Indian Institute of Science
Bangalore

NP-TEL
National Programme on
Technology Enhanced Learning

Course Title

Finite element method for structural dynamic And stability analyses

Lecture – 01
Equations of motion using Hamilton's principle

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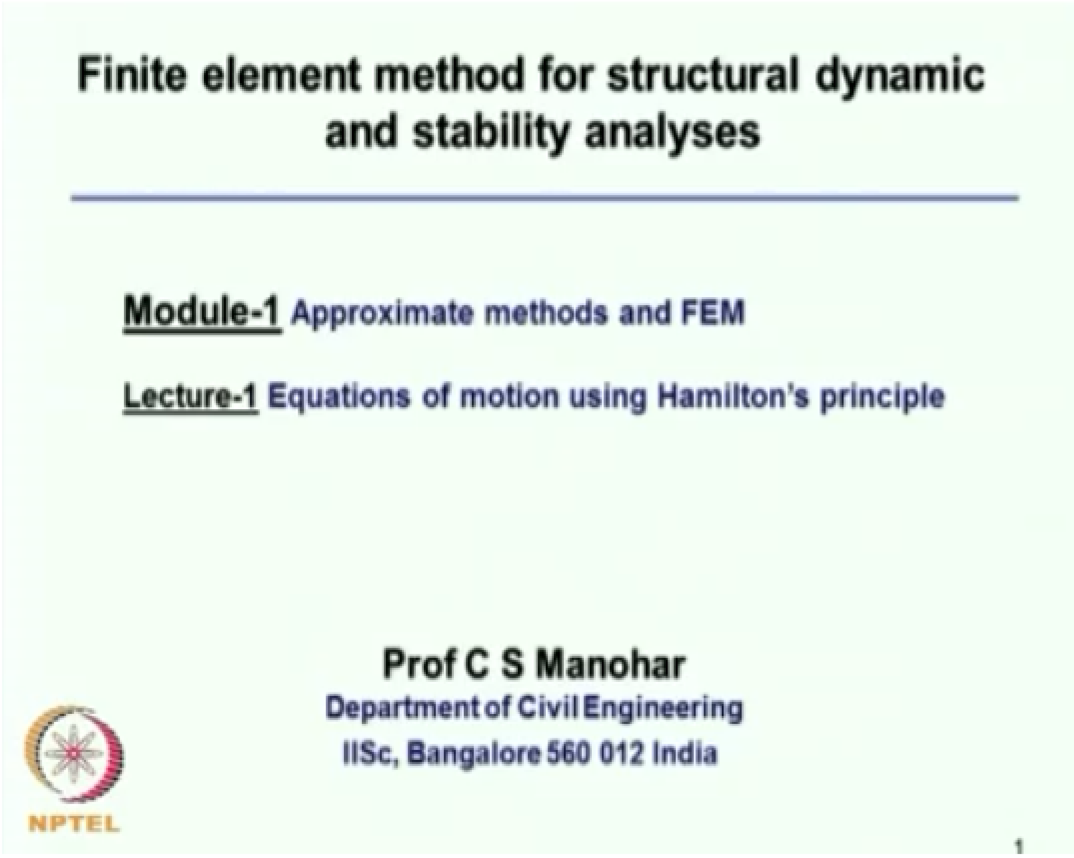
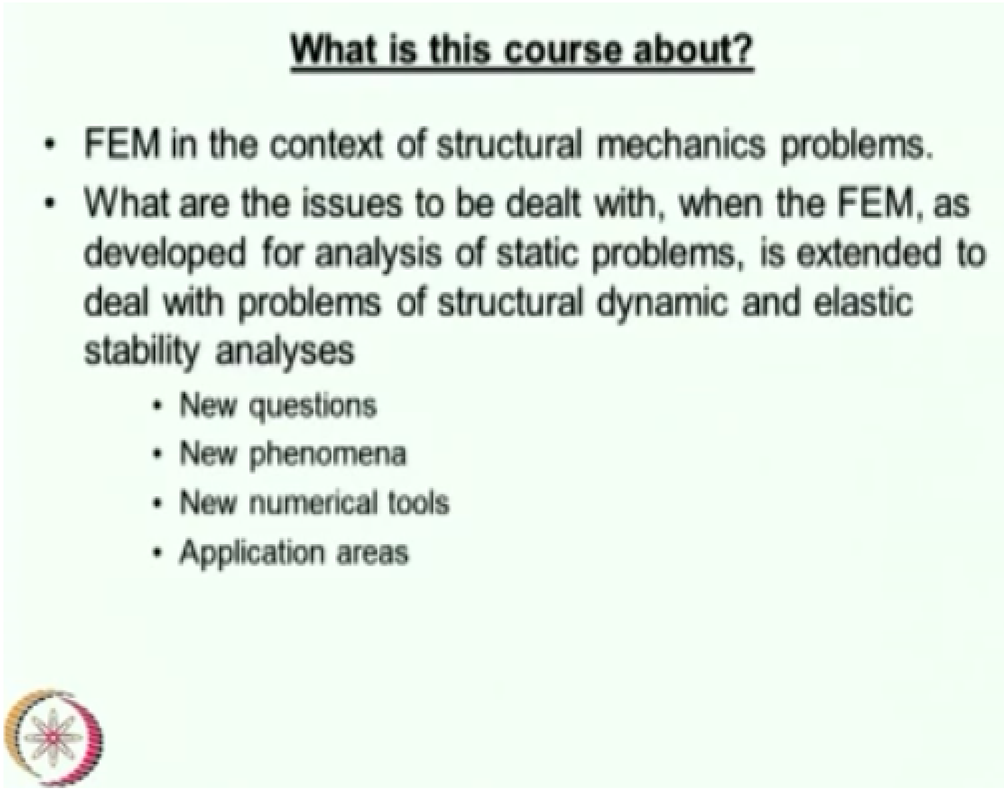
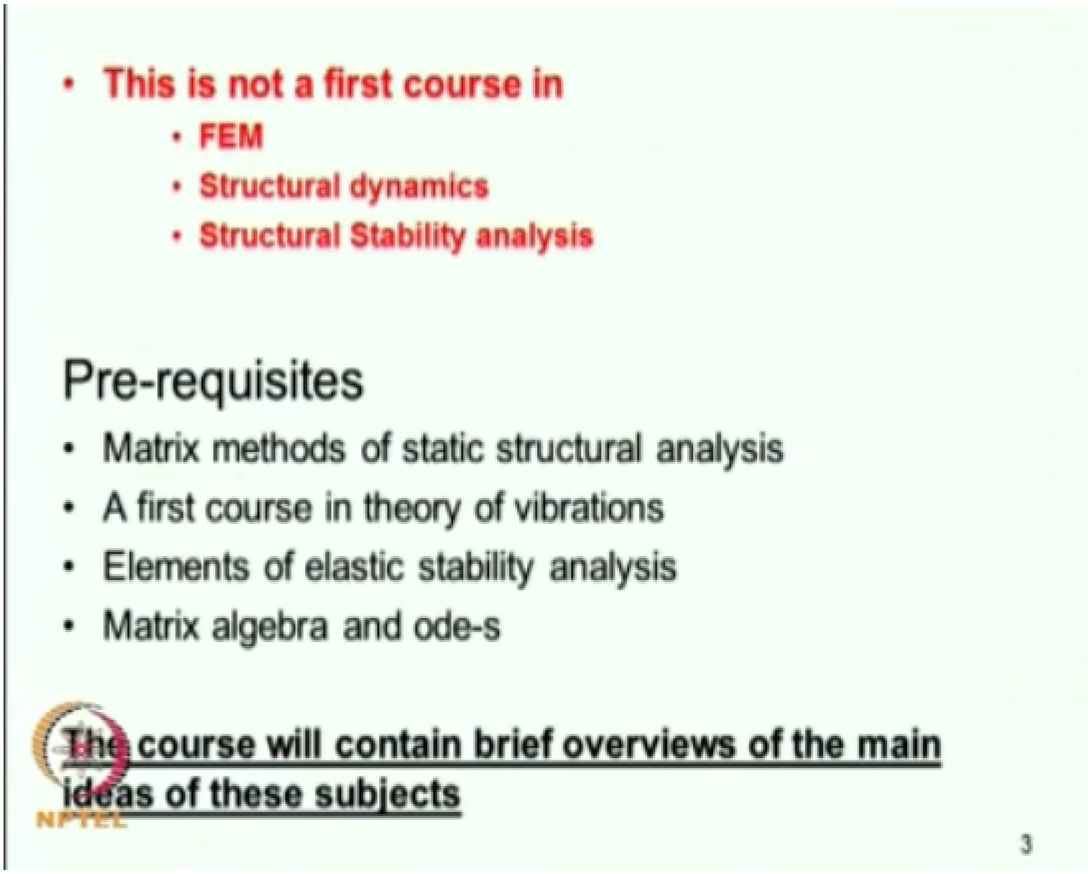
This is the first lecture and the course on Finite Element Method for Structural Dynamic and Stability Analysis.



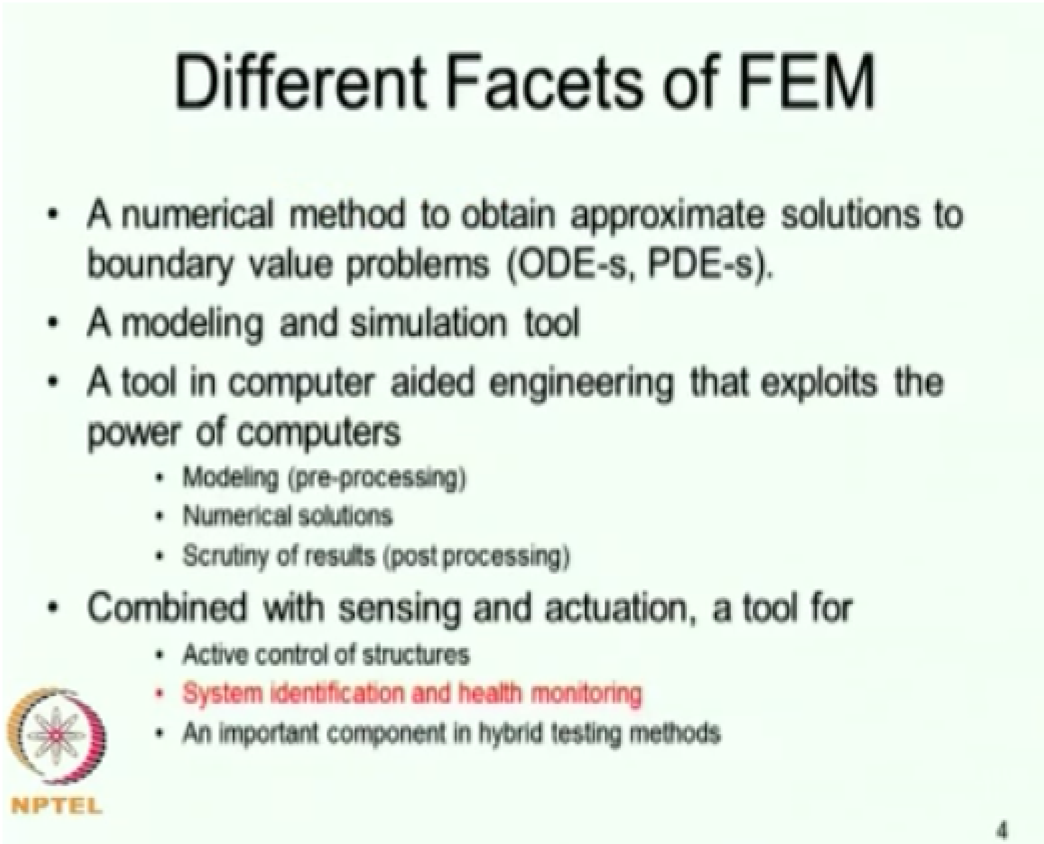
So in this course, in this lecture what I will do is before we get into details of the course I will provide an outline of what I intend to cover in this course, and what you may expect from attending this course.



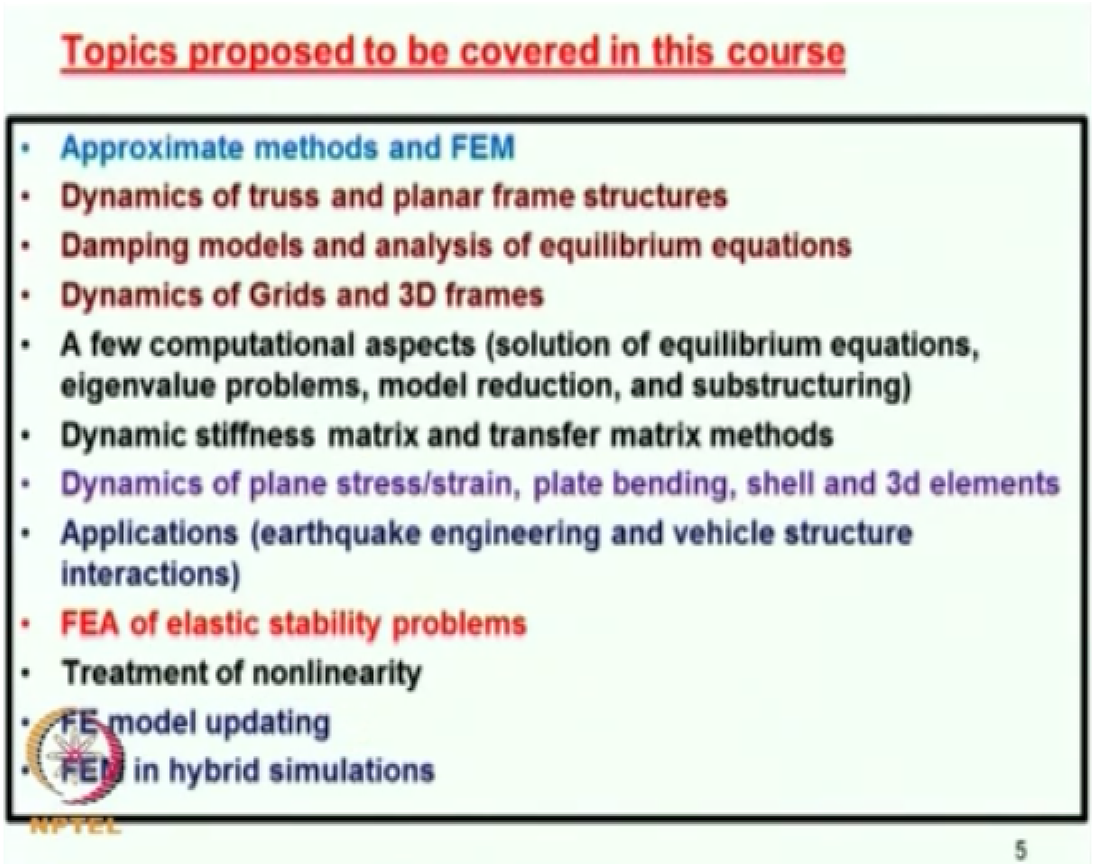
So this course basically deals with the finite element method in the context of structural mechanics problems, now this is not a first course in structural dynamics, nor a first course in finite element method, nor a first course in stability analysis. The focus of this course is to address the issues that we need to deal with when the finite element method has developed for analysis of static problems is extended to deal with the problems of structural dynamic and elastic stability analysis, so beyond the static analysis we would be asking what are the new questions that we need to deal with, are there any new phenomena that we have to model, and what are the new numerical tools that we need to develop and where all we can apply these tools in structural engineering.



So as I said this is not a first course in FEM, nor a first course in structural dynamics, nor a first course in stability analysis. So obviously the prerequisites for this course is that you should have had some exposure to matrix methods of static structural analysis, first course in theory of vibrations and some idea about elements of elastic stability analysis, and familiarity with the mathematics of matrices and ordinary differential equation. Having said this I would like to emphasize that the course will indeed contain brief overviews of the main ideas of these subjects at suitable places.

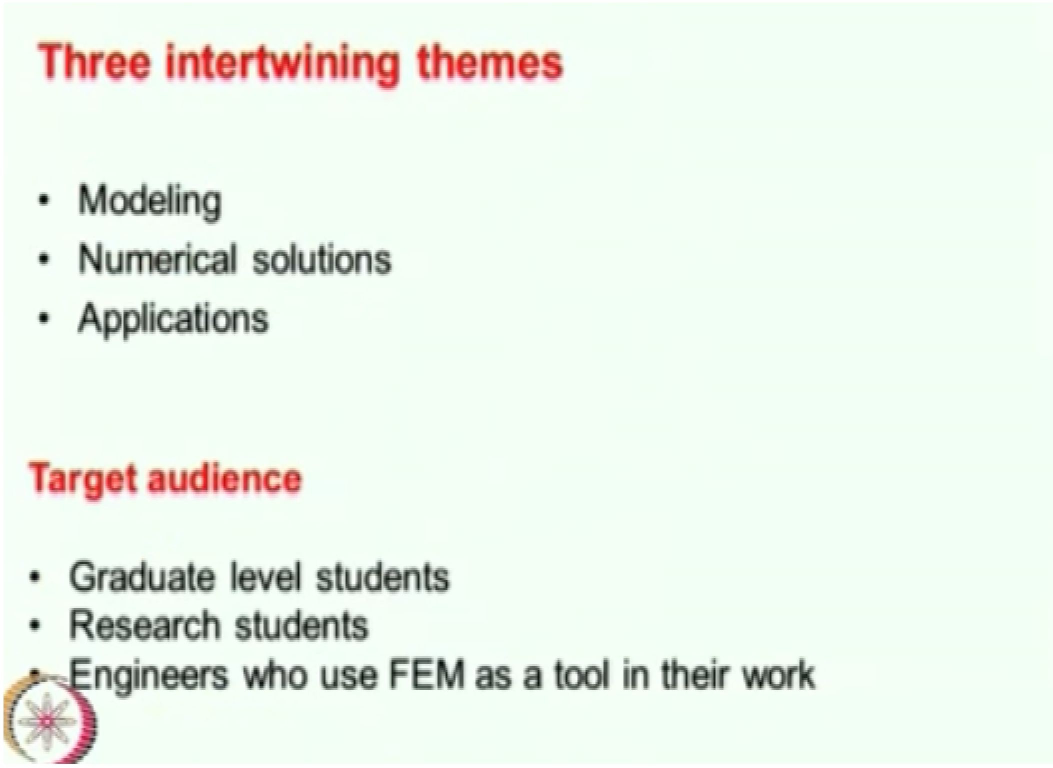


Now the finite element method has different facets, it can be viewed as a numerical method to obtain approximate solutions to boundary value problems, they could be ordinary differential equations or partial differential equations, or it can be viewed as a modeling tool, modeling and simulation tool also as a tool in computer-aided engineering that exploits the power of computers this includes pre-processing, that means how to create models for the structural geometry so on and so forth and numerical solutions that is includes manipulation of algebraic equations and differential equations, and how to scrutinize the results that we outline that is post-processing. In altogether different context if we combine sensing and actuation with computational methods the finite element method becomes a more powerful tool especially in dealing with existing structures or structures with active control elements built into them and in problems of condition assessment where we address problems to system identification and health monitoring of existing structures, and also it can also serve as an important component in modern testing methods known as hybrid testing methods. So we hope to address some of these issues as we go along in this course.



Now a quick overview of topics that I hope to cover in this course, as provided in this slide so we'll begin with approximate methods and how finite element method emerges from a discussion of approximate methods, then we'll address problems of dynamics of truss and planar frame structures, we will address issues about damping models and analysis of equilibrium equations, then we move on to grids and 3D frames, then we revert back to certain computational aspects that is how to solve equilibrium equations, eigenvalue problems, model reduction, and substructuring and so on and so forth. There are other computational methods known as dynamic stiffness matrix method and transfer matrix method, we'll briefly touch upon that. After having done this we'll move on to two dimensional problems, we will consider plane stress and plane strain problems and then plate bending and shell and 3D elements. As far as applications of these ideas are concerned we will consider some problems in earthquake engineering and problems or vehicle structure interactions that we encounter in bridge engineering problems.

Then we will formulate the problem of finite element analysis of elastic stability problems and you know we will discuss again certain computational methods related to this problem, towards the end of the course I will touch upon how to deal with nonlinearity and how to update a finite element model when data on measured responses of structures become available. Finally I will briefly discuss the role of finite element method in qualification testing using methods known as hybrid simulations. So this is the overview of the course, let's see how far we can achieve all



this, so the course will essentially have three intertwining themes one will address modeling, the next will address numerical solutions, and the last will address applications, so these three themes will get intertwined as we go along in the course.

Now the target audience for this course I perceive that it could be graduate level students pursuing masters taught courses or research programs, research students MS and PhD students and engineers who use finite element method as a tool in their work and those who want to use this tool for vibration analysis and stability analysis.

Books

1. M Petyt, 1990, Introduction to finite element vibration analysis, CUP, Cambridge.
2. W Weaver and P R Johnston, 1987, Structural dynamics by finite elements, Prentice-Hall, Englewood Cliffs.
3. M Geradin and D Rixen, 1997, Mechanical vibrations, 2nd Edition, Wiley, Chichester.
4. R W Clough and J Penzien, 1993, Dynamics of Structures, 2nd Edition, McGraw-Hill, New York.
5. K J Bathe, 1996, Finite element procedures, Prentice Hall of India, New Delhi.
6. I H Shames and C L Dym, 1991, Energy and finite element methods in structural mechanics, Wiley Eastern Limited, New Delhi.
7. R R Craig, 1981, Structural dynamics: an introduction to computer methods, Wiley, NY
8. C H Yoo and S C Lee, 2011, Stability of structures, Butterworth-Heinmann, Burlington.
9. G J Simitses and D H Hodges, 2006, Fundamentals of structural stability, Elsevier, Amsterdam.

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Here I have listed a few graphs so in the website the PDF file of this PPT files will be made available so these details will be available to you, so I will not run into the details of this, and these are some of the basic textbooks and there are certain additional references,

Additional references

1. L Meirovitch, 1997, Principles and techniques of vibrations, Prentice-Hall, New Jersey
2. O C Zienkiewicz and R L Taylor, 1989, The finite element method, Vols-I and II, 4th Edition, McGraw-Hill, London.
3. R D Cook, D S Malkus, and M E Plesha, 1989, Concepts and applications of finite element analysis, 3rd Edition, John Wiley, New York
4. J N Reddy, 2006, An introduction to the finite element method, 3rd Edition, Tata-McGraw-Hill, New Delhi.
5. S S Rao, 1999, The finite element method in engineering, 3rd Edition, Butterworth-Heinemann, Boston.
6. T J R Hughes, 2000, The finite element method, Dover, Mineola.
7. M Paz, 1985, Structural dynamics, 2nd Edition, CBS Publishers, New Delhi
8. W McGuire, R H Gallagher, and R D Ziemian, 2000, Matrix structural analysis, 2nd Edition, John Wiley, New York.
9. P Seshu, Textbook on finite element analysis, 2003, Prentice Hall India, New Delhi.
10. G Strang and G J Fix, 2008, 2nd Edition, An analysis of the finite element method, Wellesley-Cambridge Press, Wellesley.

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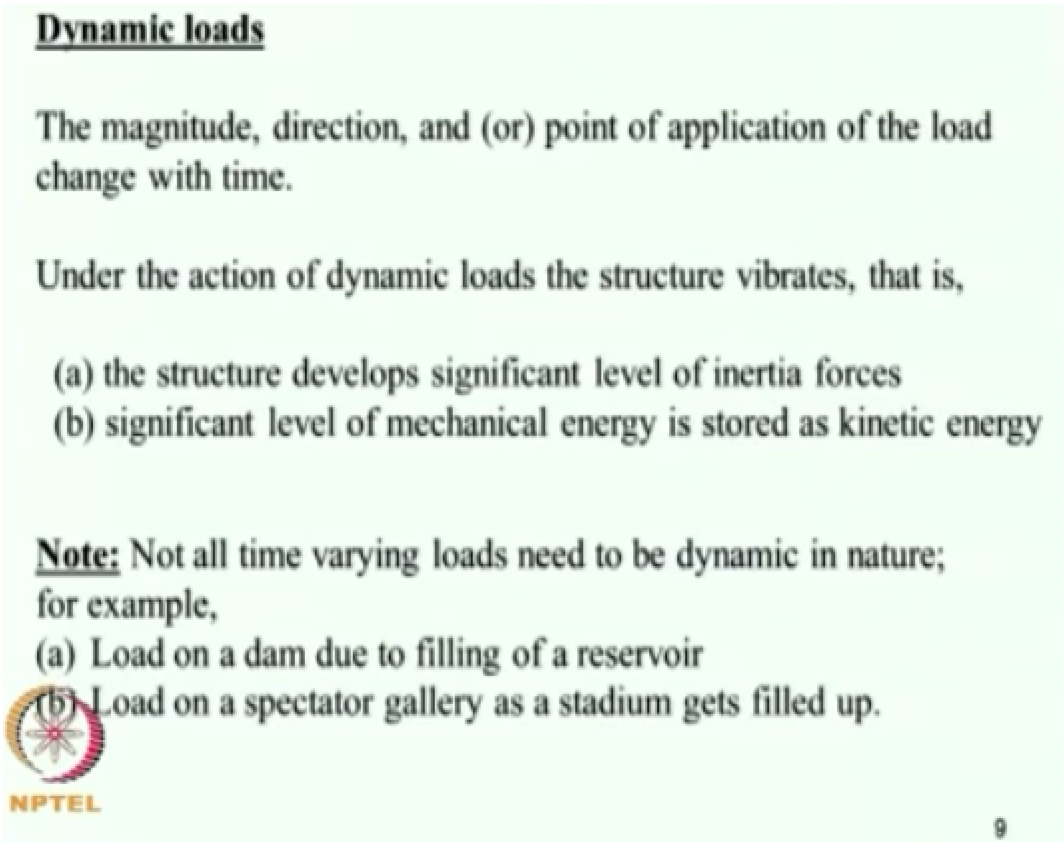
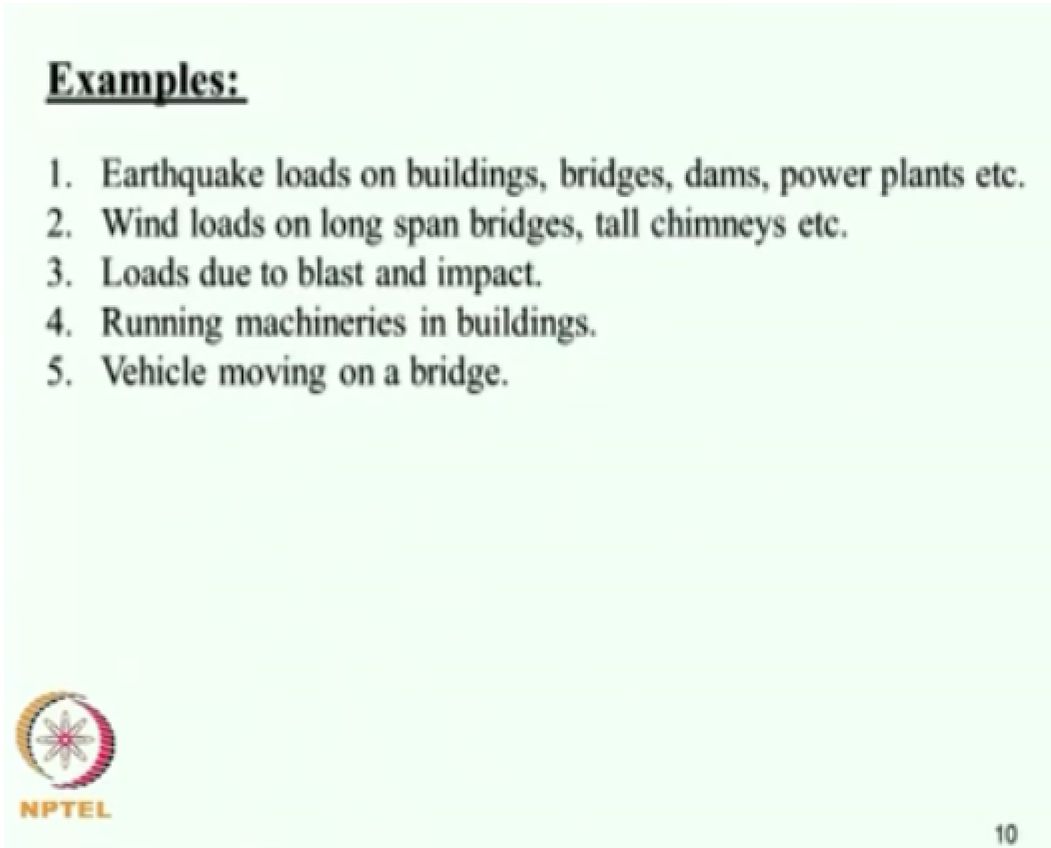
this books by Hughes and Strang and Fix provide certain mathematical basis of finite element method, there are many good books in this area and I have tried to list a few of them here.

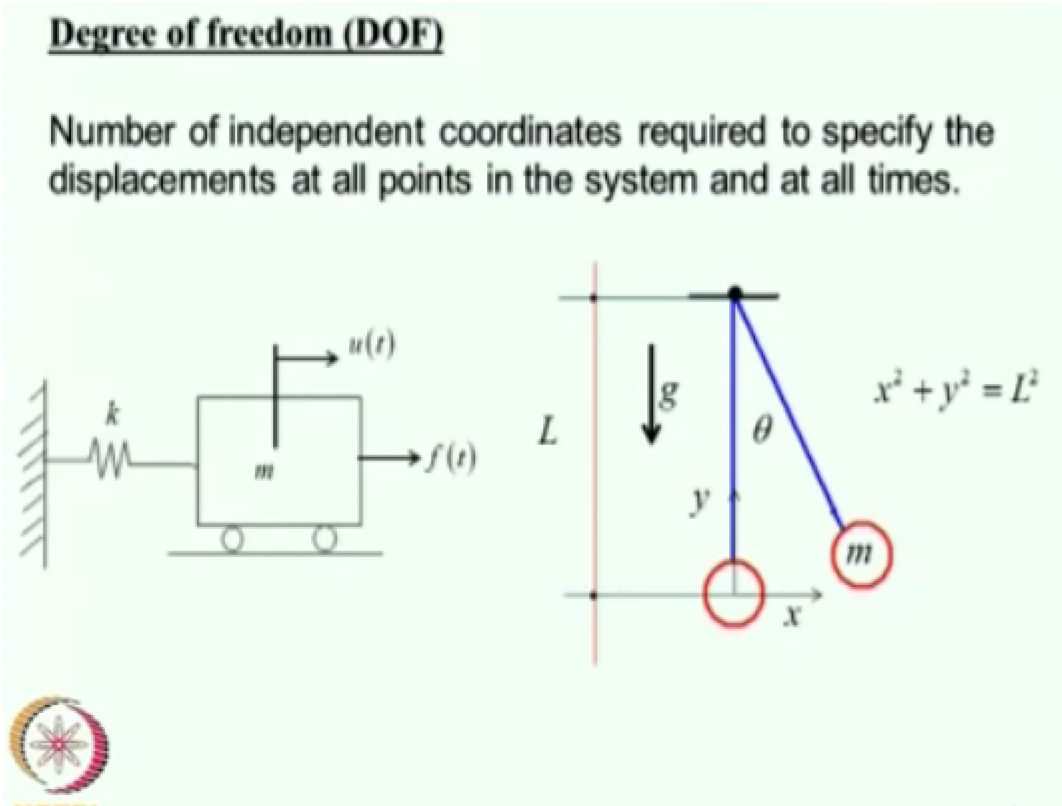


Now with this preamble let us start talking about the subject matter, so I will begin by quickly reviewing certain elementary notions about structural dynamics so we will begin by asking what are dynamic loads? So dynamic loads can be defined as those loads in which the magnitude direction and or point of application of the load change with time, so under the action of the dynamic loads the structure vibrates, what that means? The structure develops significant level of inertial forces and significant level of mechanical energy stored as kinetic energy, in a static problem the work done by external forces is stored as strain energy, but in dynamic problems the work done by the time varying load gets stored as not only potential energy, but also kinetic energy and there is a perpetual exchange of one form of the energy that is kinetic energy to potential energy, and that is what we call as vibration. Added to this there is one more source of energy, namely the energy dissipation so that we call it as damping.

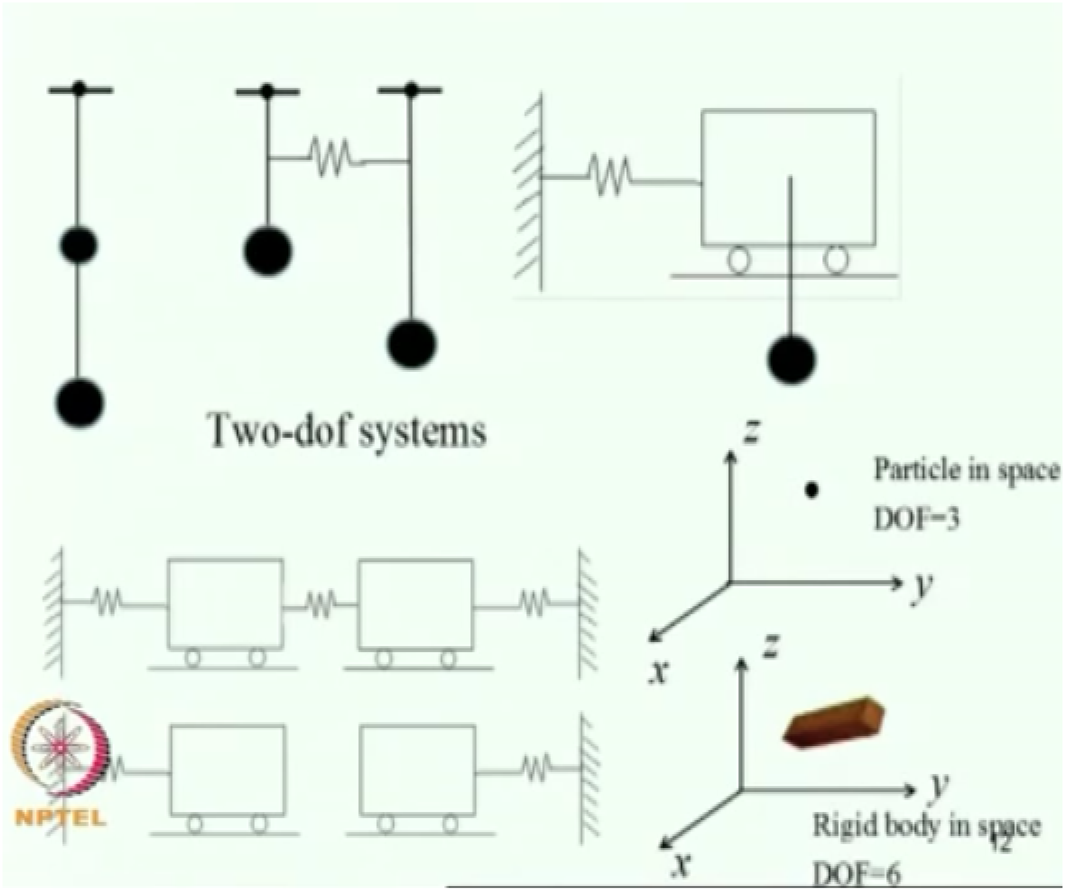
Now I want to emphasize that not all time varying new loads need to be dynamic in nature, for example load on a dam due to filling of a reservoir, it takes place so slowly that the accelerations that are developed in the structure are so low that the corresponding inertial forces will be significantly less than the static loads, so it is not a vibration problem in this context. Similarly load on a spectator gallery as a stadium gets filled up, here to the time varying nature of the displacement and hence consequent velocity and accelerations are low that means the time variation of displacement is so slow that acceleration levels are very small therefore little of inertial forces are mobilized in these problems, therefore we cannot call these problems as problems in structural dynamics.



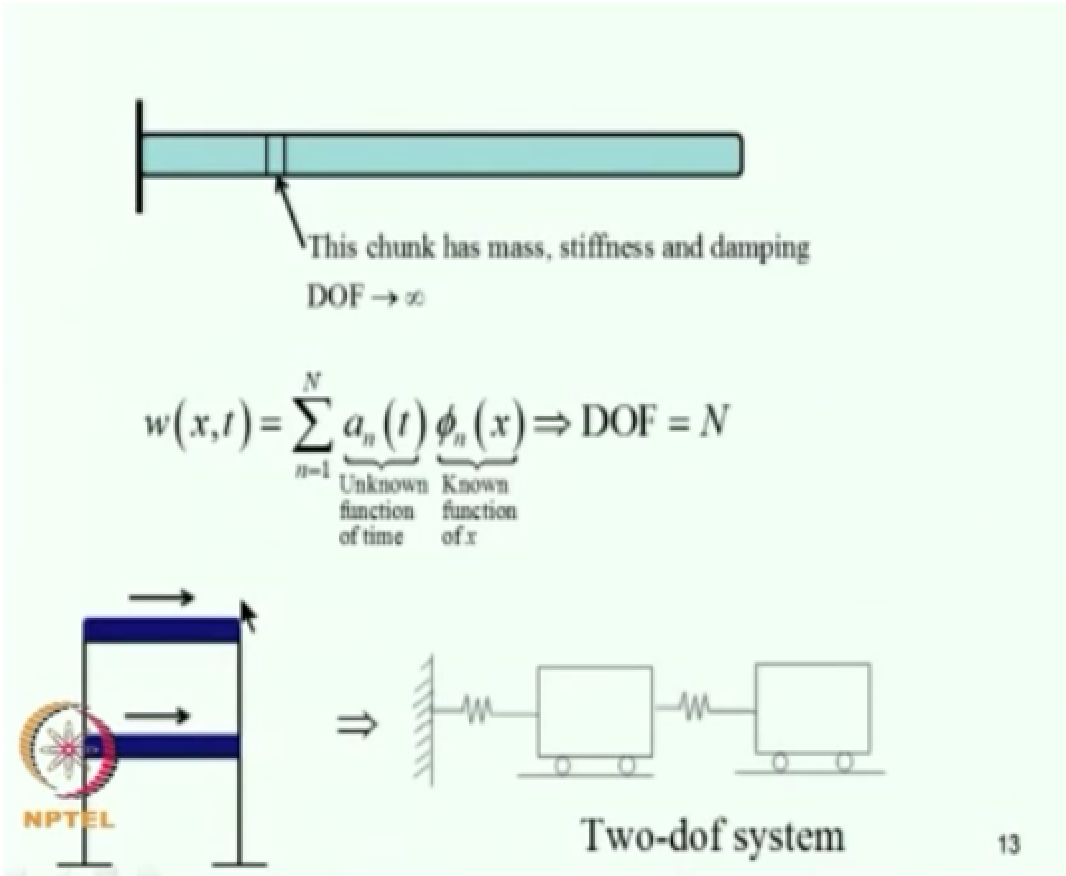
Some of the examples of problems in structural dynamics earthquake loads happened to be one major source of dynamic loads on buildings, bridges, dams, and power plants, similarly wind loads on long span bridges, tall chimneys, etcetera produce dynamic action, then loads due to blast and impact is another source of dynamic actions on civil engineering structures, and if we have machineries in the buildings due to the imbalance that those machines transferred to the supporting structure there will be vibrations and similarly whenever a vehicle moves on a bridge, the bridge vibrates and there is a problem of vehicle structure interaction, this is again is an example of a problem in structural dynamics.



Now the first technical term that we talk about in structural dynamics is the degree of freedom, so degree of freedom is a number of independent coordinates required to specify the displacement at all points in the system and at all times, so it can quickly be understood if you consider now a point mass connected to a support like this through a spring and this is made to oscillate in this plane and the only degree of freedom is U of T, that means at any time T, if I know U of T I would know where this mass is, therefore this is a one degree freedom system, although I show a rectangle here this mass is a point mass it has no dimension, similarly this spring has no length it's a point spring so this is an idealization. Similarly if a pendulum swings at any time T, I would like to know where this mass is so I can specify the X and Y coordinates of this mass, but if the length of the strain doesn't change then the X and Y coordinates of the pendulum are constrained by this equation X square + Y square = L square, therefore the number of independent coordinates is 1, therefore this is also a single degree freedom system.



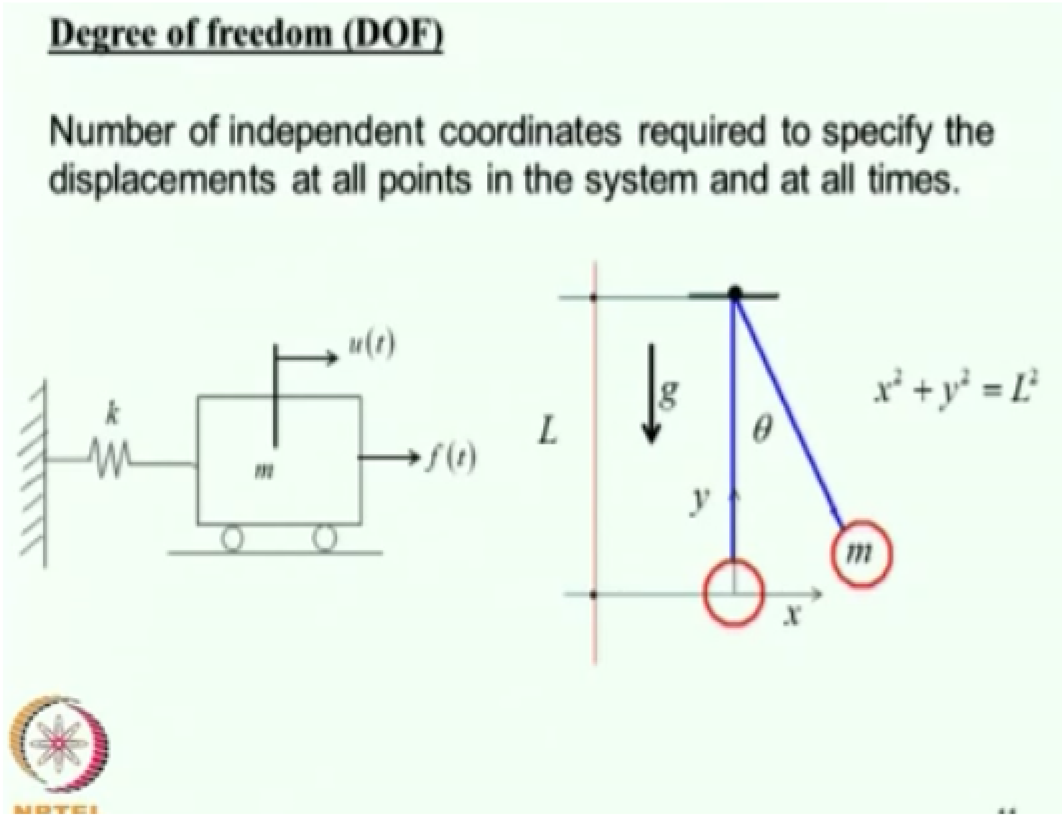
Here I have shown few system, this is a 2 degree freedom system, this again a standard pendulum its swing so I need to know where this mass is, where this mass is, so I can use theta 1 and theta 2 as degrees of freedom. Similarly here this is another example of two degrees of freedom system, this is a combination of a mass spring system and a pendulum this again is a two degree freedom system, this type of models are encountered in modeling of sloshing of liquids in flexible tanks, so maybe at some point in this course we can return to this issue.



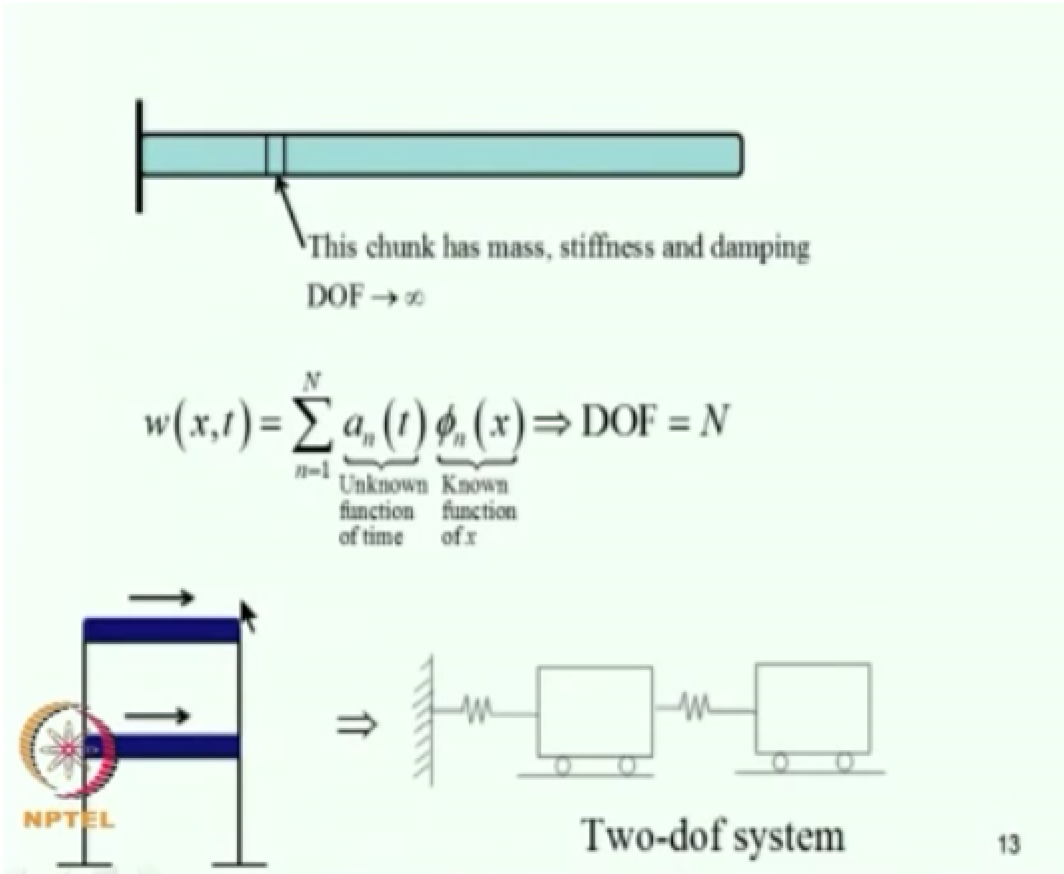
Now this is a two degree freedom system there are two masses so at any point of time I need to know where is this mass, and where is this mass. This system is also a two degree freedom system, because at any time T if you know, need to know where these masses are you have to specify the position of this mass, and position of this mass, so it is also a two degree freedom system, but there is a basic difference between these two systems in the sense that the degree of freedom, the displacement here is coupled to the displacement here in this problem whereas this oscillator is uncoupled from this oscillator, but the fact remains that this is also a two degree freedom system.

Now let us consider this example, this is a particle in space so if you want to know where and suppose it is moving in a Cartesian coordinate system we need to know at any time T where this particle is therefore, this is a three degree freedom system. On the other hand if this particle is actually a rigid body like this which is idealized as a particle in this model but if you want to include the geometric details of this object then we need to know not only the coordinates position of the center of gravity of this say XYZ but also the direction cosines of the orientation of this rigid body, therefore this is a sixth degree of freedom system.

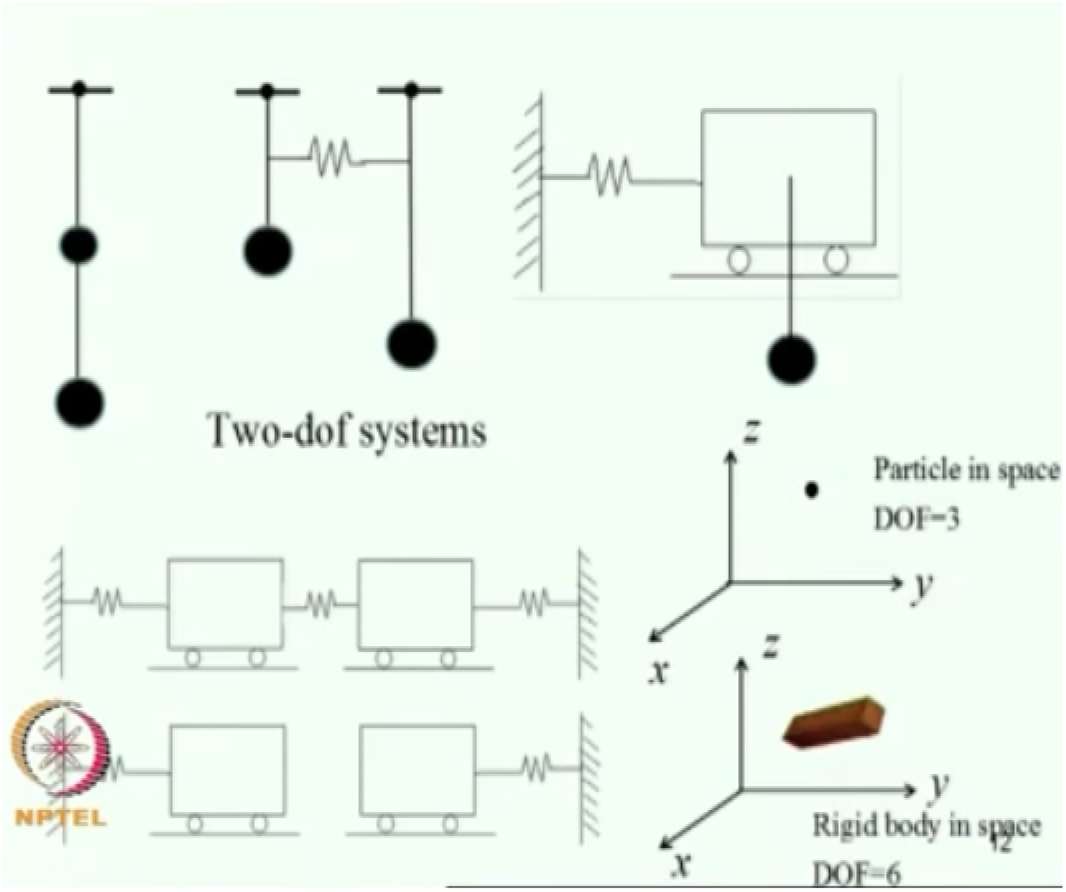
Now when I look at this type of systems or maybe we can consider a simpler one,



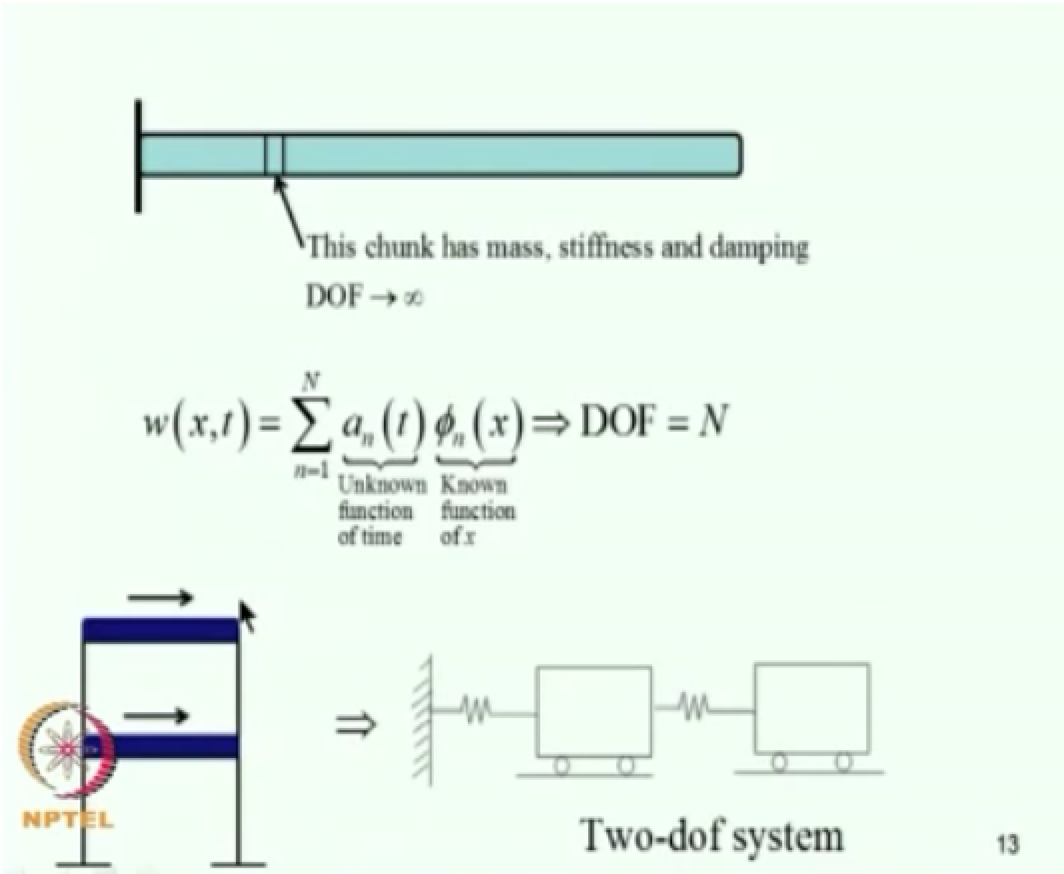
see this mass stores kinetic energy it doesn't contribute to the stiffness of the system, so this entity is only a storage element for kinetic energy, similarly this spring doesn't have any mass it contributes to only potential energy of the system so that means the parameters here are lumped, so this has a distinct role to play, this has a distinct role to play, but on the other hand if



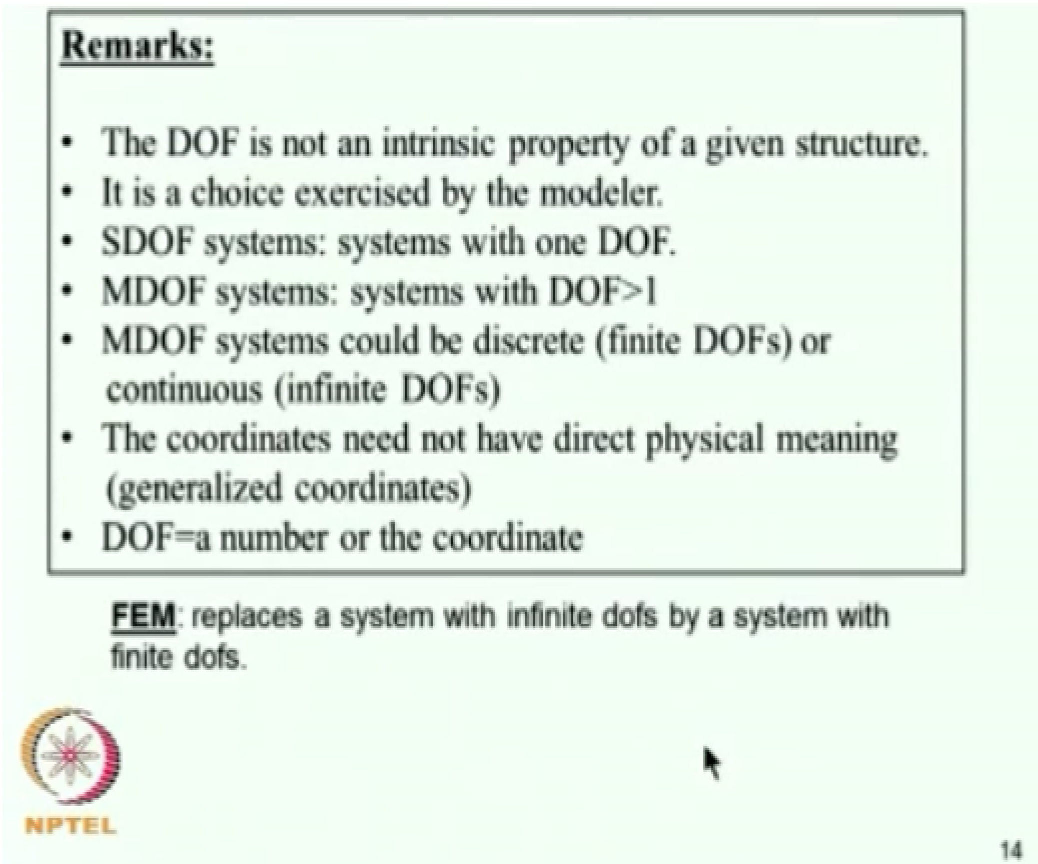
you consider say a beam like this you take a chunk of this mass, now beam this chunk has mass stiffness as well as damping that means it stores kinetic energy, it stores potential energy, it also dissipates mechanical energy so if you want to know where the configuration of the beam you need to tell where different points on the beam are at any given time T, and consequently this is an infinite degrees of freedom system. In this system such type of systems are called continuous



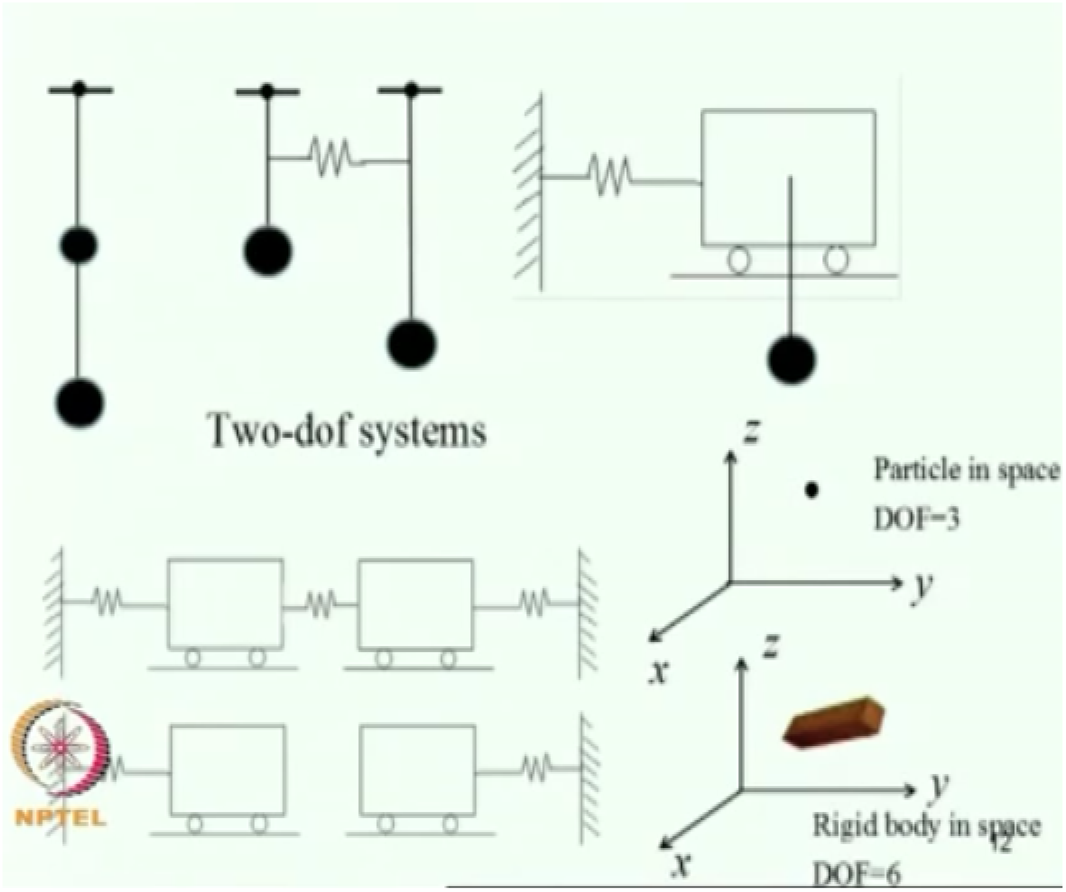
systems they are typically, they are governed by a set of partial differential equations whereas this type of systems are called discrete multi-degree freedom systems and they are gone by a set of ordinary differential equations and they have finite degrees of freedom whereas this type of systems have infinite degrees of freedom.



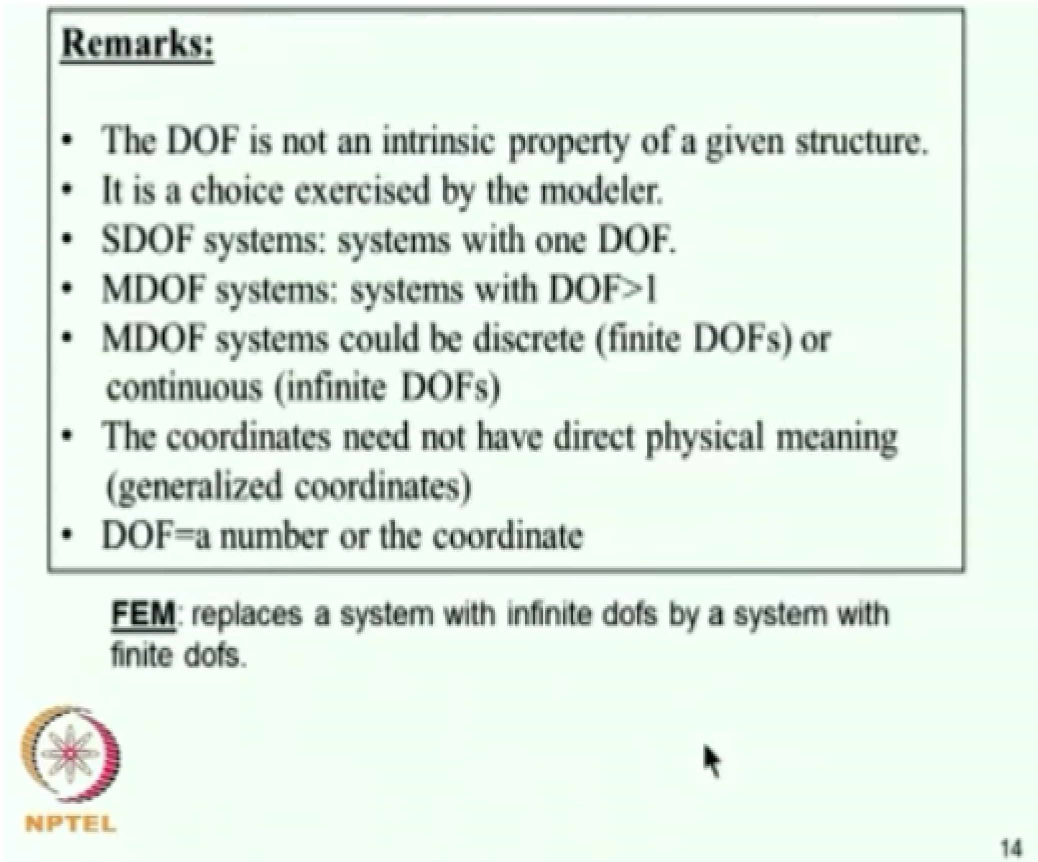
Now if the displacement field of this beam if I were to represent in a series like this with capital N number of terms then suppose it is written as a function of time into a function of space and this function suppose is known then the unknown in this representation are this A1 A2 A3, capital N, so the degree of freedom in this representation is capital N, that means this system can be approximated through this approximation as a N degree freedom system. Now if you take now a two storey building frame and if you assume that the slabs are, the mass of this slab is very large compared to the mass of the columns and they're infinitely rigid in their own plane we can approximate the behavior of this system through a two degree freedom model shown here, so these two masses correspond to mass of these two floors and this spring is the contribution to stiffness from these columns, and this spring is contribution to stiffness from these columns and the displacement of this mass and displacement of this mass correspond to the displacement of this floor and displacement of this floor, so this is a two degree freedom system.



Now so the, some remarks we can make at this stage, see here if you take a look at this cantilever beam we agreed that this system has infinite degrees of freedom, whereas this frame which is made up of beams like this we are willing to model it as two degree freedom system,

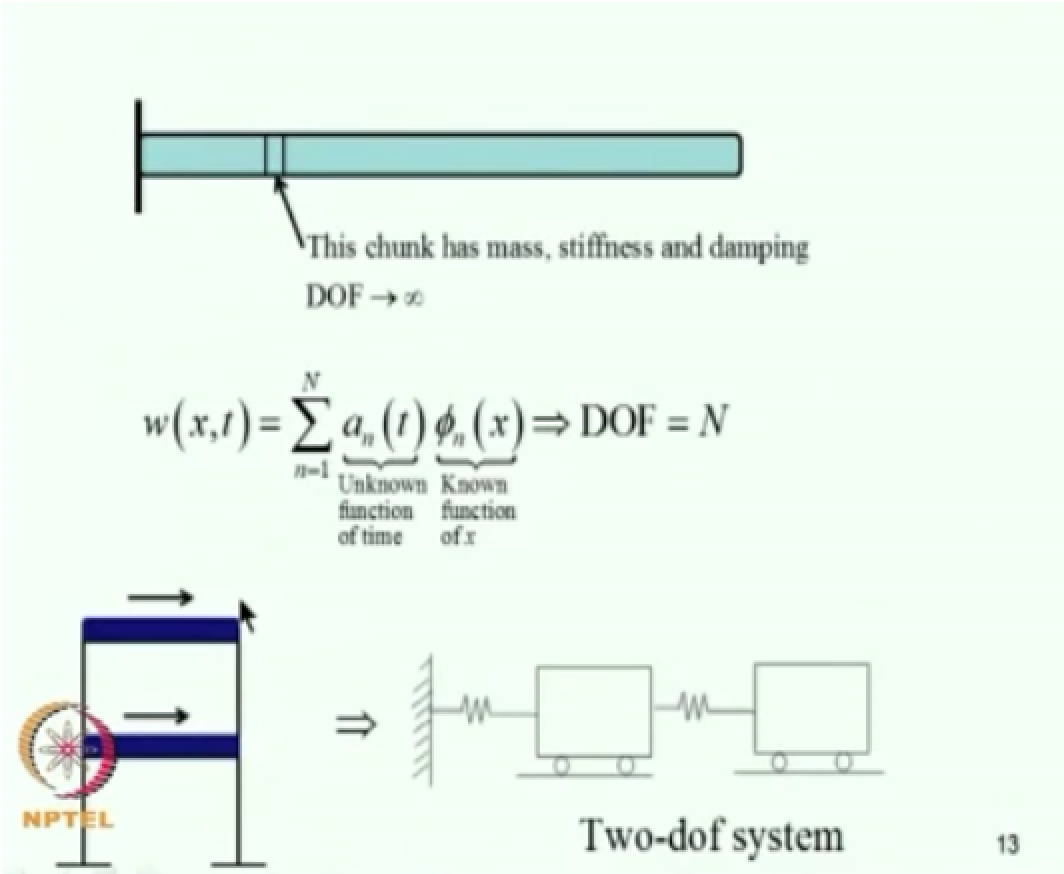


similarly here if we model the object as a particle this has three degrees of freedom, but on the other hand if you model it as a rigid body it has six degrees of freedom, but if you go ahead and further model it as a flexible beam or a flexible object it will have infinite degrees of freedom, so what does that mean? It means that the degree of freedom is not an intrinsic property of a given structure, it is a choice that we exercise in modeling so we call a system to be a single degree of freedom system if it has only one degree of freedom, multi degree of freedom systems are those systems where degree of freedom is greater than one, these multi degree freedom systems could be discrete in which case they have finite degrees of freedom and they are governed by a set of ordinary differential equations or continuous where they have infinite

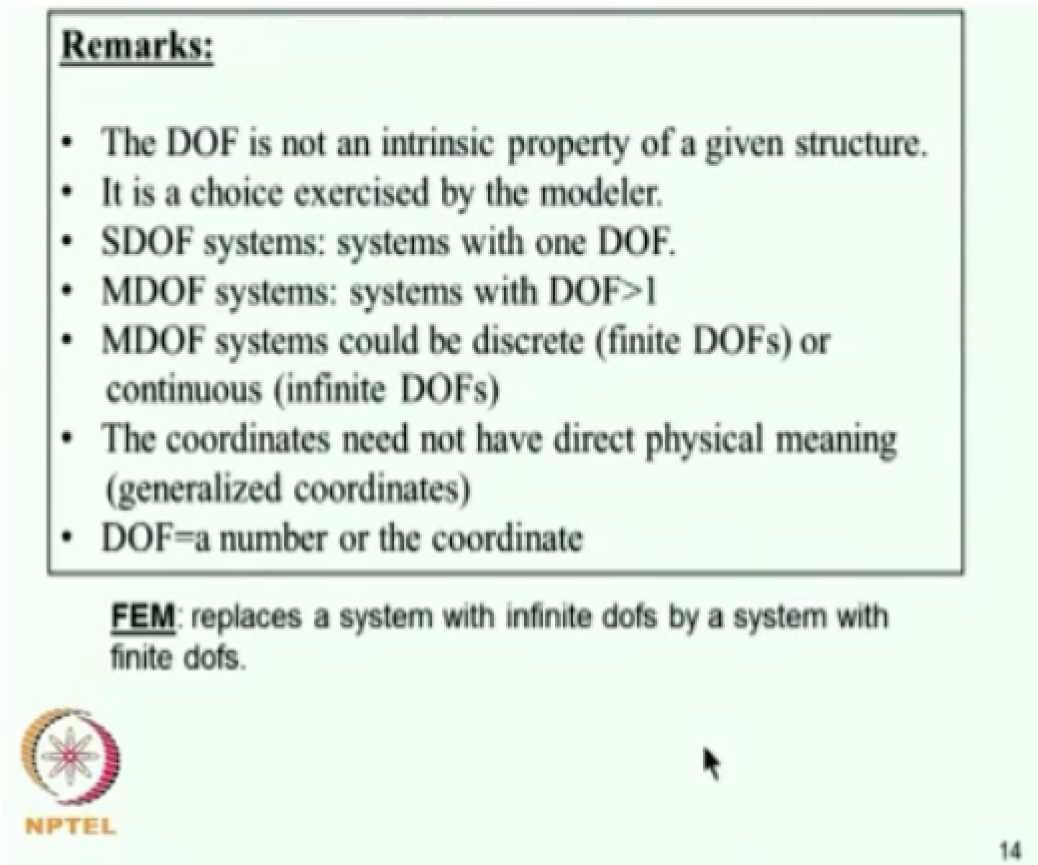


degrees of freedom, the system parameters or properties are distributed, and they are governed by a set of partial differential equations.

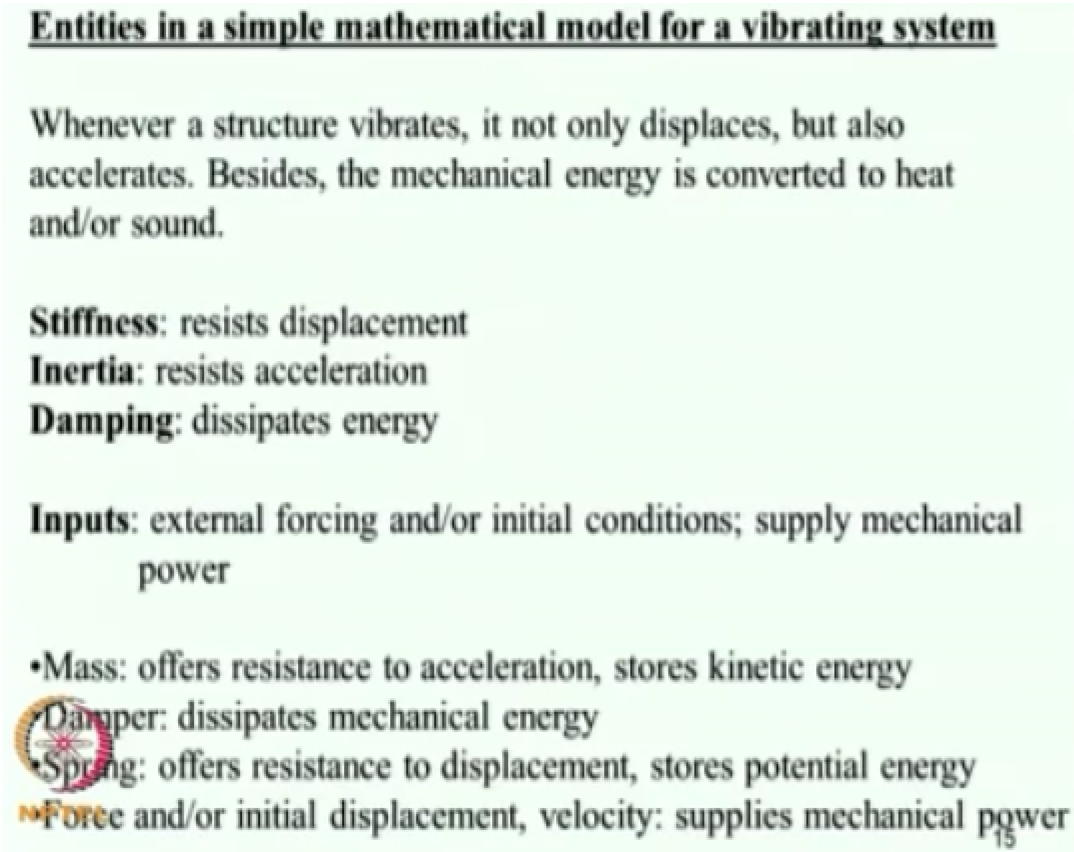
Now when I define degree of freedom the coordinates that I'm using need not have direct physical meaning, for example in the representation of the displacement field in a cantilever



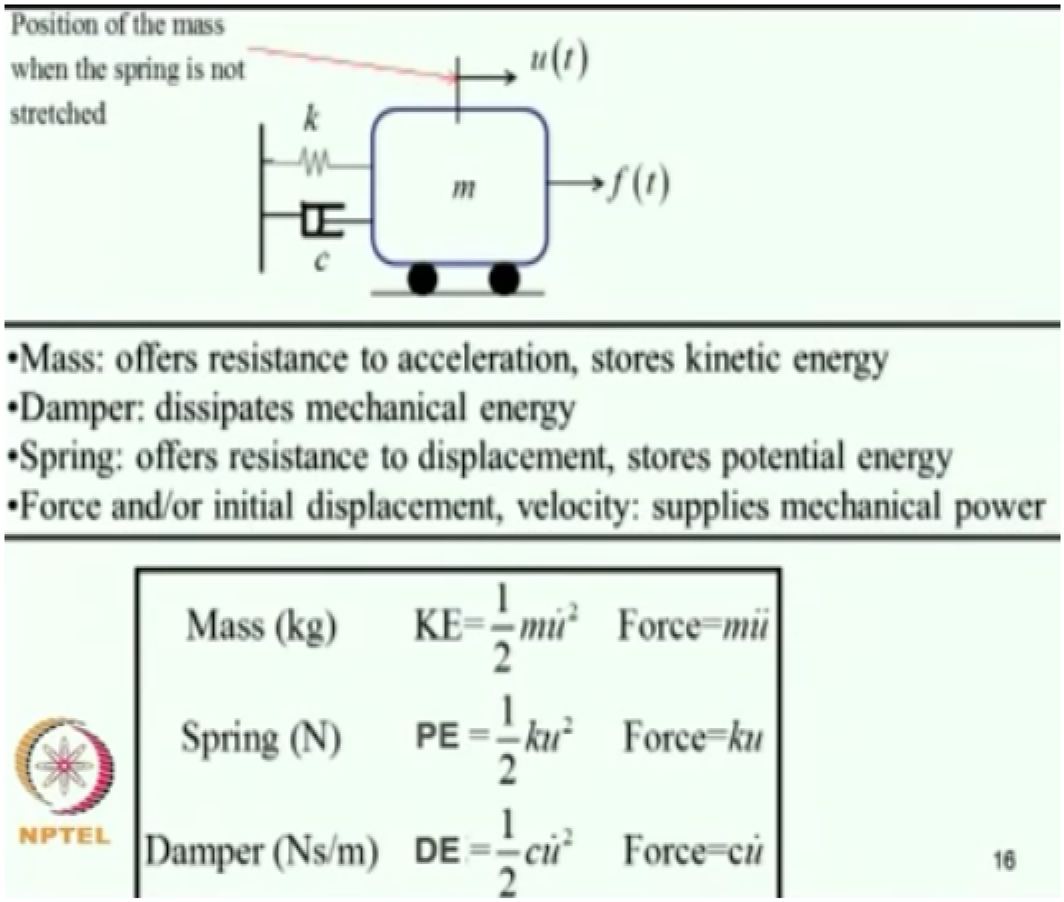
beam, these terms A1, A2, A3 and N are, you know are the unknowns which will help us to define W(X,T) but individual terms here do not have a physical you know direct interpretation, so these kind of coordinates are called generalized coordinates.



Now degree of freedom, the word degree of freedom is used in a dual sense here, in one sense it is a number that is number of independent coordinates as we define, in another sense they are also the coordinates with which we specify the configuration of the system, so it has dual meaning. Now finite element method is an approximate method in which we replace a system infinite degrees of freedom by a system with finite degrees of freedom, so it is a systematic framework to achieve this modeling simplification, so we'll come to that.

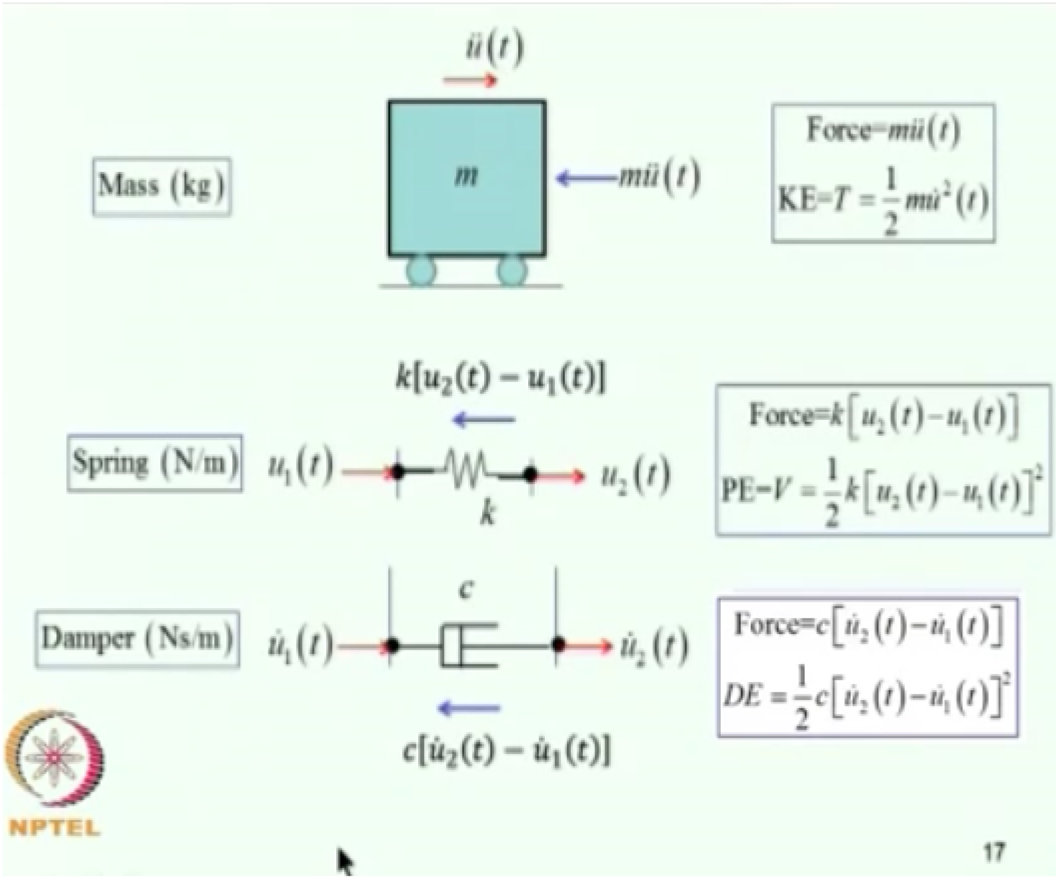


Now what are the entities in a simplest possible mathematical model for a vibrating system, so whenever a structure vibrates it not only displaces but also it exhilarates besides the mechanical energy is converted to heat and or and/or sound okay, so the stiffness property of the system resists displacement, the inertial properties resists the acceleration, the damping characteristics dissipates the energy. Now the external inputs that is the forcing function or initial conditions supply the mechanical power to the system that is work done per unit time, so mass we need to have essentially four entities to make a mathematical model for a vibrating system, we should have what is known as mass which offers resistance to acceleration and stores kinetic energy, a damper that dissipates mechanical energy, a spring that offers resistance to the displacement and stores potential energy, an external force or initial displacement or velocity which supplies mechanical power to the system which makes the system to vibrate in the first place, so we use these ideas and form this simple system this is the mass which stores kinetic energy and offers

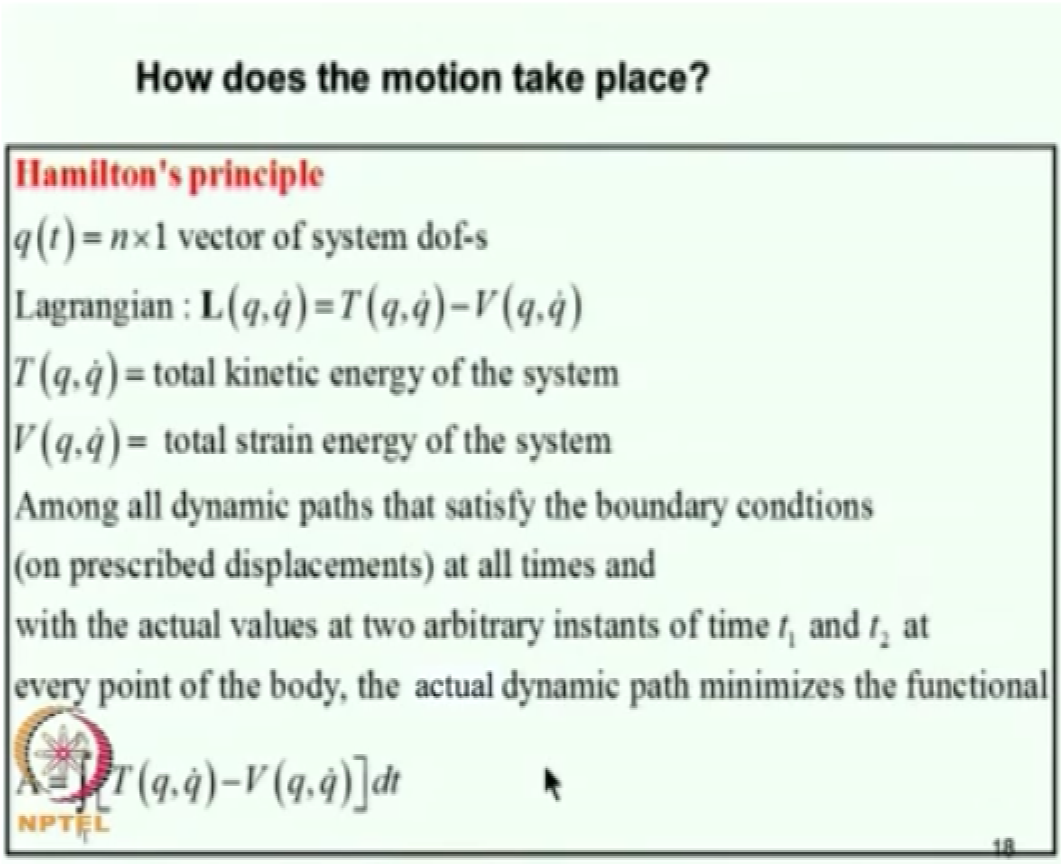


resistance to acceleration, this is a spring which stores potential energy and offers resistance to displacements, and this is an external force and this at T = 0, the mass could be pulled by some distance and release with certain velocity they are the initial conditions, this is the external force. To represent the energy dissipation mechanism we adopt a simple strategy we assume that there is a hypothetical piston with viscous fluid and that dissipates energy, we don't really address the problem of conversion of mechanical energy into heat or sound from first principles so we adopt a pragmatic approach to model energy dissipation characteristics and through a viscous damper and this type of model is known as viscous damping model.

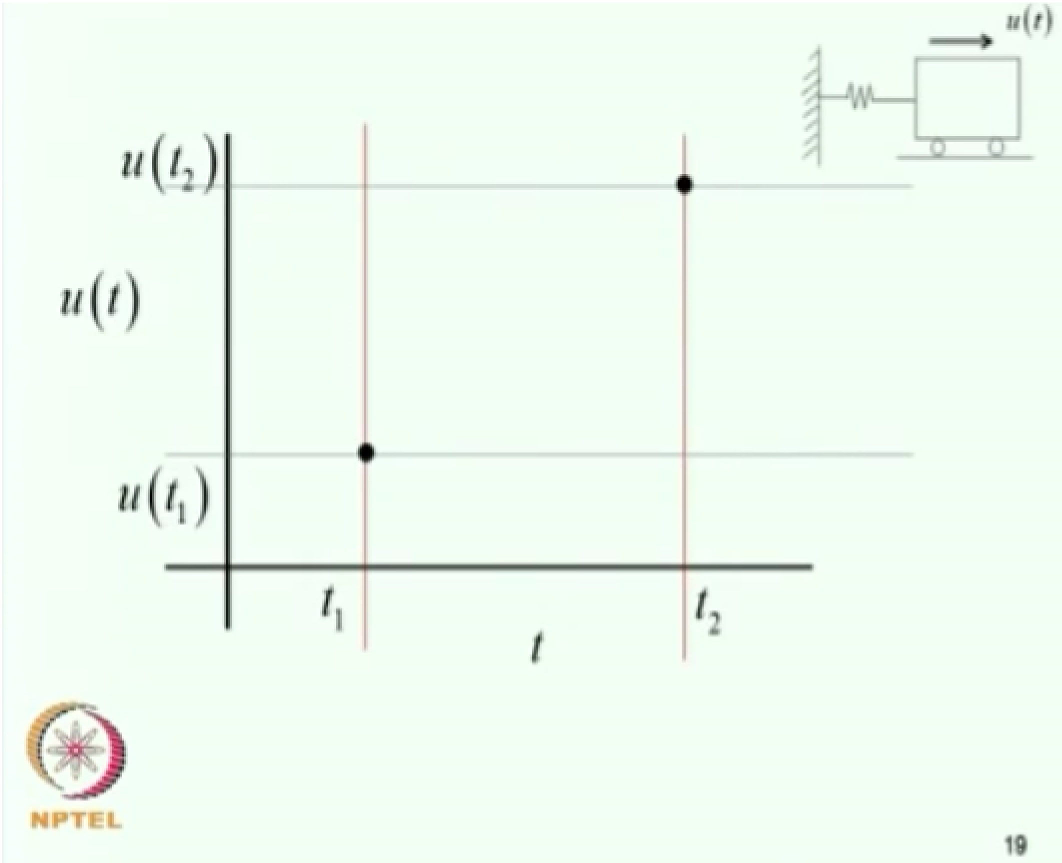
So the mass, the kinetic energy is 1/2 MU dot square, where a dot is a differentiation with respect to time, the force is MU double dot, that is the force associated with this mass, this is according to Newton's law and the spring this is potential energy which is 1/2 KU square, and the force in the spring is KU and the damper dissipation energy 1/2 CU dot square, and the force in the damper is CU dot, the units in SI system for these quantities are kg Newton, Newton second per meter, so the summary is this mass resists acceleration and the force of



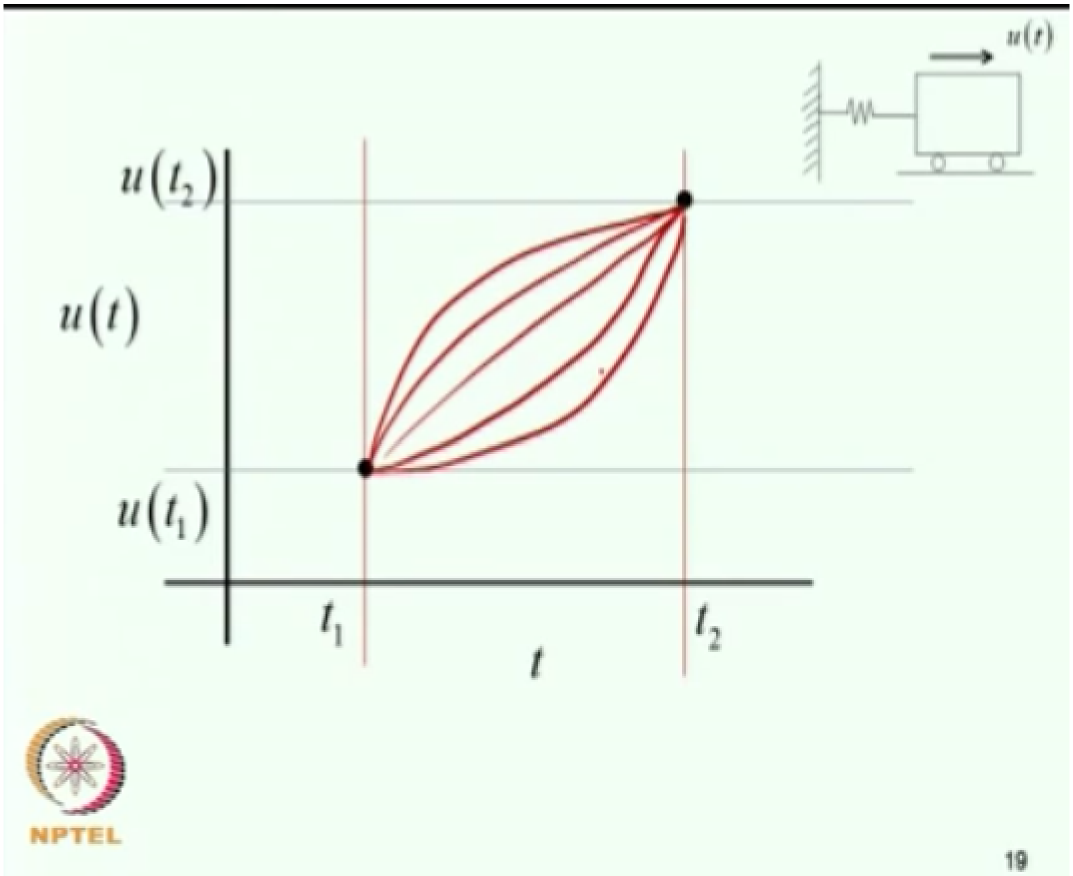
resistance is MU double dot, stores kinetic energy and this is given here, the spring, suppose you have a spring and this end is moving by U1 of T, and this spring is moving by U2 of T, and the force in the spring will be K x U2 – U1 and the potential energy stored will be 1/2 K x U2 - U1 whole square, similarly the damper, the damper force is this, and damper energy dissipated is this.



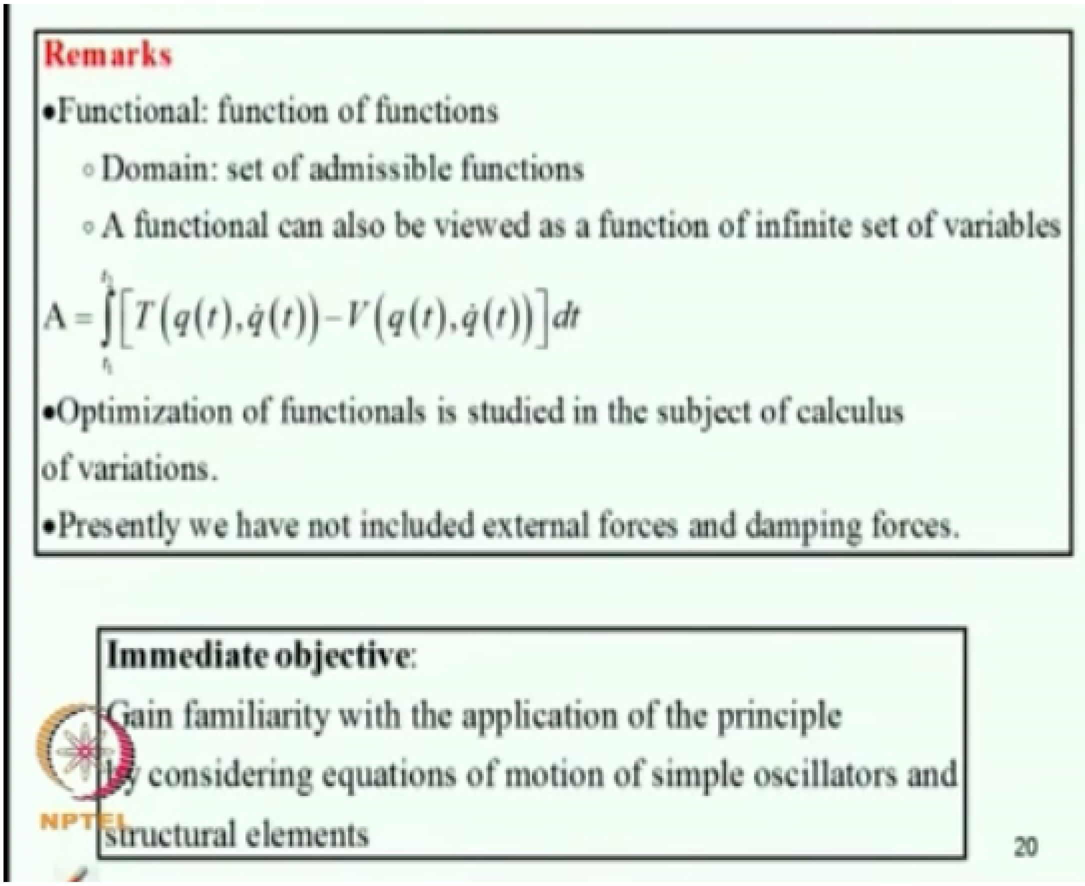
Now we have now formulated the simple mathematic physical model for the vibrating system now we can ask the question how does actually the motion take place. Now we are going to adopt, we are not going to draw free body diagrams and write forces in this course, we are going to adopt an energy based approach to characterize the motion, so the starting point for that is what is known as Hamilton's principle? Suppose you have a system with N degrees of freedom and we denote all the degrees of freedom in the vector Q of T, we define a function known as Lagrangian which is a function of Q and Q dot which is the difference of a kinetic energy and potential energy, so T is total kinetic energy of the system, V is the total strain energy of the system or the potential energy of the system. According to Hamilton's principle among all dynamic paths that satisfy the boundary conditions on prescribed displacements at all times and that start and end with the actual values at two arbitrary instants of time T1 and T2 at every point of the body the actual dynamic path minimizes the functional the quantity known as action which is the integral of T1 to T2 T - V DT, what does that mean?



Suppose at T = T1 this mass is assumed that it is here and at T = T2 it is here, so between the two time instants this object can take this path or this path or this path it has infinite options to move from this point to this point, so the question that we ask is among all these possible paths

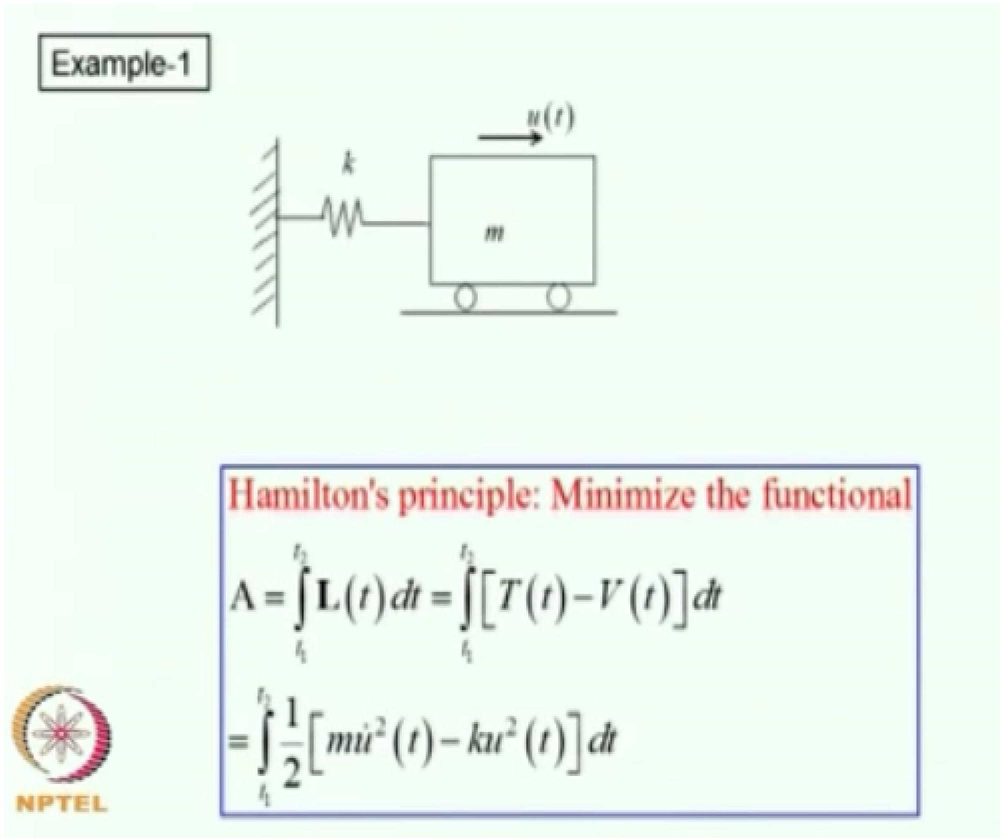


that is available to this mass M which one actually it chooses, so according to the Hamilton's principle the path that is taken by the mass is the one which minimizes that integral known as action integral.

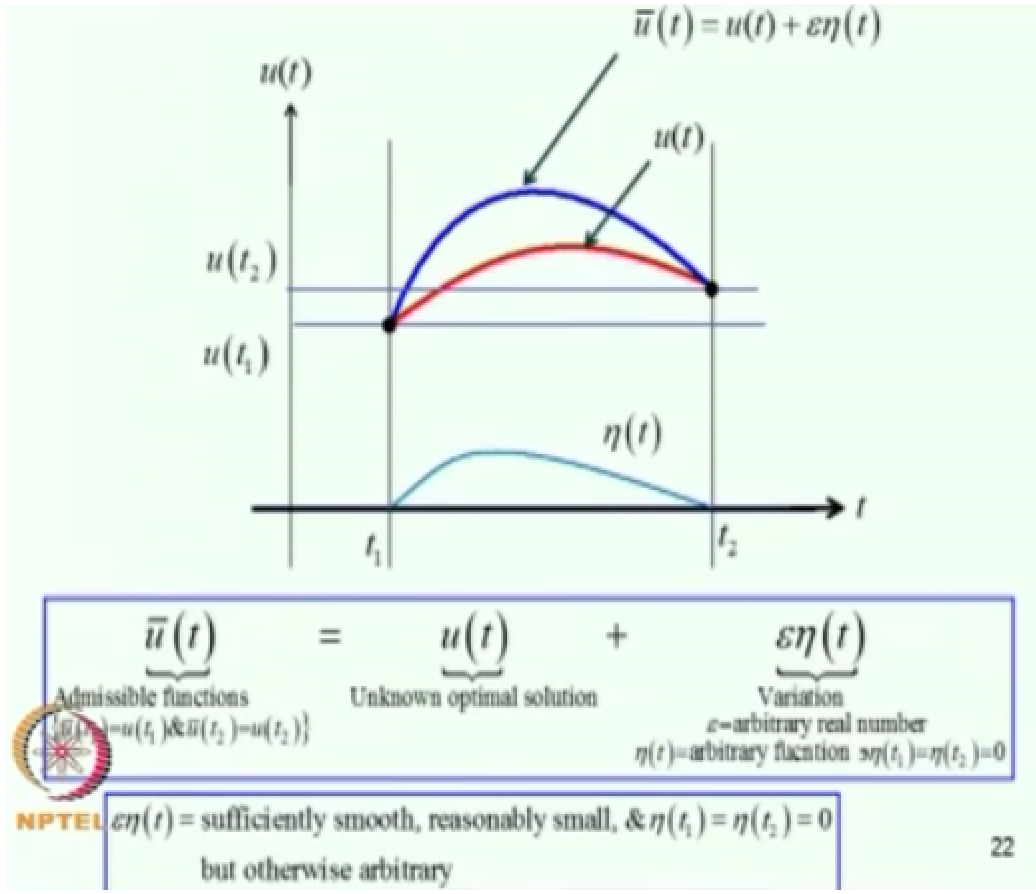


So this action integral A as it shown here is known as a functional, because for each of these paths each path is characterized by U of T and U dot of T, the U of T for this path is different from U of T for this path and so on and so forth, so each path is characterized by U of T and U dot of T and I can substitute that path in the action integral for each of the paths I can find a quantity A, this is the action. So if this is known as a functional, actually it is a function of functions that means for a functional the domain is a set of admissible functions, the admissible functions are those for example which pass through these two points are admissible for this problem, so A is a functional because for different choices of Q of T and Q dot of T I can get different values of A, so A is a function of functions because Q of T and Q dot of T are themselves functions of time.

Now the Hamilton's principle requires us to study the extremes of this action that is we need to address the problem of optimization of functionals to implement the Hamilton's principle and this subject of optimization of functionals is studied in the subject of calculus of variations. So presently in our discussion we have not included external forces and damping forces we'll come to that shortly, now an immediate object for our study would be to gain familiarity with the application of the Hamilton's principle by considering equations of motion of simple oscillators and structural elements, so after having done this we'll return to more details about how it is derived and so on and so forth.



So let us consider a simple example of a single degree freedom system consisting of a mass and a spring, and here if you apply Hamilton's principle L of T which is difference of kinetic energy and potential energy is given by T of T is 1/2 MU dot square, V of T is 1/2 KU square, so this is the action integral for this problem.



So let us see this in some detail, assume that T = T1 the mass is here and T = T2 it reaches this place, now this red line that you are seeing here is the unknown path, unknown optimal solution that is actual paths taken by the mass which is not known beforehand. Now what we do is we define a function eta of T which is shown here the property of eta of T is such that it is 0 at T = T1 and 0 at T = T2, so we construct a function U bar of T which is some of the unknown U of T which is to be determined + epsilon x eta of T where epsilon is a number, now so if you see here, we call this all functions that pass through these two points are called admissible functions, so through this representation what we are doing is we are representing the admissible functions in terms of the unknown optimal solution and a function what we call it as variation, this epsilon is arbitrary real number and eta of T say arbitrary function such that eta of T1 and eta of T2 are 0. This eta of T is taken to be sufficiently smooth and reasonably small and this condition is obeyed, but otherwise it is arbitrary, okay, so any admissible function you can see that it can be written in this form.

$$\underbrace{\bar{x}(t)}_{\text{Admissible functions}} = \underbrace{x(t)}_{\text{Unknown optimal solution}} + \underbrace{\varepsilon\eta(t)}_{\text{Variation}}$$

Admissible functions: $\{\bar{x}(t_1)=x(t_1)\ \&\ \bar{x}(t_2)=x(t_2)\}$

Variation: ε = arbitrary real number; $\eta(t)$ = arbitrary fucntion $\ni \eta(t_1)=\eta(t_2)=0$

$$A(\varepsilon)=\int_{t_1}^{t_2}\frac{1}{2}\left[m\dot{\bar{u}}^2(t)-k\bar{u}^2(t)\right]dt=\int_{t_1}^{t_2}\frac{1}{2}\mathbf{L}\left[\dot{\bar{u}}(t),\bar{u}(t)\right]dt$$

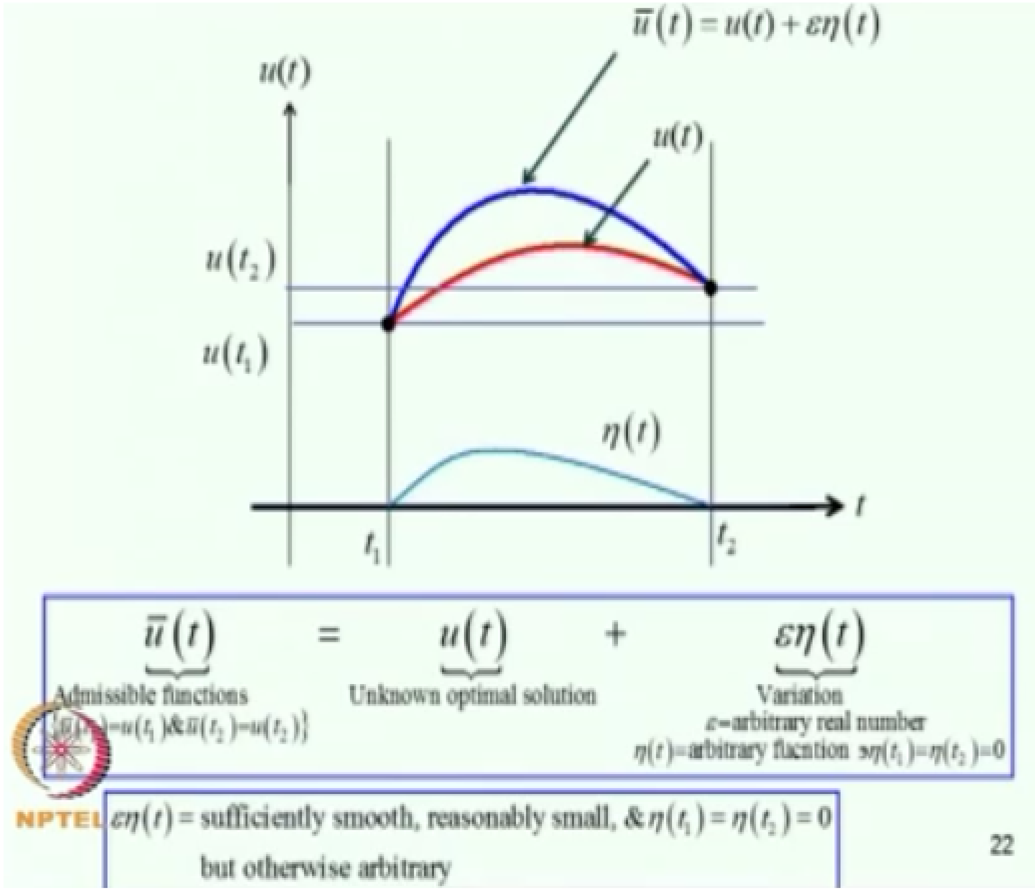
$$\Rightarrow \left.\frac{dA(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0}=0\Rightarrow\int_{t_1}^{t_2}\left[\frac{\partial\mathbf{L}}{\partial\dot{\bar{u}}}\frac{\partial\dot{\bar{u}}}{\partial\varepsilon}+\frac{\partial\mathbf{L}}{\partial\bar{x}}\frac{\partial\bar{u}}{\partial\varepsilon}\right]dt=0 \text{ for } \varepsilon=0$$

$$\Rightarrow\int_{t_1}^{t_2}\left[\frac{\partial\mathbf{L}}{\partial\dot{x}}\dot{\eta}(t)+\frac{\partial\mathbf{L}}{\partial x}\eta(t)\right]dt=0$$

$$\Rightarrow\left[\frac{\partial\mathbf{L}}{\partial\dot{x}}\eta(t)\right]_{t_1}^{t_1}+\int_{t_1}^{t_2}\left[-\frac{d}{dt}\left(\frac{\partial\mathbf{L}}{\partial\dot{u}}\right)+\frac{\partial\mathbf{L}}{\partial u}\right]\eta(t)dt=0$$

$$\Rightarrow-\frac{d}{dt}\left(\frac{\partial\mathbf{L}}{\partial\dot{u}}\right)+\frac{\partial\mathbf{L}}{\partial u}=0\left(\text{because } \eta(t) \text{ is arbitrary}\right)$$

Now what we do is we write now the action integral in terms of the admissible functions as shown here, so this is the Lagrangian corresponding to the admissible functions U bar dot and



U bar T, now the point now is if you carefully look at the formulation of the definition of these functions this action is now a function of epsilon that means we have actually parameterized the admissible function through a single parameter epsilon by varying epsilon I can get different types of admissible functions and where in mind the eta of T is also arbitrary so that would cover all possible admissible functions.

$$-\frac{d}{dt}\left(\frac{\partial \mathbf{L}}{\partial \dot{u}}\right)+\frac{\partial \mathbf{L}}{\partial u}=0$$

$$\mathbf{L}=\frac{1}{2}\left[m\dot{u}^2(t)-ku^2(t)\right]\Rightarrow \frac{\partial \mathbf{L}}{\partial \dot{u}}=m\dot{u};\frac{d}{dt}\left(\frac{\partial \mathbf{L}}{\partial \dot{u}}\right)=m\ddot{u};\frac{\partial \mathbf{L}}{\partial u}=-ku$$

$$\Rightarrow m\ddot{u}+ku=0$$

The equation $-\frac{d}{dt}\left(\frac{\partial \mathbf{L}}{\partial \dot{u}}\right)+\frac{\partial \mathbf{L}}{\partial u}=0$ is called Lagrange's equation.

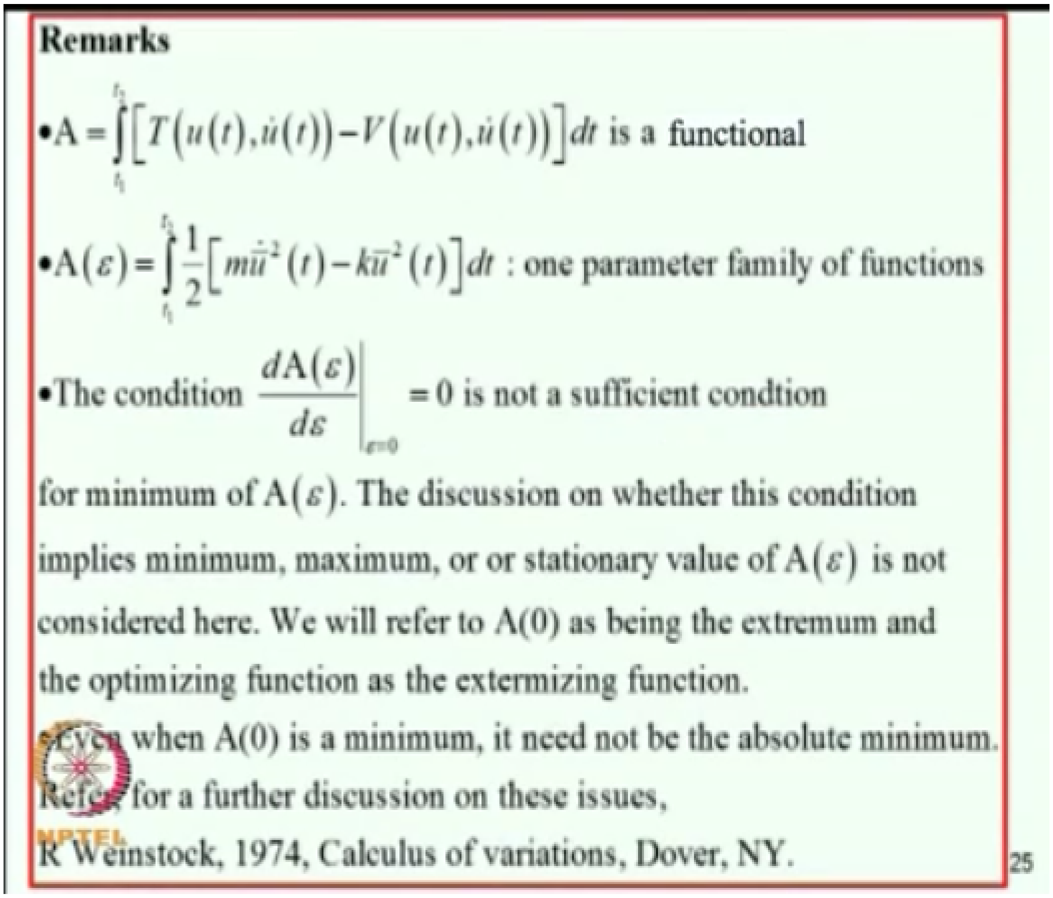
For discrete systems, with n dofs, Lagrange's equation can be generalized as

$$-\frac{d}{dt}\left(\frac{\partial \mathbf{L}}{\partial \dot{u}_i}\right)+\frac{\partial \mathbf{L}}{\partial u_i}=0; i=1,2,\cdots,n$$

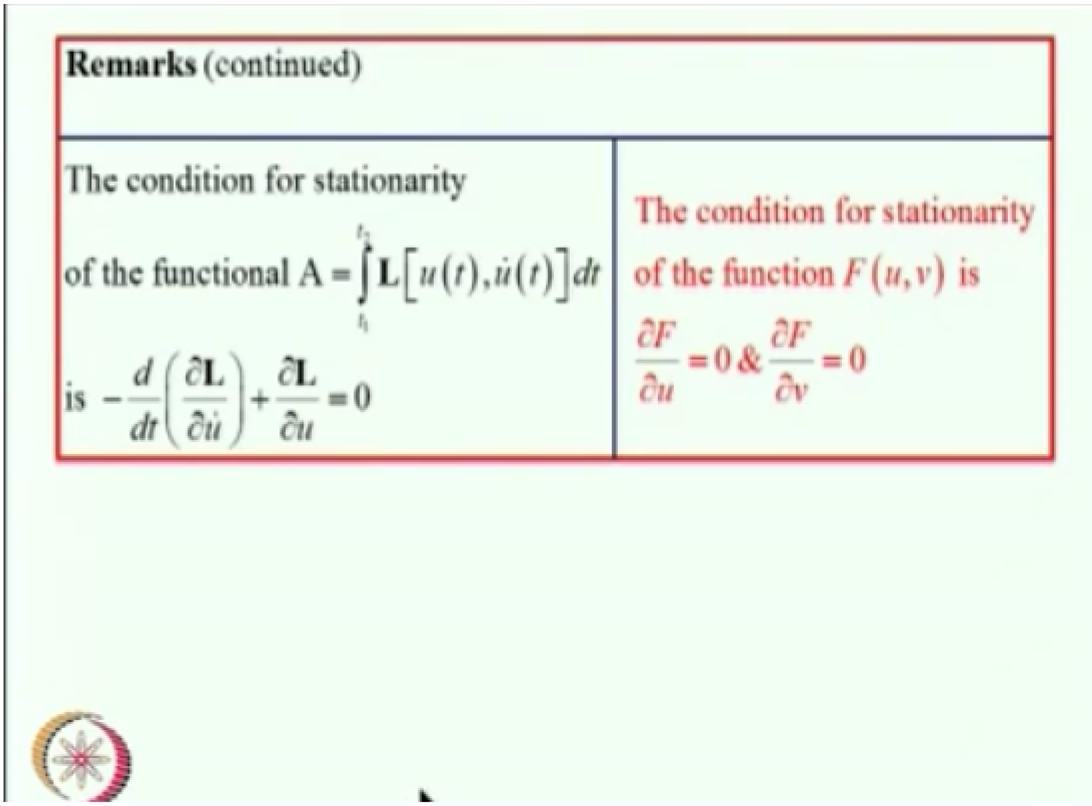
where $\mathbf{L}=\mathbf{L}\left[q_i(t),\dot{q}_i(t); i=1,2,\cdots,n\right]$.

Now I will now interpret action as a function of epsilon, then look for optimal values of A as a function of epsilon by the very definition of my U bar of T it is clear that the optimal value of A as a function of epsilon will be reached to an epsilon equal to 0, or in other words DA by D epsilon will be 0 at epsilon equal to 0, because that is a condition for extreme of the action integral. Now if you implement that so I have to differentiate this with respect to epsilon, so L is a function of U bar dot and U bar, so I will differentiate that with respect U bar dot and then U bar dot with respect to epsilon and similarly with respect to U bar.

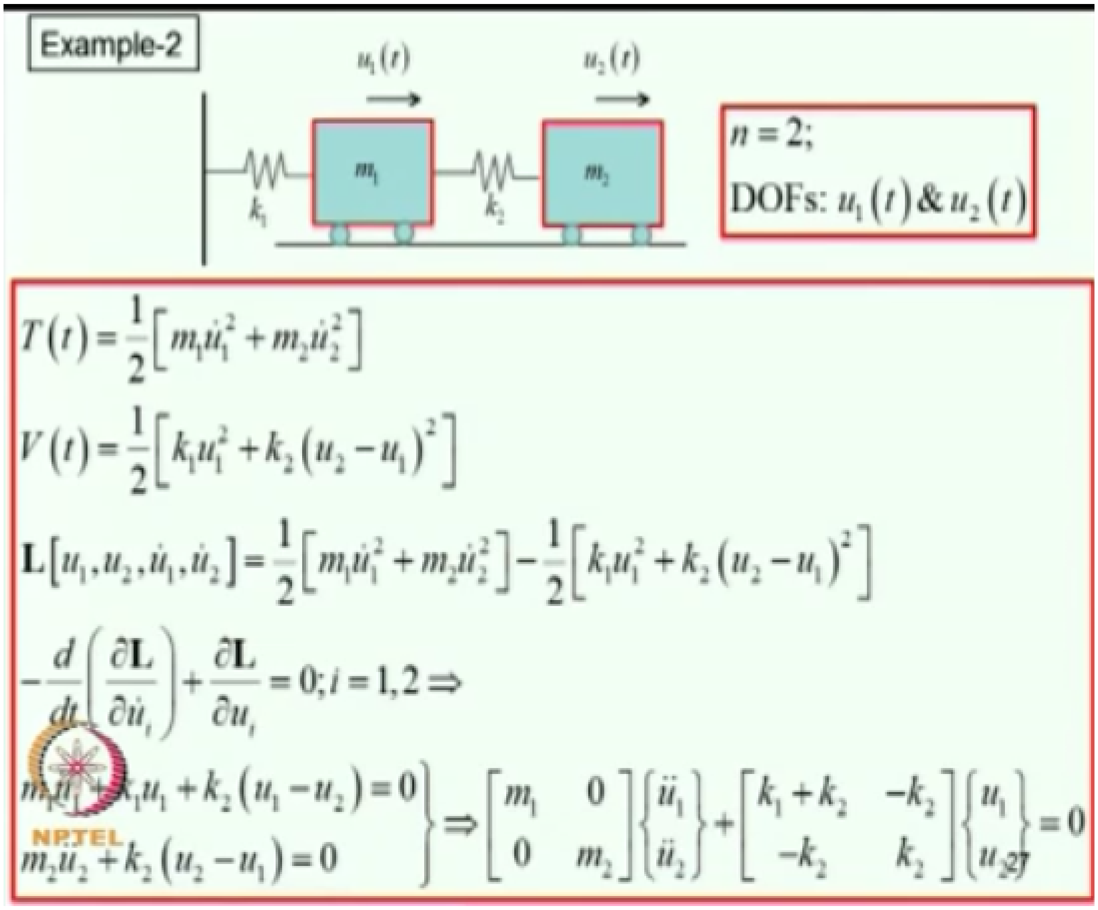
So now and this is 0 for epsilon equal to 0, since I am going to impose epsilon equal to 0, I need not distinguish between U bar and U, so consequently what happens if you differentiate U bar dot with respect to epsilon I will get eta dot of T, so I get these equations here which is equal to 0. Now I can manipulate these terms, I can do an integration by parts so I will get this term as it is and eta dot integral is eta of T, T1 to T2 plus the remaining terms we can show that we will get in this form. Now if you look at this term by the definition of eta of T it is 0 at T = T1 and it is 0 at T1 and T2, therefore this function will be 0, the terms inside the bracket would be 0. Now if you pay attention to this term since eta of T is arbitrary and this has to be 0 for all eta of T we can show that the term inside the bracket must be equal to 0, so this equation is known as the Euler Lagrange equation for this system, so I can now implement this for example dual by Dou U dot is MU dot, Dual by Dou U is D by DT of Dual by Dou U dot is MU double dot, Dual by Dou U is minus KU, therefore if I substitute this I get the familiar equilibrium equation, this we could have easily constructed in the free body diagram of the mass. Now as I said this is the, this equation is called the Lagrange equation or the Euler Lagrange equation, now this I have shown it for single degree freedom system so for discrete systems with multiple degrees of freedom with N degrees of freedom, the same equation Lagrange equation can be derived and that takes this form, for every coordinate I can write this equation and I will get any questions for the N unknown degrees of freedom. So we have to formulate the Lagrangian for the N degree of freedom system in terms of this generalized coordinates.



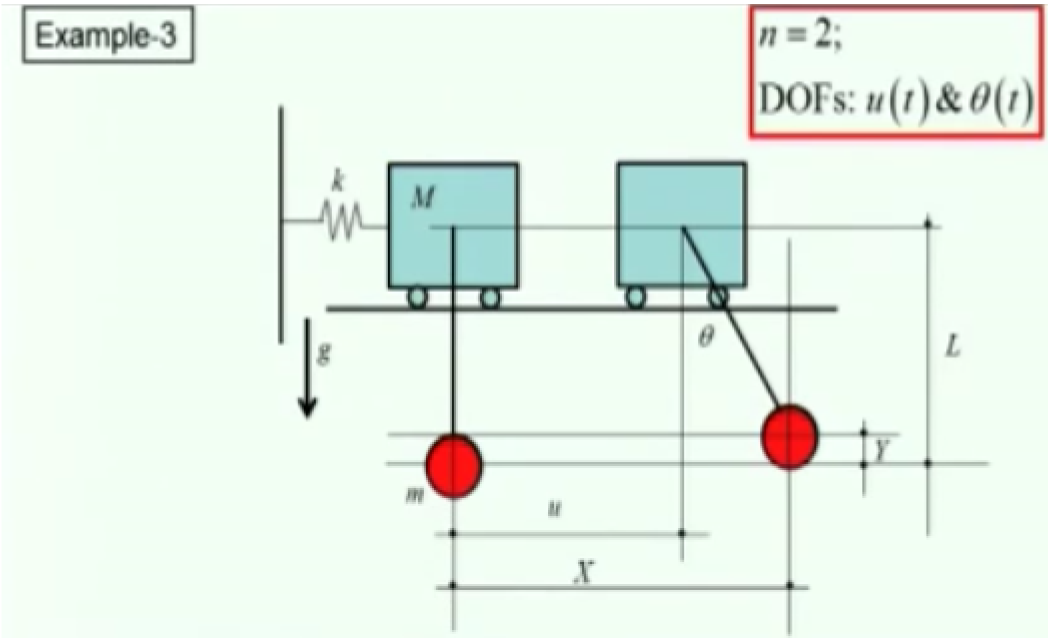
So let us quickly make some observations, this A which is difference of T - V as I said is a functional, whereas A of epsilon is now one parameter family of functions, so we are converting the problem of optimization of a functional to the optimization of a function, so the condition DA/D epsilon at, epsilon = 0, 0 is not a sufficient condition for minimum of epsilon it's a necessary condition, now the discussion on whether this condition implies minimum or maximum or stationary value of A of epsilon we are not going to consider in this lecture, we refer to A of 0 as being the extremum and the optimizing function as the extermizing function, so we will not discuss whether it is minimum or maximum, I refer you to see this book by Weinstock on book on Calculus of variations by Weinstock for further details on this.



Now let's continue the remarks, so if you have a function of two variables U and V, the condition for stationarity of the function is Dou F by Dou U = 0, and Dou F by Dou V = 0, in a similar vein the condition for stationarity of a functional like this is the Lagrange equation.



Now what I plan to do now is we'll formulate the Lagrangian for few simple systems and derive the equations of motion using the Lagrange equation, so this is a two degree freedom system M1 and M2 are the masses, K1 and K2 are the springs, the two degrees of freedom are U1 and U2, what is the total kinetic energy in the system? It consists of contribution from this mass and contribution from this mass and that is given by this expression. Similarly what is the potential energy, the potential energy in this spring plus the potential energy in this spring, potential energy in this spring is 1/2 K1 U1 square, in this spring it is 1/2 K2 U2 - U1 whole square, so this is V of T, so the Lagrangian now which is function of U1 U2, U1 dot and U2 dot is now given by this, so now I can run the Lagrange equation on U1 and U2 and I get two equations which I have shown here, this is the equation for the first mass, this is equation for the second mass, and this equation can also be written in the matrix form as we all know this quantity is known as the mass matrix, this is known as the stiffness matrix, they are symmetric in this case we will talk about properties of these matrixes in due course.



$$T(t) = \frac{1}{2}M\dot{u}^2 + \frac{1}{2}m\dot{X}^2 + \frac{1}{2}m\dot{Y}^2$$

$$V = \frac{1}{2}Ku^2 + mgY$$

$$n = 2; q = (u, \theta)$$

$$X = u + L\sin\theta; Y = L(1-\cos\theta);$$

$$\dot{X} = \dot{u} + L\dot{\theta}\cos\theta; \dot{Y} = L\dot{\theta}\sin\theta$$

Another example a combination of a mass spring system and a pendulum, so this is the original configuration at some time T if I take a snapshot the mass would have moved here and the pendulum would have swung up like this, so and I assume that this card is inextensible therefore there are only two degrees of freedom, one is position of this mass and the other one is this angle theta so this is a two degree freedom system. Now how do I construct the kinetic energy and potential energy for this system, so I've introduced some variables lowercase u capital X, capital Y, now let us see how do we write? Now the kinetic energy is given by of this mass is 1/2 MU dot square, now if capital X and Y are the coordinates of this mass then the kinetic energy is 1/2 MX dot square + 1/2 MY dot square, the potential energy is the one that is stored in the spring and the work done by the gravity on moving this mass by this distance and that is given by 1/2 KU square + mgY, now it has two degrees of freedom, now you see here in this expression I have three variables U X and Y, but actually X and Y are not independent there is a constraint because the length of this chord remains unchanged, therefore I should use that, so how do I put that? So capital X is actually U + L sin theta and from this I get and similarly Y is L 1 - cos theta and from this I get the velocities U dot differentiate this with respect to T I get this, Y dot from this, so I can substitute this expression for X, Y, X dot, and Y do into these expressions and I get the Lagrangian, this is what I am doing here, this is the T,

$$T(t) = \frac{1}{2}M\dot{u}^2 + \frac{1}{2}m\dot{X}^2 + \frac{1}{2}m\dot{Y}^2$$

$$V = \frac{1}{2}Ku^2 + mgY$$

$$n = 2; q = (u, \theta)$$

$$X = u + L\sin\theta; Y = L(1-\cos\theta);$$

$$\dot{X} = \dot{u} + L\dot{\theta}\cos\theta; \dot{Y} = L\dot{\theta}\sin\theta$$

$$T = \frac{1}{2}M\dot{u}^2 + \frac{1}{2}m\left[\dot{u} + L\dot{\theta}\cos\theta\right]^2 + \frac{1}{2}m\left[L\dot{\theta}\sin\theta\right]^2$$

$$V = \frac{1}{2}Ku^2 + mgl(1-\cos\theta)$$

$$\mathbf{L} = \mathbf{L}(u, \theta, \dot{u}, \dot{\theta})$$

$$\mathbf{L} = \frac{1}{2}M\dot{u}^2 + \frac{1}{2}m\left[\dot{u} + L\dot{\theta}\cos\theta\right]^2 + \frac{1}{2}m\left[L\dot{\theta}\sin\theta\right]^2 - \frac{1}{2}Ku^2 + mgL(1-\cos\theta)$$

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this is the V, and this is the Lagrangian. So it is now function of U and theta U, theta U dot and theta dot.

So I will now run apply the Lagrange's equation on this function and I get once on U, once on theta, if I apply I will get these two equations.

$$\mathbf{L} = \frac{1}{2}M\dot{u}^2 + \frac{1}{2}m\left[\dot{u} + L\dot{\theta}\cos\theta\right]^2 + \frac{1}{2}m\left[L\dot{\theta}\sin\theta\right]^2 - \frac{1}{2}Ku^2 + mgL(1-\cos\theta)$$

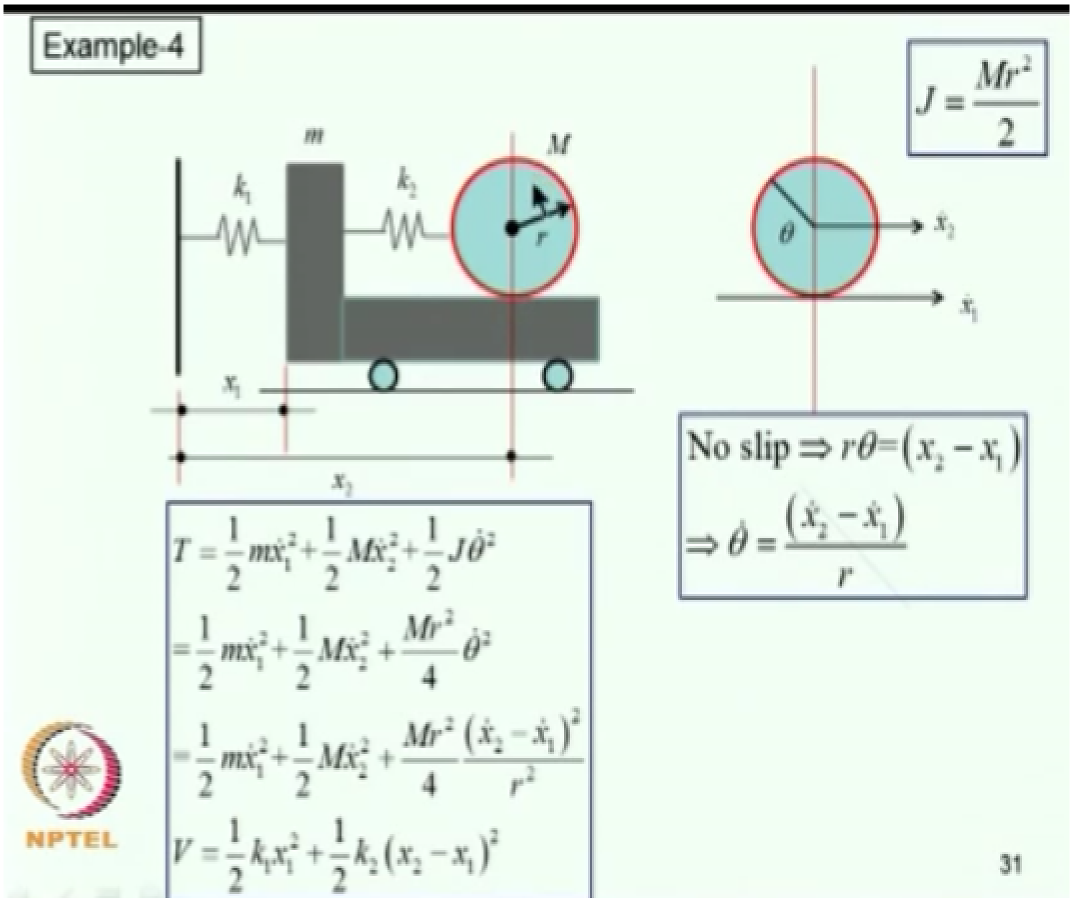
$$\frac{d}{dt}\left(\frac{\partial \mathbf{L}}{\partial \dot{u}}\right) - \frac{\partial \mathbf{L}}{\partial u} = 0 \Rightarrow (M+m)\ddot{u} + mL\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) + ku = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathbf{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathbf{L}}{\partial \theta} = 0 \Rightarrow mL^2\ddot{\theta} + mL\ddot{u}\cos\theta - mL\dot{u}\dot{\theta}\sin\theta + mgL\sin\theta = 0$$

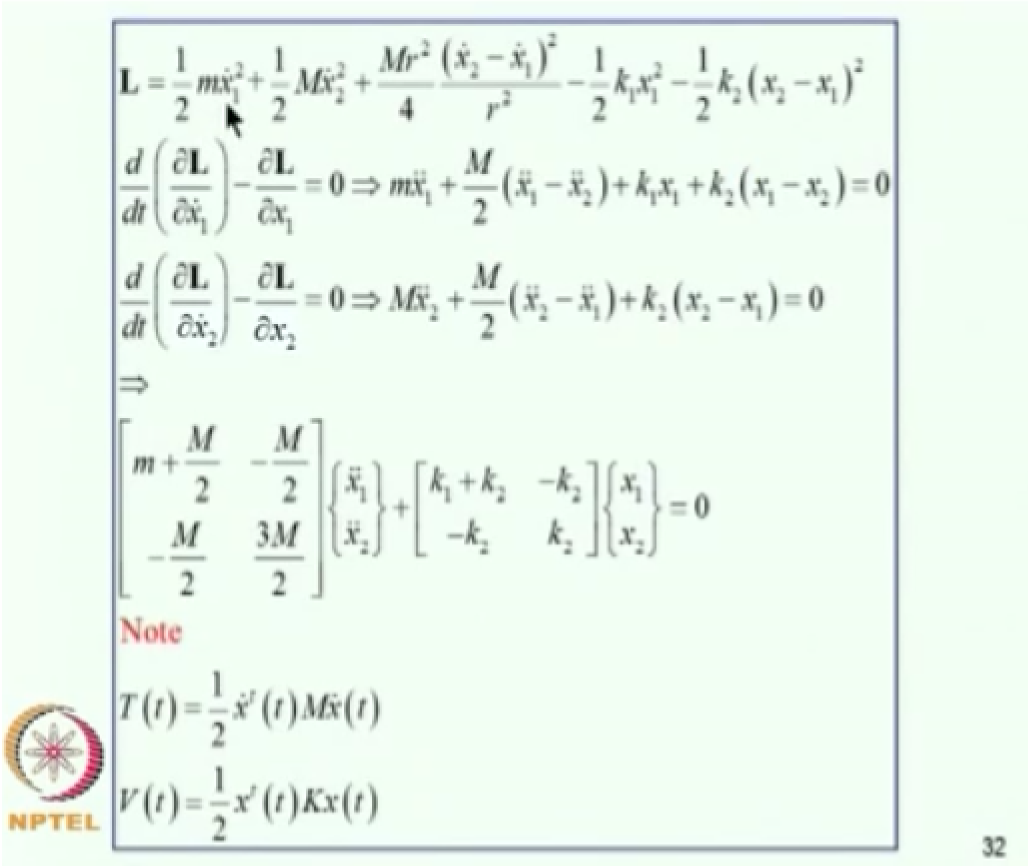
Remark

The approach can handle nonlinear systems

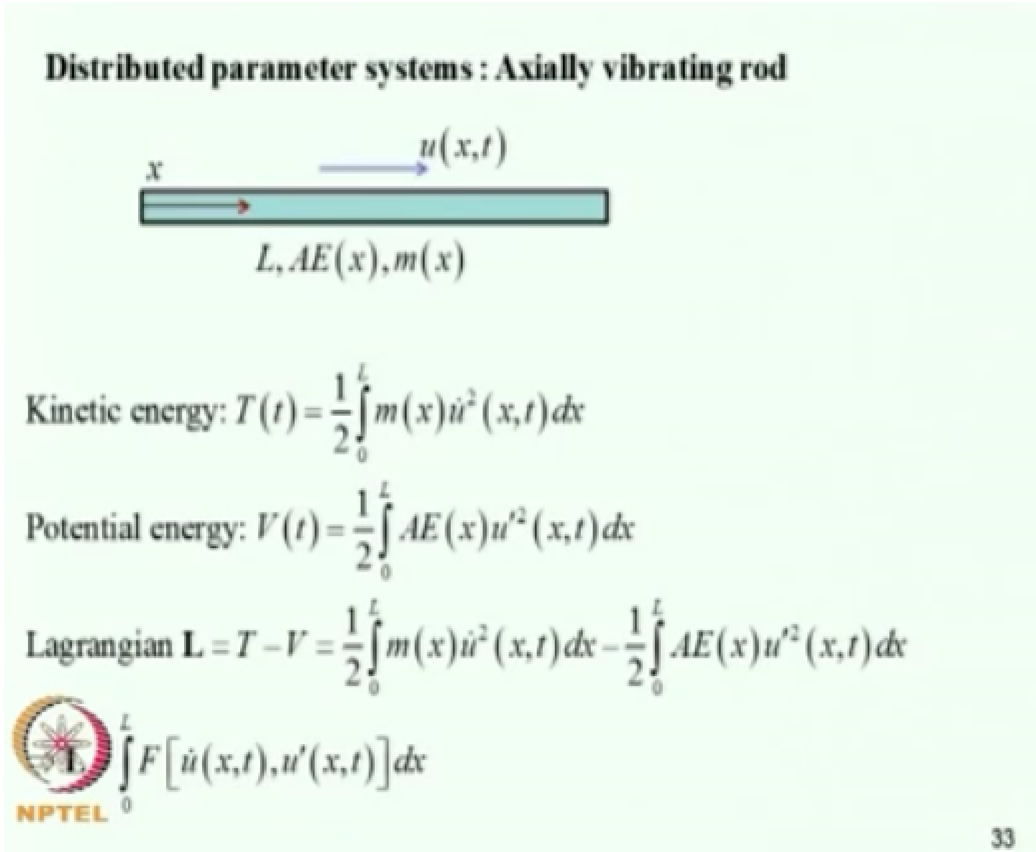
Now if you observe I have not made any assumptions on smallness of theta, so you see sin theta etcetera retained as it is here, an interesting thing in this example is that the non-linearity is also associated with the inertial forces, so the point that we can take from this illustration is that the approach that we are discussing can handle nonlinear systems.



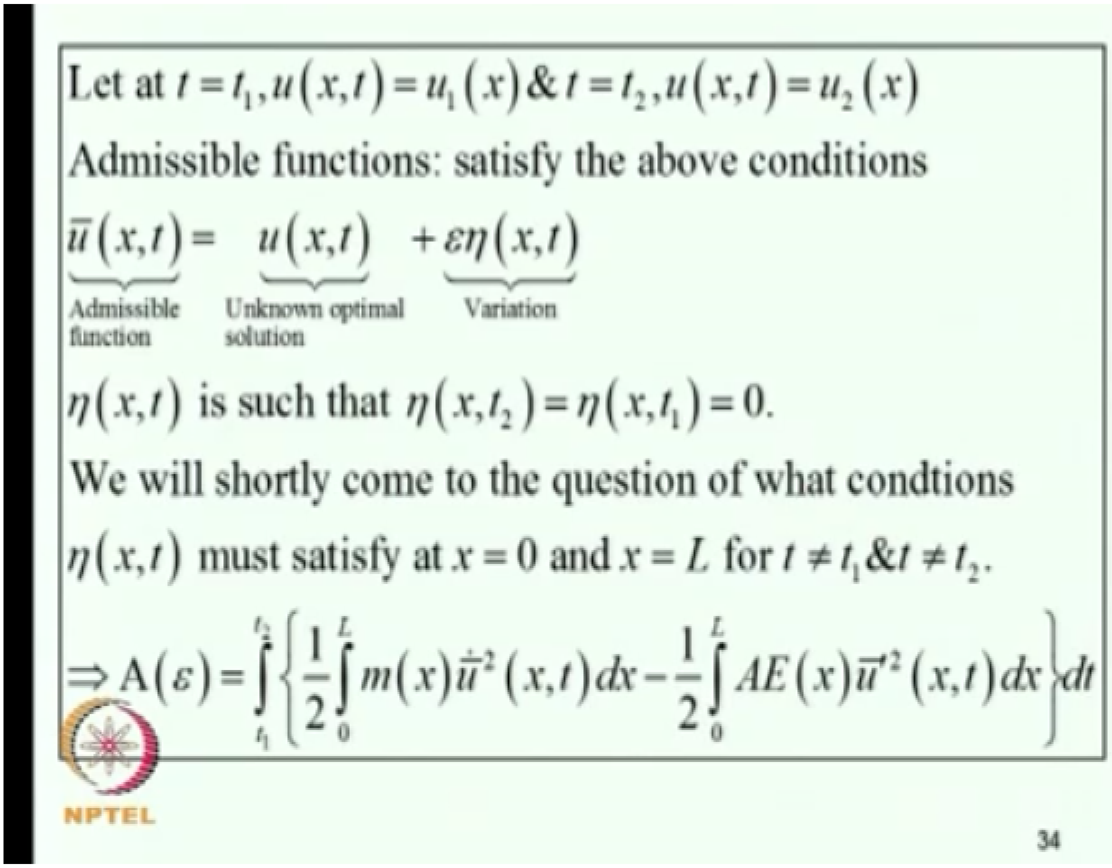
Now another example imagine that a disk is rolling on a platform which itself is moving on another horizontal platform, so I want to formulate the equation of motion for this, so this is not point mass, this is a cylindrical disk, now what is the kinetic energy? Kinetic energy due to this mass that is 1/2 MX1 dot square and then this is the capital M, which is a mass that moves X2 is its coordinate so the translation of this mass itself contributes to the kinetic energy and that is 1/2MX2 dot square, but as it moves it also rolls so there is another source of kinetic energy due to the rotation which is given like this. Now we can compute J and we can show that this MR square by 2 and we use this. Now this again is a two degree freedom system, so if you assume that there is no slip what happens is suppose R is the radius and theta is an angle through which it rotates then R theta must be equal to X2 - X1, so this serves as a constraint and that eliminates of this three variables one of the variable can be eliminated so what I do is, I write theta dot in terms of X2 and X1, theta in terms of X2 and X1 and construct the Lagrangian in terms of X1 and X2 and that reads as shown here.



Now I will run the Lagrange's equation for this function and I get the two equations of motion one for X1, another for X2 and I can also put this in the matrix form as I did in the previous example, and one point that I would like to make is if you call the vector of X1 and X2 as X and X1 dot and X2 dot as X dot, you can show that the expression for kinetic energy and potential energy that we have derived here can be obtained as 1/2 X dot transpose MX dot and 1/2 X transpose KX, so this is a structure of kinetic and potential energy for linear systems, so in this example I have assumed that the displacements are small therefore it's a linearized equation of motion.



Now I have talked about discrete systems, now how about distributed parameter system. So I'll start by discussing the case of an actually vibrating rod, so let us consider a rod element of length L axial rigidity AE of X, E is an modulus, A is area of cross section, M of X is a mass per unit length, so this rod oscillates in this direction and the degree of freedom is displacement of the rod is given by U of X, T. So this is a distributed parameter system or continuous system it has infinite degrees of freedom, so how do we use Hamilton's principle to formulate the governing equations. So we will form the Lagrangian, how do we get kinetic energy? So you take elementary chunk of material M of X into DX U dot square X, T integral 0 to L half of that is the kinetic energy, potential energy is due to the axial deformation and you can derive this using a simple model for the rod behavior and we get this. Now the Lagrangian is given by this, so this can be seen that the Lagrangian itself is an integral of a function which is a function of U dot and U prime, prime means derivative with respect to X, and dot is a derivative with respect to T.



Now what we do is let us assume that T = T1, U of X,T is U1 of X and at T = T2 U of X,T is U2 of X, now the admissible functions we take that all functions that satisfy these requirements are admissible functions, so we take admissible function in this form, so this is the unknown optimal function and this is a variation now this theta of X,T is such that at T1 and T2 it is 0, what exactly will happen at X = 0 and X = L is something of vital importance and we'll come to that shortly.

$$\frac{dA}{d\varepsilon}=0 \text{ at } \varepsilon=0$$

$$\Rightarrow \int_{t_1}^{t_2}\int_0^L [m(x)\dot{u}(x,t)\dot{\eta}(x,t) - AE(x)u'(x,t)\eta(x,t)]dxdt = 0$$

$$\text{Consider } \int_{t_1}^{t_2} m(x)\dot{u}(x,t)\dot{\eta}(x,t)dt =$$

$$\left[m(x)\dot{u}(x,t)\eta(x,t)\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{\partial}{\partial t}\left[m(x)\dot{u}(x,t)\right]\eta(x,t)dt$$

$$\text{Similarly } \int_0^L AE(x)u'(x,t)\eta'(x,t)dx =$$

$$\left[AE(x)u'(x,t)\eta(x,t)\right]_0^L - \int_0^L \frac{\partial}{\partial x}\left\{AE(x)u'(x,t)\right\}\eta(x,t)dx$$

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Now I will form the action integral as a function of this parameter epsilon as we did earlier and then again use the condition that the optimal value of A as a function of epsilon will be reached at epsilon equal to 0, so if you carry out the differentiation with respect to epsilon and put epsilon equal to 0, I will get this equation, so there is a eta dot here, and there is a eta prime will be there.

Now let us consider the first part of this M of X U dot ETA dot X,T now I will integrate by parts, so the first term will be this, the second term will be this, now this ETA of X,T is 0 at T = T1, and T = T2, therefore this entire term drops off it is 0, fine, now I have still this second term I will consider that separately again now this integration is with respect to X so again I do integration by parts so I get this is the first term and this is the second term. Now if we put back all these terms into the original equation I get this as the equation, this must be equal to 0,

$$\frac{dA}{d\varepsilon} = 0 \text{ at } \varepsilon = 0 \Rightarrow$$

$$\Rightarrow \int_0^L \left[m(x)\dot{u}(x,t)\eta(x,t) \right]_{t_1}^{t_2} dx - \int_{t_1}^{t_2} \left[AE(x)u'(x,t)\eta(x,t) \right]_0^L dt$$

$$+ \int_{t_1}^{t_2}\int_0^L \left(\frac{\partial}{\partial x}\left\{ AE(x)u'(x,t) \right\} - m(x)\frac{\partial^2 u}{\partial t^2} \right)\eta(x,t)\,dx\,dt = 0$$

Since $\eta(x,t)$ is chosen arbitrarily, we take that each of these terms are separately equal to 0. That is

$$\int_{t_1}^{t_2}\int_0^L \left(\frac{\partial}{\partial x}\left\{ AE(x)u'(x,t) \right\} - m(x)\frac{\partial^2 u}{\partial t^2} \right)\eta(x,t)\,dx\,dt = 0$$

$$\int_0^L \left[m(x)\dot{u}(x,t)\eta(x,t) \right]_{t_1}^{t_2} dx = 0$$

$$\int_{t_1}^{t_2} \left[AE(x)u'(x,t)\eta(x,t) \right]_0^L dt = 0$$

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so this we already seen this is 0 because eta is 0 at T = T1 and T = T2, so that is guaranteed.

Now since eta of X, T is chosen arbitrarily we take that each of these terms are separately equal to 0, so that is this term that's integral is 0, and this term is 0, and this term is 0, this is because eta of X, T is arbitrary. Now let us consider one of these terms now, that is this, Consider

$$\int_{t_1}^{t_2}\int_0^L\left(\frac{\partial}{\partial x}\{AE(x)u'(x,t)\}-m(x)\frac{\partial^2 u}{\partial t^2}\right)\eta(x,t)\,dxdt=0$$

$\eta(x,t)$ is arbitrary $\Rightarrow\left(\frac{\partial}{\partial x}\{AE(x)u'(x,t)\}-m(x)\frac{\partial^2 u}{\partial t^2}\right)=0$

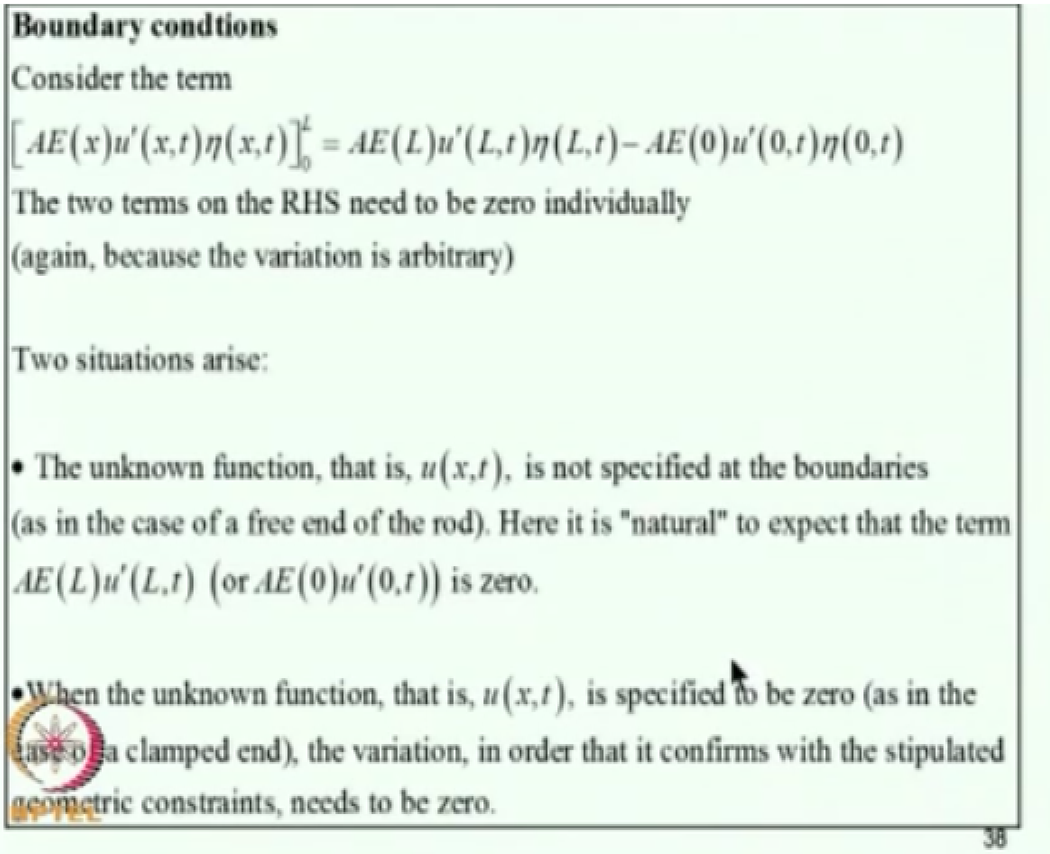
[Valid over the length of the bar]

Consider

$$\left[m(x)\dot{u}(x,t)\eta(x,t)\right]_{t_1}^{t_2}=m(x)\dot{u}(x,t_2)\eta(x,t_2)-m(x)\dot{u}(x,t_1)\eta(x,t_1)$$

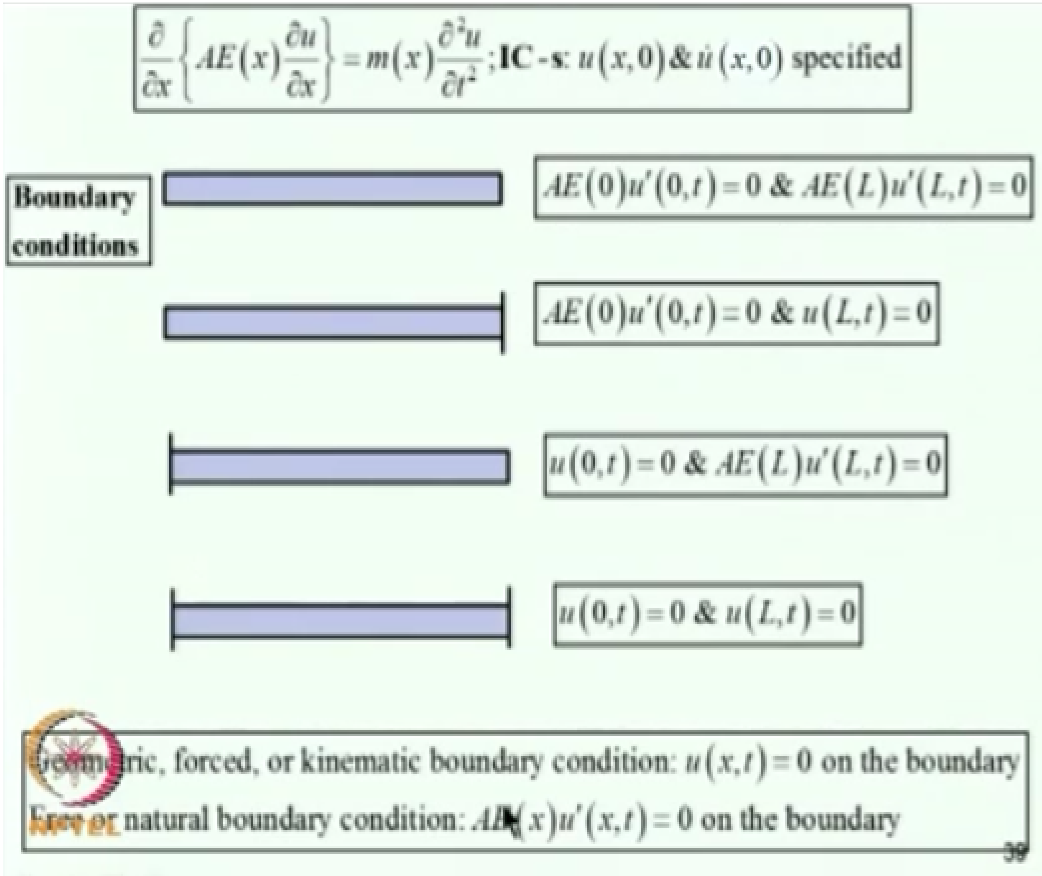
$$\eta(x,t_2)=\eta(x,t_1)=0\Rightarrow\left[m(x)\dot{u}(x,t)\eta(x,t)\right]_{t_1}^{t_2}=0$$

now here this has to be 0 for any choice of eta of X, T so that choice itself is arbitrary therefore the only way this can be 0 for arbitrary choice of eta of X, T we can show that the term inside this bracket must be equal to 0, so this is the condition which is valid for the length of the bar where X takes values from 0 to L. Now I already mentioned this M of X, U dot X, theta, X, T at T1 T2 it can be expanded like this, but since this is equal to 0, this term drops off.



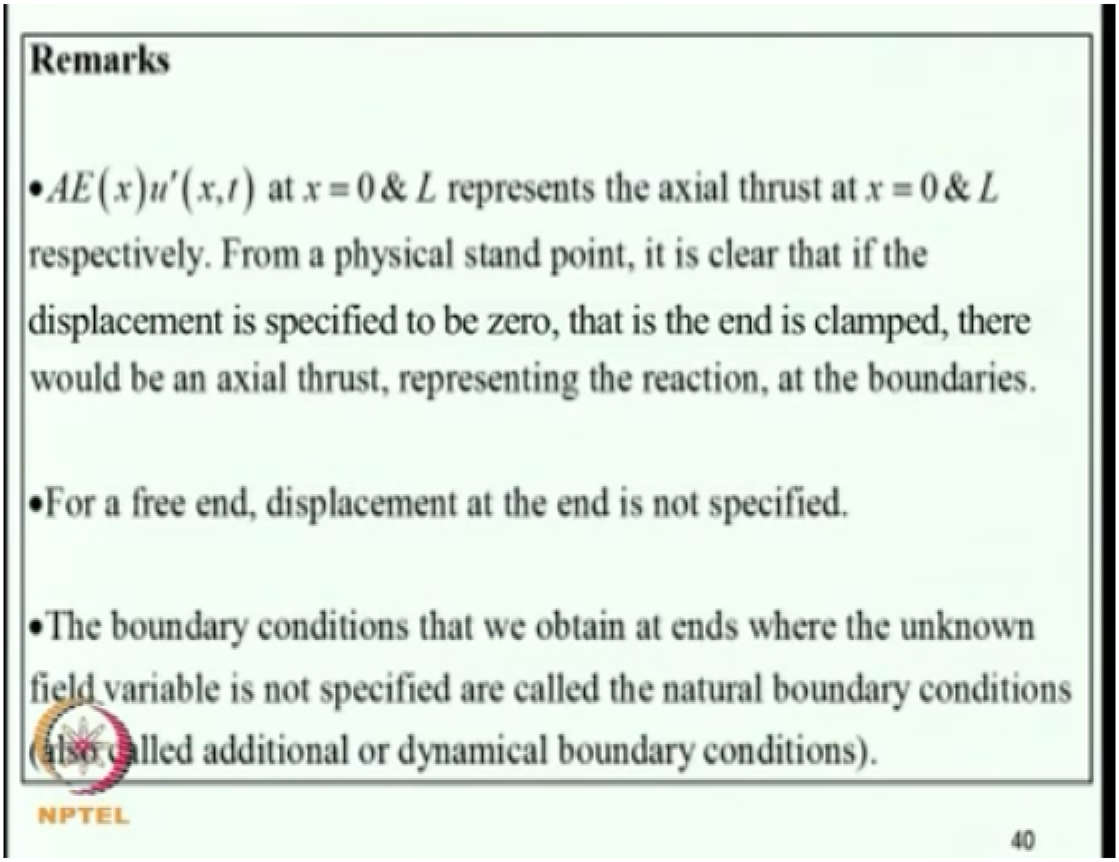
Now this takes us to the question on what happens at X = 0 and X = L, so for this we can consider this term which is the remaining term that we need to handle, so if I expand this out I get AE of L U bar L, T, et of L, T - AE of 0, U bar 0, T, eta of 0, T. Again these two terms need to be independently equal to 0 because eta is arbitrary for the same reason. Now there are two situations that will arise now, if you look at these functions so for example at X = 0, if the field variable is not specified then we cannot specify the value of the variation, now if this term has to be 0 the only way that can happen is this coefficient must be equal to 0, so this condition is known as natural boundary condition, okay. So that is the unknown function that is U of X of T is not specified at the boundaries as in the case of a free end of a rod, it is natural to expect that the term that is this multiplier term or this multiplier term is 0.

On the other hand if you are dealing with a fixed rod say that is clamped at one end, then the unknown function of U of X, T is specified to be 0, so therefore variation has to be 0 at those points, so then the variation in order that it conforms with the stipulated geometric constraints needs to be 0, so this leads us to the specification of boundary conditions and we can, for this



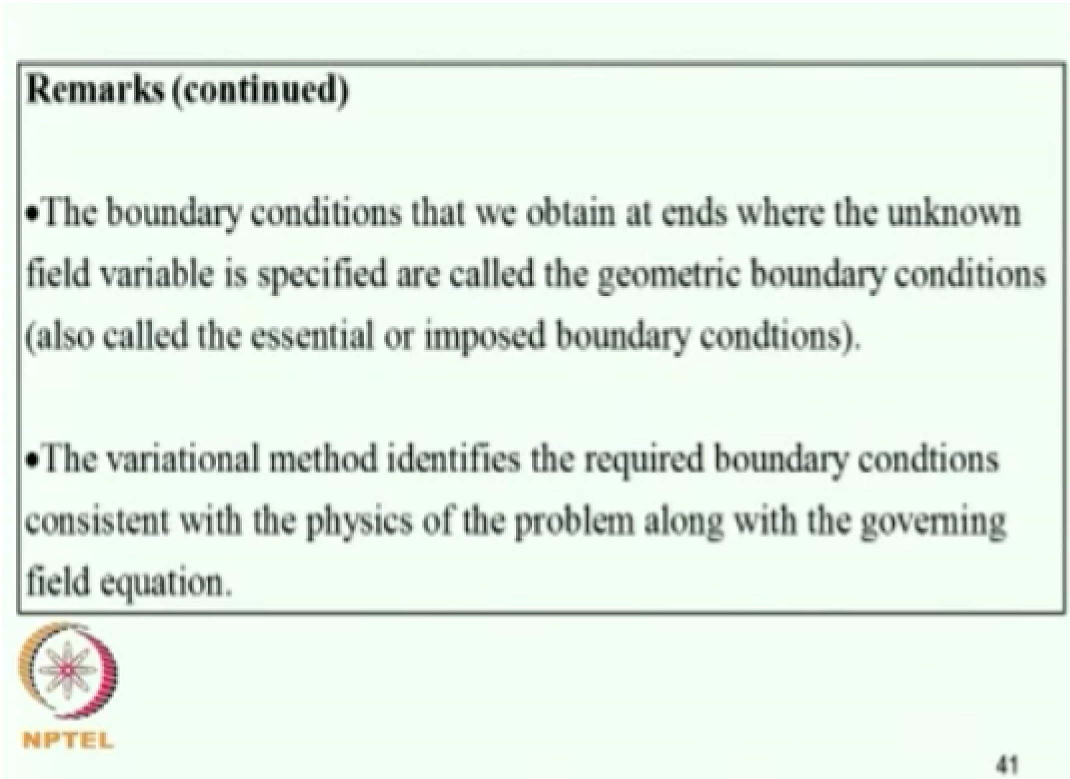
problem we can think of four possible configurations of the rod, so this equation which is valid for X = 0 to L and for T = 0 to infinity is called the field equation, so here this is the partial differential equation and associated with this, there are two initial conditions.

Now how about the boundaries? If you have a rod which is free at both ends then this quantity must be 0 at the two ends, you can see that this is nothing but the resulting axial thrust which is 0, because it's a free end, and the displacements here are unknowns which have to be determined by solving this equation. On the other hand if you consider a bar which is clamped here and free here at this place U is 0, that is U of L,T is 0, but at this end the actual thrust is 0 and this is the natural boundary condition that we have got through the variational argument, similarly for this we get, for this case we get U of 0 T is 0 and the axial thrust AEL, U prime, L, T is 0, of course here both are, both displacement at the both ends are specified to be 0, so this conditions on U we call it a geometric, forced, or kinematic boundary conditions that is U is 0 on the boundary, free or natural boundary condition that is the axial thrust AE of X U bar X, T is 0 on the boundary. So the important aspect of applying Hamilton's principle is that it not only provides us with the governing field equation but also it provides us with the necessary boundary conditions that has to be used in solving the governing partial differential equation, whereas this may be obvious for one-dimensional problem that we are seeing here for many other more complicated problems what should be the boundary condition is not something that can easily be specified by inspecting the problem, geometry of the other details of the problem it needs to be derived by applying the variational arguments.



So let's say in summary make some remarks, so these AE of X U prime of X, T at X = 0 and L represents the axial thrust at X = 0, and L respectively, from a physical standpoint it is clear that if the displacement is specified to be 0, that is the end is clamped, there would be an axial thrust which is a reaction at the boundaries. At a free end displacement of the end, displacement at the end is not specified therefore there won't be any, you know, stress result ends like axial thrust there, the boundary conditions that we obtain at the ends where the unknown field variable is not specified are called the natural boundary conditions, also called additional or dynamical boundary condition. There is a few range of terminologies that is used here so if you read different books you come across different types of terminologies.

The boundary conditions that we obtain at the ends where the unknown field variable is specified are called geometric boundary conditions, also called essential or imposed boundary



conditions, so the variational method identifies the required boundary conditions consistent with the physics of the problem along with the governing field equation, so that is one of the major takeaway from applying the Hamilton's principle to this class of problems, so what we will do now is, we will close this lecture at this point, in the next lecture I will consider few more problems in associated with vibration of bars and then we'll also consider problems of flexural vibration of bars, and after that we'll start looking at approximate methods, so at this point we'll close this lecture.

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