Water Resources Systems

Prof. P.P.Mujumdar

Department of Civil Engineering

Indian Institute of Science, Bangalore

Module No # 02

Lecture No # 08

Linear Programming: Graphical method

Good morning and welcome to this the lecture number 8 of the course Water Resource Systems Modeling Techniques and Analysis. In the previous lectures we have been discussing optimization of Functions of multiple variables with constraints; that means, we have now entered into the domain of constraint optimization and specifically in the last lecture I talked about the Kuhn Tucker conditions commonly called as the K-T conditions, which are necessary conditions for any optimization problem if it it has to have a optimal solution.

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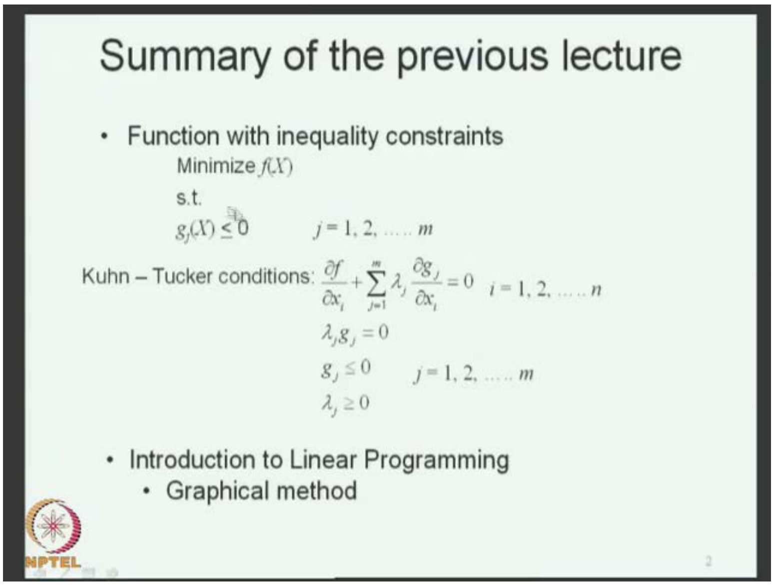

So, we discussed about a general problem of this type minimize f of X and X is a vector of decision variables X 1, X 2, etcetera X n, there are n number of variables subject to g j of X less than or equal to 0 these are the constraints; we have n number of constraints. So, n number of variables and n number of constraints for such a problem the K - T conditions are the following; so, we have discussed this at length, so you have n equations corresponding to the first first set of conditions, then n number of equations associated with the second set of conditions lambda j g j equal to 0, we use this n plus m number of equations to solve for X 1, X 2, etcetera X n and lambda 1, lambda 2, etcetera lambda n.

So, n plus m equations we have n plus m number of variables, we solve them solve for the n plus m variables and associated with each of these sets of solutions, that we obtain we check the conditions g j less than or equal to 0 which is the original set of constraints; and lambda j greater than or equal to 0 which is the sign of the lambda j s and as I have mentioned depending on maximization or g j of X greater than or equal to 0 the lambda j s sign will be changing according to which combination. We are talking about here and then we also examined through a numerical example, how we actually apply the Kuhn Tucker conditions to obtain the stationary points of an optimization problem.

Then, we introduce the most important topic of this course namely the Linear Programming. Recall that we said if f of X which is objective function is, in fact, a linear function of the decision variables and all the constraints g j of X are all linear functions of the decision variables and then additionally, if the decision variables are all non negative, then it constitute a general form of linear programming problem but, the essential requirement for a linear programming problem is that, the objective function must be a linear function of the decision variables and the constraints must be linear functions of the decision variables, then towards the end of the previous lecture I just started introducing the graphical methods, so we will continue with the graphical method of solution remembers the Graphical method of solution.

In fact, provides us with motivation for the more rigorous algorithmic way of solving the linear programming problem which is, in fact, the simplex algorithm, simplex method of solution, so we will understand the graphical method correctly, although it is a very simple procedure, but the motivation for the simplex algorithm. In fact, comes from the graphical method, that I will be discussing now, so we will revisit the problem that I discussed towards the end of the previous lecture and from that, will go on to explain the complete graphical procedure.

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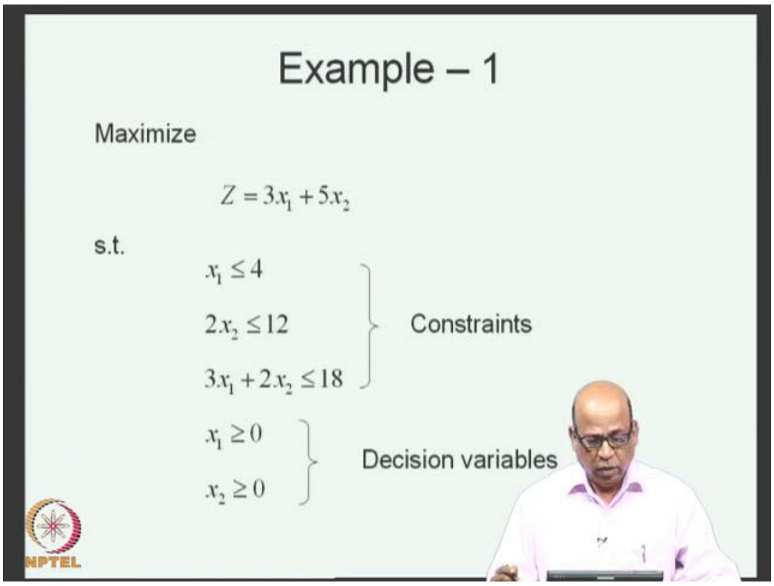

So, we started with the example maximize Z is equal to 3 x 1 plus 5 x 2, subject to x 1 less than or equal to 4, 2 x 2 less than or equal to 12. 3 x 1 plus 2 x 2 less than or equal to 18, these are the constraints.

And then, we have the non negativity of the decision variables x 1 greater than or equal to 0 and x 2 greater than or equal to 0, these are the non negativity of the decision variables. Recall that or observe that, the objective function is a linear function of the decision variables; the decision variables are x 1 and x 2 all the constraints are linear functions of the decision variables x 1 and x 2. So, all of these are linear functions additionally, we have the non negativity of the decision variables x 1 greater than or equal to 0, x 2 greater than or equal to 0, so because, this is the simple problem with only two decision variables, we can demonstrate with a graphical solution; here the fact that x 1 is greater than or equal to 0, x 2 greater than or equal to 0 leads to the solution being in the first quadrant.

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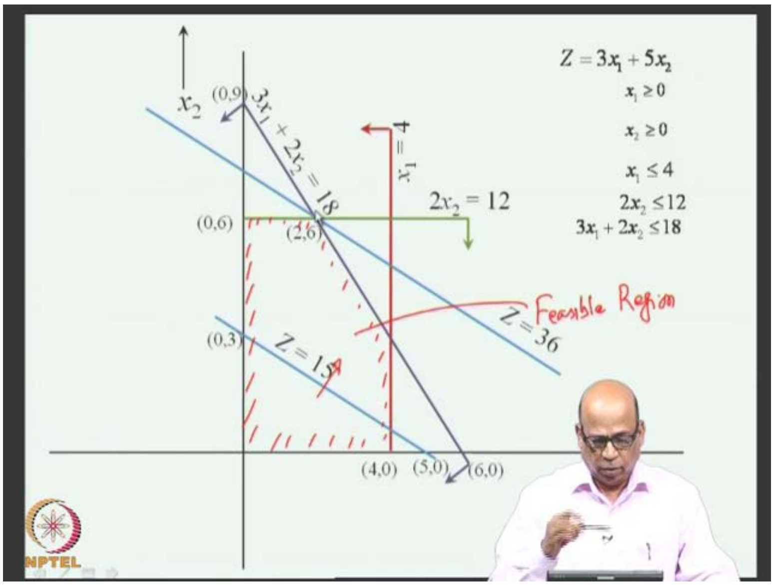

So, first we look at x 1 greater than or equal to 0 and x 2 greater than or equal to 0, which defines this region, in fact this is the first quadrant, so the solution must lie in the first quadrant then we look at each of the constraints. Let us say that, we look at x 1 is less than or equal to 4, so this is the constraint we are looking at the binding value for this will be x 1 equal to 4, we have x 1 greater than or equal to 0 and x 1 is less than or equal to 4, so we draw a line x 1 equal to 4 which, in fact shows the bound for this constraint, so you cannot have a value x 1 greater than 4. So, you cannot be looking at this region as for as this constraint is concerned because, x 1 is less than or equal to 4 it also means that, we have to look to the left of this constraint this line x 1.

Is equal to 4 so, we are looking at to the left of this line, similarly we look at the next constraint which is 2x 2 is less than or equal to 12 or x 2 is equal to 6, will become the binding line for this, so you draw a line x 2 is equal to 6 and then you point the arrow downwards to indicate that, you want 2x 2 to be less than or equal to 12. See what happens, these two together have defined a region, now that, any point here we violate this constraint 2x 2 is equal to 12, 2x 2 less than or equal to 12. So, it violates this constraint and therefore, it will not be a feasible, similarly any point here will violate the constraint x 1 is less than or equal to 4 and therefore, this will not be a feasible point, so among these two Constraints, themselves we have now identified that, you cannot be looking beyond this particular region for a solution.

Then we also add the third constraint namely 3x 1 plus 2x 2 is less than or equal to 18, so you draw a line 3x 1 plus 2x 2 equal to 18; how you do draw, you just need two points, so put x 1 is equal to 0, you will get x 2 is equal to 9, so 0 9 is 1 of the points you put x 2 is equal to 0 you will get x 1 is equal to 6, so 6 0 is another point. So, simply join these two points, you will draw the line 3x 1 plus 2x 2 is equal to 18, because you want 3x 1 plus 2x 2 to be less than or equal to 18, we look at, look to the left side of this, so this constraint says that, you look to the left side of this. So in the earlier region, that was defined by these two constraints, if you look at any point here, now we are still in that region defined by the other earlier two constraints, but if you look at any point here, that will violate this constraint 3x 1 plus 2x 2 less than or equal to 18.

Therefore, along with this constraint, now we have a new region defined which can be shown like this; so, this is the region in which all the constraints are satisfied and this region is called as the feasible region. So, feasible region is a region, where any point within that region satisfies all the constraints of the problem, so these are the three constraints, so any point you take within this region or on the boundary of this region satisfies all the three constraints, so in the graphical method the first step is to identify the feasible region, so this is the feasible region.

What does that mean, now this means that, you have virtually infinite number of points, any point within this is the possible solution to this problem, so we have infinite number of points which satisfy all these constraints, from among these infinite number of points we want that particular point or that particular solution which, in fact maximizes our objective function Z is equal to 3x 1 plus 5x 2, so is that this point, is that this point, is that this point, this point, this point, etcetera (refer side 6:01). So, virtually 100, virtually infinite number of solutions exists and out of those infinity number of problem points, we want to choose one point which maximizes 3x 1 plus 5x 2, to see, which is this point what we then do is.

We draw a Z line let us say we put an arbitrary value of Z Z is equal to let us say 15 some convenient value of Z you choose and then draw a Z line, so Z is equal to 15 which is 3x 1 plus 5x 2 is equal to 15, how do I draw this line? I put x 1 is equal to 0, I get x 2 is equal to 3, so 0 3 is one line, one point then, I put x 2 is equal to 0, x 3 is equal to 5, x 1 is equal to 5, so you get 5 0. So, this is how you draw this line Z is equal to 15, now Z is equal to 15, you on the line Z is equal to 15, there are several points which are lying within the feasible region and all of these points correspond to a value of Z is equal to 15; what is your objective? Your objective is to increase Z to the best extent, possible you want the maximum value of Z.

So, let us say Z is equal to 15, I go to Z is equal, to let us say the 18 or 21 and so, on, so, I get another line so, I keep increasing Z, the Z line moves parallel to itself as I keep increasing Z it moves parallel to itself in this direction in this particular case, so Z is moving in this direction. Now, as I increase Z it moves parallel to itself in this direction I further increase Z it goes here, so like I keep on increasing Z as long as there is at least one point within this Z line; which lies within the feasible region or on the boundary of the region we have a feasible solution possible and we are increasing Z, so we keep on increasing the Z until we hit such a point.

Where any further increasing Z, increasing Z will make the Z line leave the feasible space altogether; so, as I was increasing Z let us say we we were here we were here earlier, so you keep increasing Z the line moves parallel to itself and then finally, when you reach this point, when Z is equal to 36; any further increasing Z will make it leave the feasible space at Z is equal to 36. You have exactly one point in contact with the feasible region and therefore, that is still a feasible point, any further increasing Z will make it, make the Z line leave the feasible region and therefore, there will be no point on the Z line which are feasible by feasible, I mean they satisfy all these conditions all the constraints and therefore, this becomes the last point.

Z is equal to 36 is, in fact, the optimal solution because any further increasing Z is not possible without violating 1 of the at least 1 of the constraints and therefore, it becomes infeasible and therefore, Z is equal to 36 becomes the optimal point, so this is what we do in the graphical procedure. First you identify the feasible region; the feasible region is the region obtained by intersection of all these constraints and then it should also satisfy the non negativity conditions and therefore, it lies in the first quadrant, so we identified the feasible region, then you draw an arbitrary Z line and then start increasing the Z, because you are looking at the maximum value of Z, if you are looking at minimization of Z then, it will be moving in the other direction, so you have to look at smaller value of Z, so because we are looking at maximization of Z you would like to achieve as high a value of Z as possible. Yet at the same time maintaining all these constraints satisfying all these constraints and therefore, you are looking for a feasible solution which is also a which also leads to a maximum value of Z and therefore, you move the Z line parallel to itself Z keep increasing the value of Z until you hit a point; where the Z line is just in touch with the feasible region at one point. Beyond which no further increase in the value of Z is possible without violating at least one of these constraint, because you will leave the feasible region the moment you cross this point you will leave the feasible region and therefore, this becomes the optimal solution, so Z is equal to 36 becomes optimal solution x 1 is equal to 2 and x 2

is equal to 6 is the optimal point. How did you get this x 1 is equal to 2, x 2 is equal to 6? You look at this this point is at the intersection of this constraint, namely 2x 2 is equal to 12 and this constraint which is 3x 1 plus 2x 2 is equal to 18. So, solve for those you will get 2 and 6 as the solution, so, you obtain the point 2 6 and you got the associated value of Z is equal to 36, verify that 3x 1 plus 5x 2 with x 1 is equal to 2 and x 2 is equal to 6. In fact, leads to Z is equal to 36, so this is how we solve a problem with graphical method identify the feasible region, look at the Z line and then identify the point at which the optimal solution occurs.

Now, lets us look at your objective function here, was Z is equal to 3x 1 plus 5x 2 and you were maximizing this, so we are looking at in this example, maximize Z is equal to 3x 1 plus 5x 2 and you obtain this Z which was moving parallel to itself and therefore, finally it touch this particular point. Let us say, I change the objective function returning all the constraints and the non negativity conditions, the same instead of saying 3x 1 plus 5x 2, I will make it 3x 1 plus 2x 2.

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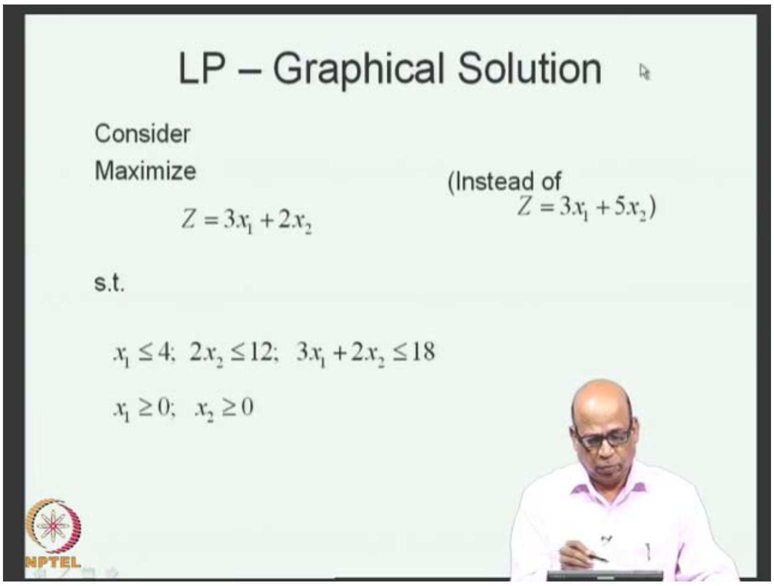

So, let us look at maximize Z is equal to 3x 1 plus 2x 2, instead of 3x 1 plus 5x 2 retain all the conditions are same which means that, your feasible region still remains the same, because we are not change the constraints and therefore, the feasible regions remains the same.

What did I do in this by changing 3x 1 plus 5x 2 to 3x 1 plus 2x 2, I change the slope of the Z line.

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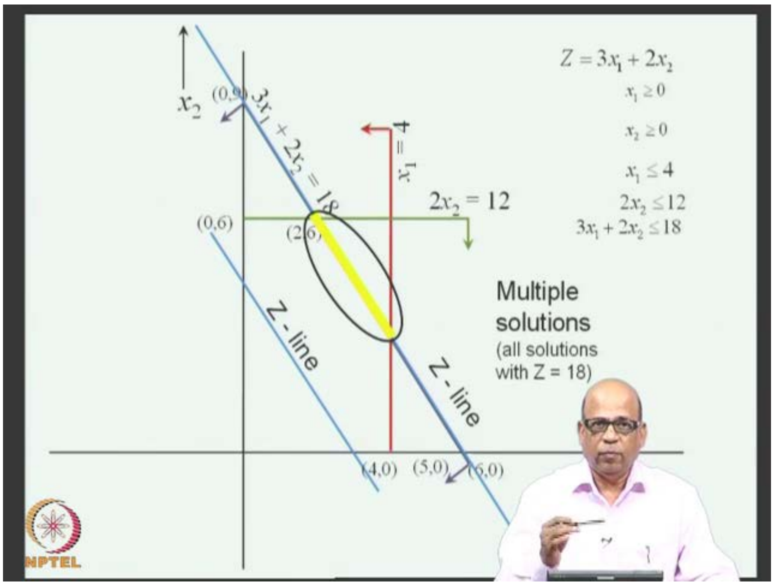

So, Z line, the inclination of the Z line changes and therefore, the Z line becomes this now, instead of this, the Z line now changes the slope and then it is starts moving parallel to itself in this direction. At the last point of contact to the feasible region the Z line will coincide with 3x 1 plus 2x 2 is equal to 18, I have 3x 1 plus 2x 2 as the objective function, so it just coincides with 3x 1 plus 2x 2 is equal to 18; any further increasing Z line will make it leave the feasible space and therefore, this becomes the optimal solution see that, in the previous case we got.

Exactly one point in which the Z line was in touch with the feasible space whereas, here it is coinciding with the boundary of the feasible region, coinciding with 1 of the boundaries of the feasible region and therefore any point on this line here this yellow line, here is a optimal solution; which means you do not have one optimal solution, but multiple optimal solutions. In fact, infinite number of optimal solutions all along this yellow yellow line, each of which leads to the same value of Z which is 18, Z is equal to 18, so this case, we will lead to multiple solutions, so you need not have one solution you may have multiple solutions to LP problem.

And the multiple solutions occur when the Z line corresponds to one of the boundaries of the feasible region. The multiple solutions offer an infinite flexibility in decision making, because you have the five maximized value Z corresponding to 18, but you do not have one point, but you have infinite number of points all of which will lead to Z is equal to 18. Which is a optimal solution and therefore, you can choose any of these infinite number of points for your decision and therefore, we must be alert to a situation that certain problems leads to multiple solutions and we should be able to capture the multiple solutions. In fact, we will see if you get one solution, here I may be another solution at this point, you should be able to generate all the infinite number of solutions all of which remember will lead to the same optimal solution Z is equal to 18.

So, this is the first special case, that we consider you identified the feasible region and then see that Z line corresponds to one of the boundaries, beyond which it use the feasible space and therefore, that becomes the optimal solution and therefore, you do not have one solution, but you have infinite number of solutions. All of these points here on the boundary become an become an optimal solution, lead to an optimal solution and therefore, you have multiple solutions if you have two solutions you have infinite number of solutions, this is a solution this is a solution therefore, any point in between is a solution is a optimal solution. So, you have multiple optimal solutions, another special case is that your constraints may be such that, you do not have a bounded region in this particular case. What happen, you have let us look at this you have a region which is bounded in all the directions, so you have a bounded region; let us say that, some of the constraints are such that, you do not have a region which is bounded.

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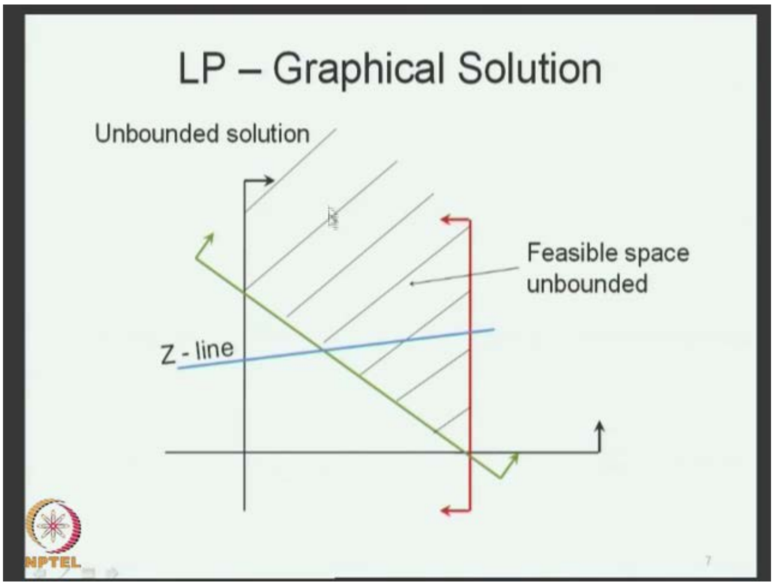

Let us say your constraints that is, such that one constraint is saying you look at this direction and that it is saying you look at this direction and therefore, anything in this region is a feasible solution and there is and Z line is increasing in this direction; let us say Z is increasing in this direction. So, Z can be increased until infinity without violating any constraints and therefore, the region is unbounded, the feasible space of the feasible region becomes unbounded and such solutions are unbounded.

Solutions in general, when you have an unbounded solution, the problem is ill formulated because, obviously any physical problem you cannot have, you cannot increase the objective function value to infinity; if it is a value formulated problem and therefore, when you get an unbounded solution the problem is ill ill formulated problem. Typically you will have another constraint which will limit the increase in the objective function, so you saw multiple solutions, you saw an unbounded solutions, your set of constraints may be such that, you may not get a feasible space at all or feasible region at all, you may have several constraints and then you are plotting constraints, sets of constraints; some constraints will define some region, some constraints will define yet another region, there is no intersecting point at all, intersecting region at all.

Such problems lead to what are called as a in feasible solutions or in feasible problems.

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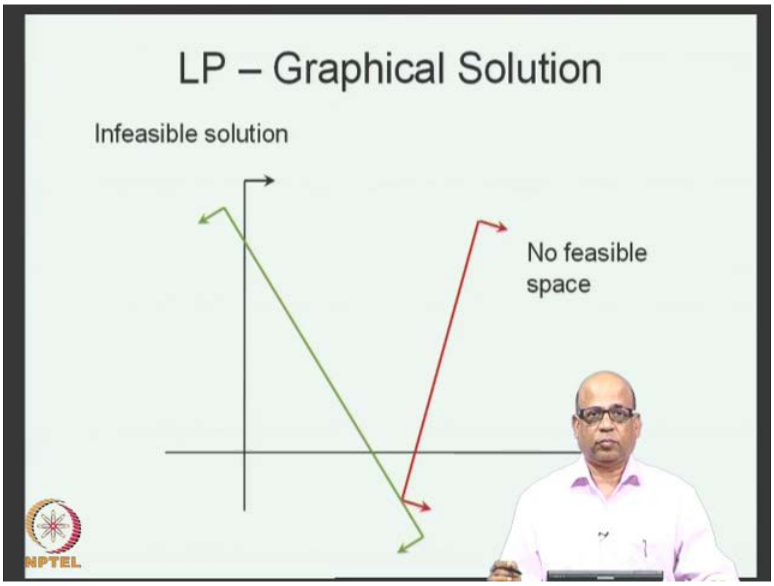

So, you may have a constraint, which says you look at this direction, another constraint which says you look at this this direction, which means that, to satisfy this constraint you should be in this region to satisfy this constraint, you should be in this region therefore, there is no region where you can satisfy both of this simultaneously and therefore, there is no feasible space possible, such problems are called as in feasible problems or the solutions are in feasible.

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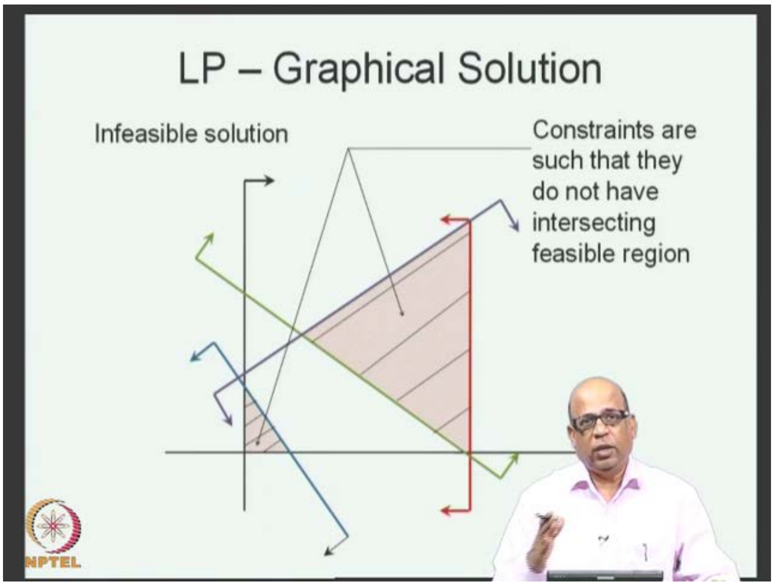

Another case, where you may have some intersection possible with certain constraints some other intersection possible with certain other sets of constraints, yet together they do not form a common intersecting region, such problems also lead to infeasibility.

In general when you have infeasible solutions, which means again that, some of the constraints that, you have formulated are in consistent with other constraints and therefore, you need to look at the physical problem and see whether the constraints have been correctly formulated associated with the physical problems.

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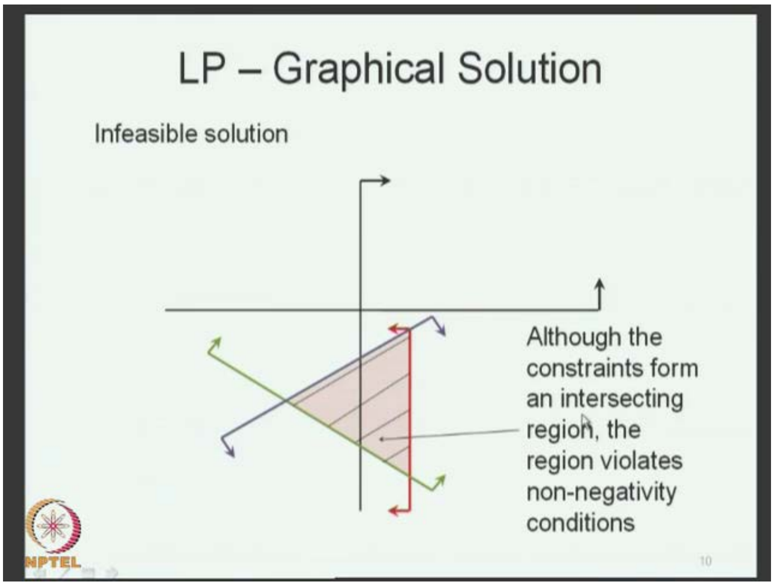

There may be yet another case, where you may have a good feasible, good region of intersection among all the constraints, but this region may not lie in the first quadrant and therefore, it violates the region, violates the requirement of non negativity of the variables and therefore, still it becomes infeasible. So, infeasibility will indicate that, at least certain constraints are violated or the non negativity condition is violated, so in this case although the constraints form an intersecting region, the region violates non negativity conditions and therefore, it becomes in Feasible.

So, in the graphical method, we have now identified three special cases and in fact, these three special cases pertain to any LP problem not necessarily, because we are solving it using the graphical method; the first one is that, you may have multiple solutions when the Z line is just coinciding with one of the boundaries of the feasible regions and therefore, any point on these boundary becomes an optimal solution therefore, you have infinitely many solutions, infinitely many optimal solutions all leading to the same optimal value of the objective function, these are multiple solutions, then you have the case where you may not have a bounded feasible region, so Z can be increased up to infinity you keep on increasing.

Z it does not violate any of the constraints and therefore, it is still feasible, so you can increase the Z value to infinity without leaving the feasible space which means that the Feasible space is unbounded direction in which the Z is increasing for a maximizing maximization problem and therefore, it becomes an unbounded solution, in general when you whenever you have an unbounded solution you will have to look at the problem formulation the problem formulation will be wrong then you have the case of infeasible solutions, where the set of constraints is such that, you will not be able to identify a intersecting region which also satisfies the non negativity conditions which means, there is no point in the region the of interest which satisfies all the constraints and therefore, the problem becomes infeasible.

In general infeasible problems are also in formulated problems, because you may have introduces in certain constraints, which violate other constraints in the in the sense that, if you meet this particular constraint, the other constraints are not satisfied and therefore, the problems are ill ill formulated. We must be able to capture all these situations that is multiple solutions, we must know how to capture the multiple solutions, we must know when the problem is unbounded, we must able to identify when the problem is unbounded and we must able to identify infeasible solutions as as we saw this this was simple problem with only two variables and only two Constraints and, so on. So, it was easy for us to identify in practical problems; you have hundreds and thousands of variables and thousands of Constraints therefore, we must have a mechanism to identify each of these different types of solutions which we presently see.

So, from the graphical solution, then we must be able to move to higher size of the problems of the LP problems and therefore, we must have an algebraic way of solution of these LP problems, so let us start introducing an algebraic method of solution which is also an algorithmic method of solution which is called as the simple method; to do that,

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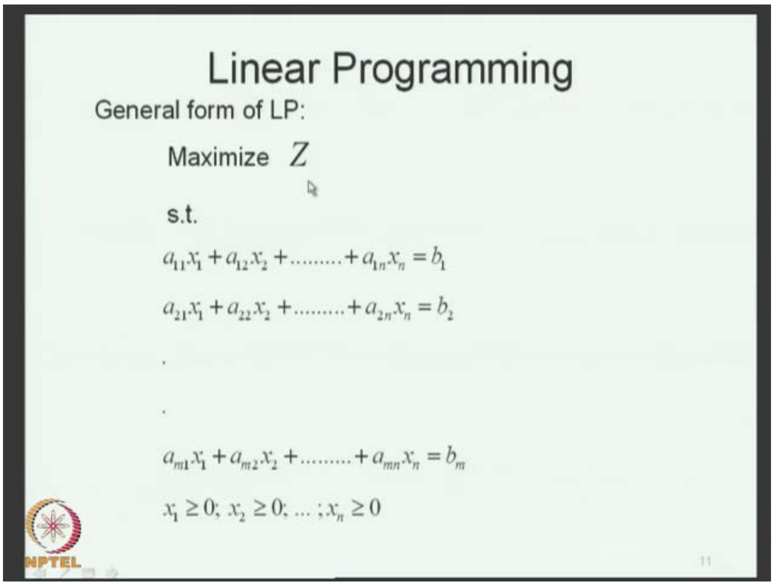

We will look at the general structure of the problem LP problem, we typically write the objective function as maximize Z and Z is a linear function of the decision variables. The decision variables are x 1, x 2, etcetera x n and Z is a linear function of x 1, x 2, etcetera x n, then you have a set of constraints n number of constraints.

In the general form of LP, we write them as equality constraints, so a 1 1, x 1, a 1 2, x 2, etc a 1 n, x n is equal to b 1, this is for the first constraint, so the right hand values are denoted as b 1, b 2 etc b m, a 1 1, a 1 2, a 1 3, a 1 n, etc. These are the coefficients of the decision variables in the constraints, so this is the first constraints and the coefficient for the first constraints, first variable is a 1 1 first constraint, second variable is a 1 2 first constraint, nth variable is a 1 n, that is the coefficient a 1 n like this a 2 1, a 2 2, etc a 2 n.

So, you have coefficient matrix we will write in the matrix form subsequently, some other lectures, but you can identify that.

These are in fact, the coefficients which are constants and x 1, x 2, etc x n are the decision variables for which solutions are sort b 1, b 2, etc are the right hand side of the constraints these are also known known values, additionally you have the non negativity condition x 1, x 2, etc must be greater than or equal to 0. So, this is the way, we write one general form of LP in which all the constraints can be expressed as equality constraints and we typically look at maximize value of Z, this is one general form you can also write in the another form for example, you may look at minimization and then still you can retain less than or equal to and so on, but we will strict to one general form and then see, how we develop the simplex algorithm corresponding to this general form.

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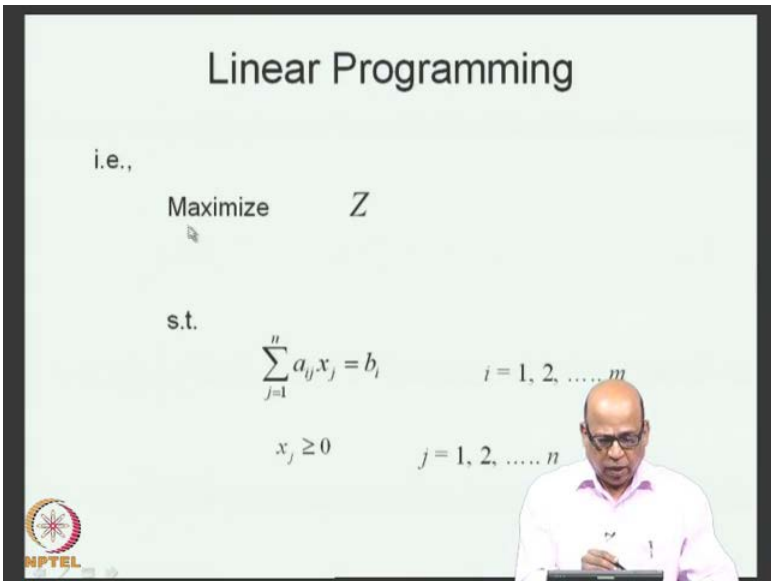

Which means your objective function must be of the type maximization, so we write this in a more compact form as maximize Z, subject to all the set of constraints I will write it as summation, in the summation form you have a i j, x j is equal to b i which means it constraints I will write as summation j is equal to 1 to n, there are n number of variables a i j, x j is equal to b i and i is equal to 1, 2 etc m, so I am writing this set of constraints here in a compact for using the summation notation and x j greater than or equal to 0, j is equal 1 to n there are n number of variables, m number of Constraints, Constraints I am denoting it as by i and the variable by j here.

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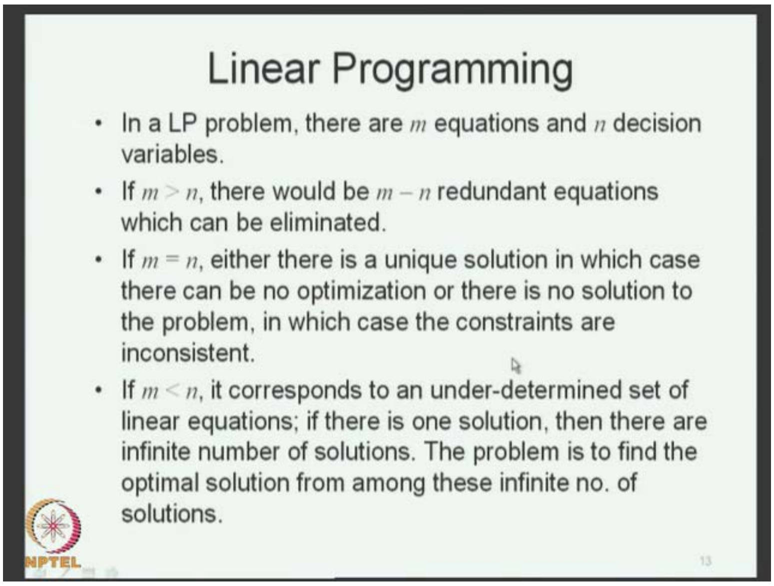

So, this is compact form of LP still in the scalar notation, so we write this in the scalar notation as shown here. Now, let us look at various features of such a problem you have n number of variables, m number of constraints and m number of equations, actually because, we have made all the constraints to be equality constraints, we will see how to handle the in equality constraints, later on, but in the general form we are expressing all the constraints to be equality constraint, which means what you have a m number of equations and n number of variables if m is greater than n, m is the number of equations is more than the number of variables, then the generally problem is over defined.

And therefore, you should be able to identify redundant constraint, which I have introduced earlier, so there must m minus n number of redundant constraint, which you must able to identify and eliminate. So, that you will have a maximum of m number of m is equal to n number of constraints, so if you are original number of constraints is more than the number of variables it indicates, that there are certain non certain redundant constraint, we must be able to identify the redundant constraint and eliminate the redundant constraint then, the next question if m is equal to n; that means, your original set of equations is, in fact, the number of variables, then you may have a unique solutions; that means, if there is a solution you you have now m number of equality constraint, which means n number of equations you solve n number of equations.

Simultaneously, you will get a unique solutions, so if you have a solution it may be a unique solution, which means here is a no question of optimization, because you have a exactly one solution for all the n number of equations or you may not have a solution at all. So, you may n number of equations which may not have a solution at all if you have solution it will be a unique solution therefore, m is equal to n is not a case to considered, because there is no question of optimization. So, the only co condition we need to consider is, where m is less than n; which means n is the number of equations that you have is less than the number of variables, in which case you may have infinite number of possibilities out of which you would look at that particular problem, where that particular solution which leads to an optimal solution, so this is the case that we will be talking about if m is less than n.

It corresponds to an under determined set linear equations, which means if there is one solution then, there are infinite number of solutions possible and out of these infinite number of solutions, we need to look at that particular solution, which also maximizes or optimizes the objective function, so this is the case, that we will be interested in the linear programming problem.

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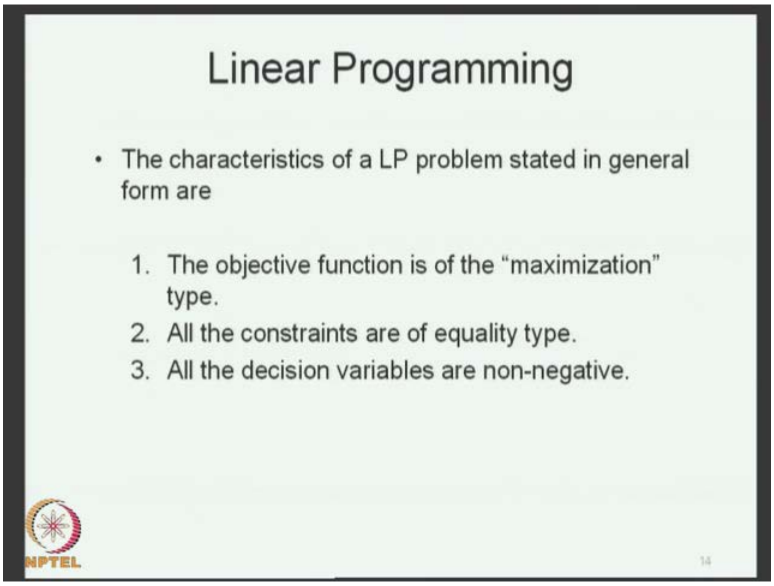

which means in the general form, the way which stated we said the objective function is of maximization type we will strict to that, as a as a standard form, we will retain the objective function as maximization type then, all the constraints of equality type and all the decision variables are non negative, if you have any optimization problem, then we should be able to express that optimization problem, we will satisfy these three conditions; which means that you may have a minimize objective function as minimization.

You must be able to express as a maximization problem and you may have certain constraint or most of the constraints as in equality type, some of them may be less than or equal to, some of them may be greater than or equal to and some of them may be equality type; which may we should be able to express all these constraints as equality type then, we must also be able to express all the variables as non negative.

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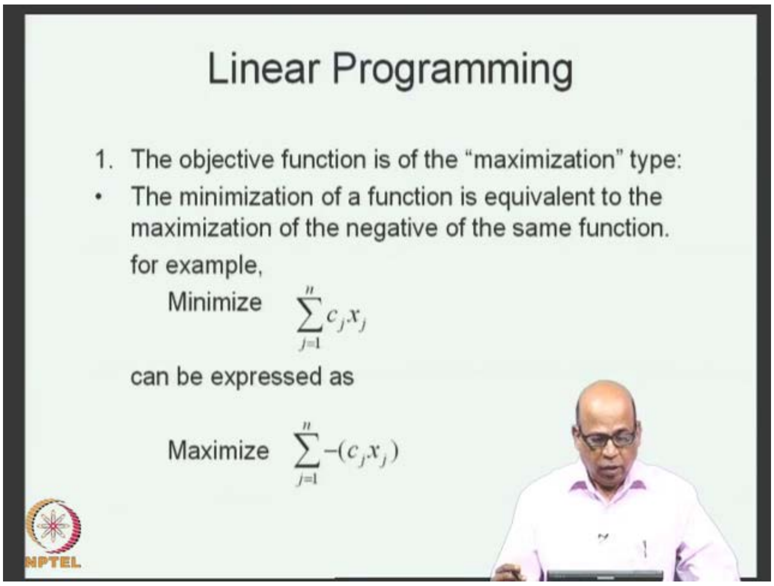

Let us say, how we do this if you have objective function as minimization; let us say that, your objective function is minimize j is equal to 1 n, C j, x j this may be cost coefficient and there are n number of variables and only you are writing the objective function. Here the objective function is of the minimization type this can be written as maximize minus C j, Z x j summation j is equal to 1 n as you recall in one of the earlier lecture.

I have shown that, if you want to maximize f of x, it is the same thing as maximizing minus f of x, so simply take Z minus C j, x j and maximize that, so a minimization objective function can be written as maximization objective function by simply taking the negative of the function itself.

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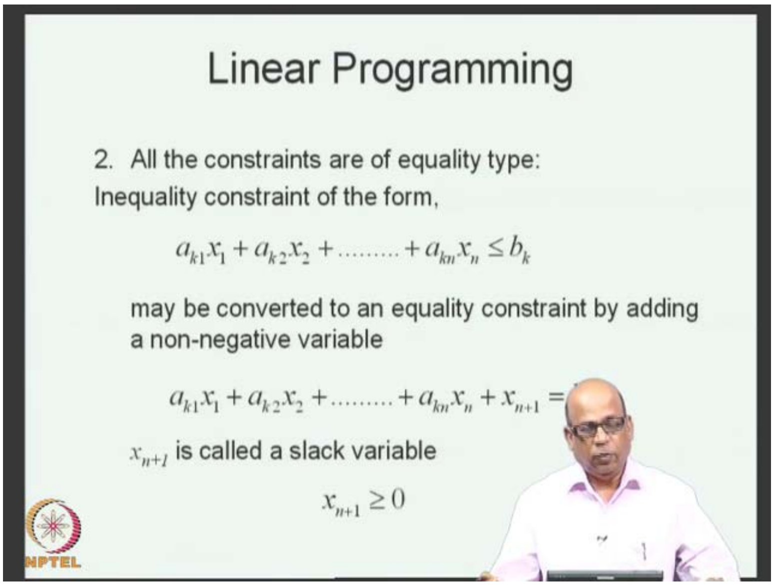

Then the second one is if you have less than or equal to constraints, we need for our general form equality constraints, but our constraint may be of this form at the kth constraints; let us say a k, 1 x, 1 a, k 2, x 2, etc a k n, x n less than or equal to b n, this is the kth constraint; let us say that, I want to convert this in to equality constraint, now the left hand side is less than the right hand side.

And I want it to be equal to which means; what I have to add certain value here certain non negative value to the left hand side, that is what we do, we add a additional variable here there, where n variables in the initial constraint, we add another non negative variable x n plus 1 to make it equal to b k and this additional variable is called as Slack variable. There was a slack, that was existing you fill up the slack with the slack variable, so the in an equality constraint of the type less than or equal to, we convert it in to an equality constraint by adding a non negative variable and this non negative variable is called as the slack variable and this non negative variable will be I am sorry this slack variable will be non negative, so x n plus 1 which you added.

Which was not there in the original problem, but you added an additional variable and that is called as slack variable, that is x n plus 1 and that will non negative.

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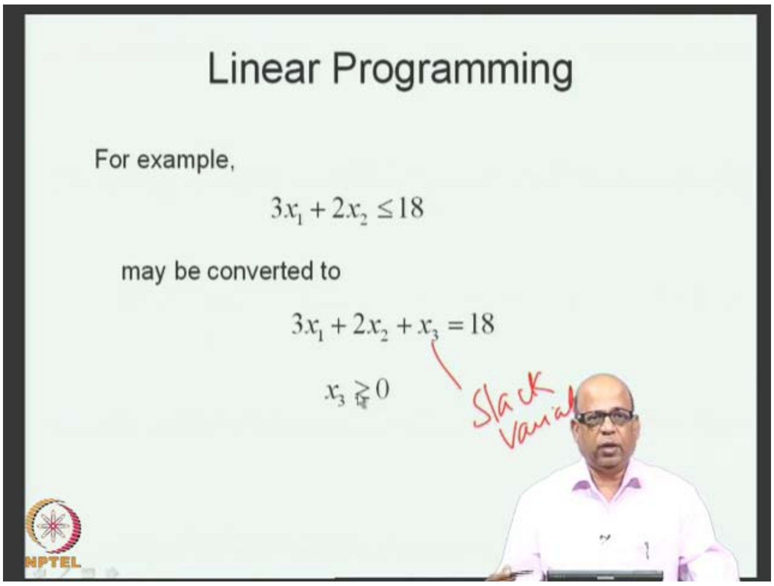

Similarly, if you have a greater than or equal to constraint let us say we will look at a small example 3x 1 plus 2x 2 is less than or equal to 18, which means what your left hand side is less than 18 you want to make it equal to 18 by adding 1 non negative variables. So, I will write 3x 1 plus 2x 2 plus x 3 is equal to 18 and this is called as a slack variable and your x 1 and x 2 anywhere greater than equal to 0, the addition variable, that you are adding is also greater than equal to 0, so you are satisfying the non negativity condition.

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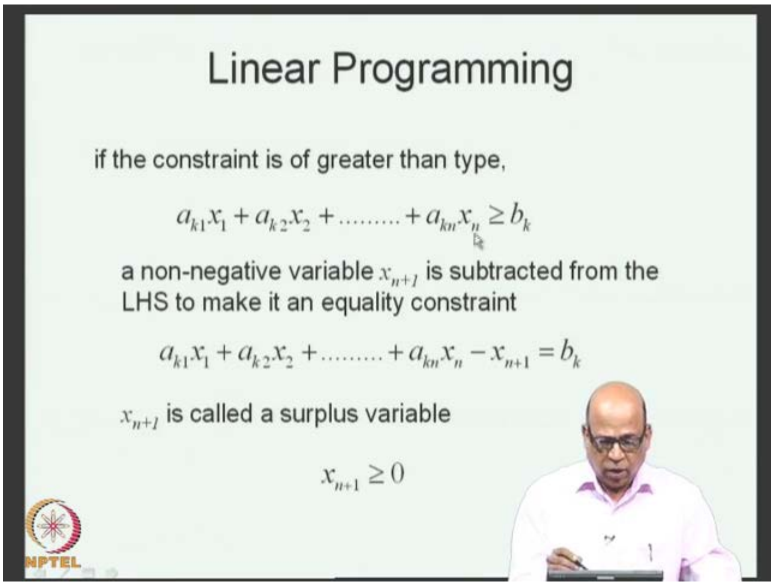

Let us say you have a equality constraint instead of the you have a greater than or equal to constraint, instead of the less than or equal to constraint, which means what the left hand side is more than the right hand side and you want make it equal and therefore, you should deduct some value from that and therefore, you subtract a non negative variable from that, so a k 1 minus x n plus 1, so you add another variable with negative sign now in the constraint. So, the variable is non negative, but you will deduct from the left hand side of that and this is called as a surplus variable, there was a surplus on the left hand side you reducing that, by deducting that surplus variable and this surplus variable is also non negative, so if you have a less than or equal to constraint you add a non negative value, non negative variable, that is called as a slack variable, if you have a greater than or equal to constraint you deduct a variable.

Deduct a certain value using a surplus variable and both the slack variable and the surplus variable that you, so introduce, where not there in the original problem and you introduce them to make sure that, the constraints are expressed as a equality constraint and both the slack variable as well as surplus variables are non negative, so if you have a let us say, x 1 plus x 2 greater than or equal to 5 as your original constraint you will put x 1 plus x 2 minus x 4 is equal to 5 and this x 4 is, in fact the surplus variable and the surplus variable is also non negative, you will get greater than or equal to 0 x 4 greater than or equal to 0 then the last requirement is that all the decision variables must be non negative.

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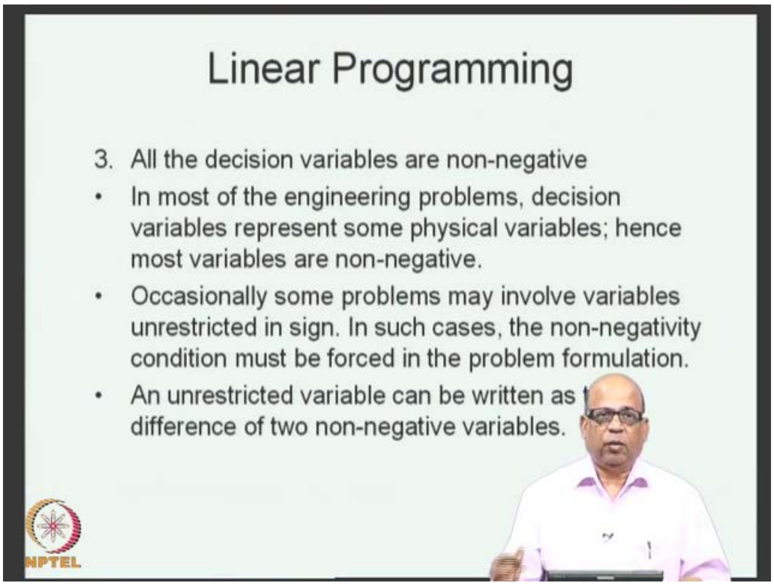

So, we looked at the objective function, we want that objective function to be maximization, so if your objective function is minimization you can convert it in to maximization, by looking at minus f of x if you are looking at minimize f of x, you can look at maximize minus f of x, then you wanted all the constraints to be of equality type we just now say, how to change the less than or equal to type of constraint to equal to constraint by adding that are called as slack variables, which are non negative then if you have greater than or equal type of constraints you deduct a surplus variable which is also non negative, then the third case is when you have a certain variables, which can also assume negative values. Remember in most of the decision making problems the variables, that we are talking about are either physical variables for for example.

You may be talking about dimensions; physical dimensions of fled control storage you may be talking about the amount of hydropower, that you generated amount of water that is applied to a particular given to a particular city and so, on. So, these are physical variables that we are talking about and most of the cases they are non negative, we were be talking about negative variables; however, there will be a certain cases, were you may have to account for non negative account for un variables that are unrestricted in sign for example, you may be talking about temperature, temperature can assume negative values and in certain in such situation there must be a way of handling these variables in the general linear programming problem. When you have a variable which is unrestricted in sign which cannot assume negative values also it can take positive values it can also take negative values in such situations you must be able to convert that for the problem formulation into a non negative variable not the original variable itself.

But we will convert, we will use those variables which are all non negative how do we do this, let us say, you have a variable x which is which is unrestricted sign inside which means in the problem solution it can assume negative values also, if you express the x as difference of two non negative, non negative variables. let us say that, I will put x is equal to x 1 prime minus x 2 prime and the x 1 prime and x 2 prime will go as decision variables, so x 1 prime and x 2 prime can be non negative, so, that if x is negative then x 2 prime is greater than x 1 prime let us look at it more formally, so an unrestricted variable can be written as a difference of two non negative variables.

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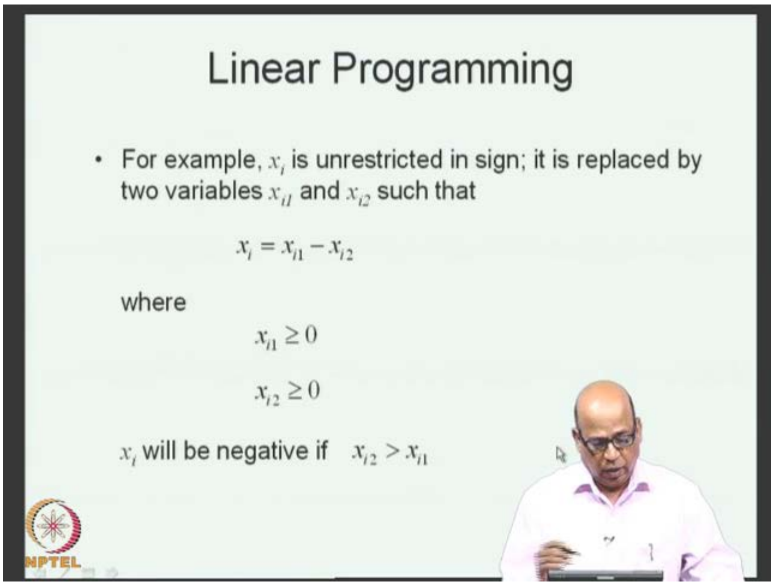

Let us say that, you have x i is unrestricted in sign, than we can we can replace x i by a difference x i 1 minus x i 2, so I will write x i as x i 1 minus x i 2 with both x i 1 and x i 2 non negative, so if in your final solution x i has to negative, what happens x i 2 will be greater than x i 1 and if x i has to be positive x i 1 will be greater than or equal to x i 2. In fact, in solutions you will have see that, only one of them will be non 0; however, we will come to that later, but right know you just understand that by expressing x i has a difference of two variables two non negative variables, what you have achieved is that in the final solution, if x i has to be positive then, this will be more than this if x i has to be negative then this will be more than this.

Both of them being non negative and therefore, in the LP problem we still be dealing with non negative variables, so you are able to convert the non negative function into a maximization type, you are able to convert the less than or equal to into equality constraints greater than or equal to constraints in to equality constraints and you are able to convert the unrestricted variables into non negative variables and therefore, any given problem optimization any given LP problem can be expressed in the general form, which requires the objective function to be maximization type which requires all the constraints to be of equality type constraints and which requires, that all the variables in the LP problem are non negative. So, this how we achieve that and then express a given LP problem in the general form now, we this now we will start introducing the algebraic solution which is the simple method.

Now, I i repeat this many times that, you can do a way without knowing the way the algorithm works, because there are elegant useful and very easily commonly available software's for linear programming, in fact in library in that math lab you can easily solve any of this optimization problems, however, for a student it is extremely important to know and understand how the algorithm works; in fact, in the class room type of examples you should be able to solve this problems with hang calculators and therefore, it is important for us to know, how the algorithm itself functions. So, we will start with we will introduce over the today's class as well as the next class we will discuss the simple method, how it works before going in to the actual Functioning of the algorithm itself, let us look at the motivation.

Form for the simple method we will re visit the Graphical solution where we obtain the optimal solution

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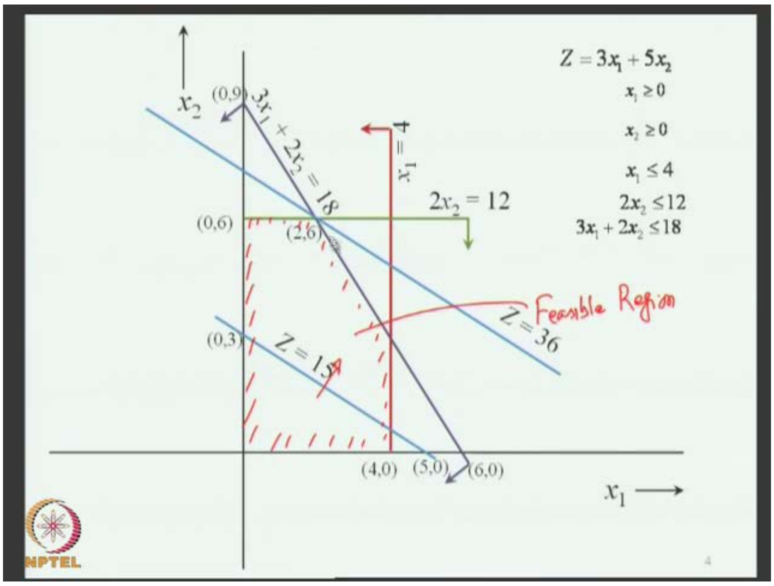

Just look at this what what did we achieve here, first we form the objective of space or the feasible region, in the feasible region the solution lies somewhere, we do not know where, so some where the optimal solution lies and then we started with looking at how the Z value increases. So, starting with the Z value we kept on increasing Z value ensuring that, it still lies within the feasible region and we achieve that particular point beyond which the Z can know further being increased without violating any of the constraint and this we say lies at a particular corner.

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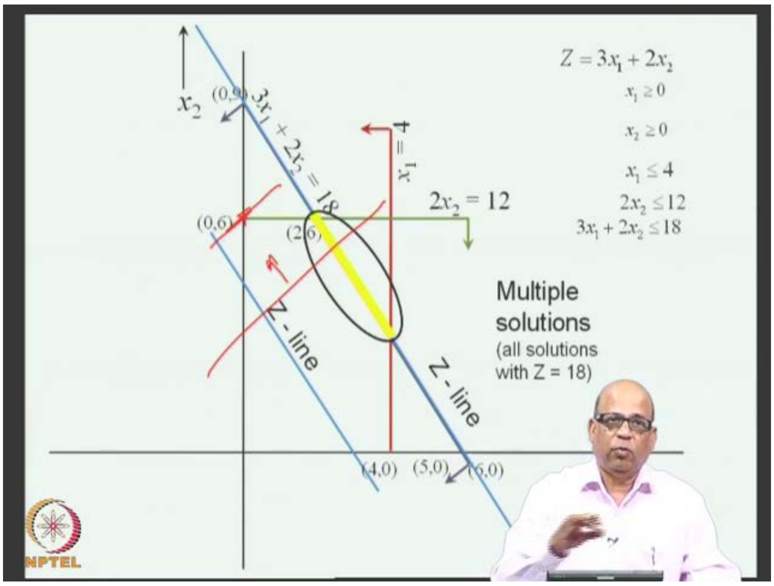

So, the one observation we made is that, the optimal solution in this particular case was at a corner; if your Z direction increases changes then, still you are hitting at a corner, so the solution will be at this corner and at this corner and then this particular case it, so happens at it will be all along this line, let us say your Z line was some other was increasing in some other direction let us look at lets Z line was here, then it keeps on increasing in this direction. Assume that, it is increasing in this direction, then the optimal solution will be at this point, so the observation that we are making is that the optimal solution lies at one of the corner of the feasible space.

So, it can be either this corner, it can be this corner, I am sorry it can be this corner or this corner or this corner or this corner etc. So, if we start the solution with some assumed value let us say that 0 0 was here x 1 is equal to 0, x 2 is equal to 0, was here which would have been Z is equal to 0 and moved from this point, to this point, this point, to this point, this point, to this point, this point, to this point, this point, to this point. Which means lets the enumerated all the corner points solutions, at all the corner points and then picked up the one that leads to the maximum, then we are safe we know that, the solution has to occur at one of the corner points of the feasible space; however, as the number of constraints increases as the number of variables, increases the problem the dimension of this feasible space will be quite quite large and therefore.

Enumerating all the solutions at all the corner point, will still be and therefore, we have a mechanism by which starting with a particular point, we must know whether I go to this point or I should go to this point, in in a multi dimensional space you imagine that, you start with a particular point and then you must be able to see whether, I should go to a neighboring point in this direction, a neighboring point in this direction, this is what we achieve in the simplex algorithm, that means we start with the feasible solution, we we have identify the feasible region, we start with the feasible solution and then instead of enumerating all possible feasible solution, we will know starting with the particular feasible solution we will know in which direction the solution should proceed, so that we will hit the optimal solution in the least number of iterations this is what we do, so will go to the motivation for the simplex problem x

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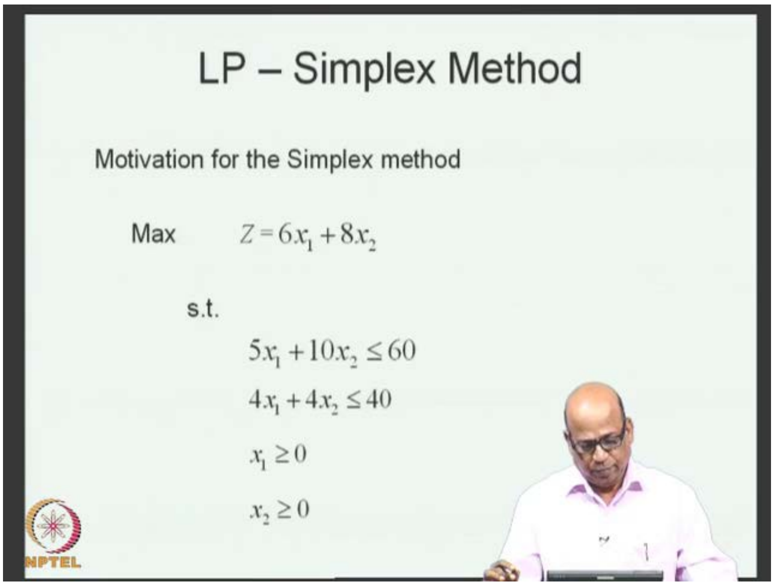

Now, again starting with a simple graphical solution, let us say that you take a problem maximize Z is equal to 6x 1 plus 8x 2 subject to 5x 1 plus 10x 2 less than or equal to 64 x 1 plus 4x 2 less than or equal to 40 and x 1 and x 2 both are non negative, if we solve this with graphical procedure.

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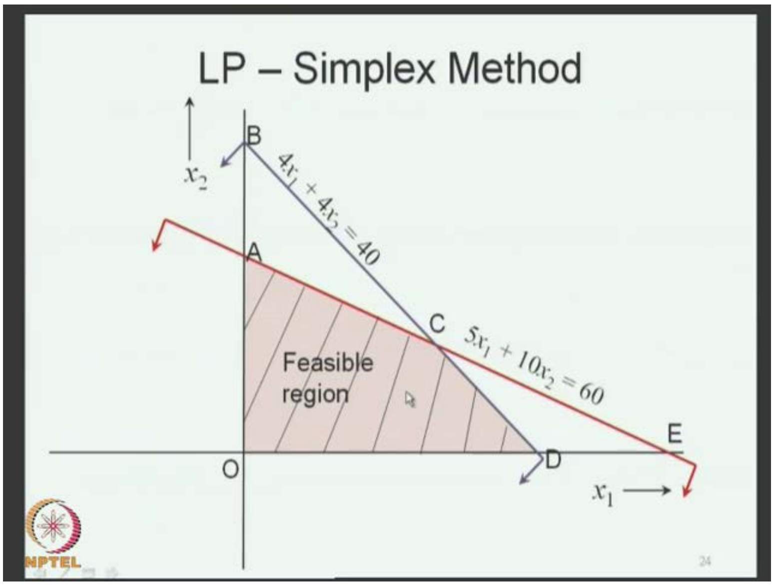

we get this feasible region, let us say you just look at constraints, now identify the feasible region, how do we identify, you draw this constraint 4x 1 plus 4x 2 is equal to 40 and then it is because less than this you are looking at this direction similarly 5x 1 plus 10x 2 is equal to 60 these are the two constraints.

You draw this and identify the feasible region, because x 1 is greater than or equal to 0 x 2 is greater than or equal to 0, this becomes the feasible region, so you have the points o A C and D as the corner point of the feasible region, but there are also other points you look at point B here point E here. So, you have six points here on this diagram, lets us see what are all the features of these six points you have two constraints 4x 1 plus 4x 2 is equal to 45 x 1 that is less than or equal to 40 and 5x 1 plus 10x 2 less than or equal to 60 these are the two constraints.

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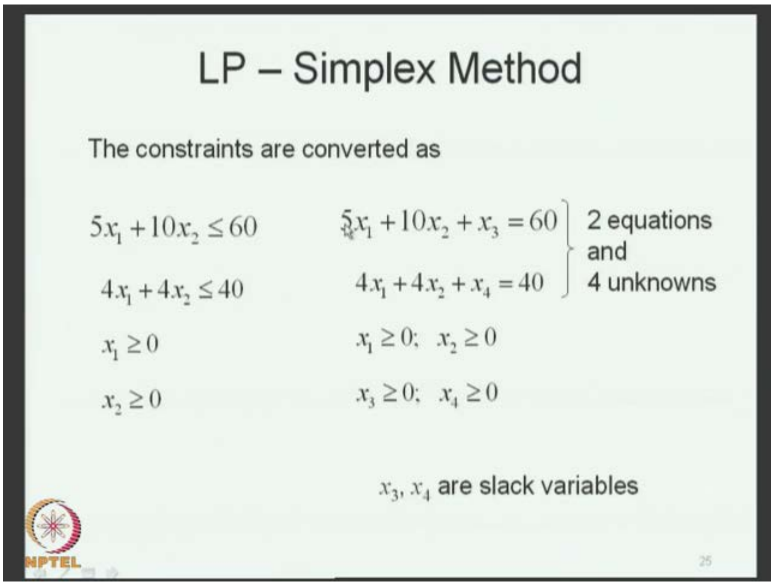

Lets say the inequality constraint now 5x 1 plus 10x 2 less than or equal to 60 I convert it into inequality constraint by adding a slack variable x 3.

Is equal to 60 and 4x 1 plus 4x 2 less than or equal to 40 I convert it in to an equality constraint by adding another slack variable, remember this slack variable must be different from this slack variable 4x 1 plus 4x 2 plus x 4 is equal to 40, so what happen in the in the process we have two equations and four unknowns, so m is less than n and all your variables are greater than or equal to 0, which is the non negativity condition. So, x 3 and x 4 are the slack variables, let us look at what are what all these corner points and the points which are intersecting with the x 1 and x 2 axis correspond to, so from this then we start developing, start looking at motivation for the simplex algorithm.

And introduce what are called as a basic solutions, basic feasible solutions, basic variables, non basic variables and so on and that, leads to an algorithmic way of solving the LP problems this we will continue discussing in the next class. So, in today class essentially what we did is we started with graphical solution we identified the feasible region and then in the feasible region any point is the feasible solution and therefore, you have infinitely many points which are all feasible out of these infinitely many points, we want to identify the optimal solution for which, what we did is we took as a Z line and then move the Z line to parallel to itself in the direction in which it is increasing for an optimal maximization problem and then identify, that particular point beyond which the

Z cannot be increased at any further without violating any of the at least one of the constraints

And that becomes the optimal solution, we saw that this optimal solution, in fact lies at one of the corners or coincides with one of the edges of the feasible space and the we saw the special cases where you have multiple solutions, multiple optimal solutions if you have two optimal solutions you have infinite number of optimal solutions then, there is case of unbounded solutions where your set of constraints may be such that, you will not have a bounded feasible region and therefore, the Z value can be increased for n for a maximization problem am until infinity without violating any of the constraints then you may have in feasible solutions. In general, when you have un bounded solutions or in feasible solutions the problem is in ill formulated, so you will have to look at the set of constraints then we just started looking at the general form of the LP and motivation for the simplex method.

In the general form of the LP, that we used in this course we express the objective function as a maximization function and all the constraints as equality constraints and all the variables must be non negative if you have a minimization objective function. You can convert it in to maximization by taking negative of that, function if you have less than or equal to constraint you add a slack variable to make it equality constraint if you have a greater than or equal to constraint, you have deduct a surplus variable both the slack variable and the surplus variables are non negative and convert them in to equality constraints if you have variables, that are unrestricted in sign you express those variables as difference of two variables both of which are non negative to convert them in to to use them as non negative variables in the LP problem, so we shall continue this discussion and in the next class I will introduce

The Simplex method or at least the motivation for the simplex method and how we identify basic variables, non basic variables and so on, thank you for your attention.