Indian Institute of Science
Bangalore

NP-TEL
National Programme on
Technology Enhanced Learning

<u>Course Title</u>

Finite element method for structural dynamic And stability analyses

Lecture – 11
Twisting of circular bars and rectangular bars. Analysis of grids.

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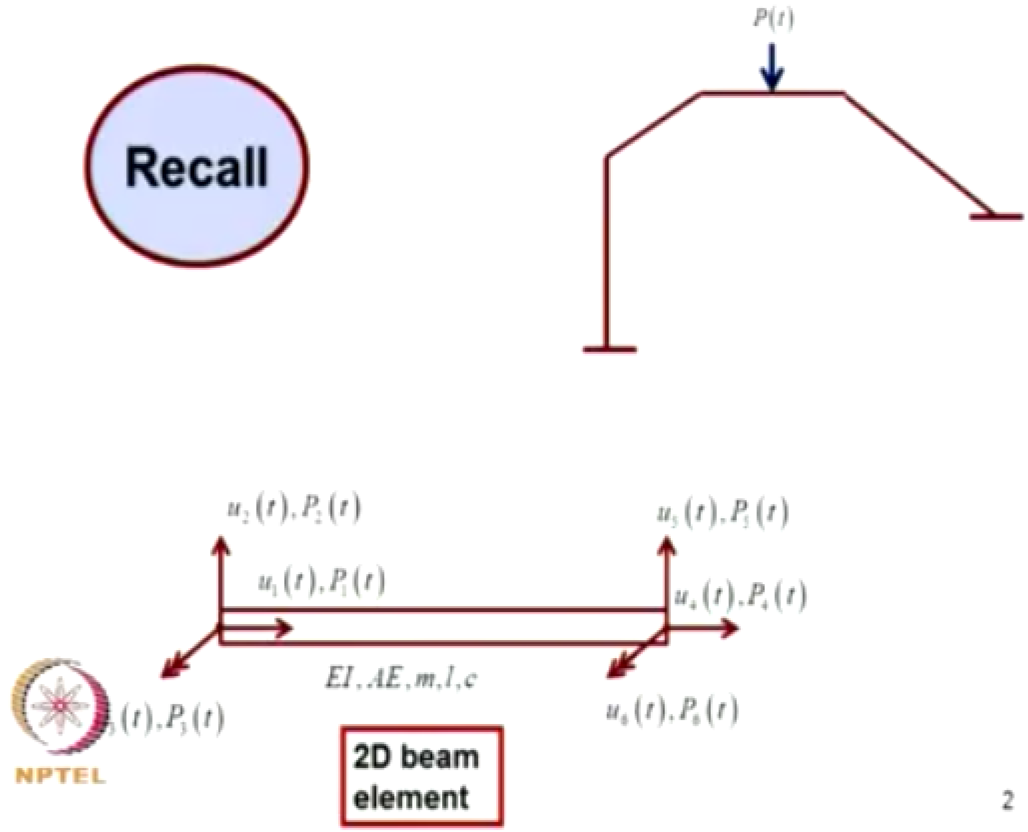
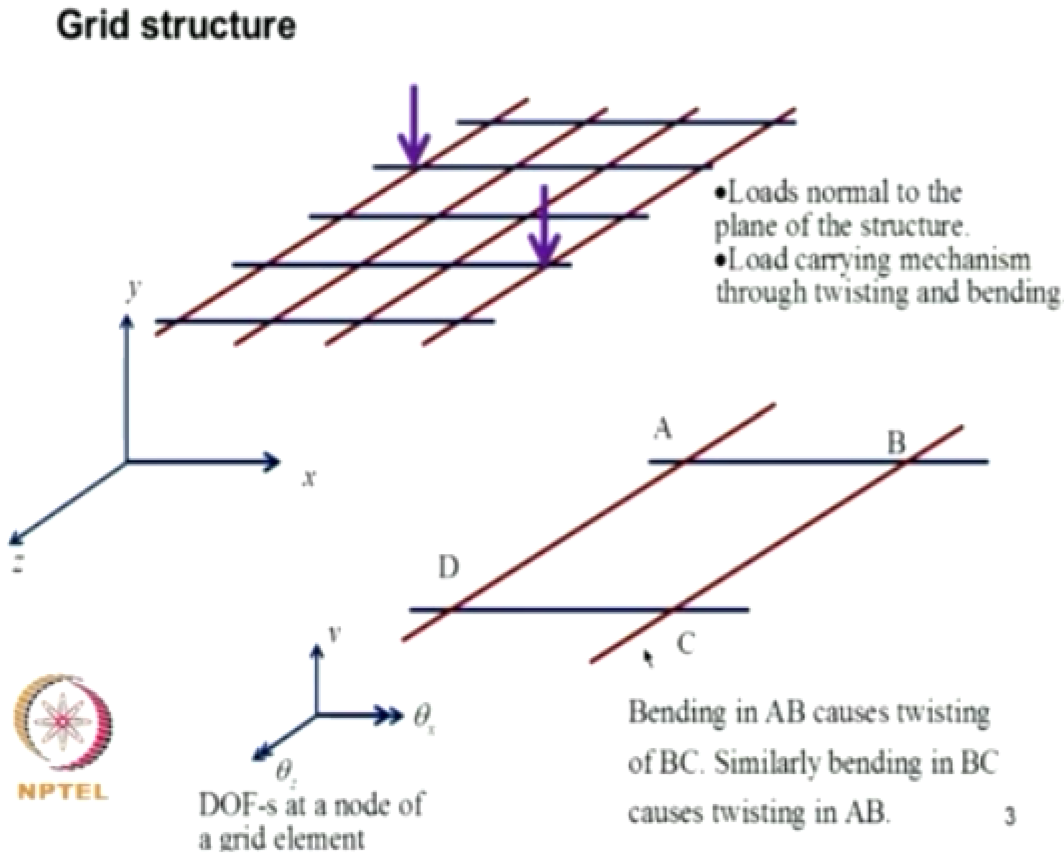
We will start with a new module today that is analysis of grids and 3D frames. We have talked about analysis of planar frames, so we will continue from there, so in the analysis of planar



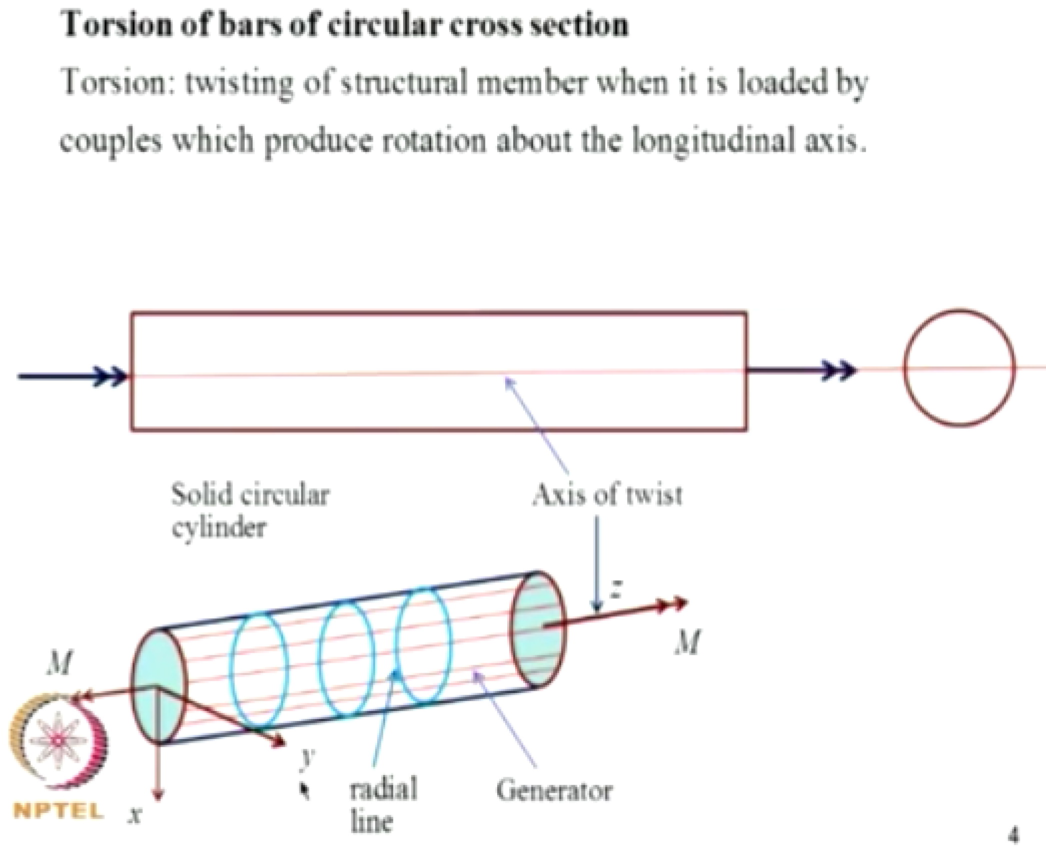
frames if you recall we had line elements carrying in plane loads and the main mechanism for carrying load was through bending and axial deformation, so the typical beam element had two axial degrees of freedom, two translations, and two rotations, so it had 6 degrees of freedom.



Another class of planar structures are known as grid structures, and typically it appears something like this, so there are girders in the XZ plane as shown here, and the loads act along Y axis that is normal to the plane of the structure. Whereas in plane frames all the elements were lying on the same plane, but load was acting in the same plane as the elements, here the load is acting transverse to the plane on which the elements reside. So an example for this would be a bridge deck, these girders can be thought of as longitudinal girders and this can be thought of as transverse girders, so grid can be viewed as a discrete version of plate, we will come to plate later, but to start with we will consider a structure like this made up of essentially line elements.

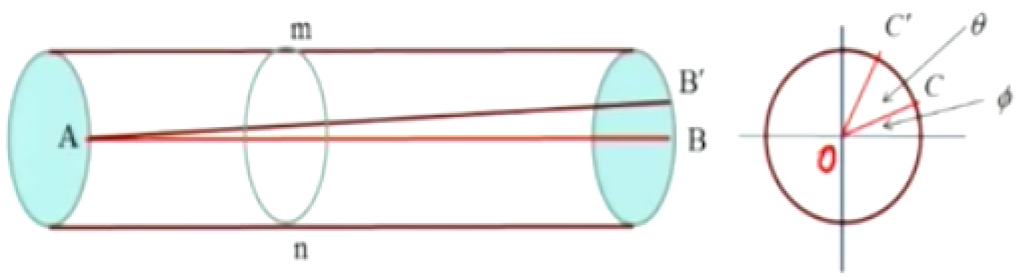
So the load carrying mechanism in this case is again through bending and in addition to that there is a new mechanism that is twisting to see that we can consider one segment of this say A, B, C, D, now if a load acts on AB, AB will bend, because of bending here BC will twist, so a bending action in AB results in twisting of BC. Similarly if there is a load on BC, it will bend like this, and bending in BC will be resulting in twisting in AB as well as DC, so the actions here are there for twisting and bending, so this is in contrast with planar frames where action was through axial deformation and bending.

So now at any node therefore we need to have one translation and one rotation due to bending and another one twisting, so for this element AB for example there will be twisting along the longitudinal axis which is say the X axis, that is theta X and it will displace along Y axis and that is V and also it will rotate about Z axis that is theta Z, so these three are the degrees of freedom. So now we have analyzed the bending action earlier now we need to consider the twisting action, so let us revisit that topic now and we will start the discussion by considering torsion of bars of circular cross section, so torsion is twisting or structural member when it is



loaded by couples which produce rotation about the longitudinal axis, so this is the line element so in this case also we assume as in beams the transverse dimensions that a depth direction, the dimensions along and say the depth and breadth are small comparison to the length of the member so it is a line element, and the loading is through couples that are acting at the ends shown here and this as a consequence of application of this couple the member twist, so what happens due to that? So let's consider, let M be the end couple and we call this axis, the longitudinal axis as the axis of twist, so we will call that as Z axis, this is Y axis, and this is X axis, so Z axis is the axis of twist, and we call lines that are parallel to Z axis as shown here as generators and these lines which are shown here are called radial lines.

Now we are basically considering a solid circular cylinder that we should notice, so now let's see what kind of deformation we can postulate for this structure.



AB=generator in the undeformed state; a straight line.

AB′=generator after deformation relative to $z = 0$.

OC=radius vector before deformation

OC′=radius vector after deformation

Aim: to construct a semi-inverse solution by making a set of suitable assumptions.

Now AB is a generator in the undeformed state, it is a straight line, so because of twisting let point B move to B prime, and we call AB prime is a generator after deformation relative to Z = 0, similarly on the side view OC that is this is O, OC is the radius vector before deformation and an account of rotation C moves to C prime and the radius vector goes to OC prime, OC prime is the radius vector after deformation. So now aim of our analysis is to construct a semi inverse solution by making a set of suitable assumptions on the displacement field, so we make these following assumptions this is our own assumptions on displacements on material

Assumptions on displacements

- Plane cross sections normal to z-axis before deformation remain plane and normal to the z-axis after deformation.

- For small displacements, a radius vector OC of a given section remains straight and inextensible. $\Rightarrow$ The couple M causes each section to rotate approximately as a rigid body about the axis of couple, that is the axis of twist, z-axis.

- Let the amount of rotation of a section relative to the plane z=0 be denoted by θ. It is assumed that amount of rotation of a given section depends linearly upon its distance z from z=0. $\Rightarrow \theta = \beta z$ with β = twist per unit length.

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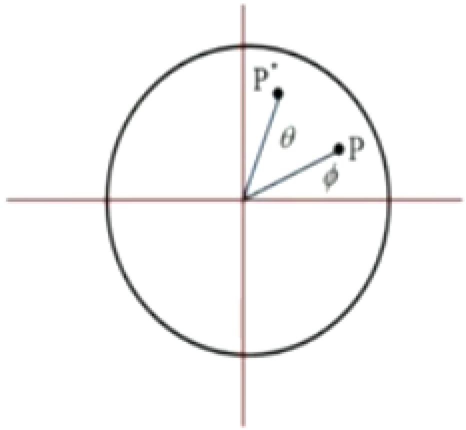
behavior, we're assuming system is behaving elastically, material is homogeneous and isotropic, and I also spelt out the assumption of the geometry that the length of the member is large in comparison to the depth and breadth, so those assumptions are already stated, but now we will consider assumptions on displacements.

Now we assume that plane cross-sections normal to Z axis before deformation remain plane and normal to the Z-axis after deformation. Next for a small displacements a radius vector OC of a given section remains straight and inextensible that is this OC after deformations remains straight and length OC is same as length OC prime, so the length there is no change in length the couple therefore, the couple M therefore causes each section to rotate approximately as a rigid body about the axis of couple, that is the axis of twist is Z axis, that is the axis of twist which is a Z-axis about that it twists.

Now let the amount of rotation of a section relative to the plane Z = 0 be denoted by theta, so this angle is I am calling as theta, so C is at an angle phi from the horizontal, and theta is the rotation. Now if we assume that the amount of rotation of a given section depends linearly upon its distance from Z = 0, that means theta will be beta into Z where beta is a twist per unit length, okay, so now based on that suppose if we now consider a point P which is at XY before



$P = P(x, y)$ [Before deformation]

$P^* = P^*(x^*, y^*)$ [After deformation]

OP = radius vector before deformation

OP^* = radius vector after deformation

$$OP = OP^* \quad (\text{Assumption } 2)$$

$$x = R\cos\phi$$

$$x^* = R^*\cos(\theta+\phi) = R\cos(\theta+\phi)$$

$$y = R\sin\phi$$

$$y^* = R^*\sin(\theta+\phi) = R\sin(\theta+\phi)$$

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deformation and let it move to P star after deformation, so coordinates of P be X, Y and coordinates of P star be X star Y star. Now OP is a radius vector before deformation OP star is a radius vector after deformation, now according to assumption 2 OP is OP star. Now by considering the geometry of this we can see that X is R cos phi, X star is this is position vector P star is actually X star Y star the coordinates, so X star R star cos theta + phi, so since R = R star, this is R cos theta + phi, then Y is R sin phi, so Y star is R star sin theta + phi, but again because of the assumption OP = OP star this is R sin theta + phi. Now we can rewrite this by expanding this cos theta + Y and sin theta + Y, so the displacement along the X direction is X star – X, so that is R cos theta + phi - R cos phi, so you expand, rewrite, we get this as - X 1 - cos theta - y sin theta, this is U, according to the assumption this is a displacement field.

$$
\begin{aligned}
u &= x^* - x \\
&= R\cos(\theta+\phi) - R\cos\phi \\
&= R\cos\theta\cos\phi - R\sin\theta\sin\phi - R\cos\phi \\
&= -R\cos\phi(1-\cos\theta) - R\sin\phi\sin\theta \\
&= -x(1-\cos\theta) - y\sin\theta
\end{aligned}
$$

$$
\begin{aligned}
v &= y^* - y \\
&= R\sin(\theta+\phi) - R\sin\phi \\
&= R\sin\theta\cos\phi + R\cos\theta\sin\phi - R\sin\phi \\
&= R\cos\phi\sin\theta - R\sin\phi(1-\cos\theta) \\
&= x\sin\theta - y(1-\cos\theta)
\end{aligned}
$$

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Now nu is Y star – Y, so if we now go through the same steps Y star is R similar steps, Y star is R sin theta + phi, and this is sin phi, so if we now expand this so I get after simplification V to be X sin theta – Y 1 – cos theta, so this is a displacement field consistent with the assumptions

$$u = -x(1-\cos\theta) - y\sin\theta;\ v = x\sin\theta - y(1-\cos\theta)$$

Small deformation $\Rightarrow \cos\theta \approx 1$ & $\sin\theta \approx \theta$

$$\Rightarrow u = -y\theta;\ v = x\theta$$

Assumption 3 $\Rightarrow \theta = \beta z$

$\Rightarrow$ Deformation field: $w = 0;\ u = -y\beta z;\ v = x\beta z$

Strain field

$$\left.\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} = 0 \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} = 0 \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\beta z + \beta z = 0 \\ \gamma_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = -\beta y \\ \gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \beta x \end{aligned}\right\} \varepsilon = \begin{bmatrix} 0 & 0 & -\frac{\beta y}{2} \\ 0 & 0 & \frac{\beta x}{2} \\ -\frac{\beta y}{2} & \frac{\beta x}{2} & 0 \end{bmatrix}$$

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made, now if we introduce the additional assumption that deformation is small, I can introduce simplify this as cos theta is approximately 1, and sin theta is approximately theta, and I get U is - Y theta, and V is X theta.

Now assumption 3 right, and according to that theta will be beta into Z, that means the rotation at any section at a distance Z from the origin is proportional to the distance that is beta into Z, where beta is a twist per unit length, so under these assumptions the postulated displacement field is therefore W = 0, U = - Y, beta Z, V is X beta Z, now beta is an unknown which has to be still determined, now the task on hand is now to see whether this satisfies equations of velocity, so the equations of elasticity are basically strain displacement equations, stress strain relations and equilibrium equations and once we verify that this satisfy those equations then we have to satisfy boundary conditions at Z = 0 and Z = L, and along the lateral surface r = capital R for all values of theta. So let's run through that now first let us establish from the given displacement field the strain field, epsilon X axis is dou U / dou X, so this is independent of X therefore that is 0, epsilon YY is dou V / dou Y this is independent of Y that is 0, gamma XY is dou U / dou Y + dou V / dou X, so that is - beta Z + beta Z, which is again 0, gamma XZ dou W / dou X that is 0, dou U by dou Z is - Y beta, similarly gamma YZ this dou W / dou Y + dou V / dou Z, W is 0 therefore first term is 0, dou V / dou Z is beta into X, so the strain matrix therefore a tensor is given by this, so we have two shearing strains and normal strains are all 0.

$$\varepsilon = \begin{bmatrix} 0 & 0 & -\frac{\beta y}{2} \\ 0 & 0 & \frac{\beta x}{2} \\ -\frac{\beta y}{2} & \frac{\beta x}{2} & 0 \end{bmatrix} \Rightarrow \left.\begin{aligned} e &= \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = 0 \\ \sigma_{xx} &= \lambda e + 2G\varepsilon_{xx} = 0 \\ \sigma_{yy} &= \lambda e + 2G\varepsilon_{yy} = 0 \\ \sigma_{zz} &= \lambda e + 2G\varepsilon_{zz} = 0 \\ \sigma_{xy} &= 2G\varepsilon_{xy} = 0 \\ \sigma_{xz} &= 2G\varepsilon_{xz} = -G\beta y \\ \sigma_{yz} &= 2G\varepsilon_{yz} = G\beta x \end{aligned}\right\} \Rightarrow \sigma = \begin{bmatrix} 0 & 0 & -G\beta y \\ 0 & 0 & G\beta x \\ -G\beta y & G\beta x & 0 \end{bmatrix}$$

Equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} = 0 \Rightarrow 0 + 0 + \frac{\partial}{\partial z}(-G\beta y) = 0 \text{ (OK)}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} = 0 \Rightarrow 0 + 0 + \frac{\partial}{\partial z}(G\beta x) = 0 \text{ (OK)}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \Rightarrow \frac{\partial}{\partial x}(-G\beta y) + \frac{\partial}{\partial z}(G\beta x) + 0 = 0 \text{ (OK)}$$

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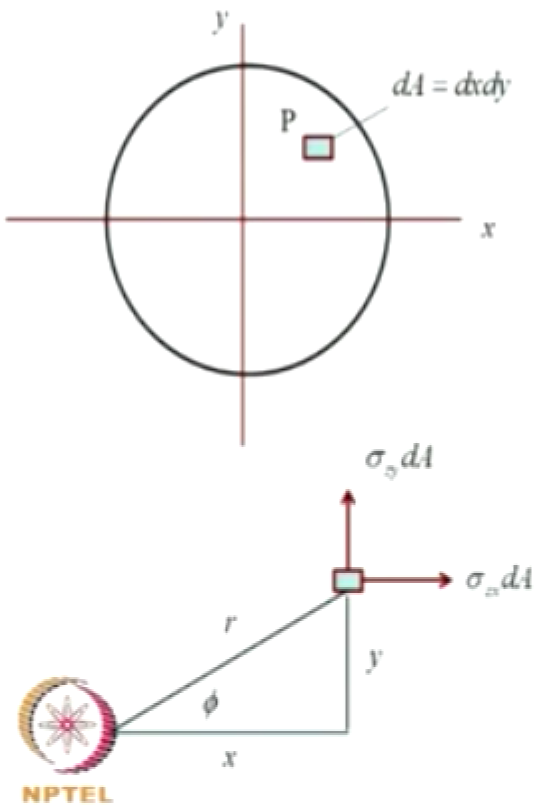
Now we will use the constitutive law valid for the isotropic material and this is a strain, stress strain relation so we know the strains therefore we can compute this E which is the first invariant, epsilon XX, epsilon YY, epsilon ZZ, that is actually 0, and then sigma XX is 0, sigma YY is 0, sigma ZZ is 0 sigma XY is 0, because the corresponding strengths are all 0 the shearing stresses we get as G into beta Y, G into beta X, G is the shear modulus. So we have now determined strains, and stresses, and displacements, now we have to check whether equilibrium conditions are satisfied, so we have the stress, field available to us suppose if we substitute into this we are considering static problem right now, so you can check one by one dou sigma XX / dou X + dou sigma YX / dou Y + dou sigma ZX / dou Z = 0 is the first equilibrium equation, the first term is 0 because sigma XX is 0, second term is again 0 because sigma YX is 0, the third term is dou / dou Z of sigma XZ which is again 0, so the first equilibrium equation is satisfied. Similarly we can see that the remaining two equilibrium equations are also satisfied, okay, so at the end of this we have satisfied now the strain displacement relation, stress strain relations, and equilibrium conditions, beta is still unknown so we need to now impose boundary conditions and determine beta, so to find that we consider



$$\int_A \left[\sigma_{zy} x - \sigma_{zx} y\right] dA = M$$

$$x = r\cos\phi$$

$$y = r\sin\phi$$

$$\sigma_{zy} = G\beta x = G\beta r\cos\phi$$

$$\sigma_{zx} = -G\beta y = -G\beta r\sin\phi$$

$$\Rightarrow \int_A G\beta\left[r^2\cos^2\phi + r^2\sin^2\phi\right] dA = M$$

$$\Rightarrow \int_A G\beta r^2 dA = M$$

$$\Rightarrow G\beta I_0 = M$$

$$\text{with } \int_A r^2 dA = I_0 = \int_0^{2\pi}\int_0^{R} r^2 r\, d\theta\, dr = \frac{\pi R^4}{2}$$

$$\Rightarrow \beta = \frac{M}{GI_0}$$

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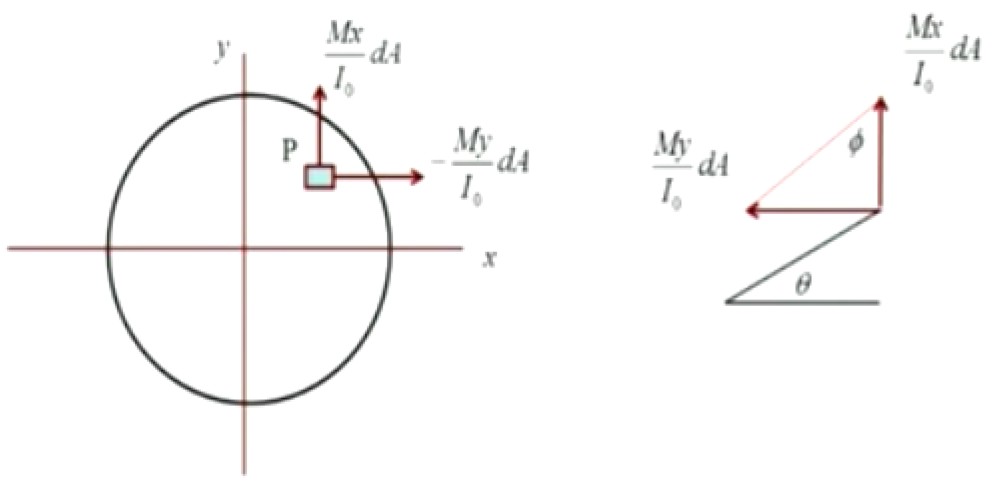
now the boundaries is Z = 0, and Z = L, and on those faces we know that there is a moment that is acting, so that is M, so the boundary condition for that would be, you can construct that there will be a stress acting in X direction that will be sigma ZX which will be acting here, that is shown here sigma ZX into DA, and on this phase there will be sigma ZY into DA.

Now we have to take moments about the origin and equate it to the applied moment and integral over the area should be equal to the applied moment couple, now this is a boundary condition, now let's run through this calculation we have now X is R cos phi, and Y is R sin phi, this is a polar coordinate system, so sigma, stresses can be expressed in terms of R phi by writing for X and Y these relations and we get this, now substituting into this equation I get this instead of, for sigma ZY I am writing G beta X which is G beta R cos phi and into X is X cos phi, so R cos phi and I get this, so the condition would be area integral G beta R square DM must be equal to M.

So now if I call I naught as R square DA integral over the area, the condition for the satisfaction of the boundary condition is G beta I naught must be equal to M, therefore beta is given by M by, sorry M / G I naught, we can compute the resultant stress also this is not crucial for our



Resultant stress

$$\tau = \frac{M}{I_0}\sqrt{x^2 + y^2} = \frac{Mr}{I_0}$$

$$\tan\phi = \frac{My / I_0}{Mx / I_0} = \tan\theta \Rightarrow \phi = \theta$$

$\tau = \frac{Mr}{I_0} = G\beta r =$ is the resultant stress and acts normal to the radius vector OA.

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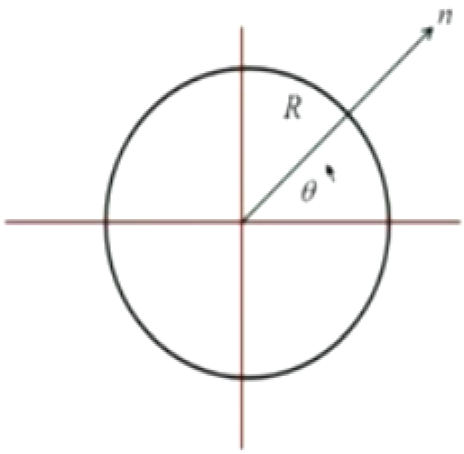
analysis but we will complete this, so at any point again at P these are the stresses moment I find beta I can find now the stresses in terms of I naught and M, so this is the forces acting on this element and if we now see the resultant would be tau is M / I naught square root X square + Y square which is MR / I naught, and if you want this angle phi we can take tan phi and that through this argument we can by studying this figure we can show that tan theta is tan phi which would mean phi is theta, so that would mean the shear stress tau is G beta R, it is the resultant stress and act normal to the radius vector OA at a every point.



Check on lateral surface

$$n_1 = \cos\theta = \frac{x}{R}$$

$$n_2 = \sin\theta = \frac{y}{R}$$

$$n_3 = 0$$

$$\overset{n}{\underset{\sim}{T}} = \sigma n$$

$$= \begin{bmatrix} 0 & 0 & -G\beta y \\ 0 & 0 & G\beta x \\ -G\beta y & G\beta x & 0 \end{bmatrix} \begin{Bmatrix} \dfrac{x}{R} \\ \dfrac{y}{R} \\ 0 \end{Bmatrix} = 0 \quad \text{(OK)}$$

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Now how about the boundary conditions on X = R, so that surface is a traction free therefore we need the, actually the stress vector on any surface is sigma into N where Sigma is the set of stress and the boundary, N is the vector of unit outward normal, so N1 is cos theta which is X /R into sin theta which is Y / R and if I substitute we can see that the boundary conditions are satisfied, so the assumed displacement field along with the condition on beta is a complete solution to the problem.

Lagrangian

$$U = \frac{1}{2}\int_0^L\int_A \left(\sigma_{xz}\gamma_{xz} + \sigma_{yz}\gamma_{yz}\right) dA\,dz$$

$$= \frac{1}{2}\int_0^L\int_A G\left(\gamma_{xz}^2 + \gamma_{yz}^2\right) dA\,dz$$

$$= \frac{1}{2}\int_0^L\int_A G\left(y^2\left(\frac{\partial\theta}{\partial z}\right)^2 + x^2\left(\frac{\partial\theta}{\partial z}\right)^2\right) dA\,dz$$

$$= \frac{1}{2}\int_0^L GJ\left(\frac{\partial\theta}{\partial z}\right)^2 dz \text{ with } J = \int_A \left(y^2 + x^2\right) dA$$

$$T = \frac{1}{2}\int_0^L\int_A \rho\left(\dot{u}^2 + \dot{v}^2\right) dA\,dz$$

$$= \frac{1}{2}\int_0^L\int_A \rho\dot{\theta}^2\left(y^2 + x^2\right) dA\,dz$$

$$= \frac{1}{2}\int_0^L I_{\bar{m}}\dot{\theta}^2 dz \text{ with } I_{\bar{m}} = \int_A \rho\left(y^2 + x^2\right) dA = \rho J$$

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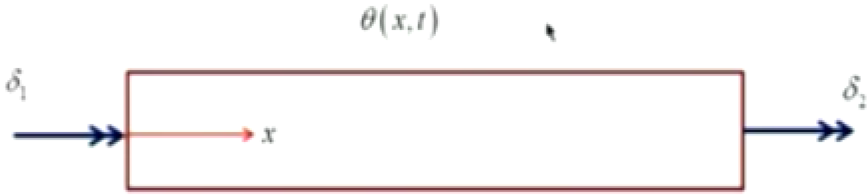
So now let us formulate the Lagrangian for the system, this is the elasticity solution now we need to develop the finite element solution for this, for under vibrating conditions, so the strain energy since there are 2 nonzero strain components and nonzero stress components strain energy is given by this, so by writing sigma XZ as G into gamma XZ, and sigma YZ as G into gamma YZ I get this equation. Now we have already determined gamma XZ and gamma YZ beta is dou theta / dou Z, which is the twist per unit length so this is dou theta / dou Z, so if we simplify now we get the strain energy as GJ dou theta / dou Z whole square DZ with J being given by this quantity, which is actually IXX + IYY + IXX this is a strain energy. Now the kinetic energy is given by half rho DA is mass per unit length, rho DA DZ is the mass per unit volume, U dot square + V dot square, so we can write for U and V in terms of the assume displacement field and theta and we get this, and if I now denote IM bar as area integral rho into X square + Y square DA which is rho J, I get the kinetic energy in this form.



$$\theta(x,t) = \delta_1(t)\left(1-\frac{x}{l}\right) + \delta_2(t)\frac{x}{l}$$

By analogy with the case of axially vibrating rod we get

$$M = \frac{I_{\theta}l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \;\&\; K = \frac{GJ}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

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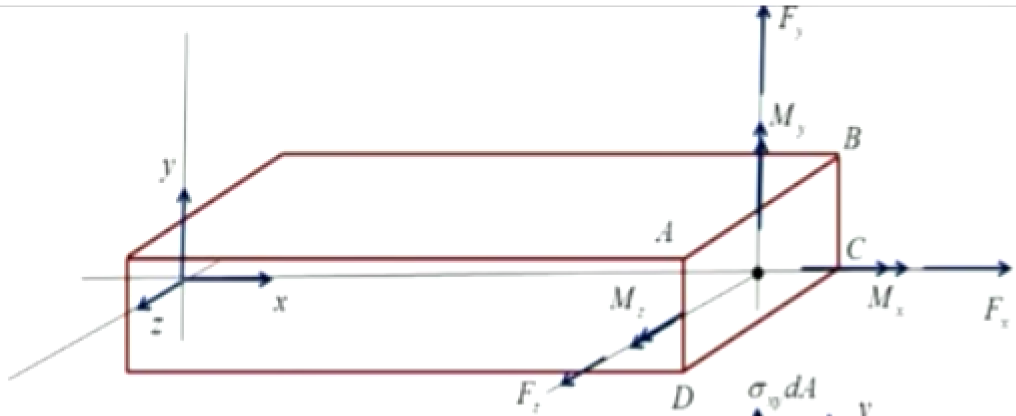
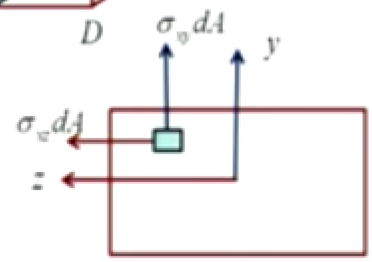
So now we have the Lagrangian which is T - U and by treating theta of X, T as a field variable and by discretizing the bar into a two noded element with one node here, and other node here, with delta 1 and delta 2 being the nodal twist, I can interpolate the twist in at any point within the bar in terms of the nodal values through this relation, so by analogy with the case of axially vibrating rod we see that the structure of the kinetic energy and strain energy are same for both axial vibration and twisting, and I get the mass matrix to be this and stiffness matrix to be this, if you recall for axially vibrating rod K was AE / L, AE was the axial rigidity, now GJ will be the torsional rigidity, and instead of mass per unit length MI now have IM bar, IM bar into L/6 and this is same as what we got earlier, so this is the mass and stiffness matrix for a circular rod.





Boundary conditions on ABCD

$$F_x = \int_A \sigma_{xx} dA \quad M_x = \int_A (\sigma_{xy} z - \sigma_{xz} y) dA$$

$$F_y = \int_A \sigma_{xy} dA \quad M_z = \int_A y\sigma_{xx} dA$$

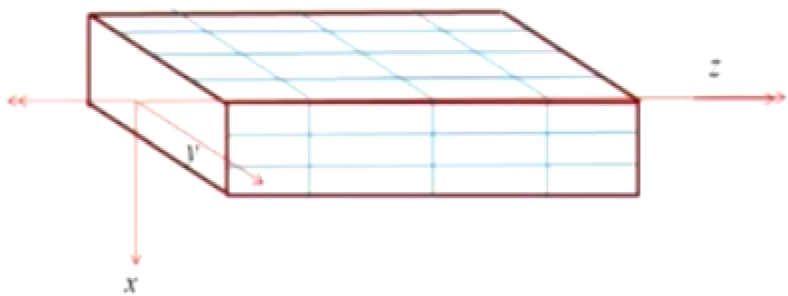
$$F_z = \int_A \sigma_{xz} dA \quad M_y = \int_A z\sigma_{xx} dA$$

Now for many civil engineering applications we come across beam elements which have rectangular cross section, so twisting of rectangular cross section is a lot more complicated problem than twisting of a circular cross section so we want to now develop a theory for torsion of a rectangular cross section, a bar with a rectangular cross section so as a prelude we can see we would need some of this if you consider a rectangular bar on the phase A, B, C, D suppose there are axial force FX and a twisting moment MX and shear force FY and FZ and twisting MY, and bending MY and MZ we will just do a simple exercise of writing the boundary conditions on A, B, C, D in terms of stresses that are acting on A, B, C, D, so this will help us to understand the notations and other conventions, so these are the axis YZ, and if I consider in a infinitesimal element here the forces acting on this element would be sigma XZ into DA, sigma XY into DA.

Now there are 6 stress resultants acting on this phase A, B, C, D, so now if you consider FX integral of sigma XX DA must be equal to 0, FX, sigma XX ends on this A, B, C, D, normal to A, B, C, D, this integral must be equal to FX. MX which is the twisting moment must be given by this, that is you have to take moment about this point sigma XY into this distance, sigma XZ into DA into this distance will give this and that must be equal to MX. Now shear force FY must be sigma XY into DA integral or A, MZ is Y sigma XX DA and FZ is sigma XZ DA integral or A, MY is Z sigma XX DA, so you can you know find out why all these are true by drawing simple sketches this will prove to be important as we proceed.

Torsion of bars of rectangular cross section

Saint Venant's solution



It is observed in experiments that $w(x,y) \neq 0$

The assumption that

plane cross sections normal to z-axis before deformation remain plane and normal to the z-axis after deformation

is not valid.

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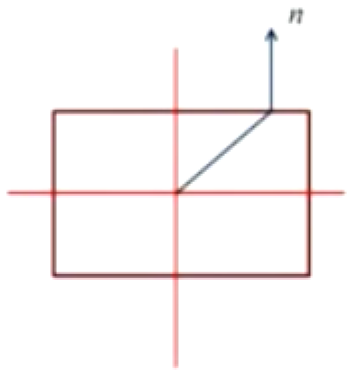
Now so let us consider the problem of torsion of bars of rectangular cross section, rectangular cross sections are often encountered in frame structures most of the beam cross sections are rectangular in civil engineering application, so this is a problem of considerable importance, so we consider a rectangular shaft as shown here and we again assume that the lateral dimensions the depth and the width are small compared with the length so that it can be treated as a line element, in actual experiments conducted on this type of elements it is observed that W X,Y is not equal to 0, the consequence of that is that the assumption that which we have made earlier for a circular bar with circular cross section, that plane cross section normal to Z axis before deformation remain plane and normal to Z axis after deformation is no longer valid, okay, this function W(X,Y) is known as warping function.



The stress field $\begin{bmatrix} 0 & 0 & -G\beta y \\ 0 & 0 & G\beta x \\ -G\beta y & G\beta x & 0 \end{bmatrix}$ does not satisfy $\overset{n}{\underset{\sim}{T}} = \sigma n = 0$. That is

$$\overset{n}{\underset{\sim}{T}} = \begin{bmatrix} 0 & 0 & -G\beta y \\ 0 & 0 & G\beta x \\ -G\beta y & G\beta x & 0 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \neq 0 \end{Bmatrix} \neq 0$$

For example

$$\begin{bmatrix} 0 & 0 & -G\beta y \\ 0 & 0 & G\beta x \\ -G\beta y & G\beta x & 0 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -G\beta y \end{Bmatrix} \neq 0$$

NPTEL

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Now we need to account for warping of cross-sections and develop the appropriate solution, moreover if we now simply assume that stress field that we obtain for circular cross-section which was of this form continues to be valid for rectangular cross-section also you can verify by simple or you know verification that this cannot satisfy boundary conditions on the boundaries, that is a lateral surfaces you can verify this for example the stress vector T tilde N is sigma N must be equal to 0 on the outer surface, so if I write this if you see here the third equation will be – G beta Y N1 + G beta X N2, there is no reason why that should be 0, okay so this stress field again will not continue to be applicable, for example if you take this N which is shown here, suppose the outward normal is 1, 0, 0, you can see that the first two equations are satisfied, but the third equation will lead to - G beta Y into 1, which is not 0, so the point is this stress field is not applicable for this problem, so you have to start from first principles through certain other arguments.

Saint Venant's Soultion

$u = -\beta yz$

$v = \beta xz$

$w = \beta\psi(x, y)$

$\Rightarrow$

- A section at a distance z from the origin rotates as a rigid body
- $\beta\psi(x, y)$ = amount by which a point in z=constant plane displaces in z direction: warping function.
- All sections undergo same amount of warping.

Unknowns: β & $\psi(x, y)$

Conditions to be satisfied

- Strain-displacements (6)
- Stress-strain (6)
- Equillibrium (3)
- Boundary conditions: $z = 0$ & $z = l$; Lateral surfaces

The solution due to Saint Venant's what it does is it assumes that U and V continue to be of the same form as was valid for circular cross section, but there is a new function of W which is nonzero, which is taken to be beta into sai X, Y, so if you assume this displacement field what it means is a section at a distance Z from the origin rotates as a rigid body, because W is now independent of Z, so no matter what is Z all sections undergo the same type of W deformation, so this is what it means. So beta into sai XY is amount by which a point in Z equal to constant plane displaces in Z direction, this is known as warping function.

Now according to this postulated displacement field all sections undergoes same amount of warping, now what are the unknowns in this assume displacement field, beta and sai XY, so now how do we determine what are the condition to be satisfied as before we have strain displacement relations 6 of them, stress strain relations 6 of them, equilibrium 3 of them and boundary conditions on Z = 0 and Z = L and on lateral surfaces, so we have to run through these calculations and we should be able to establish for this solution to be valid, what are the conditions beta and sai must satisfy.

$$u = -\beta yz$$
$$v = \beta xz$$
$$w = \beta\psi(x, y)$$

Strain field

$$\left.\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} = 0; \varepsilon_{yy} = \frac{\partial v}{\partial y} = 0 \\
\varepsilon_{zz} &= \frac{\partial w}{\partial z} = 0 \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\beta z + \beta z = 0 \\
\gamma_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \beta\left(-y + \frac{\partial \psi}{\partial x}\right) \\
\gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \beta\left(x + \frac{\partial \psi}{\partial y}\right)
\end{aligned}\right\} \varepsilon = \begin{bmatrix} 0 & 0 & \frac{\beta}{2}\left(-y + \frac{\partial \psi}{\partial x}\right) \\ 0 & 0 & \frac{\beta}{2}\left(x + \frac{\partial \psi}{\partial y}\right) \\ \frac{\beta}{2}\left(-y + \frac{\partial \psi}{\partial x}\right) & \frac{\beta}{2}\left(x + \frac{\partial \psi}{\partial y}\right) & 0 \end{bmatrix}$$

So we will start with the displacement field, assume displacement field and compute the strains, so epsilon XX is 0 because this is independent of X, epsilon YY this is independent of Y therefore that is 0, epsilon ZZ W is independent of Z that is 0, gamma XY turns out to be 0, gamma XZ and gamma YZ are now obtained, earlier we had - beta Y and beta X, now in addition to this we have now beta into dou sai by dou X, beta into dou sai by dou Y, so the strain field now is given by this matrix, the strain tensor is this. So now the new entry is essentially this terms originating from sai, okay.

$$
\varepsilon = \begin{bmatrix} 0 & 0 & \frac{\beta}{2}\left(-y+\frac{\partial\psi}{\partial x}\right) \\ 0 & 0 & \frac{\beta}{2}\left(x+\frac{\partial\psi}{\partial y}\right) \\ \frac{\beta}{2}\left(-y+\frac{\partial\psi}{\partial x}\right) & \frac{\beta}{2}\left(x+\frac{\partial\psi}{\partial y}\right) & 0 \end{bmatrix} \Rightarrow \begin{cases} e = \varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz} = 0 \\ \sigma_{xx} = \lambda e + 2G\varepsilon_{xx} = 0 \\ \sigma_{yy} = \lambda e + 2G\varepsilon_{yy} = 0 \\ \sigma_{zz} = \lambda e + 2G\varepsilon_{zz} = 0 \\ \sigma_{xy} = 2G\varepsilon_{xy} = 0 \\ \sigma_{xz} = 2G\varepsilon_{xz} = G\beta\left(-y+\frac{\partial\psi}{\partial x}\right) \\ \sigma_{yz} = 2G\varepsilon_{yz} = G\beta\left(x+\frac{\partial\psi}{\partial y}\right) \end{cases}
$$

NPTEL

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Now for this strain tensor we can compute the stress field again using the constitutive law valid for isotropic material and we get stress components sigma XX, sigma YY and sigma ZZ and sigma XY are all 0, and the two shearing stress components are obtained in terms of beta and sai as shown here.

$$\Rightarrow \sigma = \begin{bmatrix} 0 & 0 & G\beta\left(-y+\frac{\partial\psi}{\partial x}\right) \\ 0 & 0 & G\beta\left(x+\frac{\partial\psi}{\partial y}\right) \\ G\beta\left(-y+\frac{\partial\psi}{\partial x}\right) & G\beta\left(x+\frac{\partial\psi}{\partial y}\right) & 0 \end{bmatrix}$$

Equillibrium equations

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}=0 \Rightarrow 0+0+\frac{\partial}{\partial z}\left\{G\beta\left(-y+\frac{\partial\psi}{\partial x}\right)\right\}=0 \ (\text{OK})$$

$$\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{zy}}{\partial z}=0 \Rightarrow 0+0+\frac{\partial}{\partial z}\left\{G\beta\left(x+\frac{\partial\psi}{\partial y}\right)\right\}=0 \ (\text{OK})$$

$$\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}=0$$

$$\Rightarrow \frac{\partial}{\partial x}\left\{G\beta\left(-y+\frac{\partial\psi}{\partial x}\right)\right\}+\frac{\partial}{\partial z}\left\{G\beta\left(x+\frac{\partial\psi}{\partial y}\right)\right\}=0$$

$$\Rightarrow \nabla^2\psi=0$$

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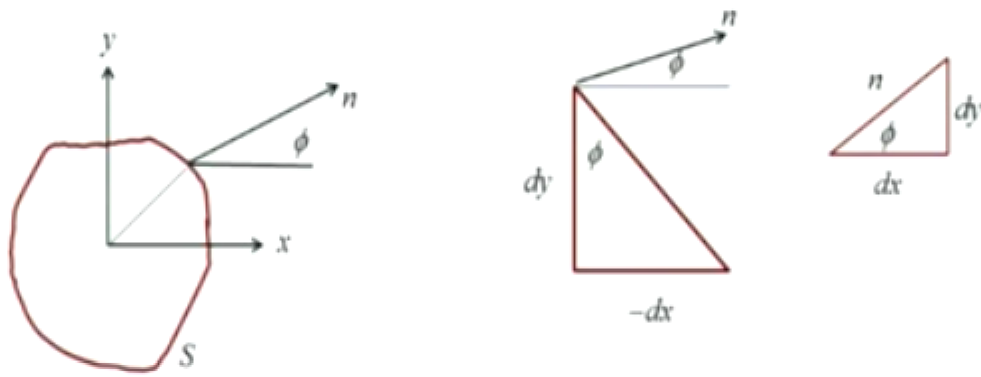
Now we go to now the equilibrium equation this is the stress matrix, so we can see that we consider individual terms in the equilibrium equations sigma XX is 0 therefore this is 0, YX is 0 therefore this is 0, ZX is independent of Z therefore this is also 0, therefore the first equilibrium equation condition is satisfied, similarly you can verify the second equilibrium equation also is satisfied. When we come to the third equilibrium equation we have dou sigma XZ / dou X, that is dou / dou X of this term G beta - Y + dou sai / dou X + sigma YZ terms, sigma ZZ is 0 therefore this term contributes this is 0. Now this first two terms get cancelled with each other I mean they are zeroes, and we get the equation for sai is del square sai = 0, that means the function associated with warping must satisfy this Laplace equation, for equilibrium to be satisfied.



$$n_1 = \cos\phi = \frac{dy}{ds} = \frac{dx}{dn}$$

$$n_2 = \sin\phi = -\frac{dx}{ds} = \frac{dy}{dn}$$

$$n_3 = 0$$

$$\overset{n}{\underset{\sim}{T}} = \sigma n$$

$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & G\beta\left(-y+\frac{\partial\psi}{\partial x}\right) \\ 0 & 0 & G\beta\left(x+\frac{\partial\psi}{\partial y}\right) \\ G\beta\left(-y+\frac{\partial\psi}{\partial x}\right) & G\beta\left(x+\frac{\partial\psi}{\partial y}\right) & 0 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

Now how about the boundary conditions, so boundary conditions on the lateral surfaces again we consider the surface traction is given by sigma N and we consider the geometry, we consider a point on the surface as shown here, and N is a unit outward normal so this won't lie on this plane as it was for, as it was happening for a circular rod because this outward normal can be different, so we have represented this data in a slightly expanded diagram here, and this is again some of the detail that will be needing, so using these details shown in this figure we can see that N1 is cos phi which is DY/DX which is DX/DN, N2 which is sin phi - DX /DS which is DY/DN, N3 is 0 because Z axis coincide for the outward normal is having no components there.

$$\overset{n}{\underset{\sim}{T}} = \sigma n = 0$$

$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & G\beta\left(-y + \frac{\partial \psi}{\partial x}\right) \\ 0 & 0 & G\beta\left(x + \frac{\partial \psi}{\partial y}\right) \\ G\beta\left(-y + \frac{\partial \psi}{\partial x}\right) & G\beta\left(x + \frac{\partial \psi}{\partial y}\right) & 0 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

$$\Rightarrow G\beta\left(-y + \frac{\partial \psi}{\partial x}\right) n_1 + G\beta\left(x + \frac{\partial \psi}{\partial y}\right) n_2 = 0$$

$$\Rightarrow \frac{dx}{dn}\frac{\partial \psi}{\partial x} + \frac{dy}{dn}\frac{\partial \psi}{\partial y} = n_1 y - n_2 x$$

$$\frac{d}{dn}\psi(x,y) = y\frac{dy}{ds} + x\frac{dx}{ds} = \frac{1}{2}\frac{d}{ds}\left(x^2 + y^2\right) = f(s)$$

So now the boundary condition to be satisfied is given by this okay, now N1, N2, N3 are as shown here so if I now substitute for that and expand that out I get this must be equal to 0, so now the first two equations are satisfied, the second one we need to, the last one we need to consider that is given by this. Now a slight amount of, slight rearrangement of these terms and simplifying we will see that the boundary condition actually is given by D / DN of sai is some F(s) where F(s) is this function N1Y- N2X on the surface S, what is S? S is the boundary, okay on which I am imposing the boundary conditions. So now the sai equation therefore the equation governing sai therefore is given to del square sai = 0, and D sai / DN is F(s), so this are

$$\left.\begin{aligned}&\nabla^2\psi = 0\\&\frac{d\psi}{dn} = f(s) \text{ on S}\end{aligned}\right\}\text{Neumann problem}$$

Remark

If the section is circular, we have $x^2 + y^2 = c \Rightarrow \frac{d}{ds}\left(x^2 + y^2\right) = \frac{d}{ds}c = 0$

$\nabla^2\psi = 0$ with $\frac{d\psi}{dn} = 0$ on S $\Rightarrow \psi = 0 \Rightarrow w = 0.$

This agrees with the previous solution.

Boundary condtions on $z = 0$ and $z = l$

$$\int_A \sigma_{xz}\, dA = 0$$

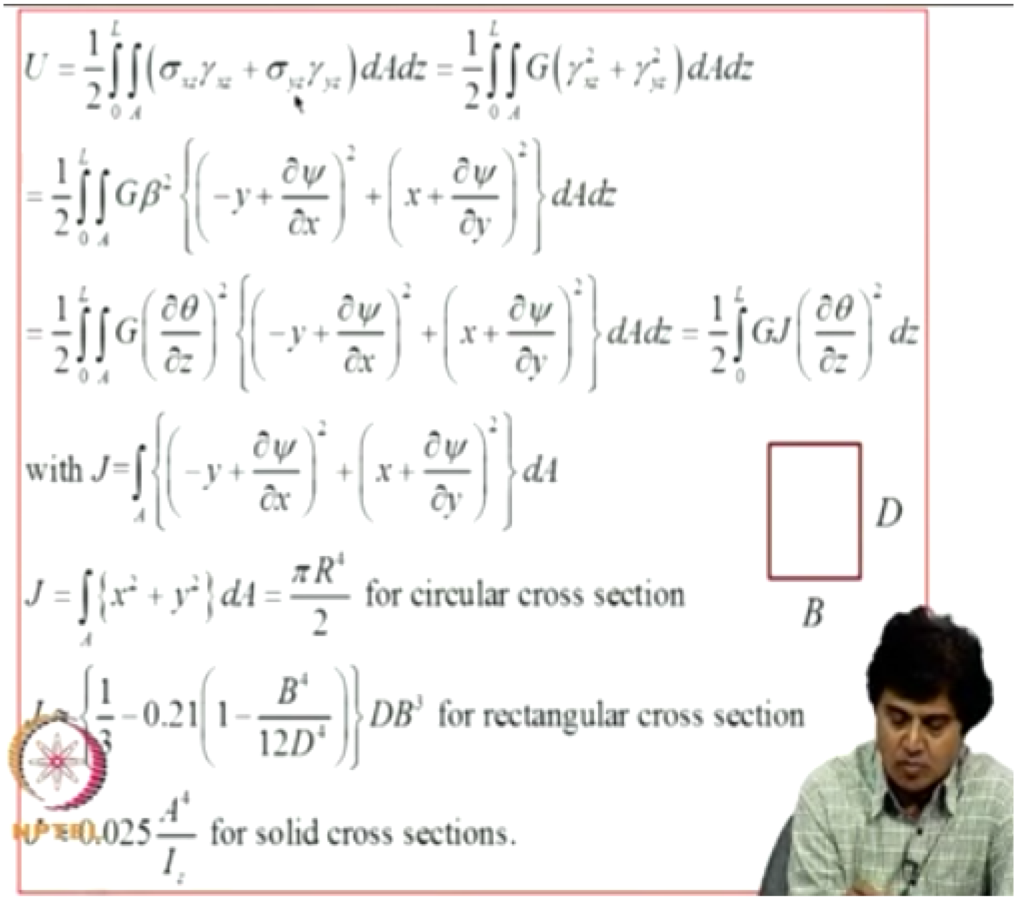
$$\int_A \sigma_{yz}\, dA = 0$$

$$\int_A \left(x\sigma_{yz} - y\sigma_{xz}\right) dA = M$$

NPTEL

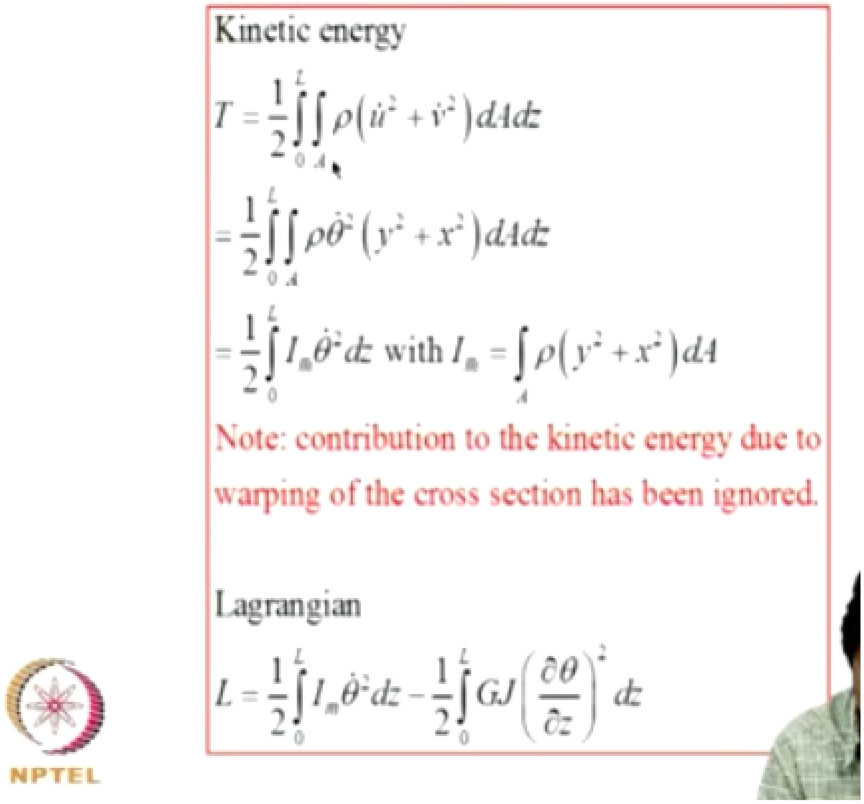
Neumann problem, in partial differential equations and there are techniques available to tackle this problem.

Now if the section is circular that is when X square + Y square is C, we see that D / DX of X square + Y square is 0, and the governing equation will become del square sai = 0, DC/DN = 0 on S, so this would mean sai = 0 which means W = 0, so that means we recover the earlier solution that we obtained in the in the previous discussion, the boundary condition on Z = 0 and Z = N, axial thrust is 0 therefore this is 0, this is sigma YZ DA is 0, because there is no shear, this is 0 now the moment is this, okay, so from this I will be able to get beta which is still an unknown, so this completes the elasticity solution, but now let us return to the finite element

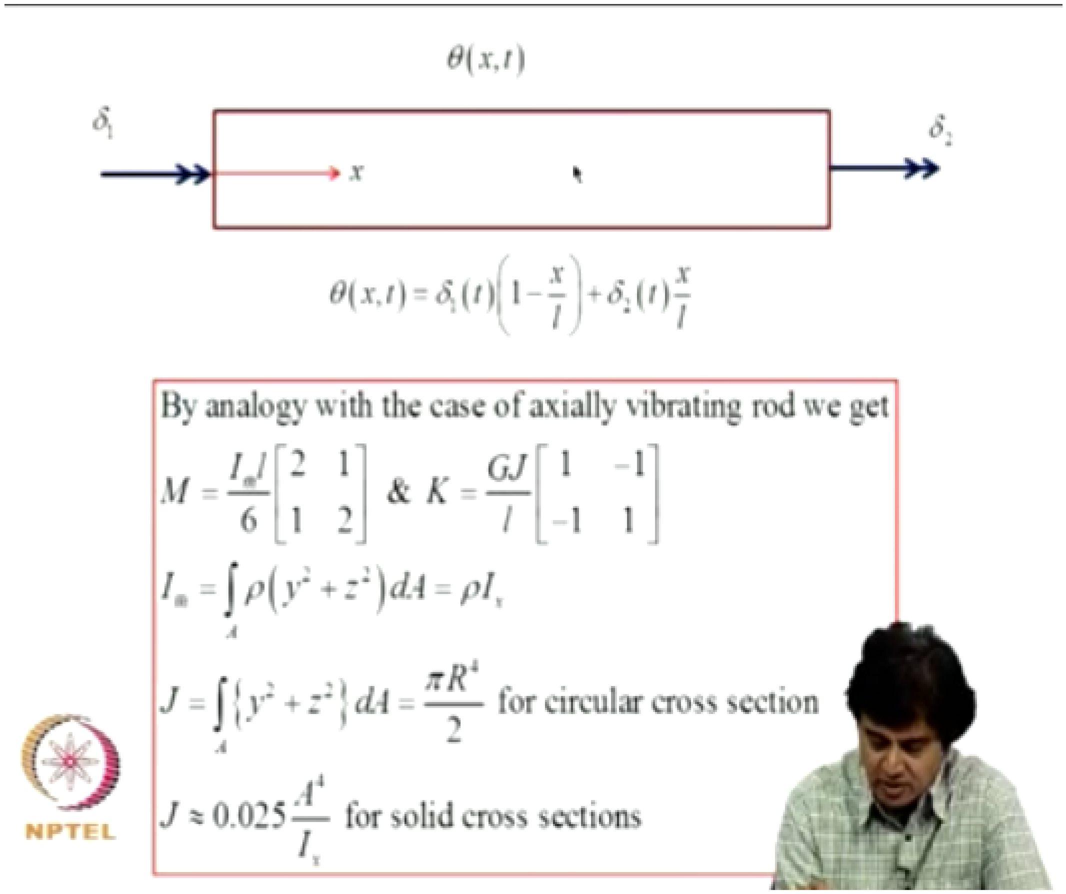


formulation based on the information that we have, I mean the formulation that we have made for using elasticity equation, so again there are two nonzero strain components and two nonzero stress components, so if you substitute that the expression for strain energy is given by this as before but now gamma XZ will be given, there will be warping function that will appear in this, beta is dou theta by dou Z which is the rate of twist, that is twist per unit length so for beta square I will write dou theta by dou Z square and I will retain the other terms here.

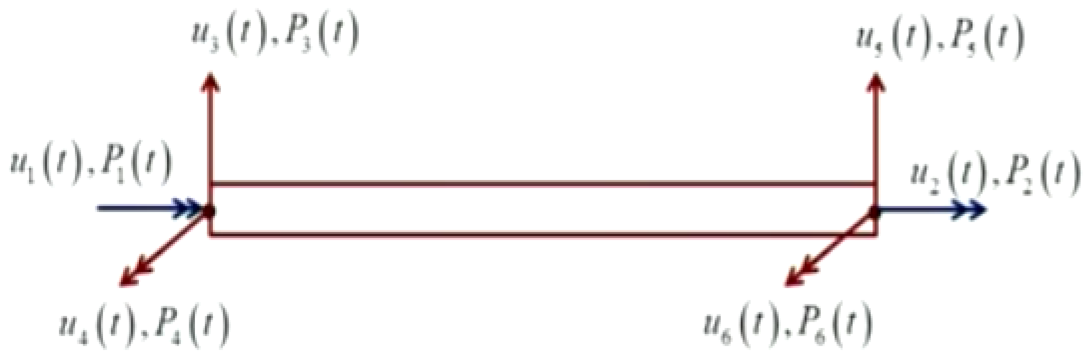
Now I will write this expression as 1/2 0 to L GJ dou theta / dou Z whole square DZ where this J is now given by this expression, this includes now warping function, so you should have solved that Neumann equation before you come to this for finding this so-called torsional constant. For a circular cross-section we have seen that we derived that J is PI R to the power of 4/2, for rectangular cross-section with dimensions B and D an approximate expression for the torsional constant is given here, so you can use this, this actually involves as I said the solution to the Neumann’s problem, okay and for others solid sections you can use this as an approximation.



Now how about kinetic energy? So kinetic energy we actually neglect the contribution to the kinetic energy due to warping, so under that assumption the kinetic energy is given in terms of velocity is associated with U and V, and this is this and I get the same expression as we got for circular rod except that this IM bar is now to be integrated or the actual cross section which is not necessarily circular, so again let us note that contribution to kinetic energy due to warping of the cross section has been ignored, so now the Lagrangian is at T - V which is given by this, so if you compare this to the Lagrangian obtained for actually vibrating rod it is quite similar IM bar is M, for axially vibrating rod it was 1/2 0 to L MU dot square DX - AE instead of GJ we had AE and theta was U, okay, so it is quite similar so by analogy therefore again I can



write the mass matrix and stiffness matrix for rod with noncircular cross section to be given by this. J is now given by this okay and for a square sections, I have rectangular section I have given the additional this one, so you have to use this, anyway J is mathematically given by this, and IM bar is rho into IX, okay, so this completes the formulation of stiffness matrix for a rod which is twisting, if rod is circular, the analysis is simpler, broad is rectangular you have to go through some details based on elasticity theory.



2-noded element with 3 dofs per node

$$\theta(x,t) = u_1(t)\psi_1(x) + u_2(t)\psi_2(x)$$
$$v(x,t) = u_3(t)\phi_1(x) + u_4(t)\phi_2(x) + u_5(t)\phi_3(x) + u_6(t)\phi_4(x)$$

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So now we can go back to the grid structure and construct element that can be used for analyzing grids, so the analysis of grids require combination of actions due to twisting and action due to bending, so now U1 and U2 are the twisting degrees of freedom and U3, U4, U5, U6 are the degrees of freedom due to bending, so we can assume theta which is the field variables are theta and V, so this can be interpolated in terms of the nodal values so I get for theta U1(t) into sai 1(x) + U2(t) into sai 2(x), whereas V(x,t) the degrees of freedom are U3, U4 U5, U6, and phi 1, phi 2, phi 3, phi 4 are the cubic polynomials that we are used for developing the beam element, sai 1 and sai 2 are linear polynomials.

$$U=\underbrace{\frac{1}{2}\int_0^L GJ\left(\frac{\partial\theta}{\partial x}\right)^2 dx}_{\text{Twisting}}+\underbrace{\frac{1}{2}\int_0^L EI\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx}_{\text{Bending}}$$

$$\left(\text{with } J=\int_A\left\{\left(-y+\frac{\partial\psi}{\partial x}\right)^2+\left(z+\frac{\partial\psi}{\partial z}\right)^2\right\}dA\right)$$

$$T=\underbrace{\frac{1}{2}\int_0^L I_m\dot{\theta}^2 dx}_{\text{Twisting}}+\underbrace{\frac{1}{2}\int_0^L m\dot{v}^2 dt}_{\text{Bending}}$$

$$\left(\text{with } I_m=\int_A\rho\left(y^2+z^2\right)dA\right)$$

$$\theta(x,t)=u_1(t)\psi_1(x)+u_2(t)\psi_2(x)$$

$$v(x,t)=u_3(t)\phi_1(x)+u_4(t)\phi_2(x)+u_5(t)\phi_3(x)+u_6(t)\phi_4(x)$$

$$L=T-U;L=L\left\{u_1(t),u_2(t),\cdots,u_6(t),\dot{u}_1(t),\dot{u}_2(t),\cdots,\dot{u}_6(t)\right\}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{u}_i}\right)-\frac{\partial L}{\partial u_i}=0;i=1,2,\cdots,6\Rightarrow M\ddot{U}+KU=0$$

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So I will now substitute that we have the expression for strain energy and kinetic energy and I will substitute this and found the Lagrangian, now the Lagrangian will have 6 generalized coordinates U1, U2, U3 up to U6 and their derivatives so if I run the Euler-Lagrange's equation for I = 1 to 6, I get the element equation of motion as MU double dot + KU = 0.

$$K = \begin{bmatrix} \frac{GJ}{l} & -\frac{GJ}{l} & 0 & 0 & 0 & 0 \\ -\frac{GJ}{l} & \frac{GJ}{l} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{12EI}{l^3} & \frac{6EI}{l^2} & -\frac{12EI}{l^3} & \frac{6EI}{l^2} \\ 0 & 0 & \frac{6EI}{l^2} & \frac{4EI}{l} & -\frac{6EI}{l^2} & \frac{2EI}{l} \\ 0 & 0 & -\frac{12EI}{l^3} & -\frac{6EI}{l^2} & \frac{12EI}{l^3} & -\frac{6EI}{l^2} \\ 0 & 0 & \frac{6EI}{l^2} & \frac{2EI}{l} & -\frac{6EI}{l^2} & \frac{4EI}{l} \end{bmatrix}$$

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So now that M and K matrices can be written out so if I now the numbering scheme that we had adopted 1 and 2 are twisting degrees of freedom and 3, 4, 5, 6 were bending so the K matrix in this case has this you know form and you can see that this is the contribution due to twisting, and this is contribution due to bending. Now actually in application and this is a mass matrix

$$M = \begin{bmatrix} \frac{I_m l}{3} & \frac{I_m l}{6} & 0 & 0 & 0 & 0 \\ \frac{I_m l}{6} & \frac{I_m l}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{156ml}{420} & \frac{22ml^2}{420} & \frac{54ml}{420} & -\frac{13ml^2}{420} \\ 0 & 0 & \frac{22ml^2}{420} & \frac{4ml^3}{420} & \frac{13ml^2}{420} & -\frac{3ml^3}{420} \\ 0 & 0 & \frac{54ml}{420} & \frac{13ml^2}{420} & \frac{156ml}{420} & -\frac{22ml^2}{420} \\ 0 & 0 & -\frac{13ml^2}{420} & -\frac{3ml^3}{420} & -\frac{22ml^2}{420} & \frac{4ml^3}{420} \end{bmatrix}$$

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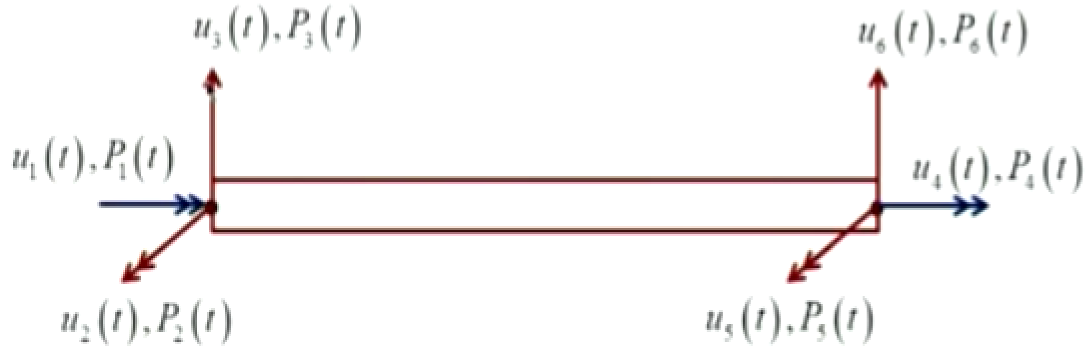
again this is the contribution due to twisting and this is the contribution due to bending, now in actual application we won't be using this type of numbering scheme because we need to know transform the matrices K and M to the global coordinate system it would be helpful if we number the degrees of freedom node wise, that is I will call this as 1, 2, 3 and 4, 5, 6 that means



2-noded element with 3 dofs per node

$$\theta(x,t) = u_1(t)\psi_1(x) + u_4(t)\psi_2(x)$$
$$v(x,t) = u_3(t)\phi_1(x) + u_2(t)\phi_2(x) + u_6(t)\phi_3(x) + u_5(t)\phi_4(x)$$

NPTEL

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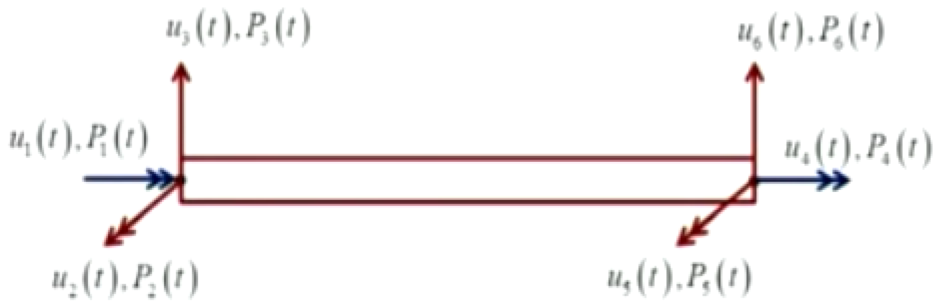
the sequence is 1, 2, 3 here, 4, 5, 6 there, if we do that we get the same argument it is just the rearrangement of terms nothing conceptually new, in which case the K matrix and M matrix will have this form which is more conventionally used, this is what you will find in many

$$K = \frac{EI}{l^3}\begin{bmatrix} \frac{JGl^2}{EI} & 0 & 0 & -\frac{JGl^2}{EI} & 0 & 0 \\ 0 & 4l^2 & 6l & 0 & 2l^2 & -6l \\ 0 & 6l & 12 & 0 & 6l & -12 \\ -\frac{JGl^2}{EI} & 0 & 0 & \frac{JGl^2}{EI} & 0 & 0 \\ 0 & 2l^2 & 6l & 0 & 4l^2 & -6l \\ 0 & -6l & -12 & 0 & -6l & 12 \end{bmatrix}$$

$$M = \frac{ml}{420}\begin{bmatrix} \frac{140I_{\theta}}{m} & 0 & 0 & \frac{70I_{\theta}}{m} & 0 & 0 \\ 0 & 4l^2 & 22l & 0 & -3l^2 & 13l \\ 0 & 22l & 156 & 0 & -13l & 54 \\ \frac{70I_{\theta}}{m} & 0 & 0 & \frac{140I_{\theta}}{m} & 0 & 0 \\ 0 & -3l^2 & -13l & 0 & 4l^2 & -22l \\ 0 & 13l & 54 & 0 & -22l & 156 \end{bmatrix}$$

textbooks, so this is what these are the stiffness and mass matrices. So there are 6 by 6 matrices symmetric and so on and so forth.

Equivalent nodal forces



$$P_1(t) = \int_0^L m_x(x,t)\psi_1(x)dx; P_4(t) = \int_0^L m_x(x,t)\psi_2(x)dx$$

$$P_3(t) = \int_0^L f_y(x,t)\phi_1(x)dx; P_2(t) = \int_0^L f_y(x,t)\phi_2(x)dx$$

$$P_6(t) = \int_0^L f_y(x,t)\phi_3(x)dx; P_5(t) = \int_0^L f_y(x,t)\phi_4(x)dx$$

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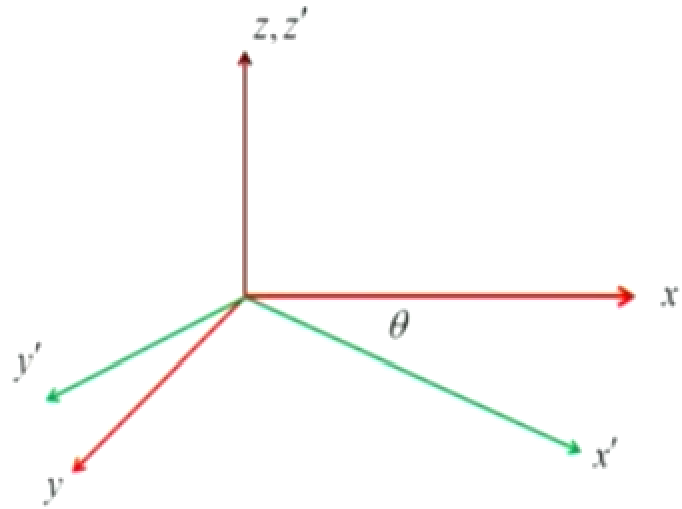
Now how do we compute equivalent nodal forces? The logic is similar to what we did if there is a distributed to twisting moment MX(x,t) throughout the rod the P1 and P4 are the couples at the two ends they are given by this, P3 and P2 are the shearing forces they are given by here these quantities and similarly the bending moments are given here okay.

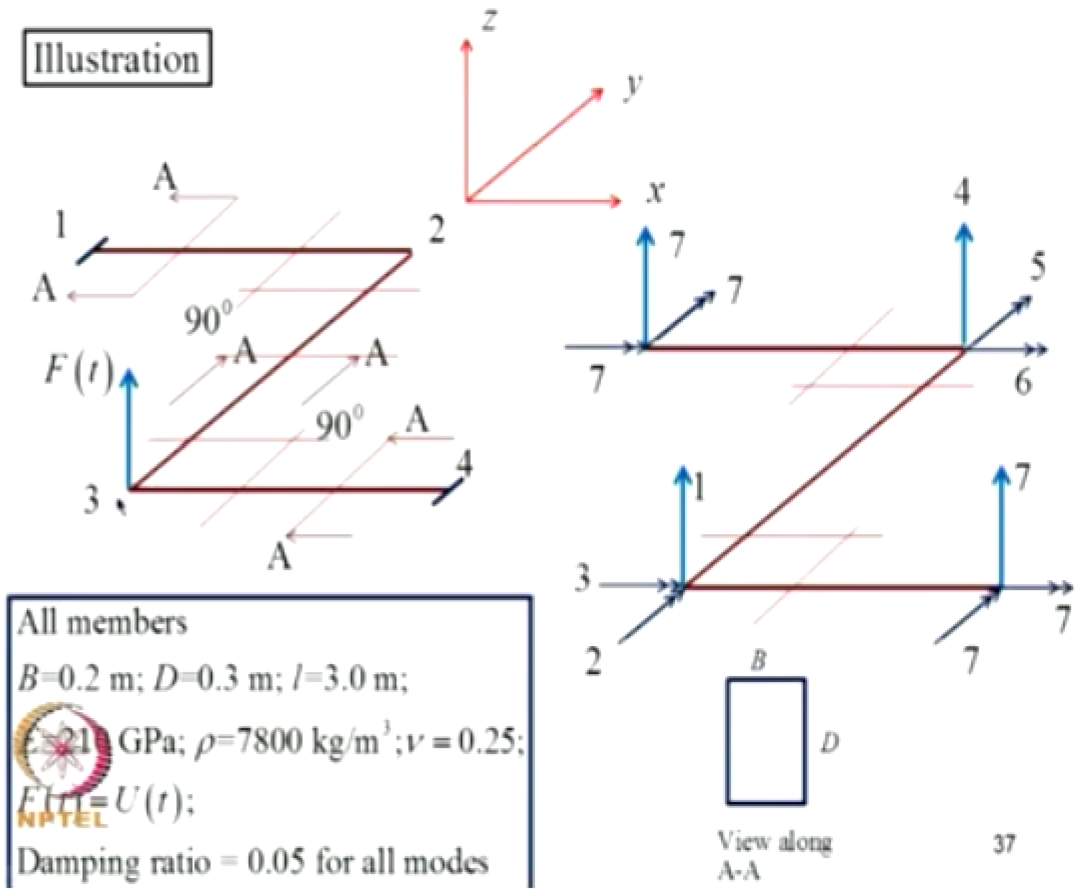
Coordinate transformation



$$T = \begin{bmatrix} T_0 & 0 \\ 0 & T_0 \end{bmatrix}; T_0 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\bar{u} = T^t u; \bar{K} = T^t K T; \bar{M} = T^t M T$$

Now the coordinate transformation matrix from going from local to coordinate, local to global coordinate system will have 2 segments T naught and 3 by 3 here, 3 by 3 here, and T naught itself is cos theta, sin theta, - sin theta cos theta and these are 0, this is 1, so the displacement field U bar is T transpose U, K bar is T transpose KT, M bar is T transpose MT, so this is quite similar to what we have done for a planar, a beam element.



Now a quick illustration we will consider a grid structure which is made up of 3 members shown here 1 2, 2 3, 3 4 and these 3 members lie in the XY plane and Z is an outward normal, and suppose if this is carrying a load F(t) in the Z direction and these are all at 90 degrees and the properties are mentioned here and the cross-section is this, and this is a nomenclature for the degrees of freedom so the degrees of freedom here are 3 degrees of freedom at node 2, and 3 degrees of freedom at node 3, node 1 and 4 are all, displacements are all 0 because the beams are clamped there, so I have named all those degrees of freedom as 7, so 1, 2, 3, 4, 5, 6 are the degrees of freedom, so you can compute the various sectional properties IX, IY, IZ, J, JG,

$I_x = 4.5000e\text{-}004 \text{ m}^4$

$I_y = 2.0000e\text{-}004 \text{ m}^4$

$I_z = 6.5000e\text{-}004 \text{ m}^4$

$J = 4.6953e\text{-}004 \text{ m}^4$

$JG = 3.9441e{+}007 \text{ Nm}^2$

$I_m = 5.0700 \text{ kgm}$

$EI = 9.4500e{+}007 \text{ Nm}^2$

$m = 468 \text{ kg/m}$

etcetera, we have to, these we will be needing in our formulation and I have formulated the K and M matrices, I leave it as an exercise for you to verify that these are correct, and after imposing the boundary conditions this is the reduced stiffness and mass matrices, and if you do

Element matrices

$$K = 10^8 \begin{bmatrix} 0.1315 & 0 & 0 & -0.1315 & 0 & 0 \\ 0 & 1.2600 & 0.6300 & 0 & 0.6300 & -0.6300 \\ 0 & 0.6300 & 0.4200 & 0 & 0.6300 & -0.4200 \\ -0.1315 & 0 & 0 & 0.1315 & 0 & 0 \\ 0 & 0.6300 & 0.6300 & 0 & 1.2600 & -0.6300 \\ 0 & -0.6300 & -0.4200 & 0 & -0.6300 & 0.4200 \end{bmatrix}$$

$$M = \begin{bmatrix} 5.0700 & 0 & 0 & 2.5350 & 0 & 0 \\ 0 & 120.3429 & 220.6286 & 0 & -90.2571 & 130.3714 \\ 0 & 220.6286 & 521.4857 & 0 & -130.3714 & 180.5143 \\ 2.5350 & 0 & 0 & 5.0700 & 0 & 0 \\ 0 & -90.2571 & -130.3714 & 0 & 120.3429 & -220.6286 \\ 0 & 130.3714 & 180.5143 & 0 & -220.6286 & 521.4857 \end{bmatrix}$$

the eigenvalue analysis the natural frequency that I have got for the 6, the 6 natural frequencies

Global matrices after imposition of boundary conditions

$$K = 10^8 \begin{array}{c} \\ \end{array} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$$K = 10^8 \begin{bmatrix} 0.8400 & 0 & 0.0000 & -0.4200 & -0.6300 & 0.0000 \\ 0 & 2.5200 & -0.0000 & 0.6300 & 0.6300 & -0.0000 \\ 0.0000 & -0.0000 & 0.2629 & -0.0000 & -0.0000 & -0.1315 \\ -0.4200 & 0.6300 & -0.0000 & 0.8400 & 0 & -0.0000 \\ -0.6300 & 0.6300 & -0.0000 & 0 & 2.5200 & -0.0000 \\ 0.0000 & -0.0000 & -0.1315 & -0.0000 & -0.0000 & 0.2629 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$$M = 10^3 \begin{bmatrix} 120.3429 & -0.0000 & 220.6286 & -90.2571 & 0.0000 & 130.3714 \\ -0.0000 & 5.0700 & -0.0000 & 0.0000 & 2.5350 & -0.0000 \\ 220.6286 & -0.0000 & 521.4857 & -130.3714 & 0.0000 & 180.5143 \\ -90.2571 & 0.0000 & -130.3714 & 120.3429 & -0.0000 & -220.6286 \\ 0.0000 & 2.5350 & 0.0000 & -0.0000 & 5.0700 & 0.0000 \\ 130.3714 & -0.0000 & 180.5143 & -220.6286 & 0.0000 & 521.4857 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

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for the system are shown here and this is the model matrix,

$$\Phi = \begin{bmatrix} 0.0192 & -0.0230 & -0.0076 & 0.0000 & -0.0117 & 0.0000 \\ -0.0068 & -0.0061 & -0.0391 & 0.0000 & 0.0615 & -0.0000 \\ -0.0000 & 0.0000 & -0.0000 & -0.1986 & 0.0000 & -0.2564 \\ 0.0192 & 0.0230 & -0.0076 & 0.0000 & 0.0117 & 0.0000 \\ 0.0068 & -0.0061 & 0.0391 & -0.0000 & 0.0615 & 0.0000 \\ -0.0000 & 0.0000 & -0.0000 & -0.1986 & 0.0000 & 0.2564 \end{bmatrix}$$

Natural frequencies in Hz

19.8349 55.5402 129.1772 162.0904 256.7160 362.4451

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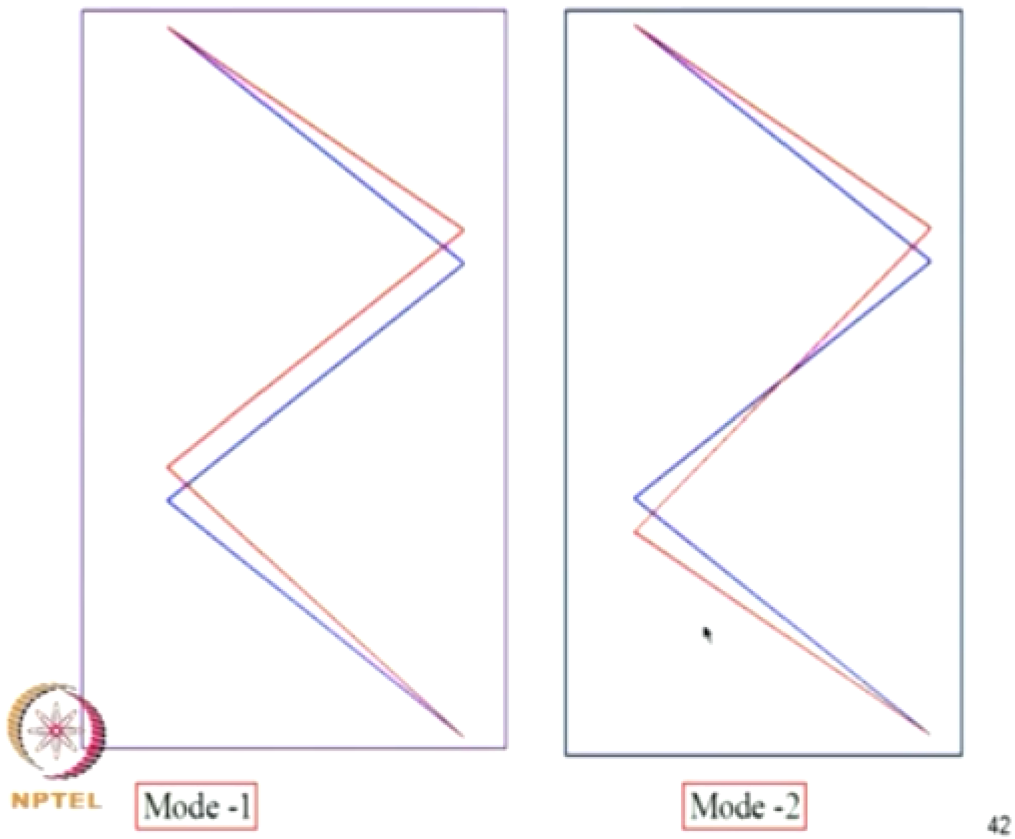
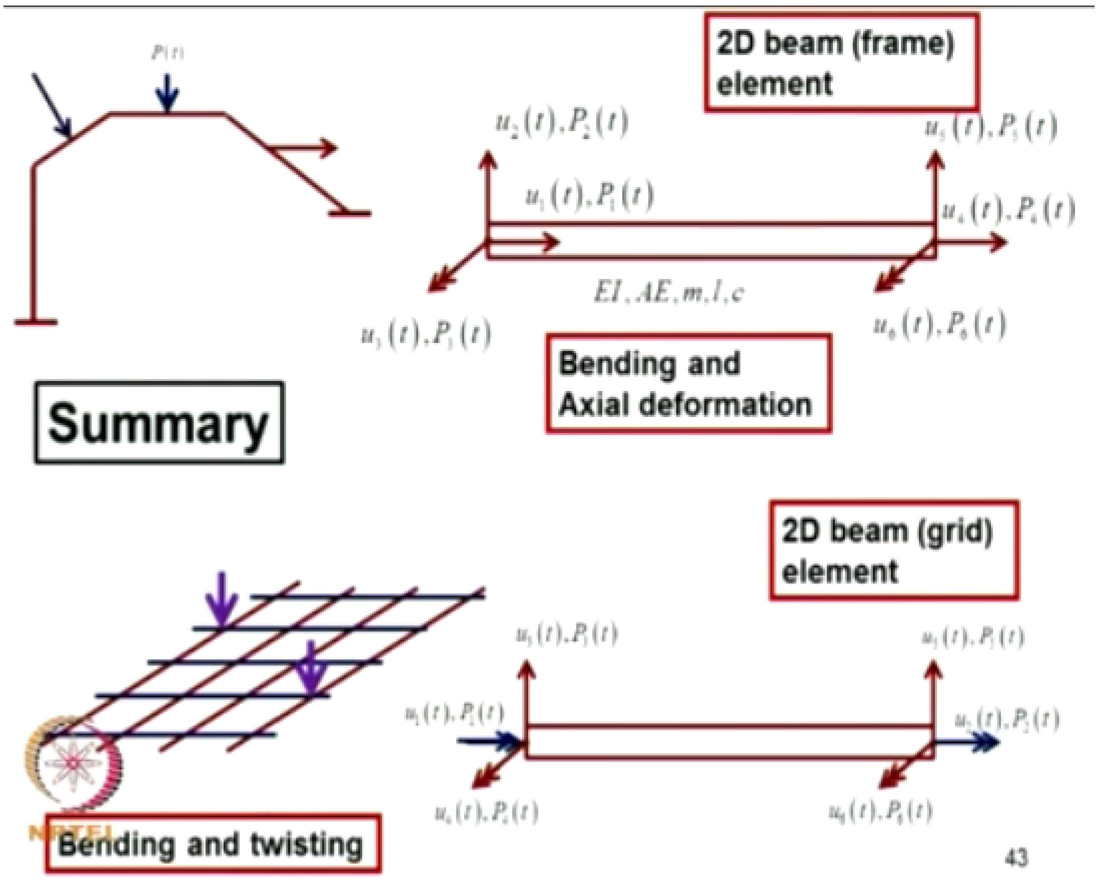
so this information is made available to you, so that if you yourself attempt to solve the problem this will provide as a check.

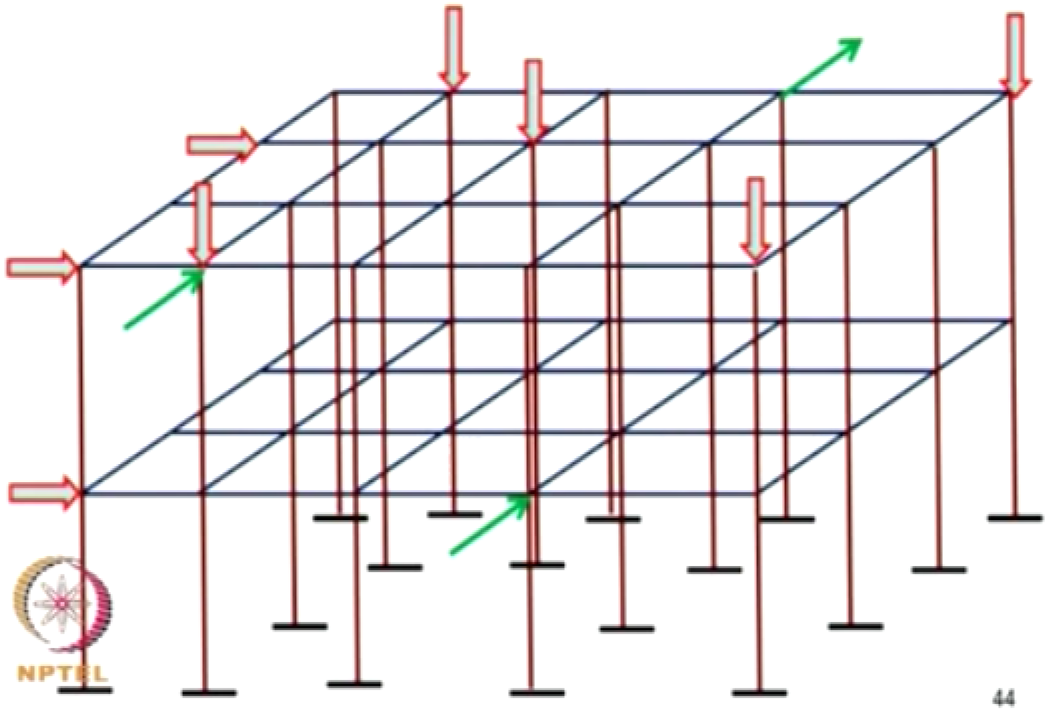


Now this is the plots of the first two modes, you can see the blue line is the un-deformed geometry and red one is a deform geometry, so this is the structure is vibrating in first mode, the structure here is vibrating in the second mode.

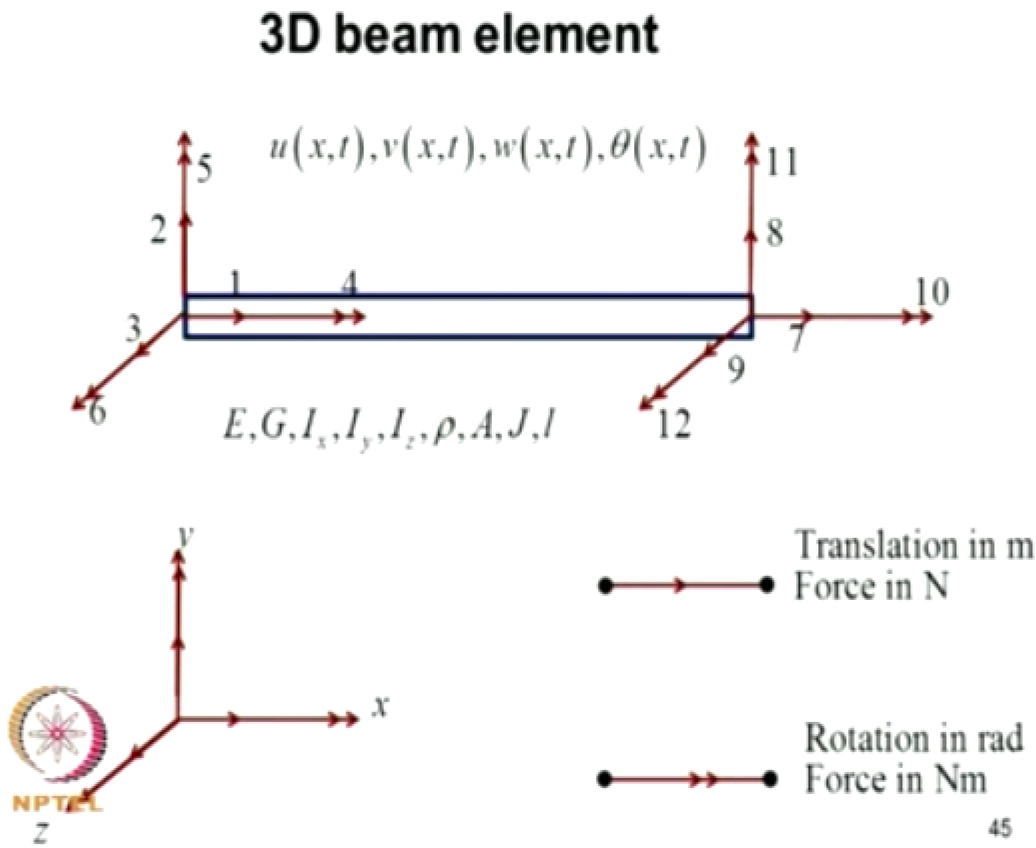


So now what we have done till now is that we have analyzed planar structures, we started with planar frames where all members were in the same plane but loads where, the loads were also in the same plane, and a typical beam element too unlike this type of frames had two axial degrees of freedom, and that was the axial deformations, and bending degrees of freedom which were U2, U3, And U5, U6, so the load carrying mechanism was through bending and axial deformation, in the discussion that we had just now we considered the so-called grid structures which are again planar structures, but here the loads act transverse to the plane of the structure, so here the load carrying mechanism is through bending and twisting, so typical grid element will have 2 twisting degrees of freedom, that is U1 and U2, and 4 bending degrees of freedom that is U3, U4, and U5, and U6, for grids we have bending and twisting, for planar frames we have bending and axial deformation.

3D beam element



Now equipped with this we can now think of how to analyze three-dimensional structures, so this is a plot of a typical building frame, idealized building frame so here the elements lie, not necessarily in the same plane, and the loads can act in all directions, for example loads are acting in plane here of these frames, outer frame, whereas these green loads are transverse to this, suppose if I draw the, so loads are in X direction, loads are in Y direction, loads are in Z direction, and the members also are lying in XZ plane, XY plane, as well as YZ plane, they can also lie in other plane for example this member could have diagonal bracing, okay, so how do we tackle this type of situation. So this is now in a way combination of planar frame and grid action, so what are this, what is a typical 3d beam element. So the load carrying mechanism



here would consist of bending in 2 planes, that is bending in YX plane, and bending in YZ plane and twisting about X, right, and axial deformations along XYZ, right, so there is axial deformation, there is bending in 2 planes and there is twisting, so a typical member therefore will have several you know field variables U(x,t) is the axial deformation, so with the degrees of freedom at the nodes being 1 and 7, V(x,t) is the bending in the plane of this figure and the degrees of freedom for that are 2, 4, 8, 10, okay, no 2 sorry, 2, 6, and 8, 12, and bending in a plane which is normal to this plane is the degrees of freedom are 3, 5, 9 and 11, okay, so there are four field variables U, V, W and theta, and this is a coordinate system I am talking about, so this indicates a single error is translation in meters or force in Newton's, double arrow is rotation in radians or force in Newton meter.

$$U = \underbrace{\frac{1}{2}\int_0^L AE\left(\frac{\partial u}{\partial x}\right)^2 dx}_{\text{Axial deformation}} + \underbrace{\frac{1}{2}\int_0^L GJ\left(\frac{\partial \theta}{\partial x}\right)^2 dx}_{\text{Twisting}}$$

$$+\underbrace{\frac{1}{2}\int_0^L EI_z\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx}_{\text{Bending@z}} + \underbrace{\frac{1}{2}\int_0^L EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx}_{\text{Bending@y}}$$

$$\left(\text{with } J = \int_A \left\{\left(-y + \frac{\partial \psi}{\partial x}\right)^2 + \left(z + \frac{\partial \psi}{\partial z}\right)^2\right\} dA\right)$$

$$T = \underbrace{\frac{1}{2}\int_0^L m\dot{u}^2 dx}_{\text{Axial deformation}} \; \underbrace{\frac{1}{2}\int_0^L I_{\bar{m}}\dot{\theta}^2 dx}_{\text{Twisting}} + \underbrace{\frac{1}{2}\int_0^L m\dot{v}^2 dt}_{\text{Bending@z}} + \underbrace{\frac{1}{2}\int_0^L m\dot{w}^2 dt}_{\text{Bending@y}}$$

$$\left(\text{with } I_{\bar{m}} = \int_A \rho\left(y^2 + z^2\right) dA\right)$$

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Now how do we analyze this problem? So what are the different sources of energies? There is energy due to axial deformation, there is energy due to twisting, energy due to bending about Z, and energy due to bending that is in plane and out-of-plane bending for the beam element, associated with this now here there is a J which is the torsional constant which is given by this expression, okay, and we have now IY and IZ also here, so J is you know GJ is the torsional rigidity AE is axial rigidity, EIZ and EIY are the flexural rigidities.

Now the kinetic energy we have kinetic energy due to axial deformation and kinetic energy due to twisting and bending about Z axis, and bending that is in plane and out of plane bending, okay, now here IM bar is actually the, this integral rho into Y square + Z Square D, now for each of these field variables U theta, V and W we can use now appropriate interpolation function, so for U(x,t) the nodal degrees of freedom are 1 and 7, so I need to interpolate to find U(x,t) for any value of X, I should use U1 and U7. Similarly for V the degrees of freedom are

$$\theta(x,t) = u_4(t)\psi_1(x) + u_{10}(t)\psi_2(x)$$
$$u(x,t) = u_1(t)\psi_1(x) + u_7(t)\psi_2(x)$$
$$v(x,t) = u_2(t)\phi_1(x) + u_6(t)\phi_2(x) + u_8(t)\phi_3(x) + u_{12}(t)\phi_4(x)$$
$$w(x,t) = u_3(t)\phi_1(x) + u_5(t)\phi_2(x) + u_9(t)\phi_3(x) + u_{11}(t)\phi_4(x)$$
$$\psi_1(x) = 1 - \frac{x}{l}; \psi_2(x) = \frac{x}{l}$$
$$\phi_1(x) = 1 - 3\frac{x^2}{l^2} + 2\frac{x^3}{l^3}; \phi_2(x) = x - 2\frac{x^2}{l} + \frac{x^3}{l^2};$$
$$\phi_3(x) = 3\frac{x^2}{l^2} - 2\frac{x^3}{l^3};; \phi_4(x) = -\frac{x^2}{l} + \frac{x^3}{l^2}$$
$$L = T - U; L \equiv L\{u_1(t), u_2(t), \cdots, u_{12}(t), \dot{u}_1(t), \dot{u}_2(t), \cdots, \dot{u}_{12}(t)\}$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_i}\right) - \frac{\partial L}{\partial u_i} = 0; i = 1, 2, \cdots, 12 \Rightarrow M\ddot{U} + KU = 0$$

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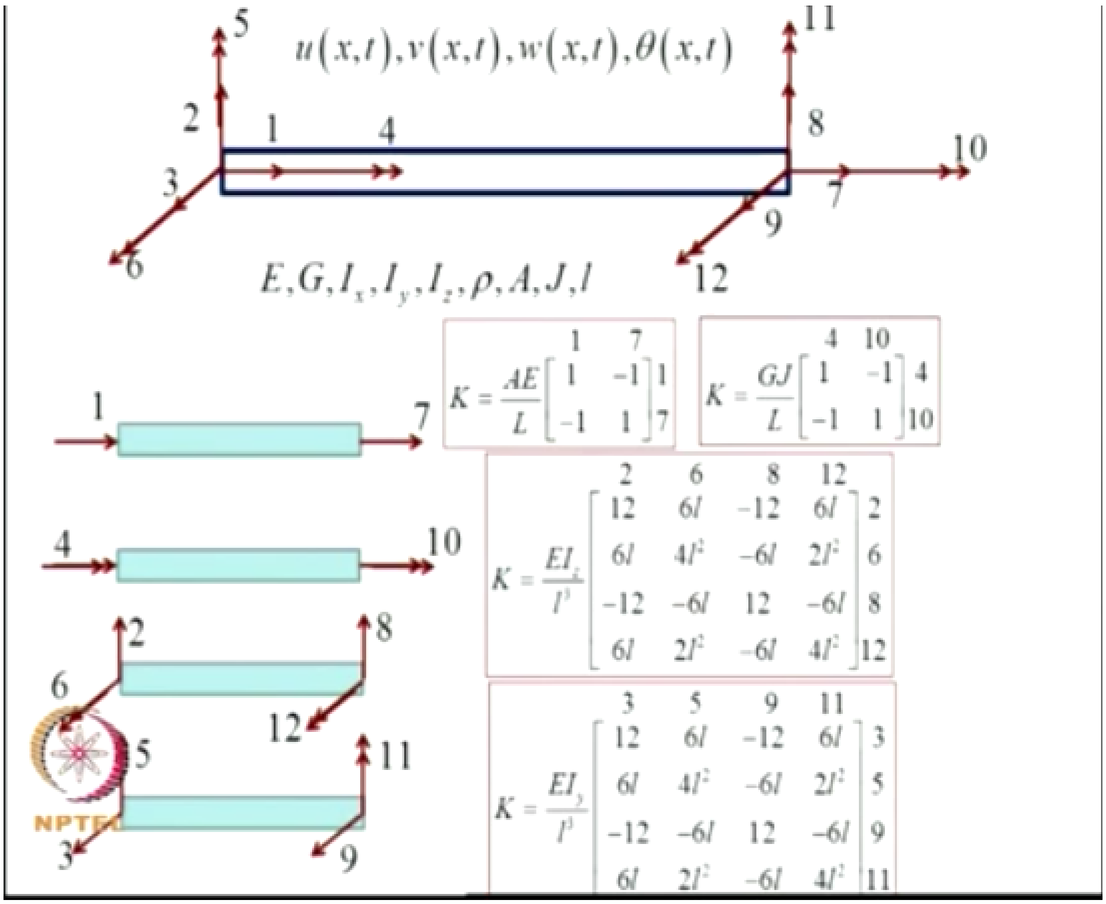
2, 6, 8 and 12, so if I do that for theta and U, theta is 4 and 10, so I get U4 sai 1(x) + U10 sai 2(x), for U which is axial deformation, U1 sai 1 + U7 sai 2. For V that is 2, 6, 8, 12, phi 1, phi 2, phi 3, phi 4, for W it is 3, 5, 9, 11 again phi 1, phi 2, phi 3 and phi 4 where sai 1 and sai 2 are the linear interpolation functions and phi 1, phi 2, phi 3, phi 4 are the cubic polynomials.

So now we can substitute these expressions into the expression for strain energy and kinetic energy, and we can form the Lagrangian. Now the Lagrangian will be now function of U1, U2 up to U12, U1 dot, U2 dot and U12 dot, so there are 12 degrees of freedom so we need to run the Euler-Lagrange's equation, for I running from 1 to 12 and that leads to the equation MU double dot + KU = 0. So we can write that 12 by 12 matrix but we can understand the structure



of that matrix by considering the individual action separately. For example if we have for axial degrees of freedom we have 1 and 7, 1 and 7 the stiffness matrix will be AE/L into this, that is for 1 and 7, for 4 and 10 we have the twisting and the stiffness matrix is GJ/L into this 1, –1, –1, 1, for bending 2, 6, 8, 12 that is in plane bending, the stiffness matrix is given by this. I is now IZ. For out-of-plane bending we have 3, 5, 9, 11, so 3, 5, 9, 11 are here this is the stiffness matrix for bending, now I is IY, so if we can club all this and put it in the 12 by 12 matrix we

$$
\begin{array}{c}
\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\
K = \begin{bmatrix}
A & & & & & & A & & & & & \\
 & B_z & & & & B_z & & B_z & & & & B_z \\
 & & B_y & & B_y & & & & B_y & & B_y & \\
 & & & T & & & & & & T & & \\
 & & B_y & & B_y & & & & B_y & & B_y & \\
 & B_z & & & & B_z & & B_z & & & & B_z \\
A & & & & & & A & & & & & \\
 & B_z & & & & B_z & & B_z & & & & B_z \\
 & & B_y & & B_y & & & & B_y & & B_y & \\
 & & & T & & & & & & T & & \\
 & & B_y & & B_y & & & & B_y & & B_y & \\
 & B_z & & & & B_z & & B_z & & & & B_z
\end{bmatrix}
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array}
\end{array}
$$

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can populate this matrix by the actual values that we can do, we'll be doing in the next class, but at the for the present we can see now what is the structure of this matrix, if you now look at 1, 7 and this, so 1 7, 7, 7, 7 1, this is axial degrees of freedom, so you can see here 1 and 7 are axial degrees of freedom.

Now similarly if you now look at 4, 10, if we now look at 4, 10 we have twisting, so you can see here, now for 2, 6, 8, 12 it is bending about Z, that is 2, 6, 8, 12 so that populates BZ, you will see BZ that, and finally if you see now 3, 5, and 9, 11 you get BY so they're here,

$$
\begin{array}{c|cccccccccccc|c}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \\
\hline
 & A & & & & & & A & & & & & & 1 \\
 & & B_z & & & & B_z & & B_z & & & & B_z & 2 \\
 & & & B_y & & B_y & & & & B_y & & B_y & & 3 \\
 & & & & T & & & & & & T & & & 4 \\
 & & & B_y & & B_y & & & & B_y & & B_y & & 5 \\
 & & B_z & & & & B_z & & B_z & & & & B_z & 6 \\
K = & A & & & & & & A & & & & & & 7 \\
 & & B_z & & & & B_z & & B_z & & & & B_z & 8 \\
 & & & B_y & & B_y & & & & B_y & & B_y & & 9 \\
 & & & & T & & & & & & T & & & 10 \\
 & & & B_y & & B_y & & & & B_y & & B_y & & 11 \\
 & & B_z & & & & B_z & & B_z & & & & B_z & 12
\end{array}
$$

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okay, so we can understand the structure of this matrix so there is essentially axial deformation twisting and bending in plane and out-of-plane bending, so these elements fall into place as I have indicated, so we will conclude this lecture at this point, in the next lecture we'll continue from here and we will actually populate this matrix with the actual numerical the correct values and then see how to proceed with assembling and trans coordinate transformations and so on and so forth for 3-dimensional structures. So we will conclude this lecture here.

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