

**Indian Institute of Science
Bangalore**

**NP-TEL
National Programme on
Technology Enhanced Learning**

Copyright

1. All rights reserved. No part of this work may be reproduced, stored or transmitted in any form or by any means, electronic or mechanical, including downloading, recording, photocopying or by using any information storage and retrieval system without prior permission in writing from the copyright owner.

Provided that the above condition of obtaining prior permission from the copyright owner for reproduction, storage or transmission of this work in any form or by any means, shall not apply for placing this information in the concerned institute's library, departments, hostels or any other place suitable for academic purposes in any electronic form purely on non-commercial basis.

2. Any commercial use of this content in any form is forbidden.

Course Title

**Finite element method for structural dynamic
And stability analyses**

**Lecture – 15
Second order implicit methods**

**By
Prof. CS Manohar
Professor
Department of Civil Engineering**

**Indian Institute of Science,
Bangalore-560 012
India**

Finite element method for structural dynamic and stability analyses

Module-5

Time integration of equation of motion

Lecture-15 Second order implicit methods



Prof C S Manohar
Department of Civil Engineering
IISc, Bangalore 560 012 India

1

We have been discussing solution of equilibrium equations in dynamics using time stepping methods, so in today's lecture we will talk about second order implicit methods, so far we have

Recall

- Forward Euler scheme: Explicit. Conditionally stable for damped system. Unstable for undamped system. First order accuracy.
- Backward Euler scheme: Implicit. Unconditionally stable. First order accuracy.
- Central difference scheme: Explicit. Conditionally stable. Second order accuracy. Not self-starting.



discussed this forward Euler scheme which we have seen to be an explicit scheme, it is conditionally stable for damped systems, and it is unstable for undamped system for any step size, when stable it has first order accuracy.

The backward Euler scheme is implicit, it is unconditionally stable and it poses the first order accuracy, the central difference scheme is explicit, it is conditionally stable it has second order accuracy it is not a self-starting scheme, whereas these 2 schemes are self-starting. Now we will continue with this discussion we are interested in developing methods that possess second order accuracy and let's consider if we can develop methods which are implicit in nature.

Critical step size

$$\Delta t_{cr} = \frac{2\eta}{\omega} = \frac{\eta T}{\pi} \quad [\text{Forward difference scheme}]$$

$$\Delta t_{cr} = \frac{2}{\omega^*} = \frac{T}{\pi} \quad [\text{Central difference scheme}]$$

When applied to MDOF systems, Δt must be selected such that these conditions are met for all participating modes.

⇒ For direct integrations

$$\Delta t_{cr} = \min_{1 \leq n \leq N} \frac{2\eta_n}{\omega_n} = \frac{2\eta}{\omega_N} \quad \text{if } \eta_n = \eta \forall n \quad [\text{Forward difference scheme}]$$

$$\Delta t_{cr} = \min_{1 \leq n \leq N} \frac{2}{\omega_n} = \frac{2}{\omega_N} \quad [\text{Central difference scheme}]$$



⇒ For modal superposition method, each mode can be integrated, in principle, with different step size with higher modes requiring finer step sizes.

3

Now before we take up that topic we'll revisit some issues about choice of critical step size in conditionally stable schemes, see we had for forward difference scheme the critical step size was $2\eta/\omega$, so in terms of period $D\eta/\phi$, for central difference method which is again conditionally stable, the critical step size was T/ϕ , where T is the period. Now if we apply any of these schemes to multi degree freedom systems it means that Δt must be selected such that these conditions are satisfied for all the participating modes, that means all the modes in the system, in the N degree of freedom system there will be N modes and this condition should be satisfied by all the N modes, if you are doing direct integration the question arises how do you estimate the maximum frequency, so Δt critical is actually for forward difference scheme, the minimum of this taken over all the possible modes, so if damping is constant for all modes it is $2\eta/\omega_N$, if η is the same for all modes.

For central difference scheme it is $2/\omega_N$, which is $2/\Omega$ capital N , if we use central difference scheme. For mode superposition method each mode can be integrated at least in principle with different step sizes with higher modes requiring finer step size, but when we construct back the solution in the original coordinate space you need to take care of adding the modal contributions at appropriate times.

Bounds on ω_N

Let ω_N be the highest natural frequency of an assembled FE mesh with N_e elements.

Let K^e & M^e be the element matrices for $e = 1, 2, \dots, N_e$.

Consider the eigenvalue problems $K^e \phi^e = (\omega^e)^2 M^e \phi^e; e = 1, 2, \dots, N_e$.

Let ω_N^e be the largest frequency of element e .

It can be shown that $\omega_N \leq \max_{1 \leq e \leq N_e} \omega_N^e$.

Thus, for central difference method, $\Delta t = \frac{2}{\max_{1 \leq e \leq N_e} \omega_N^e} \leq \Delta t_{cr}$



Now if given a finite element model for a much, you know multi degree finite element model how does one estimate what is the highest natural frequency, typically in professional softwares you will not be able to compute the highest natural frequency, if you do an Eigenvalue analysis, suppose ω_N is the highest natural frequency of an assembled FE mesh with N_e elements, let K^e and M^e be the element matrices for $e = 1, 2, \dots, N_e$. Now what we do is we consider the Eigenvalue problem for the e th element, so $K^e \phi^e = \omega_e^2 M^e \phi^e$.

Now let ω_N^e be the largest frequency of element e , now we can show that the highest natural frequency of the model is actually the maximum of the highest natural frequency of the all the elements, thus for central difference method and a bound on Δt , the Δt critical can be obtained as shown here, so that means for each element if we were to do an Eigenvalue analysis and find the highest natural frequency for every element the maximum natural frequency of the assembled system will be less than or equal to the highest natural frequency of the elements has taken over all the elements, so how do we show that?

We have $\omega_N^2 = \frac{\phi_N^T K \phi_N}{\phi_N^T M \phi_N} = \frac{\phi_N^T \left(\sum_{e=1}^{N_e} K_e \right) \phi_N}{\phi_N^T \left(\sum_{e=1}^{N_e} M_e \right) \phi_N} = \frac{\sum_{e=1}^{N_e} \phi_N^T K_e \phi_N}{\sum_{e=1}^{N_e} \phi_N^T M_e \phi_N} = \frac{\sum_{e=1}^{N_e} U_e}{\sum_{e=1}^{N_e} I_e}$

For the e-th element, we have by property of Rayleigh's quotient

$$\frac{\phi_N^T K_e \phi_N}{\phi_N^T M_e \phi_N} = \frac{U_e}{I_e} \leq (\omega_N^e)^2 \Rightarrow U_e \leq (\omega_N^e)^2 I_e$$

\Rightarrow

$$\omega_N^2 = \frac{\sum_{e=1}^{N_e} U_e}{\sum_{e=1}^{N_e} I_e} \leq \frac{\sum_{e=1}^{N_e} (\omega_N^e)^2 I_e}{\sum_{e=1}^{N_e} I_e} \leq \max_{1 \leq e \leq N_e} (\omega_N^e)^2 \frac{\sum_{e=1}^{N_e} I_e}{\sum_{e=1}^{N_e} I_e} = \max_{1 \leq e \leq N_e} (\omega_N^e)^2 \Rightarrow \omega_N^2 \leq \max_{1 \leq e \leq N_e} (\omega_N^e)^2$$

Remarks

- Size of K_e & M_e are taken to be that of K and M
- In solving element level eigenvalue problem we have removed BCS and interactions with other elements,


5

So let's start with the Eigenvalue problem, so ω_N^2 that is the highest natural frequency is given by this, this is exact, there is no problem, now $\sum_{e=1}^{N_e} K_e$ I will write it as KE , as summed over E from 1 to NE . Now similarly for $\sum_{e=1}^{N_e} M_e$ I'll write this, so I can take now this ϕ_N^T transpose inside, I'll get $E = 1, 2, NE$, $\phi_N^T KE \phi_N$. Now let's call this $\phi_N^T KE \phi_N$ as UE , and similarly $\phi_N^T M_e \phi_N$ as IE . Now for the e th element we have by property of Rayleigh's quotient, $\phi_N^T KE \phi_N / \phi_N^T M_e \phi_N$ which is UE/IE will be less than or equal to the highest natural frequency of that element, so this would mean UE will be less than or equal to this quantity.

Now let's return to this expression for ω_N^2 and I can write this now in the denominator, since UE is less than or equal to this, for UE I will write this, so instead of an equality I'll have now the inequality.

Now if I take this term outside the summation it will be maximum of ω_N^e whole square from 1 to, V equal to 1, 2, NE that goes out, and I am left with this ratio which gets cancelled, which is equal to this basic, so from this I get the required relation that the square of the highest natural frequency of the assembled system is less than or equal to the square of the highest natural frequency of taken across all the elements. Now here you should understand that KE and ME the sizes of KE and ME are taken to be that of K and M , that means if you recall that $AS^T M_e AS$ is KE , it's because for example if structure has say 20 degrees of freedom and beam element if you are using it may have 4 degrees of freedom, but this KE will have 20 degrees of freedom, not 4 degrees of freedom, so many Eigenvalues initial, first few Eigenvalues will be 0, but the highest Eigenvalue there won't be any problem, I mean you'll get

the required result. In solving element level Eigenvalue problem we have removed the boundary conditions and interactions with other elements that also you should remember.

Axially vibrating rod	$K = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$	$M = \frac{ml}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$	$\Delta t \leq \frac{L}{\sqrt{3}} \sqrt{\frac{\rho}{E}}$
Axially vibrating rod	$K = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$	$M = \frac{ml}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\Delta t \leq L \sqrt{\frac{\rho}{E}}$
Euler-Bernoulli Beam			
$K = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$	$M = \frac{ml}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$		
	$\frac{A}{48I} L^2 \sqrt{\frac{\rho}{E}}$		

NPTEL

Now, so now we can now look at say for example axially vibrating rod, this is the stiffness matrix, and this is a consistent mass matrix, so delta T should be less than if you find the Eigenvalues of this 2/2 stiffness and mass matrix you will get delta T should be less than this, the square root E/rho is nothing but the speed of the wave, the wave velocity in the rod, so what it this requirement basically means is that the step size should not be less than the time taken by the wave to pass through the element, so if you are using lumped mass model instead of a consistent mass matrix model the delta T will be actually L/ C where C square root E/rho, this is a classical result available in mathematics literature attributed to that.

Now similarly for Euler Bernoulli beam with consistent mass matrix we can show that the delta T must be less than this, so for a given finite element model we can find out these critical step size for all the elements and pick the one which is smallest, so you need not really estimate the highest natural frequency this will provide a bound, okay.

Average acceleration, linear acceleration, and Newmark β methods


Average acceleration method

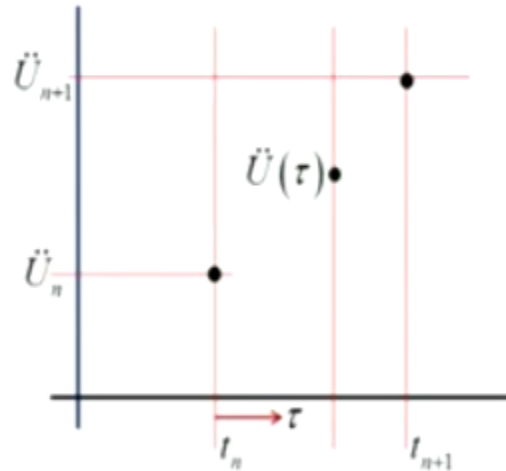
$$\ddot{U}(\tau) = \frac{1}{2}[\ddot{U}_{n+1} + \ddot{U}_n]$$

$$\dot{U}(\tau) = \dot{U}_n + \frac{\tau}{2}[\ddot{U}_{n+1} + \ddot{U}_n]$$

$$\Rightarrow \dot{U}_{n+1} = \dot{U}_n + \frac{\Delta t}{2}[\ddot{U}_{n+1} + \ddot{U}_n]$$

$$U(\tau) = U_n + \tau\dot{U}_n + \frac{\tau^2}{4}[\ddot{U}_{n+1} + \ddot{U}_n]$$

$$\Rightarrow U_{n+1} = U_n + \Delta t\dot{U}_n + \frac{\Delta t^2}{4}[\ddot{U}_{n+1} + \ddot{U}_n]$$




7

Now let's take up the problem of developing second order implicit schemes. So let's start with what are known as average acceleration, linear acceleration, and Newmark beta methods.

Now in this diagram X is the time axis, and Y is the acceleration axis, and I am considering 2 time instant T_N and T_{N+1} , so at T_N acceleration is U_N double dot, and at T_{N+1} , the acceleration is U_{N+1} double dot. The question is at a time instant τ lying between T_N and T_{N+1} , how do we approximate U double dot of τ , according to average acceleration method U double dot of τ is taken to be the average of these two values, so that is $1/2$ of U_{N+1} double dot + U_N double dot. Now to get velocity from this I integrate this, with respect to τ so it will be U_N dot initial velocity + integral of this is $\tau/2$ into this is a constant, it will remain as it is.

Now similar, from this it follows U dot of $N+1$ will be this, now if I now integrate this further to get displacement I have to integrate this now, so I will get $U_N + \tau U_N$ dot, I am integrating this and similarly I have to integrate this now $\tau^2/2$, that will be $\tau^2/4$ into this. Now for τ if you put you know you reach this point you will get this as $U_N + \Delta T U_N$ dot + $\Delta T^2/4$, and U double dot $N+1$ + U double dot N , so we have now the expression for the velocity and the displacement in which we are assuming acceleration at any point τ is average of these two, a slightly different version of this is instead of assuming that acceleration

Average acceleration, linear acceleration and Newmark's methods

Linear acceleration method

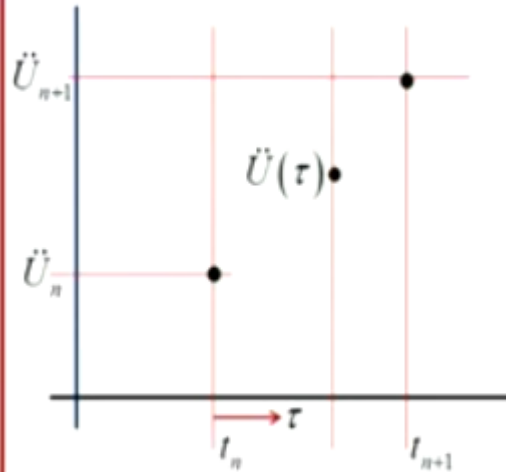
$$\ddot{U}(\tau) = \ddot{U}_n + \frac{\tau}{\Delta t} [\ddot{U}_{n+1} - \ddot{U}_n]$$

$$\dot{U}(\tau) = \dot{U}_n + \ddot{U}_n \tau + \frac{\tau^2}{2\Delta t} [\ddot{U}_{n+1} - \ddot{U}_n]$$

$$U_{n+1} = U_n + \dot{U}_n \Delta t + \frac{\Delta t}{2} [\ddot{U}_{n+1} - \ddot{U}_n]$$

$$U(\tau) = U_n + \dot{U}_n \tau + \ddot{U}_n \frac{\tau^2}{2} + \frac{\tau^3}{6\Delta t} [\ddot{U}_{n+1} - \ddot{U}_n]$$

$$U_{n+1} = U_n + \dot{U}_n \Delta t + \ddot{U}_n \frac{\Delta t^2}{2} + \frac{\Delta t^2}{6} [\ddot{U}_{n+1} - \ddot{U}_n]$$



here is average of these two, we could as well take that to be a linear, assume it is a linear variation, so in which case \ddot{U} of τ in terms of these two values I will fit a straight line and the equation for straight line is this, so this is known as linear acceleration approximation. Now from this again I can get velocity by integration, so the initial velocity \dot{U}_n dot, so if I integrate this I will get first term will lead to \dot{U}_1 double dot or τ , this is $\tau^2/2 \Delta t$ into this constant. From this if I now evaluate velocity at $N+1$ this will be the expression. And now you return to this and integrate with respect to τ I will get $U(\tau)$ so I get this expression. So again without if I put now, if I reach $N+1$, I will get this as the expression for displacement, okay.

The Newmark method

Consider $\dot{U}(t)$


$$\frac{\dot{U}(t + \Delta t) - \dot{U}(t)}{\Delta t} = \ddot{U}(t + \delta \Delta t) \text{ for some } \delta \in (0, 1)$$

$$\Rightarrow \dot{U}(t + \Delta t) = \dot{U}(t) + \Delta t \ddot{U}(t + \delta \Delta t)$$

Suppose we now approximate $\ddot{U}(t + \delta \Delta t)$ as a weighted sum

$$\ddot{U}(t + \delta \Delta t) = (1 - \delta) \ddot{U}_n + \delta \ddot{U}_{n+1}$$

$$\Rightarrow \dot{U}(t + \Delta t) = \dot{U}(t) + \Delta t [(1 - \delta) \ddot{U}_n + \delta \ddot{U}_{n+1}]$$



$$\dot{U}_{n+1} = \dot{U}_n + \Delta t [(1 - \delta) \ddot{U}_n + \delta \ddot{U}_{n+1}]$$

Now what Newmark beta method, Newmark method is also known as Newmark beta method because the parameter beta appears in its description, I will simply call it as Newmark method. Now if we consider $\dot{U}(t)$ we can write $\dot{U}(t + \Delta t) - \dot{U}(t) / \Delta t$, as $\ddot{U}(t + \delta \Delta t)$, for some δ belonging to $(0, 1)$, on this by mean value theorem, so from this I get $\dot{U}(t + \Delta t) = \dot{U}(t) + \Delta t \ddot{U}(t + \delta \Delta t)$, $\ddot{U}(t + \delta \Delta t)$ for some value of δ lying between 0 and 1. Now suppose we now approximate $\ddot{U}(t + \delta \Delta t)$ as a weighted sum, right as shown here, I will get now $\dot{U}(t + \Delta t)$ will be given by this, okay, so what I am doing is for this term I am writing an approximation, so δ is now an algorithmic parameter okay, which has to be selected. So now this means in a discrete version it is \dot{U}_{n+1} is given by this.

Now consider the Taylor's expansion

$$U(t + \Delta t) = U(t) + \Delta t \dot{U}(t) + \frac{\Delta t^2}{2} \ddot{U}(t + 2\alpha \Delta t) \text{ for some } \alpha \in (0, 0.5)$$

Suppose we now approximate $\ddot{U}(t + 2\alpha \Delta t)$ as a weighted sum

$$\ddot{U}(t + 2\alpha \Delta t) = (1 - 2\alpha) \ddot{U}_n + 2\alpha \ddot{U}_{n+1}$$

$$\Rightarrow U_{n+1} = U_n + \Delta t \dot{U}_n + \Delta t^2 \left[\left(\frac{1}{2} - \alpha \right) \ddot{U}_n + \alpha \ddot{U}_{n+1} \right]$$

Note: α & δ are treated as being independent

Now if we now consider we have to now make a model for displacement, now if we consider Taylor's expansion for displacement I get $U(t + \Delta t) = U(t) + \Delta t \dot{U}(t) + \frac{\Delta t^2}{2} \ddot{U}(t + 2\alpha \Delta t)$ for some α lying between 0 to 0.5, again by mean value theorem. Now this second term now I approximate as a weighted sum with weights 2α and $1 - 2\alpha$, as shown here. And now if I substitute back into this I will get a model for displacement, so here α is another algorithmic parameter which is taken to be independent of Δt , so in the Newmark's method what we have done is we can take a look at

<p>Average acceleration method</p> $\dot{U}_{n+1} = \dot{U}_n + \frac{\Delta t}{2} [\ddot{U}_{n+1} + \ddot{U}_n]$ $U_{n+1} = U_n + \Delta t \dot{U}_n + \frac{\Delta t^2}{4} [\ddot{U}_{n+1} + \ddot{U}_n]$ <p>Linear acceleration method</p> $\dot{U}_{n+1} = \dot{U}_n + \frac{\Delta t}{2} [\ddot{U}_{n+1} + \ddot{U}_n]$ $U_{n+1} = U_n + \dot{U}_n \Delta t + \frac{\Delta t^2}{6} [\ddot{U}_{n+1} + 2\ddot{U}_n]$ <p>Newmark method</p> $\dot{U}_{n+1} = \dot{U}_n + \Delta t [(1 - \delta) \ddot{U}_n + \delta \ddot{U}_{n+1}]$ $U_{n+1} = U_n + \dot{U}_n \Delta t + \Delta t^2 \left[\left(\frac{1}{2} - \alpha \right) \ddot{U}_n + \alpha \ddot{U}_{n+1} \right]$	<p>Newmark's method</p> $\alpha = \frac{1}{4} \text{ \& } \delta = \frac{1}{2}$ <p>\Rightarrow Average acceleration method</p> $\alpha = \frac{1}{6} \text{ \& } \delta = \frac{1}{2}$ <p>\Rightarrow Linear acceleration method</p>
	<p>Newmark's method</p> <ul style="list-style-type: none"> • Implicit • Self-starting • Single step

the 3 methods average acceleration method, linear acceleration method, and Newmark method, so in Newmark method what we have done is we have introduced 2 algorithmic parameters

delta and alpha, in addition to delta T which is a parameter that we already have. Now by adjusting this value of alpha and delta, for example for alpha = 1/4 and delta = 1/2 Newmark's method becomes average acceleration method.

Similarly, sorry, yes correct, then if alpha = 1/6 and delta = 1/2, Newmark's method become linear acceleration, so you can see these expressions and you can see that a Newmark's method is an implicit method because U dot N+1 will have U double dot of N+1 and if I write this in terms of the velocity it will involve U dot N+1 terms okay, so this is implicit and it is a self-starting scheme, it is a single step scheme.

Now we can study Newmark's method and reduce properties of the other 2 method we need not have to consider these 2 methods separately. So how do we proceed to implement Newmark's

Newmark's method

$$\dot{U}_{n+1} = \dot{U}_n + \Delta t \left[(1-\delta)\ddot{U}_n + \delta\ddot{U}_{n+1} \right] \dots (1)$$

$$U_{n+1} = U_n + \dot{U}_n \Delta t + \Delta t^2 \left[\left(\frac{1}{2} - \alpha \right) \ddot{U}_n + \alpha \ddot{U}_{n+1} \right] \dots (2)$$

$$M\ddot{U}_{n+1} + C\dot{U}_{n+1} + KU_{n+1} = F_{n+1} \dots (3)$$

$$(2) \Rightarrow \ddot{U}_{n+1} = \frac{1}{\alpha \Delta t^2} \left[U_{n+1} - U_n - \dot{U}_n \Delta t \right] - \frac{1}{\alpha} \left(\frac{1}{2} - \alpha \right) \ddot{U}_n \dots (4)$$

Substitute (4) in (1) to get

$$\dot{U}_{n+1} = \dot{U}_n + \Delta t \left[(1-\delta)\ddot{U}_n + \frac{\delta}{\alpha \Delta t^2} \left\{ U_{n+1} - U_n - \dot{U}_n \Delta t \right\} - \frac{\delta}{\alpha} \left(\frac{1}{2} - \alpha \right) \ddot{U}_n \right] \dots (5)$$



method, these 2 expressions we already got for U dot N+1 and UN+ 1, so now I write the equation of motion at T = N+1, so I get this, so I am considering basically linear systems in this discussion. So now I will solve for U double dot N+1 from equation 2, and I get this and what I get in equation 4 if I substitute in equation 1, I get this expression, so these are some simple manipulations you need to sit with pen and paper and work this through.

$$\dot{U}_{n+1} = \dot{U}_n + \Delta \left[(1-\delta)\ddot{U}_n + \frac{\delta}{\alpha\Delta^2} \{U_{n+1} - U_n - \dot{U}_n\Delta\} - \frac{\delta}{\alpha} \left(\frac{1}{2} - \alpha \right) \ddot{U}_n \right] \dots (5)$$

Note: (4) and (5) contain the unknown U_{n+1} on the RHS.

Substitute of (4) and (5) into (3) to get

$$\left[\frac{1}{\alpha\Delta^2} M + \frac{\delta}{\alpha\Delta} C + K \right] U_{n+1} = F_{n+1} + M (a_0 U_n + a_2 \dot{U}_n + a_3 \ddot{U}_n)$$

$$+ C (a_1 U_n + a_4 \dot{U}_n + a_5 \ddot{U}_n)$$

$$a_0 = \frac{1}{\alpha\Delta^2}; a_1 = \frac{\delta}{\alpha\Delta}; a_2 = \frac{1}{\alpha\Delta}; a_3 = \left(\frac{1}{2\alpha} - 1 \right); a_4 = \frac{\delta}{\alpha} - 1;$$

$$a_5 = \frac{\Delta}{\alpha} \left(\frac{\delta}{\alpha} - 2 \right); a_6 = \Delta(1-\delta); a_7 = \delta\Delta$$



Now if you look at equations 4 and 5 they contain U_{n+1} which is unknown, now if I substitute 4 into 5, 4 and 5 into 3, 3 is what, 3 is this equilibrium equation at $T = T_{n+1}$, so I have now \dot{U}_{n+1} , equation 4 is \ddot{U}_{n+1} and \dot{U}_{n+1} I have, so if I substitute that I will get an equation for U_{n+1} , so after rearranging the terms we get the equation for U_{n+1} as this, so this coefficient matrix consists of MCK matrices and the algorithmic parameters of the Newmark method, and then the right hand side we have A_1 , A_2 , A_3 and there may see A_4 , A_5 their meaning is displayed here, okay, right.

Newmark's method - I

$$(1) n = 0; \text{ Read } U_0 \text{ \& } \dot{U}_0; \ddot{U}_0 = M^{-1} [F_0 - C\dot{U}_0 - KU_0]$$

$$(2) U_{n+1} = \left[\frac{1}{\alpha \Delta t^2} M + \frac{\delta}{\alpha \Delta t} C + K \right]^{-1} \left\{ F_{n+1} + M (a_0 U_n + a_2 \dot{U}_n + a_3 \ddot{U}_n) + C (a_1 U_n + a_4 \dot{U}_n + a_5 \ddot{U}_n) \right\}$$

$$(3) \ddot{U}_{n+1} = \frac{1}{\alpha \Delta t^2} \{ U_{n+1} - U_n - \dot{U}_n \Delta t \} - \frac{1}{\alpha} \left(\frac{1}{2} - \alpha \right) \ddot{U}_n$$

$$(4) \dot{U}_{n+1} = \dot{U}_n + \Delta t \left[(1 - \delta) \ddot{U}_n + \delta \ddot{U}_{n+1} \right]$$

(5) $n \rightarrow n+1$; if $n\Delta t > t_f$, go to (2); otherwise, end.



NPTEL

14

So now how do we implement this? We can look at this expression and a simple-minded implementation would be, start with $N = 0$ read the initial conditions and derive the initial acceleration from the equation of motion, then use the expression for displacement which is given by this, which is basically this equation and then you compute U double dot $N+1$ and U dot $N+1$ moment U_{N+1} is known these terms come on the right hand side here, so you are able to get displacement, acceleration and velocity. Then increase N to $N+1$ stop if you across the final time limit otherwise go back to 2. Now this can be refined a bit because if Δt is constant, I need not have to invert this matrix and multiply this matrix with M , C etcetera at every time step, so that can be done offline, outside this loop so a more efficient implementation would be, you start with $N = 0$ read the initial conditions, compute the initial acceleration. Now compute these matrices A , right then B is A into M , D is A into C , so U_{N+1} will be now $A F_{N+1}$, B into this, + D into this, so what I am basically doing here is that I am avoiding this inversion at every time step and this matrix multiplication at every time step, so next I get acceleration and then velocity, I increment time and if we reach the final time instant I will exit, otherwise I go here. So 3, 4, 5 are the steps that will be repeatedly implemented as we advance in time, and step 2 is done outside that loop, so this is a more efficient implementation.

Stability of Newmark's method

$$\text{Consider } \ddot{x} + 2\eta\omega\dot{x} + \omega^2x = 0$$

$$\Rightarrow \ddot{x}_{n-1} + 2\eta\omega\dot{x}_{n-1} + \omega^2x_{n-1} = 0$$

$$\dot{x}_{n-1} = \dot{x}_n + \Delta \left[(1-\delta)\ddot{x}_n + \delta\ddot{x}_{n-1} \right]$$

$$\Rightarrow \dot{x}_{n-1} = \dot{x}_n + \Delta \left[(1-\delta)(-2\eta\omega\dot{x}_n - \omega^2x_n) + \delta(-2\eta\omega\dot{x}_{n-1} - \omega^2x_{n-1}) \right]$$

$$\Rightarrow (1 + 2\eta\omega\delta\Delta)\dot{x}_{n-1} + (\delta\omega^2\Delta)x_{n-1} = (1 - 2\eta\omega\Delta(1-\delta))\dot{x}_n - \Delta(1-\delta)\omega^2x_n$$

$$x_{n-1} = x_n + \dot{x}_n\Delta + \Delta^2 \left[\left(\frac{1}{2} - \alpha \right) \ddot{x}_n + \alpha\ddot{x}_{n-1} \right]$$

$$\Rightarrow x_{n-1} = x_n + \dot{x}_n\Delta + \Delta^2 \left[\left(\frac{1}{2} - \alpha \right) (-2\eta\omega\dot{x}_n - \omega^2x_n) + \alpha(-2\eta\omega\dot{x}_{n-1} - \omega^2x_{n-1}) \right]$$



Now one of the concerns that we have about integration schemes is the questions about the growth of errors, so for studying that as we have been doing we consider a test system which is basically an undamped, sorry damped single degree freedom system, so let's consider $\ddot{x} + 2\eta\omega\dot{x} + \omega^2x = 0$. Now let's implement the Newmark method on this system, so acceleration is given by this, velocity is given by this, and we can manipulate this, this is the equation for displacement we can do a bit of manipulation, and we can recast

$$\Rightarrow (2\eta\omega\alpha\Delta t^2)\dot{x}_{n+1} + (1 + \alpha\omega^2\Delta t^2)x_{n+1}$$

$$= \left(\Delta t - 2\eta\omega\Delta t^2 \left(\frac{1}{2} - \alpha \right) \right) \dot{x}_n + \left(1 - \omega^2\Delta t^2 \left(\frac{1}{2} - \alpha \right) \right) x_n$$

$$\begin{bmatrix} 1 + 2\eta\omega\delta\Delta t & \delta\omega^2\Delta t \\ 2\eta\omega\alpha\Delta t^2 & 1 + \alpha\omega^2\Delta t^2 \end{bmatrix} \begin{Bmatrix} \dot{x}_{n+1} \\ x_{n+1} \end{Bmatrix} =$$

$$\begin{bmatrix} 1 - 2\eta\omega\Delta t(1 - \delta) & -\Delta t(1 - \delta)\omega^2 \\ \Delta t - 2\eta\omega\Delta t^2 \left(\frac{1}{2} - \alpha \right) & 1 - \omega^2\Delta t^2 \left(\frac{1}{2} - \alpha \right) \end{bmatrix} \begin{Bmatrix} \dot{x}_n \\ x_n \end{Bmatrix}$$

$$\Rightarrow AX_{n+1} = BX_n$$

Characteristic equation $|B - \lambda A| = 0$

17

this equation in a matrix form if I define a velocity and displacement of the state vector, the Newmark's method leads to a matrix equation of this form, there is a coefficient matrix here let's call it as A, A into suppose the state is called as capital XN, this will be A into X dot XN+1, and B is this matrix which is on the right hand side, B into XN, this is XN, okay, so this is how the solution advances.

Now the amplification matrix is A inverse B, now we have to find Eigenvalues of A inverse B and look at the spectral radius and see whether that is less than 1, condition for stability is the spectral radius must be less than 1, we can as well consider the characteristic equation B - lambda A = 0, so this is a 2/2 matrix this characteristic equation will be a quadratic in lambda, so i can write with lambda is Z, we can use the Z transform method and I can write the

Jury's criteria

$$P(z) = a_0 z^2 + a_1 z + a_2$$

$$a_0 = 1 + 2\eta\delta\omega\Delta t + \alpha\omega^2\Delta t^2$$

$$a_1 = -2 + 2\eta\omega\Delta t(1 - 2\delta) + \omega^2\Delta t^2$$

$$a_2 = 1 + 2\eta\omega\Delta t(\delta - 1) + \Delta t^2\omega^2(0.5 + \alpha - \delta)$$

$$P(1) = \omega^2\Delta t^2$$

$$(-1)^2 P(-1) = 4 + \omega^2\Delta t^2(4\alpha - 2\delta) + 4\eta\omega\Delta t(2\delta - 1)$$



polynomial equation as $A_0 Z^2 + A_1 Z + A_2$, now by expanding this determinant I can identify these terms A_0 , A_1 , A_2 and this is the polynomial.

Now according to the Jury's criterion what we should do is we have to find $P(1)$, so $P(1)$ will be $\omega^2 \Delta t^2$ then -1 to the power of N $P(-1)$ is given by this, so the necessary

Necessary conditions

$$P(1) > 0 \Rightarrow P(1) = \omega^2 \Delta T^2 > 0 \text{ [OK]}$$

$$(-1)^2 P(-1) > 0 \text{ is satisfied if } 2\delta - 1 > 0 \text{ \& } (4\alpha - 2\delta) > 0$$

$$\Rightarrow \delta > \frac{1}{2} \text{ \& } \alpha > \frac{\delta}{2}$$

Sufficient conditions: Number of conditions=2-1=1

$$a_2 > a_0 \Rightarrow 2\eta\omega\Delta T + \omega^2 \Delta T^2 \left(\delta - \frac{1}{2} \right) > 0.$$



This is satisfied if $\delta > \frac{1}{2}$.

condition for stability is $P(1)$ must be greater than 0, that means all Eigenvalues should lie within unit circle in the complex plane, so $P(1)$ is $\omega^2 \Delta T^2$ which must be greater than 0, so this is satisfied, because ω is positive, ΔT is positive, $\omega^2 \Delta T^2$ is positive. Now the second condition if you look at this is the second condition, now this will be satisfied if $2\delta - 1 > 0$, and $4\alpha - 2\delta > 0$, so that leads to the condition $\delta > \frac{1}{2}$, and $\alpha > \frac{\delta}{2}$, okay.

Now sufficient conditions this is second order polynomial therefore there will be one sufficient condition that we need to check that is $A_2 > A_0$, so for that to happen this is the equation that we get. Now this again we can see that if $\delta > \frac{1}{2}$, this will be satisfied because all these quantities are positive, so we get the now the conclusion that the

Remarks

- The method is unconditionally stable if $\delta > 0.5$ & $\alpha > 0.5\delta$

These requirements are independent of system natural frequency and damping.

- Otherwise the following conditions need to be satisfied

$$4 + \omega^2 \Delta t^2 (4\alpha - 2\delta) + 4\eta \omega \Delta t (2\delta - 1) > 0$$

$$2\eta \omega \Delta t + \omega^2 \Delta t^2 (\delta - 0.5) > 0 \Rightarrow \omega \Delta t < \frac{2\eta}{(\delta - 0.5)}$$



requirements depend upon system natural frequency and damping.

20

method will be unconditionally stable if delta is greater than 1/2, and alpha is greater than delta/2.

Now these are algorithmic parameters alpha and delta, they have nothing to do a system natural frequency and damping, therefore these requirements if they are met by choosing the algorithmic parameter so that these conditions are met then the method becomes unconditionally stable for all damping and natural frequency of the system, otherwise this is not the only requirement we can derive the conditions for stability as given by this, so this is a stronger requirement we can relax this and still be able to get stable algorithm provided these conditions are met these are conditional in which case the step size is function of, critical step size is function of natural frequency and damping, and the method becomes conditionally stable.

• Average acceleration method ($\alpha=0.25$ & $\delta=0.5$)

is unconditionally stable

• Linear acceleration method ($\alpha=\frac{1}{6}$ & $\delta=\frac{1}{2}$) is

conditionally stable

• For $\delta>0.5$ & $\alpha>0.5\delta$ the method displays high frequency dissipation characteristics. That is, the spectral radius drops below value of 1 for larger ω (keeping Δt fixed).

• For $\delta=0.5$, the high frequency dissipation characteristics is not witnessed.

• If $\eta=0$, the above conditions read as



$$1 + \omega^2 \Delta t^2 (4\alpha - 2\delta) > 0 \text{ \& \ } \omega^2 \Delta t^2 (\delta - 0.5) > 0$$

$$\Rightarrow \delta > 0.5 \text{ \& \ } \alpha > 0.5\delta$$

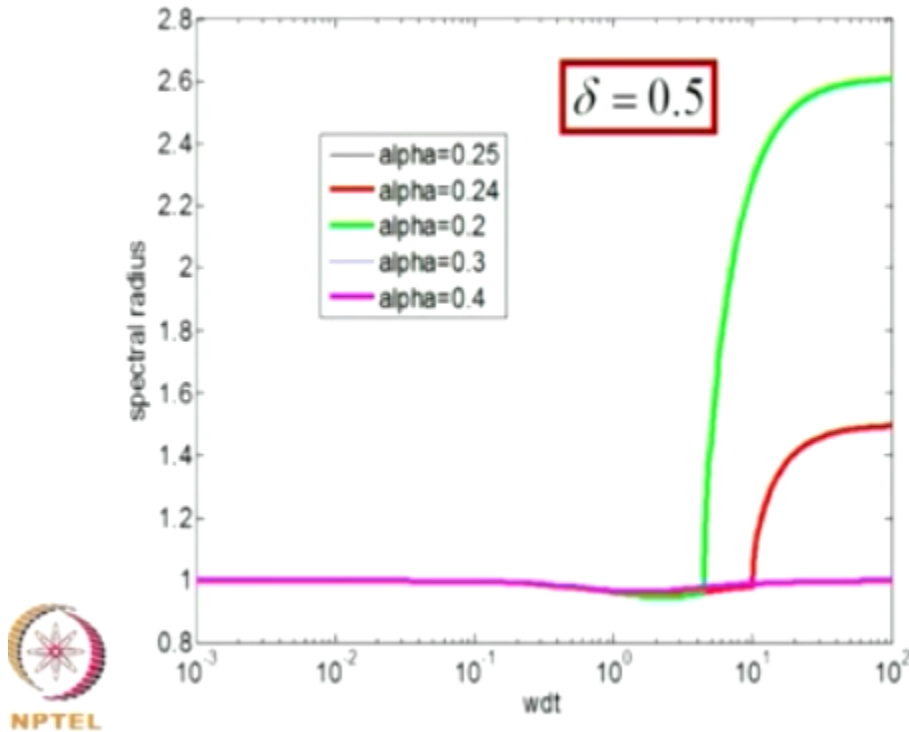
• For $\delta \neq 0.5$, the accuracy of the method drops to $O(\Delta t)$

21

Now we have already seen that the Newmark method for $\alpha = 0.25$, and $\delta = 0.5$, is average acceleration method and we can verify that this is unconditionally stable, then linear acceleration method becomes conditionally stable, for δ greater than $1/2$ and α greater than $\delta/2$ the method actually displays what is known as high frequency dissipation characteristics, that is the spectral radius drops below value of 1 for larger ω if we keep δT constant that means you look at the amplitude of the highest Eigenvalue as a function of ω by keeping δT fixed, so the spectral radius for higher ω drops below 1, so that means if you are using this scheme for direct integration for a fixed value of δT the contribution from higher normal modes which are spurious actually will be dissipated out, whereas the lower modes which are important for calculating the response correctly hopefully are not you know dissipated, so this is one of the requirements of a integration scheme that the higher order modes should be numerically dissipated especially if system response is governed by contributions from lower modes.

Now if $\delta = 0.5$, the high frequency dissipation characteristics is not witnessed, you can see that the spectral radius becomes 1 if δ is 0.5 and we don't get any high-frequency dissipation characteristics, therefore this is not a very desirable feature. Now if damping is 0, we can work through the requirements on step size and we get this as a requirement. Now one important observation which we need to make, we can make at this stage and later on substantiate is we have seen that for δ not equal to 0.5 there will be high frequency dissipation possible, but on the other hand if δ is not equal to 0.5, the order of accuracy becomes order δT , it won't be order δT square as we wished it would be, it will be order

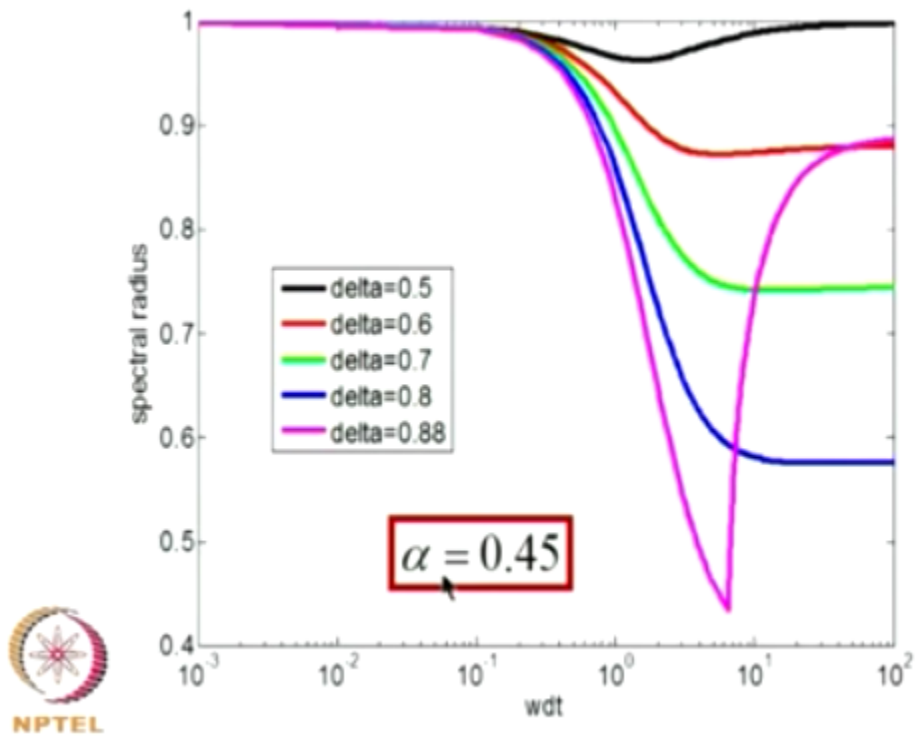
delta T, so there is a trade-off between achieving second order accuracy, and achieving numerical dissipation for higher modes, so Newmark beta method cannot achieve both, you have to sacrifice one of them, so I have plotted a few results for spectral radius on X axis we



22

have omega into delta T and here I have the spectral radius so ideally the spectral radius should be less than 1 so I have selected delta = 0.5 and I have varied alpha, alpha should be less than 0.25 for stability and we can see here so for example the green line alpha is 0.2, alpha should be greater than 0.25 for stability, if alpha 0.2 this becomes unstable, and similarly alpha is 0.4 which is okay, the spectral radius is less than 1. Red 1 alpha is 0.24, it crosses this, so the higher modes become unstable, so the errors would grow, okay, so that is the problem.

So, on the other hand 0.25 actually that color scheme shown here that graph is not clearly seen, it will be 1. Now here what I have done is I've fixed alpha and varied delta, so this is a spectral



radius on X axis it is $\omega \delta T$, suppose I fix δT and vary ω , so for low frequency up to about say 10^0 , that is 1 hertz, the spectral radius is constant, as you go higher in the frequency the spectral radius drops and the contribution from higher modes get numerically dissipated, so these are desirable feature.

Error estimates and convergence of forward Euler's method

Reference: S D Conte and C de Boor, Elementary numerical analysis, 2005, Tata McGraw-Hill, New Delhi

Consider

$$\dot{x}(t) = a[x(t), t]; x(t_0) = x_0; x(t) \text{ is } 2N \times 1.$$

Discrete time approximation: let us take $t_n = n\Delta t + t_0$

According to Euler's method

$$\frac{y_{n+1} - y_n}{\Delta t} = a[y_n, t_n], n = 0, 1, 2, 3, \dots \quad (1)$$

Let $x(t_n)$ = true solution of $\dot{x}(t) = a[x(t), t]; x(t_0) = x_0$ at $t = t_n$.

Discretization error $e_n = x(t_n) - y_n$

Since we expect to specify y_0 exactly, $e_0 = 0$.

24

Now we talked about order of convergence, the order of approximation and we can work through some details for a couple of schemes I will illustrate this with convergence of the Euler's method, and the reference for this is a book on numerical analysis by Conte and De Boor which I have mentioned here, it is instructive to go through this, this is a relatively a simple derivation we can understand quite a bit by understanding what this mean, what this tells us. So let's consider a differential equation $X \dot{(t)} = A(x(t), t)$ with initial condition X naught, so $X(t)$ is $2N$ cross 1 , so it is a vector differential equation.

Now discrete time approximation let us take T_N to be $N \Delta T + T$ naught, according to Euler's method we approximate $Y_{N+1} - Y_N / \Delta T$, as $A(Y_N, T_N)$ for $N = 0, 1, 2, 3$, etcetera, where see the notation we have used is $X(T_N)$ is a true solution of this equation at $T = T_N$, whereas Y_N is the solution obtained through Euler's method at, suppose Y_N is approximate solution to this equation at $T = T_N$, okay therefore the error would be $X(T_N) - Y_N$, that is a discretization error, since we expect to specify Y naught exactly that is initial condition, there is no truncation there, E naught would be 0 . Now the objective of this discussion is can we derive an equation for growth of E_N , right that is the first question, second question is can we find a bound on E_N by solving that equation for evolution of E_N .

$$x(t_{n+1}) = x(t_n + \Delta t) = x(t_n) + \Delta t \dot{x}(t_n) + \frac{\Delta t^2}{2} \ddot{x}(\xi_n) \text{ with } t_n \leq \xi_n \leq t_{n+1} \quad (2)$$

$$\frac{\Delta t^2}{2} \ddot{x}(\xi_n) = \text{Local discretization error}$$

= error committed in the single step from t_n to t_{n+1} assuming that $x(t)$ & $\dot{x}(t)$ are known exactly at $t = t_n$.

Assumption: neglect round off errors due to finite precision calculations.

$$\frac{y_{n+1} - y_n}{\Delta t} = a[y_n, t_n], n = 0, 1, 2, 3, \dots \quad (1)$$

From 1 and 2 we get

$$x(t_{n+1}) - y_{n+1} = \{x(t_n) - y_n\} + \Delta t \{\dot{x}(t_n) - a[y_n, t_n]\} + \frac{\Delta t^2}{2} \ddot{x}(\xi_n)$$

$$e_{n+1} = e_n + \Delta t \{a[x(t_n), t_n] - a[y_n, t_n]\} + \frac{\Delta t^2}{2} \ddot{x}(\xi_n)$$



NPTEL

25

Let's consider $X(TN+1)$ which is $X(TN + \Delta T)$ now Taylor's expansion tells us it is $X(TN) + \Delta T \dot{X}(TN) + \frac{\Delta T^2}{2} \ddot{X}(XIN)$, where XIN lies between TN to $TN+1$, this is called $\Delta T^2 \ddot{X}$, XIN is called the local discretization error, because this is where you will truncate the Taylor series so this is a discretization error, that is it is error committed in the single step from TN to $TN+1$ assuming that $X(t)$ and $\dot{X}(t)$ are known exactly at $T = TN$. Now we are assuming that in this discussion the round off errors that occurs due to finite precision calculation on computer is ignored, you may use double precision or single precision so there will be round off error at every step, I am not talking about that I am only talking about the errors that result from truncating the Taylor's expansion at a given term.

Now therefore now let's consider the Euler approximation, this is the Euler approximation, now from 1 and 2, if I now subtract if I now write an equation for $YN+1$ and subtract from this, I will get $X(TN+1) - YN+1$ would be $X(TN) - YN +$ this will be $\Delta T A$ and there will be this term, so I will get ΔT into $\dot{X}(TN) - A$ of this, plus this term, okay, fine. Now this first term is nothing but e_{n+1} , the left hand side this is e_n , what inside, what multiplies ΔT is \dot{X} which is nothing but $A(X(tn), tn) - A(y_n, tn) +$ this term, so this is equation we have got,

$$e_{n-1} = e_n + \Delta t \left\{ a[x(t_n), t_n] - a[y_n, t_n] \right\} + \frac{\Delta t^2}{2} \ddot{x}(\xi_n)$$

$$\text{Consider } a[x(t_n), t_n] = a[y_n + x(t_n) - y_n, t_n]$$

$$= a[y_n, t_n] + \{x(t_n) - y_n\} a'[\bar{x}_n, t_n] \text{ for some } \bar{x}_n \text{ between } x(t_n) \text{ \& } y_n$$

$$\Rightarrow a[x(t_n), t_n] - a[y_n, t_n] = a'[\bar{x}_n, t_n] e_n$$

$$\Rightarrow e_{n-1} = e_n + \Delta t a'[\bar{x}_n, t_n] e_n + \frac{\Delta t^2}{2} \ddot{x}(\xi_n) \text{ with } e_0 = 0 \quad (3)$$

$$\text{Let } |a'[\bar{x}_n, t_n]| < L \text{ \& } |\ddot{x}(\xi_n)| < Y \text{ with } L, Y > 0.$$

$$(3) \Rightarrow |e_{n-1}| \leq |e_n| + \Delta t L |e_n| + \frac{\Delta t^2}{2} Y$$

$$\Rightarrow |e_{n-1}| \leq |e_n| (1 + \Delta t L) + \frac{\Delta t^2}{2} Y$$



now let us consider this $A(X(t_n), t_n)$ and this I can write it as $Y_{n+1} - X(t_n) - Y_n$, okay, so then I will do a Taylor's expansion on this, I will get $A(y_n, t_n) + X(t_n) - Y_n$ into A' of this, for some \bar{X}_n between $X(t_n)$ and Y_n , again I'm using mean value theorem here, okay, so I can thus write the difference $A(X(t_n), t_n) - A(y_n, t_n)$ can be written in this form, so consequently the equation for evolution of error was given by e_{n+1} is $e_n + \Delta t A'$ into this, into e_n plus this, with $e_0 = 0$.

Now if we assume that our A is such that I can place an upper bound on modulus of A' as L , and the second derivative as capital Y , I can write that this error would now be less than or equal to, I am writing absolute values into this, where this L and capital Y are bounds on this, A' and X double dot term present here, the idea here is not so much to evaluate L and Y , but to understand how the errors behave, okay, so now the equation for the growth of error is an inequality that I get like this.

$$|e_{n+1}| \leq |e_n|(1 + \Delta t L) + \frac{\Delta t^2}{2} Y \quad (4)$$

Consider the difference equation

$$\xi_{n+1} = \xi_n(1 + \Delta t L) + \frac{\Delta t^2}{2} Y \text{ with } \xi_0 = 0 \quad (5)$$

At $n = 0$, $\xi_0 = 0$ & $e_0 = 0 \Rightarrow \xi_n \geq |e_n|$ for $n=0$.

Let $\xi_n \geq |e_n|$ be true for some $n > 0$.

$$|e_{n+1}| \leq \xi_n(1 + \Delta t L) + \frac{\Delta t^2}{2} Y$$

$$\Rightarrow |e_{n+1}| \leq \xi_{n+1}$$



Thus to obtain a bound on the error, we need to solve the difference equation (5).



Now let us consider the difference equation where this inequality is replaced by equality I get X_{N+1} is $X_N(1 + \Delta t L) + \frac{\Delta t^2}{2} Y$ with $X_0 = 0$, let's consider this difference equation. Now at $N = 0$, $X_0 = 0$, therefore E_0 is already 0, therefore at least at $N = 0$, this X_N is greater than or equal to modulus E_N , in fact equality holds good here. Now let's assume that this is true for some N greater than 0, I am using induction here so then you can see that if this is true I can write for E_N , I will write this and consequently I get this, so it follows that this error is bound by this X_{N+1} , now this is a difference which can be solved, okay, so thus to obtain a bound on the error we need to solve the difference equation which is equation 5, how do I solve that? You consider the difference

$$\xi_{n+1} = \xi_n (1 + \Delta L) + \frac{\Delta t^2}{2} Y \text{ with } \xi_0 = 0$$

Consider $\xi_{n+1} = \xi_n (1 + \Delta L)$. Let $\xi_n = c \beta^n \Rightarrow c \beta^{n+1} = c \beta^n (1 + \Delta L) \Rightarrow \beta = (1 + \Delta L)$

Try $\xi_n = c(1 + \Delta L)^n + B$

$$c(1 + \Delta L)^{n+1} + B = \left\{ c(1 + \Delta L)^n + B \right\} (1 + \Delta L) + \frac{\Delta t^2}{2} Y \Rightarrow B = -\frac{\Delta t}{2L} Y$$

$$\Rightarrow \xi_n = c(1 + \Delta L)^n - \frac{\Delta t}{2L} Y$$

$$\xi_0 = 0 \Rightarrow c = \frac{\Delta t}{2L} Y \Rightarrow \xi_n = \frac{\Delta t}{2L} Y \left\{ (1 + \Delta L)^n - 1 \right\}$$

Recall: $\exp(x) = 1 + x + \frac{x^2}{2} \exp(\zeta) \Rightarrow \exp(x) > 1 + x \forall x$

$$\Rightarrow 1 + \Delta L < \exp(\Delta L) \Rightarrow (1 + \Delta L)^n < \exp(n\Delta L)$$

$$\frac{\Delta t}{2L} Y \left\{ \exp(n\Delta L) - 1 \right\} = \frac{\Delta t}{2L} Y \left\{ \exp(t_n L) - 1 \right\}$$

$$\Rightarrow |e_n| \leq \frac{\Delta t}{2L} Y \left\{ \exp(t_n L) - 1 \right\}$$



equation, first you consider, you remove this constant term if you consider only $XIN + 1$ is XIN into some constant we can assume solution to be in the form of $C \beta$ to the power of N , and from this I get $C \beta^{N+1}$ is $C \beta^N (1 + \Delta L)$, therefore β is this.

Now what I will do is for this equation I will assume solution in this form XIN is C into $1 + \Delta L$ to the power of $N + B$, for some B , now if I substitute this back into this and find the B which satisfy that equation then I am done, indeed it is possible so I will write for $XIN + 1$ this term with $N + 1$ here, and on the left hand side XIN , I will write this I get this equation. Now after simplification I will get that this equation will be satisfied if B is given by this, so consequently I get XIN is C into $1 + \Delta L$ to the power of N minus this, now C is a constant which is yet to be determined and since we know ξ_0 is 0 by substituting this here I get C also, so the final solution would be given by this, okay.

Now you recall e^x can be written as $1 + x + \frac{x^2}{2} \exp(\zeta)$ into some exponential of ζ , where ζ lies between, ζ is the, you know again mean value theorem is applied here, 0 to infinity, or range of x , therefore from this I can conclude that exponential of x is greater than $1 + x$ for all x , so therefore for this $1 + \Delta L$, I can write it as $1 + \Delta L$ is less than $\exp(\Delta L)$, and therefore n th power of that will be less than exponential and ΔL , so using these facts I will be able to get a bound of XIN which is given by this, consequently I get a bound on the way the error grows, okay, so this is one instance where this type of analysis is possible, we are getting, this is the error and we see that this is the global error is off order

Remarks

(1) The global error in Euler's method is $O(\Delta t)$, that is, the error $\rightarrow 0$, as $\Delta t \rightarrow 0$, like $c\Delta t$ for some constant c if $t = t_n$ is held fixed. It should be noted that the local error is $O(\Delta t^2)$.

(2) $|e_n| \leq \frac{\Delta t}{2L} Y \{ \exp(t_n L) - 1 \}$ is an upper bound; it may not be realistic.

Its value lies in providing an idea about the nature of the error.



delta T here, that is the error tends to 0 is delta T tends to 0, like C delta T for some constants C if T = TN is held fixed .

In the expansion however if you look at the local error is order delta T square before integrating, before finding EN so this is a typical situation where the local error is, if it is order delta T square, global error will be of the order delta T. Now this function this inequality

Remarks

(1) The global error in Euler's method is $O(\Delta t)$, that is, the error $\rightarrow 0$, as $\Delta t \rightarrow 0$, like $c\Delta t$ for some constant c if $t = t_n$ is held fixed. It should be noted that the local error is $O(\Delta t^2)$.

(2) $|e_n| \leq \frac{\Delta t}{2L} Y \{ \exp(t_n L) - 1 \}$ is an upper bound; it may not be realistic.

Its value lies in providing an idea about the nature of the error.



NPTEL

provides an upper bound, it may not be a realistic upper bound, the objective is never to bound this error and use it as a prior estimate of error or things like that, the objective here is to understand how the error grows and what is the nature of the error, so the main conclusion from this is contained here that the global error is of order delta T.

Order of accuracy in Newmark's method

Exact solution: $U(t), \dot{U}(t), \ddot{U}(t), \dots$

Newmark's approximation: $U_n, \dot{U}_n, \ddot{U}_n$

$$\dot{U}_{n+1} = \dot{U}_n + \Delta t \left[(1-\delta)\ddot{U}_n + \delta\ddot{U}_{n+1} \right]$$

Consider $\ddot{U}(t+\Delta t) = \ddot{U}(t) + \Delta t \ddot{\ddot{U}}(\zeta) ; t < \zeta < t + \Delta t$

$$\Rightarrow \dot{U}_{n+1} = \dot{U}_n + \Delta t \left[(1-\delta)\ddot{U}_n + \delta\ddot{U}_n + \delta\Delta t \ddot{\ddot{U}}(\zeta) \right] ; t < \zeta < t + \Delta t \quad (1)$$

$$\text{We have } \dot{U}(t+\Delta t) = \dot{U}(t) + \Delta t \ddot{U}(t) + \frac{\Delta t^2}{2} \ddot{\ddot{U}}(\zeta); t < \zeta < t + \Delta t \quad (2)$$

(2)-(1) \Rightarrow

$$\left\{ \dot{U}(t+\Delta t) - \dot{U}_{n+1} \right\} = \left\{ \dot{U}(t) - \dot{U}_n \right\} + \Delta t \left\{ \ddot{U}(t) - \ddot{U}_n \right\} + \Delta t^2 \ddot{\ddot{U}}(\zeta) \left(\frac{1}{2} - \delta \right)$$



$$e_n = \dot{e}_n + \Delta t \ddot{e}_n + \Delta t^2 \ddot{\ddot{e}}(\zeta) \left(\frac{1}{2} - \delta \right)$$

Now how do we study make a similar study for Newmark's method, now let's consider the exact solution to a governing equation say $M\ddot{U} + U\dot{} + KU = F(t)$, they let the exact solution be $U(t)$, $\dot{U}(t)$ and $\ddot{U}(t)$, displacement, velocity, acceleration, let the Newmark's approximation to these quantities be respectively U_N , \dot{U}_N , \ddot{U}_N , now according to the Newmark's approximation \dot{U}_{N+1} is given by this, now let us consider \dot{U}_{N+1} is $\dot{U}_N + \Delta t \left[(1-\delta)\ddot{U}_N + \delta\ddot{U}_{N+1} \right]$, that's what I need here, let us see what we get, this is $\dot{U}_{N+1} = \dot{U}_N + \Delta t \left[(1-\delta)\ddot{U}_N + \delta\ddot{U}_N + \delta\Delta t \ddot{\ddot{U}}(\zeta) \right]$, again use of mean value theorem. So now you substitute that result into this, I will get $\dot{U}_{N+1} = \dot{U}_N + \Delta t \left[(1-\delta)\ddot{U}_N + \delta\ddot{U}_N + \delta\Delta t \ddot{\ddot{U}}(\zeta) \right]$, that will be $\dot{U}_{N+1} = \dot{U}_N + \Delta t \left[(1-\delta)\ddot{U}_N + \delta\ddot{U}_N + \delta\Delta t \ddot{\ddot{U}}(\zeta) \right]$. Similarly we have for \dot{U}_{N+1} is $\dot{U}_N + \Delta t \left[(1-\delta)\ddot{U}_N + \delta\ddot{U}_N + \delta\Delta t \ddot{\ddot{U}}(\zeta) \right]$, that is $\dot{U}_{N+1} = \dot{U}_N + \Delta t \left[(1-\delta)\ddot{U}_N + \delta\ddot{U}_N + \delta\Delta t \ddot{\ddot{U}}(\zeta) \right]$. Now if I subtract these 2, I will get now the error term this is e_{N+1} , e_{N+1} , that is $\dot{U}_{N+1} - \dot{U}_{N+1}$, this is exact solution, this is the Newmark beta solution, Newmark solution, this is again error and this is the you know terms that helps us to understand the error, so the equation I get is $e_{N+1} = \dot{e}_N + \Delta t \ddot{e}_N + \Delta t^2 \ddot{\ddot{e}}(\zeta) \left(\frac{1}{2} - \delta \right)$.

Similarly

$$U_{n-1} = U_n + \Delta t \dot{U}_n + \Delta t^2 \left[\left(\frac{1}{2} - \alpha \right) \ddot{U}_n + \alpha \ddot{U}_{n-1} \right]$$

$$U_{n-1} = U_n + \Delta t \dot{U}_n + \Delta t^2 \left[\left(\frac{1}{2} - \alpha \right) \ddot{U}_n + \alpha \{ \ddot{U}_n + \Delta t \ddot{U}(\zeta) \} \right]; t < \zeta < t + \Delta t$$

\Rightarrow

$$U(t + \Delta t) = U(t) + \Delta t \dot{U}(t) + \frac{\Delta t^2}{2} \ddot{U}(t) + \frac{\Delta t^3}{6} \ddot{U}(\zeta); t < \zeta < t + \Delta t$$

$$U(t + \Delta t) - U_{n-1} = \{ U(t) - U_n \} + \Delta t \{ \dot{U}(t) - \dot{U}_n \} + \frac{\Delta t^2}{2} \{ \ddot{U}(t) - \ddot{U}_n \} \\ + \Delta t^3 \left(\frac{1}{6} - \alpha \right) \ddot{U}(\zeta); t < \zeta < t + \Delta t$$



Now similarly I can do the analysis for U_{n+1} you can go through this and we get for U that a displacement $U(t + \Delta t) - U_{n+1}$ is given by this, so in terms of error notation E , I get from

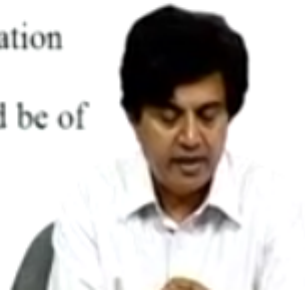
$$\dot{e}_{n+1} = \dot{e}_n + \Delta t \ddot{e}_n + \Delta t^2 \left(\frac{1}{2} - \delta \right) \ddot{U}(\zeta)$$

$$e_{n+1} = e_n + \Delta t \dot{e}_n + \frac{\Delta t^2}{2} \ddot{e}_n + \Delta t^3 \left(\frac{1}{6} - \alpha \right) \ddot{U}(\zeta); t < \zeta < t + \Delta t$$

$$\left. \begin{aligned} \delta - \frac{1}{2} = 0 &\Rightarrow \text{velocity is } O(\Delta t^3) \text{ (local)} \\ \alpha - \frac{1}{6} = 0 &\Rightarrow \text{displacement is } O(\Delta t^4) \text{ (local)} \end{aligned} \right\} \text{Linear acceleration method}$$

Summary

If $\delta > \frac{1}{2}$, the method displays high frequency dissipation characteristics (desirable) but the global error would be of order $O(\Delta t)$ (not desirable).



the first equation \dot{e}_{n+1} is this, and from the equation for displacement I get this. Now if you see now if δ is $1/2$, the velocity the error term will be of Δt^3 , because this is $1/2$, the remaining term will be of the order Δt^3 . Similarly if α equal to $1/6$, displacement is order Δt^4 , these are local errors and this is for linear acceleration method. Now if δ is greater than half, the method displays high frequency dissipation characteristics which is desirable, but the global error will be of the order Δt , if suppose this is not 0 then the local error is of the order Δt^2 , global error will be of the order Δt , okay, so for the method to display numerical dissipation I need δ to be greater than half, in which case the term inside the parentheses won't be 0, and the error, the local error becomes order Δt^2 , and the global error will become order Δt , okay, so that is, that is a precisely what I was telling is that trade-off between the desirable feature that the method should have high frequency dissipation characteristic and also it should have a higher order accuracy, so both cannot be met simultaneously.

HHT- α method and generalizations

Motivation

To develop an unconditionally stable, second order accurate method with acceptable high frequency dissipation characteristics.

Idea: Introduce a few more algorithmic parameters

H M Hilber, T J R Hughes, R L Taylor, 1977, Improved numerical dissipation for time integration algorithms in structural dynamics, Earthquake Engineering and Structural Dynamics, 5, 283-293.



Chung and G M Hulbert, 1993, A time integration algorithm for structural dynamics with improved numerical dissipation, Journal of Applied Mechanics, 60, 371-375.

33

So that takes us to the development of another method in which this weakness of Newmark beta method is overcome, this is called HHT alpha method, the HH the alphabets HHT refer to the names of the three authors Hilber, Hughes and Taylor, this is the reference that I have provided and this is another related reference on the method, so the motivation of developing this method is to develop an unconditionally stable second order accurate method with acceptable high frequency dissipation characteristics, okay, so what did we see in Newmark beta method, it was unconditionally stable all right but there was a question of either achieving second order accuracy or achieving high frequency dissipation, not both, now can we achieve both is a

HHT α method and generalizations

The system of equilibrium equations is written as

$$(1-\alpha_m)M\ddot{U}_{n+1} + \alpha_m M\ddot{U}_n + (1-\alpha_f)C\dot{U}_{n+1} + \alpha_f C\dot{U}_n + (1-\alpha_f)KU_{n+1} + \alpha_f KU_n = (1-\alpha_f)F_{n+1} + \alpha_f F_n$$

The velocity and displacement vectors are represented as in the Newmark method as

$$\dot{U}_{n+1} = \dot{U}_n + \Delta t \left[(1-\delta)\ddot{U}_n + \delta\ddot{U}_{n+1} \right]$$

$$U_{n+1} = U_n + \dot{U}_n \Delta t + \Delta t^2 \left[(0.5-\alpha)\ddot{U}_n + \alpha\ddot{U}_{n+1} \right]$$

\Rightarrow

$$\begin{bmatrix} (1-\alpha_f)K & (1-\alpha_f)C & (1-\alpha_m)M \\ 0 & 1 & -\delta\Delta t \\ 1 & 0 & -\alpha\Delta t^2 \end{bmatrix} \begin{Bmatrix} U_{n+1} \\ \dot{U}_{n+1} \\ \ddot{U}_{n+1} \end{Bmatrix} =$$

$$\begin{bmatrix} -\alpha_f K & -\alpha_f C & -\alpha_m M \\ 0 & 1 & \Delta t(1-\delta) \\ \Delta t & \Delta t^2(0.5-\alpha) & 0 \end{bmatrix} \begin{Bmatrix} U_n \\ \dot{U}_n \\ \ddot{U}_n \end{Bmatrix} + \begin{Bmatrix} (1-\alpha_f)F_{n+1} + \alpha_f F_n \\ 0 \\ 0 \end{Bmatrix}$$



NPTEL

$$\Rightarrow AX_{n+1} = BX_n + f_n$$

34

question. So indeed that is possible I will not get into all the details, I will just mention what exactly is done here, so in this method or in this class of method there are generalizations, what is done is in Newmark beta method we used this representation for velocity and displacement and we introduced algorithmic parameter in delta and alpha as we shown earlier.

Now the HHT alpha method a similar scheme is introduced in writing the equation of motion, so the MU double dot is written as a weighted sum of $1 - \alpha_m$ into MU double dot N+1 and α_m M, MU and double dot, so this is an approximation to MU double dot at N+1, so α_m M $1 - \alpha_m$ add to 1, so it is a weighted sum. Similarly the damping term is written in this form, and stiffness term is written in this form, and the external force is written in this form, so here we see now that we have now 2 more algorithmic parameters namely α_m and α_f , in addition to this delta and alpha, so that helps us to now you know examine you know make choices by suitably making choices of these parameters we can hope to achieve the objective of you know achieving unconditionally stable algorithm with second order accuracy and desirable high frequency dissipation characteristics, so we can run through this, I will not provide all the details so these 3 equations can be recast in a matrix form as shown here, and we can write this

Implementation of the generalized HHT α method

(1) Read algorithmic parameters $\Delta t, \alpha_w, \alpha_f, \alpha, \& \delta$

(2) Read initial conditions: U_0, \dot{U}_0 & forcing vector; Evaluate $\ddot{U}_0 = M^{-1}(-C\dot{U}_0 - KU_0 - F_0)$

(3) Form $A = \begin{bmatrix} (1-\alpha_f)K & (1-\alpha_f)C & (1-\alpha_w)M \\ 0 & 1 & -\delta\Delta t \\ 1 & 0 & -\alpha\Delta t^2 \end{bmatrix}$ & $B = \begin{bmatrix} -\alpha_f K & -\alpha_f C & -\alpha_w M \\ 0 & 1 & \Delta t(1-\delta) \\ 1 & \Delta t & \Delta t^2(0.5-\alpha) \end{bmatrix}$

(4) Evaluate $\tilde{A} = A^{-1}B$

(5) Set $n = 0$

(6) $X_{n+1} = \tilde{A}X_n + \tilde{A} \begin{Bmatrix} (1-\alpha_f)F_{n+1} + \alpha_f F_n \\ 0 \\ 0 \end{Bmatrix}$



$n = n + 1$; if $n\Delta t > t_f$, exit; otherwise go to step 6.



NPTEL

as $AX_{n+1} = BX_n + FN$ and we can write out the steps for implementation, it is not very difficult it can be done if you follow, if you rearrange these 3 equations in this form, we get you know these are the steps for implementation, we read algorithmic parameters I have to come to issue how to select them, I will come there shortly and read the initial conditions and evaluate the initial acceleration, and we form these matrices if Δt is constant we need to do this only once, and we evaluate this \tilde{A} which is $A^{-1}B$ which again we need to do only once if Δt is constant. Now then set $N = 0$, and X_{N+1} is given by this, and increment N stop if you reach the final time instant otherwise simply go to step 6, that means the looping is only for step 6 and 7, so all other calculations are done outside this loop, okay, so this can be done later on, later on we will consider some examples but now we are focusing on qualitative features of the integration schemes, so let's now ask the question how about the stability characteristics of

Stability analysis of the generalized HHT α method

Consider $\ddot{x} + 2\eta\omega\dot{x} + \omega^2x = 0$

$$(1 - \alpha_m)\ddot{x}_{n+1} + \alpha_m\ddot{x}_n + (1 - \alpha_f)2\eta\omega\dot{x}_{n+1} + \alpha_f2\eta\omega\dot{x}_n + (1 - \alpha_f)\omega^2x_{n+1} + \alpha_f\omega^2x_n = 0$$

$$\dot{x}_{n+1} = \dot{x}_n + \Delta[(1 - \delta)\ddot{x}_n + \delta\ddot{x}_{n+1}]$$

$$x_{n+1} = x_n + \dot{x}_n\Delta + \Delta^2[(0.5 - \alpha)\ddot{x}_n + \alpha\ddot{x}_{n+1}]$$

\Rightarrow

$$\begin{bmatrix} (1 - \alpha_f)\omega^2 & 2(1 - \alpha_f)\eta\omega & (1 - \alpha_m) \\ 0 & 1 & -\delta\Delta \\ 1 & 0 & -\alpha\Delta^2 \end{bmatrix} \begin{Bmatrix} x_{n+1} \\ \dot{x}_{n+1} \\ \ddot{x}_{n+1} \end{Bmatrix} = \begin{bmatrix} -\alpha_f\omega^2 & -2\alpha_f\eta\omega & -\alpha_m \\ 0 & 1 & \Delta(1 - \delta) \\ 1 & \Delta & \Delta^2(0.5 - \alpha) \end{bmatrix} \begin{Bmatrix} x_n \\ \dot{x}_n \\ \ddot{x}_n \end{Bmatrix}$$

$$\Rightarrow AX_{n+1} = BX_n$$

Characteristic equation $|AZ - B| = 0$

Do the roots lie within the unit circle in the complex plane?



NPTEL



this method, so again the test system is undamped single degree freedom system, so I write now the HHT alpha you know equations for acceleration, velocity and displacement, and we can assemble them in this form as we obtained earlier I will get $AX_{N+1} = BX_N$, so the characteristic equation will be some $AZ - B$, determinant equal to 0, I use Eigenvalue as Z instead of lambda which I was doing earlier, it doesn't matter, so the question is do the roots lie within the unit circle of the complex plane, that is the requirement for stability analysis.

$$f(z) = a_0 z^3 + a_1 z^2 + a_2 z + a_3$$

$$a_0 = 1 - \alpha_n + 2\eta\delta(1 - \alpha_f)\omega\Delta + \alpha(1 - \alpha_f)\omega^2\Delta^2$$

$$a_1 = -2 + 3\alpha_n + 2\eta\omega\Delta(1 - 2\delta - \alpha_f + 3\alpha_f\delta)$$

$$a_2 = 1 - 3\alpha_n + 2\eta\omega\Delta(-1 + \delta + 2\alpha_f - 3\alpha_f\delta) + \omega^2\Delta^2(0.5 + \alpha - \delta + 2\alpha_f\delta - 3\alpha\alpha_f)$$

$$a_3 = \alpha_n + 2\eta\alpha_f\omega\Delta(\delta - 1) + \alpha_f\omega^2\Delta^2\left(\frac{1}{2} + \alpha - \delta\right)$$

Necessary conditions

$$P(1) = \omega^2\Delta^2 > 0$$

$$(-1)^3 P(-1) = -4(2\alpha_n - 1) - 4\eta\omega\Delta(2\alpha_f - 1)(2\delta - 1) - 2\omega^2\Delta^2(2\alpha - \delta)(2\alpha_f - 1) > 0$$

This condition is satisfied if $\alpha_n < 0.5, \alpha_f < 0.5, \delta > 0.5$ & $\alpha > 0.5\delta$

Sufficient conditions

$$|a_3| > |a_0| \text{ \& \ } |b_0| > |b_2|$$

Parameter range for unconditional stable scheme

$$\alpha_n \leq \alpha \leq 0.5; \delta = 0.5 - \alpha_n + \alpha_f; \alpha \geq 0.25 + 0.5(\alpha_f - \alpha_n)$$

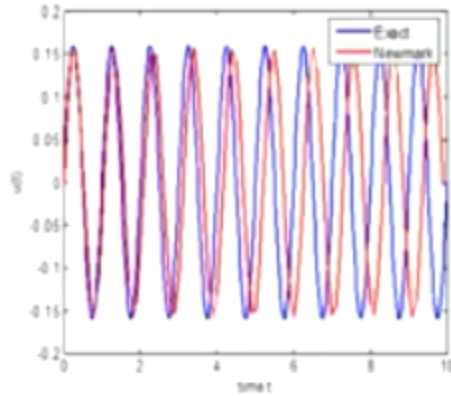


Now since this is a 3/3 equation that will have a cubic polynomial and that is given by this, $A_0 z^3 + A_1 z^2 + A_2 z + A_3$, now again we use a Jury's criterion so I have A_0, A_1, A_2, A_3 from the expansion of the determinant we can get these terms, you can use symbolic maths, you know, symbolic you know mathematics softwares to do this, this is precisely how I have done this, the necessary condition will be $P(1)$ is $\omega^2 \Delta^2$ turns out to be $\omega^2 \Delta^2$, it must be greater than 0, that is satisfied and -1 to the power of $3P(-1)$ which is given by this, this must be greater than 0. So again we can examine this equation and see that one way to satisfy this is to take the parameters to satisfy these inequalities, there will be 2 sufficient conditions now, modulus of A_3 should be greater than modulus of A_0 , and modulus of B_0 must be greater than B_2 , so we can examine all this and we can place now restrictions on the range of parameter values to achieve unconditional stability and that is summarized here, so the idea, the important the details can be understood but the point is that we are now adjusting the values of algorithmic parameters to achieve unconditional stability.

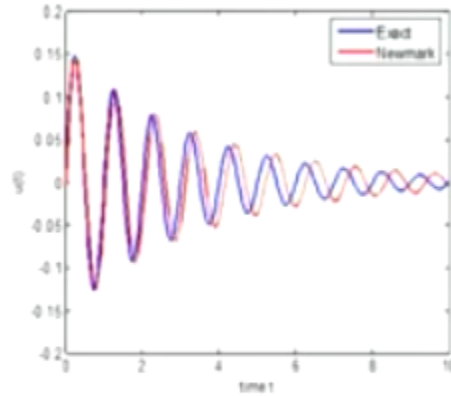
Newmark's algorithm

$$\ddot{u} + 2\eta\omega\dot{u} + \omega^2u = 0; u(0) = 0; \dot{u}(0) = 1$$

$$\omega = 2\pi \text{ rad/s}; \Delta t = 0.1\text{s}; \delta = 0.5; \alpha = 0.3$$



$$\eta = 0$$



$$\eta = 0.05$$

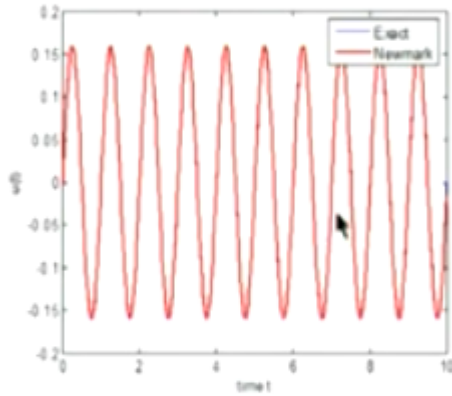


Now I will illustrate some of these algorithms with simple single degree freedom system, so I start with a system with period one second, that is ω is 2π and I am using Newmark's method in an acceptable region with 10 points in a cycle I get this solution, blue line is the exact solution and red is a Newmark's solution, this is for undamped system and this is for damped system, so 10 points in a cycle doesn't seem to give good results but if you take 100

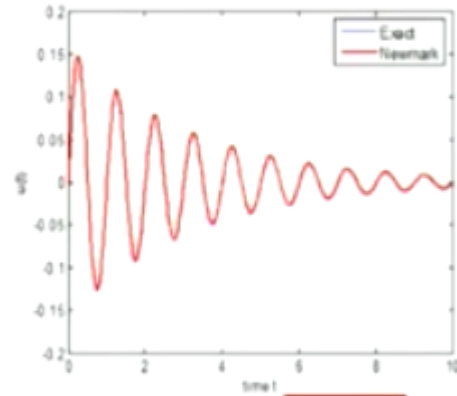
Newmark's algorithm

$$\ddot{u} + 2\eta\omega\dot{u} + \omega^2u = 0; u(0) = 0; \dot{u}(0) = 1$$

$$\omega = 2\pi \text{ rad/s}; \Delta t = 0.01\text{s}; \delta = 0.5; \alpha = 0.3$$



$$\eta = 0$$



$$\eta = 0.05$$

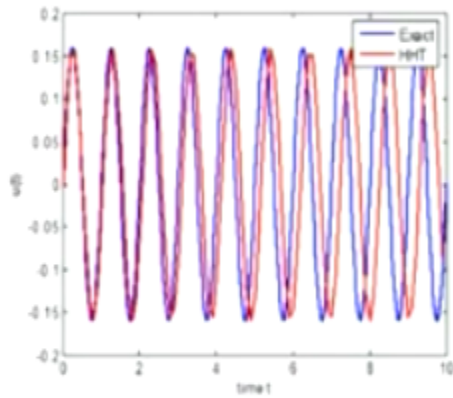


steps both give reasonably good solution, we will come to guidelines on selecting step size etcetera sometime later, but we will just observe now outcomes of this simple numerical experiments, this is now HHT alpha method again this is the parameters that I have taken which

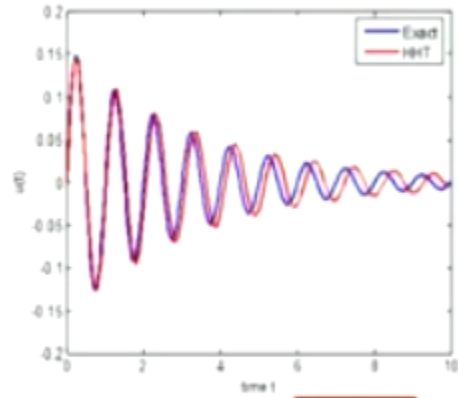
HHT- α algorithm

$$\ddot{u} + 2\eta\omega\dot{u} + \omega^2u = 0; u(0) = 0; \dot{u}(0) = 1$$

$$\omega = 2\pi \text{ rad/s}; \Delta t = 0.1\text{s}; \delta = 0.5; \alpha = 0.25; \alpha_m = 0.25; \alpha_f = 0.25$$



$$\eta = 0$$



$$\eta = 0.05$$

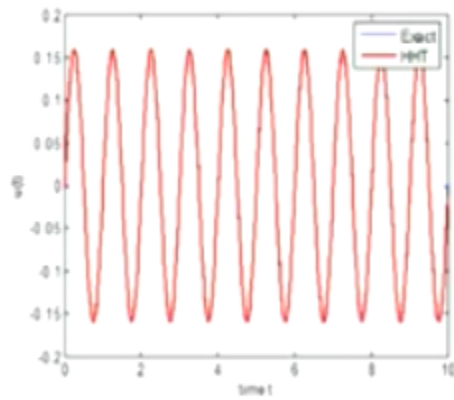


guarantees unconditional stability, 10 points in a cycle this is the result for undamped system and this is for damped system.

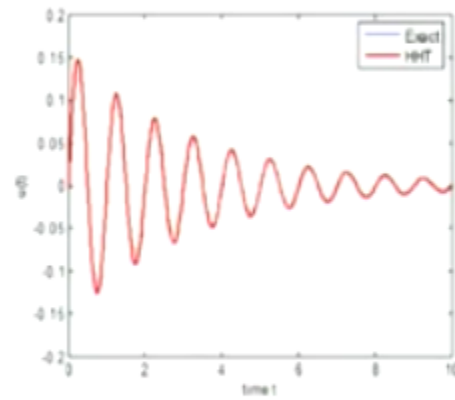
HHT- α algorithm

$$\ddot{u} + 2\eta\omega\dot{u} + \omega^2 u = 0; u(0) = 0; \dot{u}(0) = 1$$

$$\omega = 2\pi \text{ rad/s}; \Delta t = 0.01 \text{ s}; \xi = 0.5; \alpha = 0.25; \alpha_m = 0.25; \alpha_f = 0.25$$



$$\eta = 0$$



$$\eta = 0.05$$



Now with 100 points in a second both the methods I mean the HHT alpha method gives the result from HHT alpha method agrees with the exact solution as shown here.

Example of recent developments

Y K Ping, 2008, A new family of generalized- α time integration algorithms without overshoot for structural dynamics, Earthquake Engineering and Structural Dynamics, 37, 1389-1409

The system of equilibrium equations is written as

$$(1 - \alpha_m)M\ddot{U}_{n-1} + \alpha_m M\ddot{U}_n + (1 - \alpha_f)C\dot{U}_{n-1} + \alpha_f C\dot{U}_n + (1 - \alpha_k)KU_{n-1} + \alpha_k KU_n = (1 - \alpha_k)F_{n-1} + \alpha_k F_n$$

The velocity and displacement vectors are represented as

$$\dot{U}_{n-1} = \dot{U}_n + \Delta t \left[\mu \ddot{U}_n + \gamma \ddot{U}_{n-1} \right]$$
$$U_{n-1} = U_n + \dot{U}_n \Delta t + \Delta t^2 \left[\varepsilon \ddot{U}_n + \beta \ddot{U}_{n-1} \right]$$



$\alpha_m, \alpha_f, \alpha_k, \mu, \gamma, \varepsilon, \& \beta$: seven algorithmic parameters

Now this line of development of integration schemes to achieve you know desirable features in the schemes has been pursued by several authors and I just want to give an example of one such study, so in the HHT alpha method if you go back, we had a parameter alpha M to approximate the inertial forces, and the same alpha F was used for damping, stiffness, and external forces, so one can relax that, introduce more algorithmic parameters, so study to illustrate that has been carried out by this person in 2008, so what he does is introduced alpha M here, and alpha F here, alpha K here, and that alpha K is used here again, but even in writing the expressions for velocity and displacement the new parameters are introduced. Now these become algorithmic parameters, 7 algorithmic parameter, so now how do we select them so that takes us into the question of what are the desirable properties of a integration scheme that is what we have been

Desirable features of numerical integration schemes

- At least second order accuracy
- Unconditional stability when applied to LTI systems
- Controllable algorithmic damping in higher modes

Investigate spectral radius as frequency $\rightarrow \infty$

Spectral radius $\rightarrow 1$ as driving frequency $\rightarrow 0$

- No overshoot

Excessive oscillations during the first few steps

- Self-starting

• No more than one set of implicit equations to be solved at each step

NPTEL

43

talking about, so typically we want at least second-order accuracy that is at the global level and when applied to linear time invariant systems we want unconditional stability, and then controllable algorithmic damping in higher modes, so what we do is we examine the spectral radius as frequency becomes large, what we want is the spectral radius should drop, so that higher contributions from spurious higher modes are dissipated, this is desirable as I already said if response is dominated by low frequency mode.

Then for however for low frequency region the spectral radius should be 1, otherwise the contribution from lower modes also get distorted which is not desirable, we don't want numerical dissipation to affect the contributing modes, the significantly contributing modes should not be disturbed by the numerical dissipation, but the higher modes it helps if they are quickly dissipated numerical. There is another phenomenon that is observed is known as overshoot, this is excessive oscillations during the first few steps, so HHT alpha method has this deficiency and further improvements on that has overcome some of these deficiencies, and we want the algorithm to be self-starting, and we have not talked about nonlinear system, but when we talk about nonlinear systems we don't want to solve more than 1 set of implicit equations at every time step. So the idea here is that by formulating the integration scheme in different ways and introducing different algorithmic parameters we can find regions in the parameter space where if these parameters are selected from those regions these objectives are met, so this has been the approach to developing these type of schemes.

So in the next lecture what I will do is we will make some additional comments on these methods and we will try to apply these methods to some multi degree freedom systems, just for illustration and see how they perform, so we will conclude this lecture at this stage.

Programme Assistance

**Guruprakash P
Dipali K Salokhe**

Technical Supervision

**B K A N Singh
Gururaj Kadloor**

**Indian Institute of Science
Bangalore**