, so I can use the second equation here which is DSM XM + DSS XS = 0 from which I get XS to be this, there is no approximation here, okay I am not throwing out any term, so the transformation matrix that I am looking for is given by this. Now this omega, okay is now a parameter in your model reduction, okay, so you've to make a choice for this omega, that is in addition to making choices on which of the degrees of freedom should be master and which should be slaves this reduction scheme also demands that you should make a choice on omega, okay.

$$X = \begin{Bmatrix} X_m \\ X_s \end{Bmatrix} = \begin{bmatrix} I \\ -\left[K_{ss} - \omega^2 M_{ss}\right]^{-1}\left[K_{sm} - \omega^2 M_{sm}\right] \end{bmatrix} \{X_m\}$$

$$T = \frac{1}{2}\dot{X}^t M \dot{X} = \frac{1}{2}\dot{X}_m^t \Psi^t M \Psi \dot{X}_m = \frac{1}{2}\dot{X}_m^t M_r \dot{X}_m$$

$$M_r = \Psi^t M \Psi$$

$$V = \frac{1}{2}X^t K X = \frac{1}{2}X_m^t \Psi^t K \Psi X_m = \frac{1}{2}X_m^t K_r X_m$$

$$K_r = \Psi^t K \Psi$$

Remarks

- In addition to choosing slave and master dofs, here one also needs to specify the frequency ω at which the condensation has to be done.
- The method requires the determination of inverse of the matrix $\left[K_{ss} - \omega^2 M_{ss}\right]$

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Now again we can do the same steps and obtain the reduced mass and stiffness matrices as shown here, this is what I was saying in addition to choosing slave and master DOF's here one also need to specify the frequency omega at which the condensation has to be done. The method requires the determination of inverse of this matrix see here this matrix needs to be inverted. Again let me point out one more thing, the reduced mass matrix here, and the reduced stiffness matrix here now depend on the mass stiffness and the driving frequency in the original system, so these reduced matrices MR and KR you have to, they are not amenable for a direct physical interpretation, okay, this inversion of the matrix can be computationally demanding so

Approximation to $\left[K_{ss}-\omega^2 M_{ss}\right]^{-1}$

$$\left[K_{ss}-\omega^2 M_{ss}\right]^{-1}=\left(K_{ss}\left[I-\omega^2 K_{ss}^{-1}M_{ss}\right]\right)^{-1}$$

$$=\left[I-\omega^2 K_{ss}^{-1}M_{ss}\right]^{-1}K_{ss}^{-1}$$

$$=\left[I+\omega^2 K_{ss}^{-1}M_{ss}+\omega^4 K_{ss}^{-1}M_{ss}K_{ss}^{-1}M_{ss}+\ldots\right]K_{ss}^{-1}$$

$$\approx\left[I+\omega^2 K_{ss}^{-1}M_{ss}\right]K_{ss}^{-1}$$

$$\Rightarrow X=\begin{bmatrix} I \\ -\left[I+\omega^2 K_{ss}^{-1}M_{ss}\right]K_{ss}^{-1}\left[K_{sm}-\omega^2 M_{sm}\right]\end{bmatrix}\{X_m\}$$

This reduction scheme is called the improved reduction scheme (improvement over the static condensation method).

Avoids the need to invert $\left[K_{sm}-\omega^2 M_{sm}\right]$ for every ω.

we could adopt some simplifications if they're necessary, and while doing so we will also see what is the relationship between dynamic condensation and static condensation. So if we consider the inverse of this I will pull this KSS term outside, and I can write in this form, and the inverse of product is, product of inverse in the reversed order, so this becomes this, and if you use what is known as Neumann expansion, I can expand this matrix in this form, it's a expansion with infinite number of terms, only few terms are shown here.

So if I now omit this higher order terms in omega and retain only the two terms, I get this as my matrix, and if I instead of inverting the matrix now I can use this you know matrix, and this method is called the improved reduction scheme, it's an improvement over static condensation method , and it avoids the inversion of this matrix. If you are going to repeat this calculation for different values of omega then this is a simpler approach, we can also of course do something

Alternatively $\left[K_{ss} - \omega^2 M_{ss}\right]^{-1}$ can also be evaluated in terms of the solution of the eigenvalue problem

$K_{ss}\alpha = \lambda M_{ss}\alpha$

Let Φ be the matrix of eigenvectors and Λ be the diagonal matrix of eigenvalues such that

$\Phi^t M_{ss}\Phi = I \;\&\; \Phi^t K_{ss}\Phi = \Lambda.$

$\Phi \;\&\; \Lambda$: independent of ω.

$$\left[K_{ss} - \omega^2 M_{ss}\right] y = p$$

$$y = \Phi u$$

$$\Rightarrow \left[K_{ss} - \omega^2 M_{ss}\right]\Phi u = p$$

$$\Rightarrow \Phi^t \left[K_{ss} - \omega^2 M_{ss}\right]\Phi u = \Phi^t p$$

$$\Rightarrow \left[\Lambda - \omega^2 I\right] u = \Phi^t p \Rightarrow y = \Phi\left[\Lambda - \omega^2 I\right]^{-1}\Phi^t p$$

$$\Rightarrow \left[K_{ss} - \omega^2 M_{ss}\right]^{-1} = \Phi\left[\Lambda - \omega^2 I\right]^{-1}\Phi^t$$

else we can formulate a Eigenvalue problem associated with KSS and MSS, and suppose if I consider KSS alpha as lambda MSS alpha, and if phi is a matrix of eigenvectors and capital Lambda be the diagonal matrix of Eigenvalues such that phi transpose MSS is phi, and this is lambda, then if I consider the problem of inverting this I can consider the set of these equations, and if I make a transformation Y = phi U, where phi is this matrix of Eigenvectors and substitute here, and use these orthogonality relations I will be able to show that this inverse is nothing but this, and this is a diagonal matrix so it does not require inversion. So this phi can be computed, see this I am computing phi there is no omega here, so the same phi can be used for different omegas so that is the idea which affords simplification here.

Numerical example: dynamic condensation

ω_r	$\omega = 4.61$	$\omega = 13.09$	$\omega = 13.71$	$\omega = 19.93$	$\omega = 22.71$	$\omega = 29.97$
ω_1	4.61	13.09	13.45	19.93	19.79	19.07
ω_2	19.57	13.72	13.71	20.59	22.71	21.35
ω_3	21.40	20.15	20.15	25.45	28.64	29.96

Choice 1

All frequencies in rad/s

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Now we will return to the example that we considered here the same example, I will again follow the same choices of degrees of, master and slave degrees of freedom, but now what I will do is I have no additional choice to make on driving frequencies, so I have made several choices, so I have made 7 choices, in the first choice I take omega to be 4.61 which happens to be the first natural frequency of the system, and I make from a 7 degree of freedom system I get a 3 degree freedom system, so that 4.61 turns out to be the one of the Eigenvalues of the reduce system, and there are two more frequencies. If omega has taken a 13, 13 happens to be, 13.09 happens to be one of the frequencies, the other two of course are not the natural frequency of the system, so by selecting 13.71, I ensure that 13.71 is one of the natural frequencies of my reduced model and so on and so forth. So in choice 1 this is what I get. In choice 2 that means

Numerical example : dynamic condensation

ω_r	$\omega = 4.61$	$\omega = 13.09$	$\omega = 13.71$	$\omega = 19.93$	$\omega = 22.71$	$\omega = 29.97$
ω_1	4.61	7.16	7.86	10.61	9.93	8.79
ω_2	13.14	13.09	13.09	13.99	14.26	20.07
ω_3	14.96	13.75	13.71	19.93	22.71	29.97

Choice 2
All frequencies in rad/s

- Choice of ω matters.
- Those natural frequencies which are close to ω are predicted well.

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choice of master and slave degrees of freedom the same thing happens, but of course now the frequencies other than the one that are in proximity of these numbers will be different from what was there in model one, so the choice of omega do matter, and those natural frequencies which are close to omega are predicted well that is the observation that we make here, so this 4.6, 13 these are the frequencies they are captured here, the 6 degree of freedom not 7 degree of freedom, so there are 6 alternative models and each one works well in the neighborhood of the frequency chosen, and that fact is independent of choice of master and slave degrees of freedom, no matter which is master which is slave the fact that at omega, if you select omega = 19.93 the reduced model will have that as one of the frequencies in both the cases.

Remarks

- In addition to choice of master and slave dofs, the choice of ω also matters.
- Those natural frequencies which are close to ω are predicted well.
- In a harmonic response analysis, ω can be chosen to be equal to the driving frequency.
- If the FRF-s need to be traced over a frequency range, for every value of driving frequency, the condensation needs to be made separately.
- Expected to lead to acceptable results if modes are well separated and damping is light.

So we can make few comments now, so in dynamic condensation, in addition to choice of master slave DOF's the choice of omega also matters, those natural frequencies which are close to omega are predicted well. In a harmonic response analysis omega can be chosen to be equal to the driving frequency, so if you are varying omega then computationally this form of, you know the writing the transformation matrix in this form is advantageous, because you have to do only one Eigenvalue analysis, and you need not have to invert the matrix, this matrix for every omega.

If the FRF's need to be traced over a frequency range, for every value of driving frequency the condensation needs to be made separately. If omega is taken to match with the driving frequency, that is the best option that you would have, because at least at the frequency where you are driving that nearby natural frequencies are captured correctly in the reduced model, this is expected to lead an acceptable results if modes are well separated and damping is light, okay, that's very clear, because once you make a choice of omega, and you are driving a system harmonically at that frequency the response contribution is dominated by a single mode, then the method is likely to work well even for first response analysis.

System equivalent reduction expansion process (SEREP)

- Preserves a collection of normal modes during the reduction process.
- Consider a N dof FE model for a linear vibrating systems.
- Let Φ denote the $N \times p$ modal matrix that includes the first p modes.
- As before we partition X into master and slave dofs.

This induces the following partition on Φ

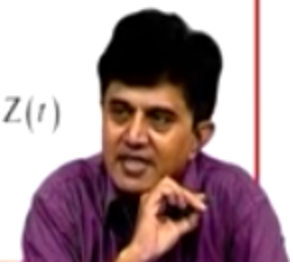
$$\Phi = \begin{Bmatrix} \Phi_m \\ \Phi_s \end{Bmatrix}$$

Sizes: $\Phi: N \times p$, $\Phi_m : n \times p$, & $\Phi_s : (N-n) \times p$

Assume: $n \geq p$.

Introduce a $n \times 1$ vector of generalized coordinates $Z(t)$

$$X(t) = \begin{Bmatrix} X_m(t) \\ X_s(t) \end{Bmatrix} = \begin{bmatrix} \Phi_m \\ \Phi_s \end{bmatrix} \{Z(t)\}$$



Now we can ask this question in dynamic condensation in the reduced model we are able to capture only one mode correctly, so the next logical question that we can ask is can we retain a subset of the natural the frequencies of the original system in the reduced model in an exact manner, okay, suppose in 100 degree freedom system there will be 100 natural frequencies and if I am looking at reduced model with 10 degrees of freedom, the reduced model will have 10 natural frequencies, the question I'm asking is can we select this 10, all these 10 natural frequencies, can they be the natural frequencies of the original system, need not be the first 10, it can be any 10 that you can arbitrarily specify, so if you can do that then you are achieving something substantial, but then you have to input lot of details into the reduction scheme, so such a scheme indeed exist and that is known as system equivalent reduction expansion process, and abbreviated as a SEREP, so I will be using the term SEREP, so the main features of this reduction scheme, and also as the name indicates it's an expansion scheme as well, but we will focus on reduction aspect of it, so it preserves a collection of normal modes during the reduction process.

Suppose we consider N degree of freedom model for a linear vibrating system, and let capital Phi denote the N cross P modal matrix that include the first P modes, okay as before we partition X into master and slave degrees of freedom, this partitioning on X induces a partitioning on the partially known modal matrix as phi M and phi S, so what are the different sizes here? This phi is N cross P, phi M will be lowercase n cross p, where this n is the size of the master degrees of freedom, and p is the number of modes retained, so in dynamic condensation P was 1, now P can be more. Phi S is N - N cross P, and also when I say P it is not just the first P, by selecting the appropriate vectors in the modal matrix I am also specifying which of the P modes I am looking at. Now we will assume that N is greater than P that means

the size of the reduced model is larger than the number of modes that you have in the global model.

Now we want to introduce an N cross 1 vector of generalized coordinates Z(t) through this relation, okay, this is X is equal to some modal matrix into Z, that part is fine, so this is the

$$X(t) = \begin{Bmatrix} X_m(t) \\ X_s(t) \end{Bmatrix} = \begin{bmatrix} \Phi_m \\ \Phi_s \end{bmatrix} \{Z(t)\}$$

$$\Rightarrow$$

$$X_m(t) = \Phi_m \{Z(t)\}$$

$$X_s(t) = \Phi_s \{Z(t)\}$$

$$\Rightarrow$$

$$\{Z(t)\} = [\Phi_m]^+ \{X_m(t)\} \text{ where } [\Phi_m]^+ = \left[\Phi_m^T \Phi_m\right]^{-1} \Phi_m^T$$

$$[\Phi_m]^+ = \text{pseudoinverse of } \Phi_m$$

$$\Rightarrow$$

$$X(t) = \begin{Bmatrix} X_m(t) \\ X_s(t) \end{Bmatrix} = \Psi X_m = \begin{bmatrix} \Phi_m \\ \Phi_s \end{bmatrix} \left[\Phi_m^T \Phi_m\right]^{-1} \Phi_m^T \{X_m(t)\}$$

$$\Psi = \begin{bmatrix} \Phi_m \\ \Phi_s \end{bmatrix} \left[\Phi_m^T \Phi_m\right]^{-1} \Phi_m^T$$

$$\Rightarrow M_r = \Psi^T M \Psi$$

$$K_r = \Psi^T K \Psi$$

relation. Now XM(t) is we can put it in the, you can expand this I get this, so from which I can, we can solve for Z(t) using this equation. Now the number of unknowns and number of equations in this case would not match, therefore I cannot use inverse directly I will have to use what is known as pseudo inverse, I will just shortly explain what should of inverse is, but if you, right now we will accept that there is an operation known as pseudo inverse as indicated here, and this is this, that is phi M + is this, where plus indicates pseudo inverse.

Now if this is acceptable then X(t) can be written in this form, that is phi M, phi S into this this, so the sai matrix which relates X to XM is now given by this. Mind you this capital Phi here is the modal matrix of the complete system, so before you do model reduction you should perform the Eigenvalue analysis in the large system, okay otherwise you cannot use this method so the reduced mass matrix and reduced stiffness matrix are obtained as shown here.

Concept of Pseudo-inverse

Pseudo-inverse

Motivation

Consider a linear algebraic equation $x_1 + 5x_2 = 1$. Since we have two variables and only one equation, no unique solution is possible. However if we decide to pick the point that is closest to the origin, then the "solution" is unique.

Consider the matrix equation

$AX = B$

where

$A \rightarrow n \times m$

$\rightarrow m \times 1$

$n \times 1$

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Now let me quickly describe what is pseudo inverse, it is not a thorough discussion but it tells you what the idea is, so the motivation is suppose if you consider a linear algebraic equation X1 + 5X2 = 1, so we have 1 equation and 2 unknowns, so there will be an infinity of solutions, you draw this line any point lying on that line is a solution to this equation, however if we decide to pick the point that is closest to the origin as the solution, by that I mean so all points lying on this straight line is a solution, but if I decide that I will take this point which is closest to the origin as the solution, this one, right? Then I get a unique solution, okay.

Case 1: ***m>n*** (Number of unknowns greater than number of equations)

The solution that minimizes the norm ||X|| is given by

$$X^0 = A^{Rt} B$$
$$A^{Rt} = A^T \left[AA^T \right]^{-1}$$

A^{Rt} = Right Pseudo-inverse of A.

Case 2: ***m<n*** (number of unknowns is less than number of equations)

The "solution" X^0 that minimizes the norm ||AX-B|| is given by

$$X^0 = A^{Lt} B$$
$$A^{Lt} = \left[AA^T \right]^{-1} A^T$$

A^{Lt} = Left Pseudo-inverse of A.

So let's consider what that means. So let us consider AX = B, where A is N cross M, and X is M cross 1, and B is N cross 1, okay, so this is a question. Now let us consider the KS where M is greater than N that is number of unknowns is greater than number of equations. Now what we do is the solution that minimizes the norm X that is a distance from the origin to the line, this is given by this, this distance, so we can show that this is given by what is known as ARM, ARM is A transpose, AA transpose inverse, so ARM is known as right pseudo inverse of A.
Now on the other hand if number of unknowns is less than number of equations then I can find A solution X naught that minimizes this norm, the error in satisfying this equation is minimized, and we can do the simple calculation and show that X naught that is the solution in this case is given by ALM into B, where ALM is given by this, and this is known as left pseudo inverse of A. So what does these things mean, what do these things mean? A simple example

$$A = \begin{bmatrix} 0.6787 & 0.6555 & 0.2769 & 0.6948 & 0.4387 & 0.1869 \\ 0.7577 & 0.1712 & 0.0462 & 0.3171 & 0.3816 & 0.4898 \\ 0.7431 & 0.7060 & 0.0971 & 0.9502 & 0.7655 & 0.4456 \\ 0.3922 & 0.0318 & 0.8235 & 0.0344 & 0.7952 & 0.6463 \end{bmatrix}$$

B = pseudo inverse of A

$$= \begin{bmatrix} 1.2713 & 1.6368 & -1.4333 & -0.3097 \\ 1.0286 & -0.6138 & -0.1202 & -0.1329 \\ 1.6711 & -0.6109 & -1.2445 & 0.7315 \\ -0.2419 & -0.5423 & 0.9348 & -0.2292 \\ -1.3539 & -0.6393 & 1.4178 & 0.4807 \\ -1.2726 & 0.6308 & 0.6671 & 0.2305 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1.0000 & -0.0000 & -0.0000 & -0.0000 \\ 0.0000 & 1.0000 & -0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & -0.0000 \\ -0.0000 & 0.0000 & 0.0000 & 1.0000 \end{bmatrix}$$

suppose I consider a 4 cross 6 matrix A and define B as pseudo inverse of A, so I will use these definitions and compute the pseudo inverse, B is computed like this. Now if I multiply A and B I get an identity matrix, so in that sense B is a pseudo inverse of A, although this matrix is not a square matrix I am able to define another B matrix so that AB is an identity matrix, that is why it is called a pseudo inverse.

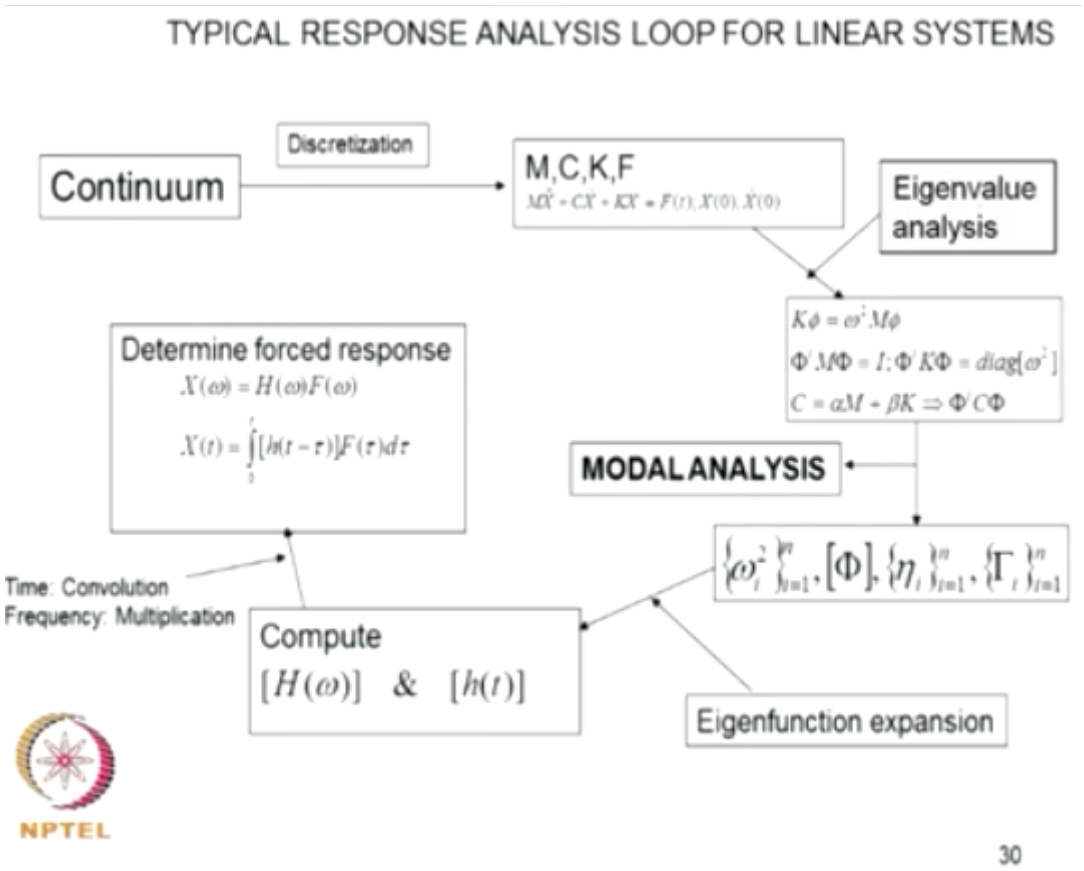
$$A = \begin{bmatrix} 0.8143 & 0.6160 & 0.9172 & 0.0759 \\ 0.2435 & 0.4733 & 0.2858 & 0.0540 \\ 0.9293 & 0.3517 & 0.7572 & 0.5308 \\ 0.3500 & 0.8308 & 0.7537 & 0.7792 \\ 0.1966 & 0.5853 & 0.3804 & 0.9340 \\ 0.2511 & 0.5497 & 0.5678 & 0.1299 \end{bmatrix}$$

B = pseudo inverse of A

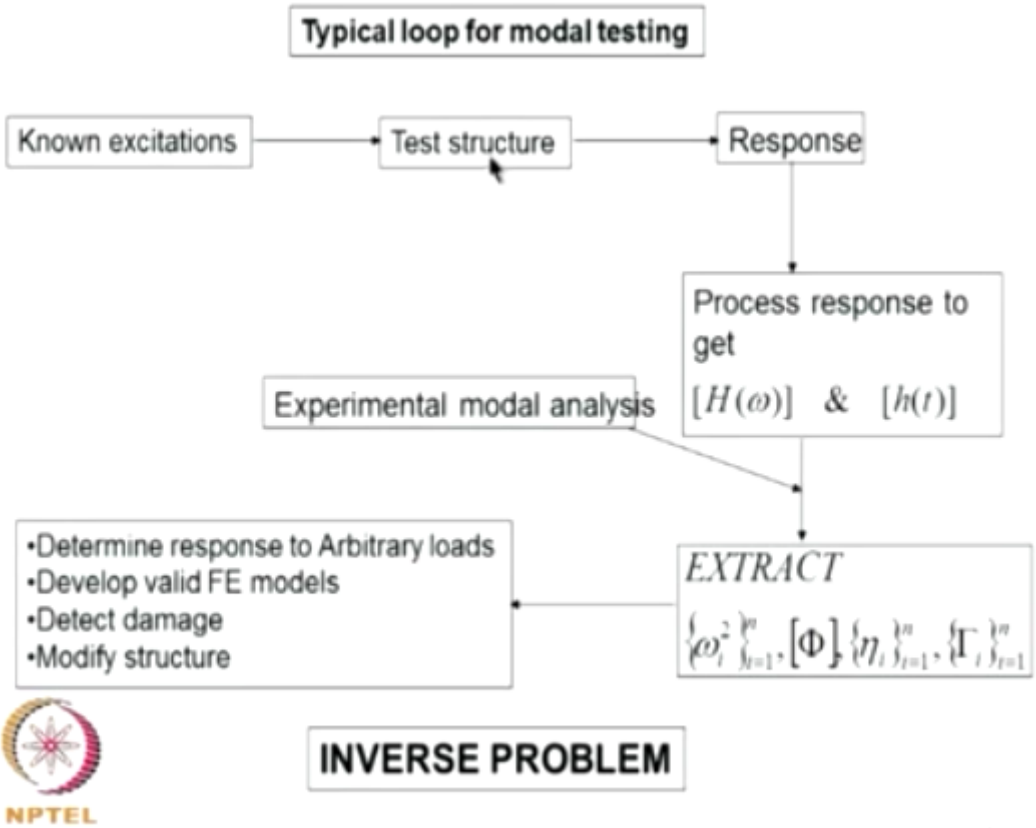
$$= \begin{bmatrix} 0.0722 & 2.4626 & 1.1602 & -1.3753 & 0.5401 & -1.4398 \\ -0.0340 & 3.2902 & -0.6433 & -0.5762 & 0.7234 & -0.4634 \\ 0.6329 & -4.1668 & -0.3440 & 1.6690 & -1.2761 & 1.9349 \\ -0.5038 & -0.7280 & 0.4489 & 0.3370 & 0.6602 & -0.3080 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1.0000 & -0.0000 & -0.0000 & -0.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & -0.0000 & 1.0000 & 0.0000 \\ -0.0000 & 0.0000 & 0.0000 & 1.0000 \end{bmatrix}$$

Now if this is 6 cross 4, instead of 4 cross 6 the pseudo inverse will be 4 cross 6, so AB in that case is again a diagonal matrix, okay, so this is a notion of pseudo inverse that we are using in developing this SEREP transformation.



Now I have been mentioning, I am referring to experimental you know models so it is better at this stage to you know understand what is the difference between modeling in an experimental work and in a computational work, so in a typical computational loop we start with the continuum, we discretize, and suppose we are dealing with time invariant linear systems, we discretize and form the structural matrices MCK and the load vector F and write this equilibrium equation with these specified initial conditions. Then I will perform the Eigenvalue analysis and determine the natural frequencies, mode shapes, modal participation factors, and modal damping ratios, so this analysis is known as modal analysis, that is given the structural matrices how to find the natural frequencies, modal matrix, damping ratios, and the participation factors, so this is the modal analysis in a computational modeling approach, where we solve an Eigenvalue problem, once this is known we have seen already how to compute the frequency response function or impulse response function, and either use this algebraic relation or this convolution relation and obtain the response either in time or in frequency domain, this we have seen, so here it is a 10 time it is a convolution, in frequency it is a multiplication, this is what we do in a computational modeling.



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In an experimental work this loop is reversed, we start by measuring the response, okay so the story is here we apply known excitation to a test structure and measure the response, and what we measure we process and get the matrix of impulse response function and frequency response function, from this we extract natural frequencies, mode shapes, damping ratios and participation factor, this process of obtaining the modal information from measured responses is known as experimental modal analysis, this is in contrast to the modal analysis in computational modeling where we knowing the structural matrices we perform an Eigenvalue analysis and find these quantities, and that we use to compute the response, here we measure the response and we extract this information from these measured responses, and from this we would like to construct models for the structure that means mass, stiffness, damping matrices, and so on and so forth. So the loop is you know the directions are reversed here, so there will be fundamental difficulty whenever we use these two alternative approach to the same problem, and that is where this question of modal reduction and expansion become crucial.

Numerical example: SEREP

Case - 1 Master dofs : 1, 2, and 3; slave dofs : 4, 5, and 6

Retain the first three modes (n=3).

$$\Phi_m = \begin{bmatrix} 0.0649 & 0.0703 & 0.0746 \\ 0.0774 & 0.0023 & 0.1565 \\ 0.0102 & 0.0005 & 0.0369 \end{bmatrix}$$

$$\Phi_s = \begin{bmatrix} 0.1040 & -0.0671 & 0.1312 \\ 0.1273 & -0.0850 & -0.0918 \\ 0.1043 & 0.1825 & -0.0439 \end{bmatrix}$$

$$\Psi = \begin{bmatrix} 1.0000 & -0.0000 & 0.0000 \\ -0.0000 & 1.0000 & 0.0000 \\ -0.0000 & -0.0000 & 1.0000 \\ -1.0018 & 3.2893 & -8.3684 \\ -1.2222 & 6.0717 & -25.7591 \\ 2.6408 & -0.0157 & -6.4561 \end{bmatrix}$$

Now let me return to the example of that 6 degree of freedom system, and now apply SEREP, so let us retain now first 3 modes, so to implement SEREP I have to again declare certain degrees of freedom as masters, and certain degrees of freedom as slaves, additionally I should specify which are the modes that I want to replicate in my reduced model, how many of them? So what I am selecting is I am taking 3 modes, and I am taking the first 3 modes so I can get phi M and phi S by partitioning the modal matrix of the 6 degree of freedom system that's what I have done, and this is a transformation matrix, okay, using this I will construct the reduced mass matrix and reduce stiffness matrix and perform the Eigenvalue analysis on the reduced system, it is a 3/3 system, so that system now has these 3 natural frequencies 4.60, 13 point this and this. Now if you look back these are exactly the 3 frequencies of the larger model, okay, so there is no, precisely mathematically exactly this, this is the reduced modal matrix, and I get the

Reduced system natural frequencies:

$$\omega_r = \begin{bmatrix} 4.6090 & 13.0911 & 13.7122 \end{bmatrix} \text{ rad/s}$$

Reduced modal matrix

$$\Phi_r = \begin{bmatrix} 0.8393 & 1.0000 & -0.4765 \\ 1.0000 & 0.0325 & -1.0000 \\ 0.1320 & 0.0071 & -0.2359 \end{bmatrix}$$

Reduced structural matrices

$$M_r = 10^4 \times \begin{bmatrix} 0.0209 & -0.0273 & 0.0729 \\ -0.0273 & 0.1283 & -0.5103 \\ 0.0729 & -0.5103 & 2.1810 \end{bmatrix}$$

$$K_r = 10^6 \begin{bmatrix} 0.0359 & -0.0464 & 0.1235 \\ -0.0464 & 0.0908 & -0.3313 \\ 0.1235 & -0.3313 & 1.4652 \end{bmatrix}$$

reduced structural matrices as shown here, actually this is nothing but this is sai transpose M sai, this is reduced, this is sai transpose K sai.

Case - 2 Master dofs : 4,5, and 6; slave dofs : 1, 2, and 3

Retain the first three modes (n=3).

$$\Phi_m = \begin{bmatrix} 0.1040 & -0.0671 & -0.1312 \\ 0.1273 & -0.0850 & 0.0918 \\ 0.1043 & 0.1825 & 0.0439 \end{bmatrix}$$

$$\Phi_s = \begin{bmatrix} 0.0649 & 0.0703 & 0.0746 \\ 0.0774 & 0.0023 & 0.1565 \\ 0.0102 & 0.0005 & 0.0369 \end{bmatrix}$$

$$\Psi = \begin{bmatrix} 1.0000 & -0.0000 & 0.0000 \\ -0.0000 & 1.0000 & 0.0000 \\ -0.0000 & -0.0000 & 1.0000 \\ 0.5210 & -0.2811 & 0.4462 \\ 0.9987 & -0.3758 & 0.2049 \\ 0.2107 & -0.1141 & 0.0271 \end{bmatrix}$$

Similarly if I now declare 4, 5, 6 as master and 1, 2, 3 as slaves, there will be certain changes in my features of the reduced model, but the 3 natural frequencies will be identical again, so this is phi M and phi S, this is a transformation matrix, this is slightly now different from this transformation matrix. So again I will perform the Eigenvalue analysis, no surprises, the first 3 natural frequency is exactly matched, the reduced modal matrix of course is now different,

Reduced system natural frequencies:

$$\omega_r = \begin{bmatrix} 4.6090 & 13.0911 & 13.7122 \end{bmatrix} \text{ rad/s}$$

Reduced modal matrix

$$\Phi_r = \begin{bmatrix} 0.5340 & -0.3164 & -0.7901 \\ 0.6539 & -0.4004 & 0.5530 \\ 0.5359 & 0.8600 & 0.2643 \end{bmatrix}$$

Reduced structural matrices

$$K_r = 10^3 \times \begin{bmatrix} 4.7019 & -3.0944 & -0.2786 \\ -3.0944 & 4.4262 & -1.6702 \\ -0.2786 & -1.6702 & 2.8490 \end{bmatrix}$$

$$M_r = \begin{bmatrix} 33.5619 & -7.5745 & 5.5088 \\ -7.5745 & 33.1687 & -2.4713 \\ 5.5088 & -2.4713 & 22.6358 \end{bmatrix}$$

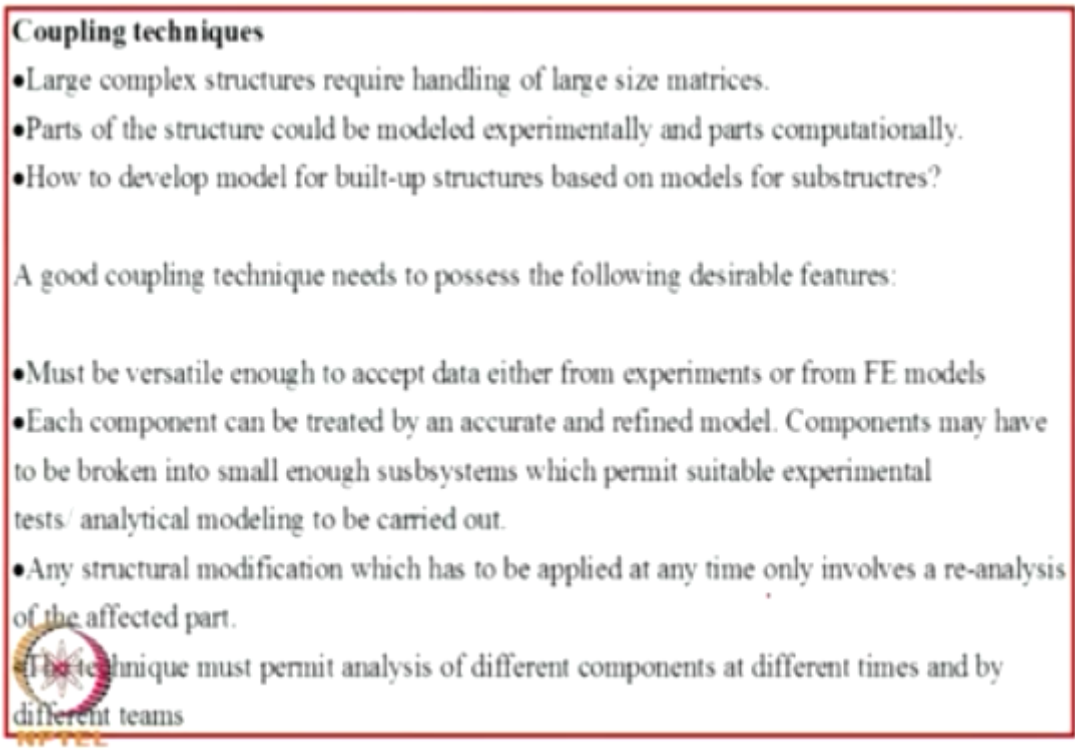
okay, and reduce structural matrices are obtained here and the interesting thing is this pair of KR and MR, and this pair of KR and MR although they are different they share the same Eigenvalues, okay, so that is the achievement of this method.

SEREP

- The user needs to specify the number of modes to be retained, the mode indices, and the slave and master dofs.
- The choice of normal modes to be included in the reduced model is arbitrary.
- The scheme preserves collection of normal modes during reduction
- The transformation matrix is deduced from the modal matrix. The modal matrix can be incomplete. Knowledge of K and M is needed. This could be of value if modal matrix is obtained experimentally.
- The natural frequencies of the reduced system matches with the full system natural frequencies irrespective of choice of master and slave dofs.
- The method can be used for model reduction or for model expansion.

So what are the features of this? To implement SEREP the user need to specify the number of modes to be retained, the more indices which modes, and also the slaves and master DOF's. The choice of normal modes to be included in the reduced model is arbitrary, for example in a 100 degree freedom system you want to select 10 modes, you can select 1, 18, 32, and 76 and so on and so forth, you need not be first 10 nor they need to be in a cluster. The scheme preserves the collection of normal modes during reduction, whatever you are identified as the natural frequencies you should be retained in the reduced model they will be faithfully retained. The transformation matrix is deduced from the modal matrix here, the modal matrix can be incomplete, you need not have a square modal matrix even you can work with rectangular modal matrix, this is what would happen if you do experimental modal analysis, in an experimental model analysis the modal matrix is seldom square, it will be always you know a rectangular matrix, so it will be incomplete.

So knowledge of K of course is needed, if you are doing computationally because you need to find the modal matrix, this could be of value if modal matrix is obtained experimentally that in which case K and M need not be known, phi can be directly measured experimental. The natural frequencies of the reduced system matches with the full system, natural frequency is irrespective of choice of master and slave degree of freedom. The method can be used for model reduction or for modal expansion, so I have not discussed what exactly happens if you use it for modal expansion, but in principle it can be done.

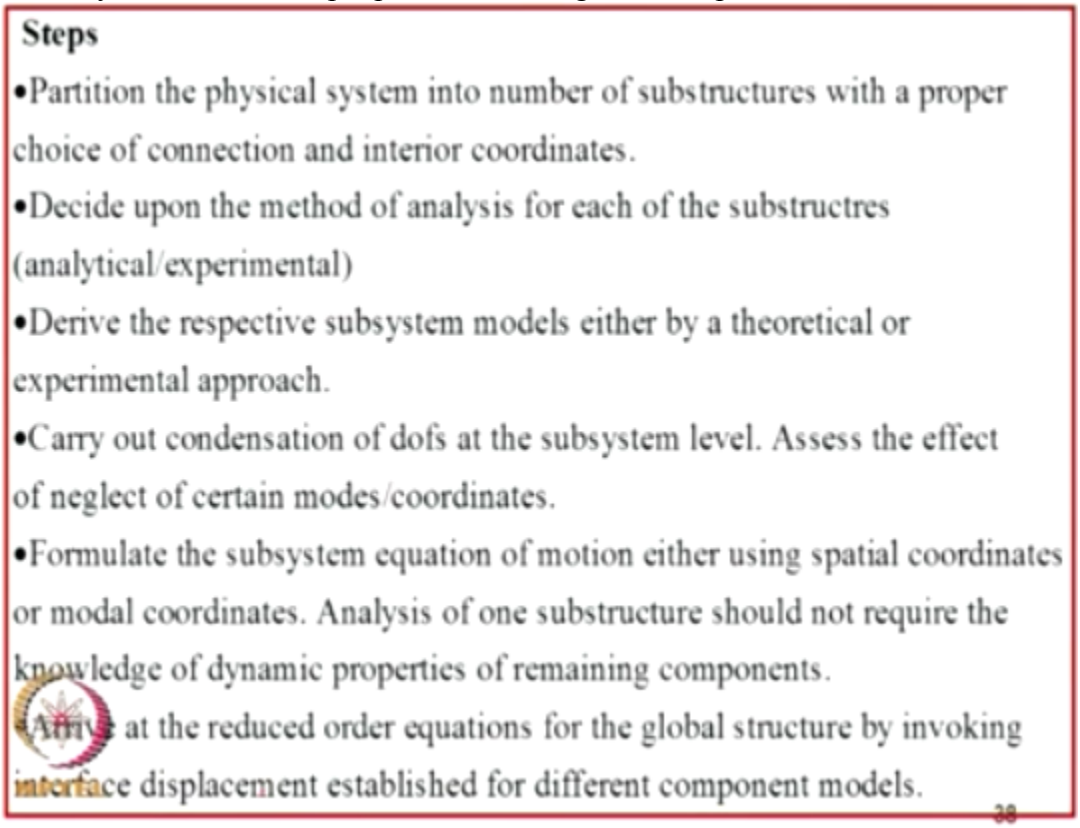


Now we have now talked about modal reduction, the next topic that is related to this type of questions is, questions or what are known as coupling techniques. Again here this is a device to treat large complex structures, and the problem is large complex structure require handling of large size matrices so can we do something about that, and similarly in a manufactured product, parts of this structure could be modeled experimentally in parts computationally.

Now the question is how to develop model for built up structures based on models for these sub structures. Now the coupling techniques answer these questions, a good coupling technique needs to process some desirable features, it must be versatile enough to accept data either from experiments or from FE model, say part of a structure can be modeled experimentally and part of the structure computationally, the computational modeler will be able to give structural matrices, mode shapes, natural frequencies, and so on and so forth. An experimental is typically would be able to give FRF's impulse response function and if model information is extracted you will be able to give the natural frequencies mode shapes, damping ratios, participation factors which is experimentally measured, but the experimentalist will have difficulty in specifying the structural matrices.

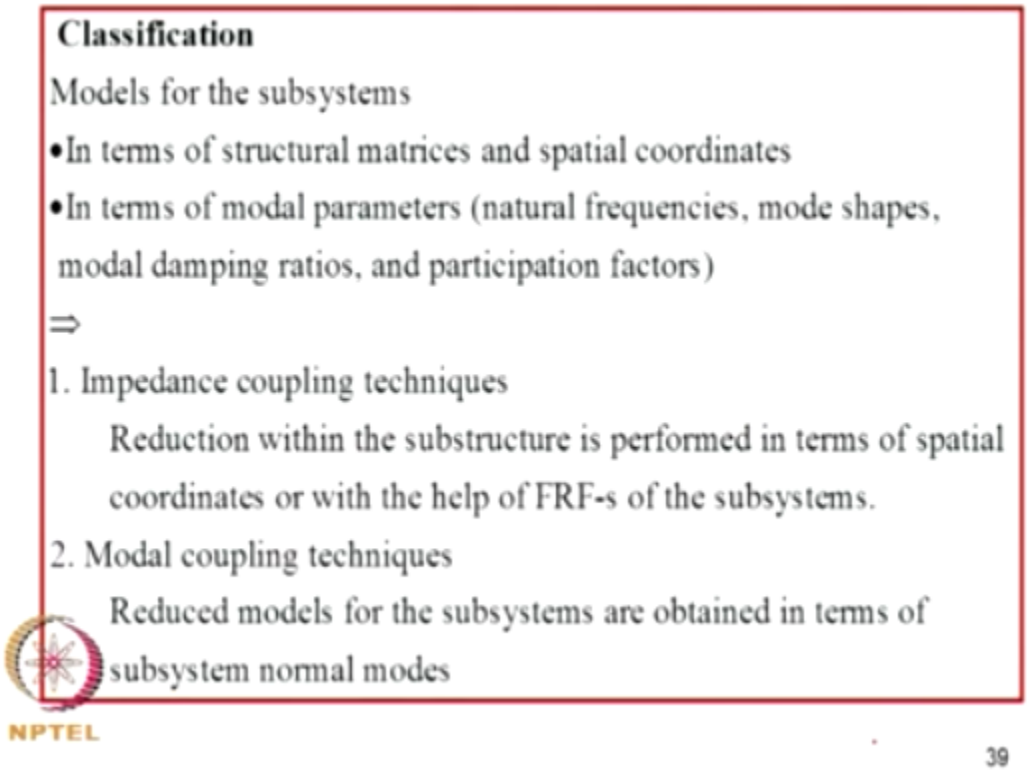
Now each component can be treated by an accurate and refined model, you can use each component, each sub-structure can be modeled with any level of refinement and you know detailed modeling, any level of detail can be included in a model, components may have to be broken into small enough subsystems which permit suitable experimental test or analytical modeling to be carried out, that means the substructure a scheme should not constrain the user in terms of you know, if user wishes to do this it should not be a constrained, any structural modification which has to be applied at any time only involves the reanalysis of the affected part, suppose there are A, B, C are three substructures, and if A is modified then we should not end up analyzing B and C, okay, then the technique must permit analysis of different components at different times and by different teams, this is what typically happens in a you

know space structures, or automotive systems, and so on and so forth, and even you know mechanical systems in civil engineering applications like a turbine, or piping, in an industrial structure, okay, so the different people will be doing different products.



Now typically what are the steps involved in this? The steps involved are, we partition the physical system into number of sub-structures with a proper choice of connection and interior coordinates, we need to decide upon the method of analysis for each of the sub-structure that means the analytical or experimental, we have to derive the respective subsystem models either by a theoretical or experimental approach, then we need to carry out condensation of degrees of freedom at the subsystem level, and we need to assess the effect of neglect of certain modes and coordinates.

Next, we need to formulate the subsystem equation of motion either using spatial coordinates or modal coordinates, analysis of one substructure should not require the knowledge of dynamical properties of remaining components, then we arrive at the reduced order equations for the global structure by invoking interface displacement established for different component models, okay, so this is what will lead to the coupled system. So the coupling techniques can be



classified based on how you model the subsystems, for example for modeling the subsystems we could use structural matrices that is mass, stiffness, and damping, and special coordinate like displacement degrees of freedom and so on and so forth, or you can model each subsystem in terms of a set of natural frequencies, mode shapes, damping ratios, and participation factors, both are equivalent, so depending on how you choose we can have different types of coupling techniques, in one of the scheme of classification, we classify this coupling techniques as the impedance complete techniques and modal coupling techniques.

In impedance coupling techniques reduction within the substructure is performed in terms of spatial coordinates or with the help of frequency response functions of the subsystems, we don't use modal information. In modal coupling techniques reduced model for the subsystems are obtained in terms of subsystem normal modes, so we need to develop the formulary for dealing with you know these coupling techniques and will take up these questions in the next class, and specifically we will be talking about method known as component mode synthesis which is widely used in practice. So with this will conclude the present lecture.

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