Water Resources Systems

Modeling Techniques and Analysis

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Lecture No. # 07

Kuhn- Tucker conditions and Introduction to Liner Programming

Good morning, and welcome to this the lecture number 7, of the course, Water Resource Systems - Modeling Techniques and Analysis. We have been talking about optimization of multiple variables initially, we started with optimization of functions with no constraints, such problems are called as unconstraint optimization problems. And then we introduced the equality constraints that is, optimization of a functions of n variables with m number of equality constraints, and in the constrained optimization where you have only equality constraints, we discussed two methods, one is the method of direct substitution, in which you will use the m equations to express m variables in terms of the remaining n minus m variables, and convert the original constraint optimization problem in to an unconstraint optimization problem with n minus m variables.

So, essentially we eliminated the m number of variables using the m equations, and then we introduce the more regress Lagrange multiplier method, where corresponding to each of the equality constraint, we introduce an additional variable called the Lagrange multiplier; and then the original problem of n variables is now converted into another problem of n plus m variables with all the constraints taken into account. For the Lagrange multiplier method, we formulated the necessary conditions and the sufficiency conditions for a stationary point to correspond to either a maximum or minimum, and then we went on to state the Kuhn-tucker conditions. So, this is what we did in the previous lecture.

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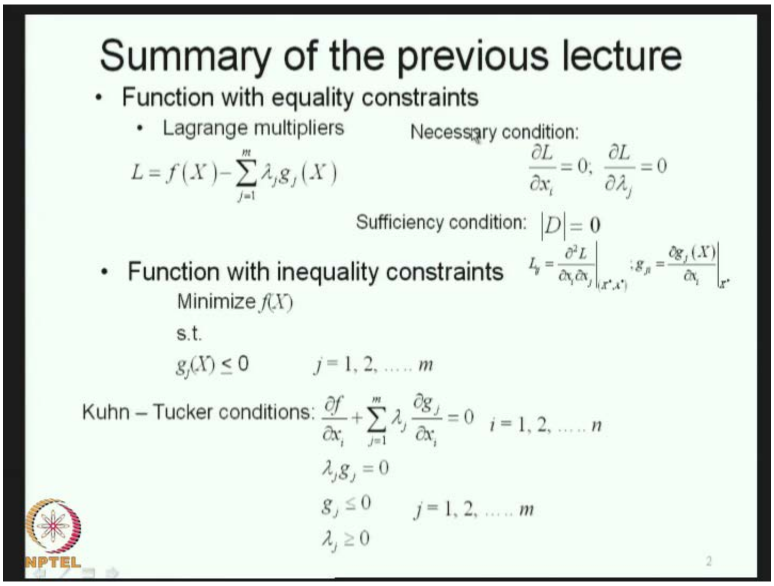

We… In the last lecture, I discussed the Lagrange multiplier method where we formulate the Lagrange Function L is equal to f of X, summation j is equal to 1 to m lambda j g j of X, the lambda j s are the Lagrange multipliers g j of X are the constraints. So, we stated g j of X is equal to 0, as a set of m constraints, m equality constraints, and we apply the Lagrange multiplier method for equality for optimization of functions with subject to equality constraints; the necessary condition for the problem will be d L by d X i is equal to 0 with respect to all is that means, we take the first derivative of the Lagrange with respect to each of the variables X i and the first derivative of the Lagrange with respect to each of the Lagrange multipliers themselves lambda j. So, you will have n conditions here plus m conditions here; so, n plus m equations, you solve for these n plus m equations simultaneously to get X 1, X 2 X 3 etcetera X n and lambda 1, lambda 2 etcetera lambda n lambda m I am sorry. So, n plus m equations, you solve for the n plus m variables.

The sufficiency condition we formulated the… For the sufficiency condition we formulated the determinant d based on the second order derivatives of the Lagrange d square L by d X i d X j evaluated at the stationary point X star lambda star, and the first order derivative of the constraints functions g j of X with respect to each of the variables X i, using these we formulated the determinant, and equated the determinant to 0, and you get a polynomial of order n minus m for Z, which is the function in which is the variable in the determinant D; solve for Z, if all the values of Z are positive, then the point X star, lambda star corresponds to a minimum; if all the values of Z are negative, the point X star, lambda star corresponds to a maximum. So, this is what we did in the Lagrange multiplier method, then we want went on to state Kuhn - tucker conditions which are necessary conditions for any optimization problem to be satisfied.

So, the more general optimization problems are stated as minimize or maximize f of X, subject to g j of X less than or equal to 0 or greater than or equal to 0 and so on. So, you have… You have inequality constraints g j of X less than or equal to 0, g j of X greater than or equal to 0 and so on; and you have a function f of X of n decision variables X 1, X 2, X 3 etcetera X n; for such a general optimization problem we will state the Kuhn - tucker conditions as follows, this is for the particular combination of minimization of objective function subject to g j of X less than or equal to 0. So, you have the objective function as minimization, and the constraints are all of the form g j of X less than or equal to 0; then the Kuhn - tucker conditions are stated as d f by d X I, this is the first derivative, first order derivative of the original function d f by d X i plus j is equal to 1 to m lambda j d g j d X i where d g j of X is the constraint Function. So, you take the first derivative with respect to each of the variables, and these conditions you write for each of the variables. So, you have n number of conditions here, and then lambda j g j is equal to 0, g j less than or equal to 0, and lambda j greater than or equal to 0.

So, you have n number of conditions associated with this, m number of conditions associated with this, m conditions associated with this and m conditions associated with this. So, you have n plus 3 m conditions here, what we do is we take the first n plus m conditions which are in fact, equations n plus m equations, to solve for the n plus m variables namely X 1, X 2, X 3 etcetera X n; lambda 1, lambda 2 etcetera lambda m. So, you use these first n plus m equations to solve for the n plus m variables; and then examine whether the solutions that you so obtain satisfy in fact, these two conditions lambda g j is less than equal to 0, which are the original constraints, and lambda j greater than equal to 0. So, we use the last two conditions to examine whether the solutions that we obtain from the first two conditions in fact, satisfy these conditions.

So, this is the way, we apply the Kuhn - tucker conditions, I again repeat Kuhn - tucker conditions are necessary conditions, they are not sufficiency conditions, expect for a specific type of problems, where the objective function is convex and you are looking for a minimization and all the constraints are convex function. So, the objective function is a convex function of the variables and all the constraints are also convex function of the variables such problems are called as convex programming problems. The Kuhn - tucker conditions that we just stated will also be sufficient conditions only in cases of convex programming problems, I also indicated the variance of the Kuhn - tucker conditions where you may have maximization of f X subject to g j of X less than or equal to 0, in which case lambda j will be negative; you may have maximization of f X subject to g of X greater than or equal to 0, in which case lambda j s is still remain greater than equal to 0 you may have minimization of f of X subject to g j of X greater than equal to 0, in which case lambda j will be less than equal to 0, which means in this general form that I have stated, you change one of the conditions, then the lambda j s will be changing their sign that is instead of minimization you have maximization then lambda j s will be the less than or equal to 0 subject to g j of X less than or equal to 0 and so on; if you change both of them the lambda j s will still retain the same sign as I have indicated here;

The g j of X less than or equal to 0 will correspond to the original constraint. So, if you have g j of X greater than or equal to 0, then that particular constraint will come as a condition here, one of the conditions here. So, when we are applying the K - T conditions, the Kuhn - tucker conditions for a optimization problem, we solve for the stationary point using the first n plus m equations, and then examine whether the solutions that we obtain satisfy the other two conditions; you may have several solutions arising out of first two conditions, we examine the satisfying satisfaction of these two conditions for each of these conditions, and then out of several solutions that you have obtain may be 1, 2 or some number of solutions satisfy all the conditions, and these correspond to the stationary points of the problem.

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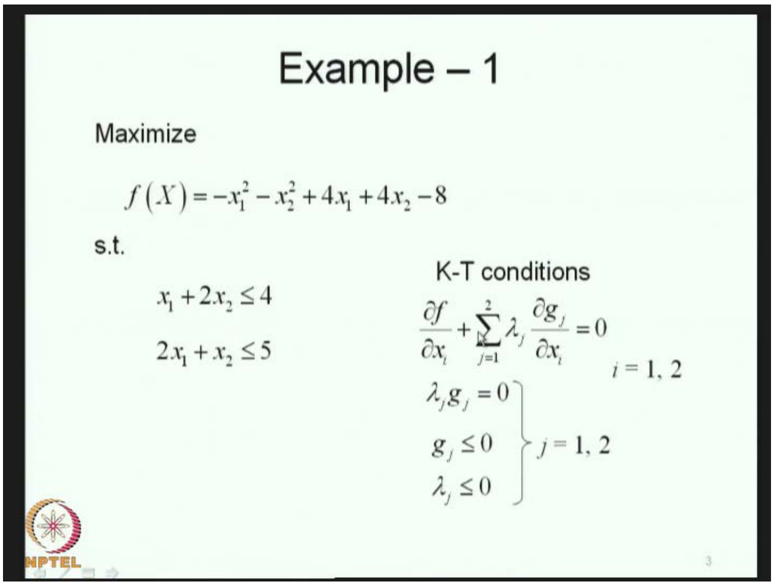

Let us look at one simple example to derive from the point of use of Kuhn - tucker conditions; let us look at maximization of a function f of X is equal to minus X 1 square minus X 2 square plus 4 X 1 plus 4 X 2 minus 8; subject to X 1 plus 2 X 2 less than or equal to 4; 2 X 1 plus X 2 less than or equal to 5; there are two constraints and two variables here, because we are talking about maximization of the function subject to g j of X less than or equal to 0, the Kuhn - tucker conditions we write as d f by d X i plus j is equal to 1 to m lambda j g j d g j d X i the partial derivative of the j eth constraint with respect to the i eth variable, this we write for i is equal to 1 comma 2. So, you get two equations corresponding to this; and then lambda j g j is equal to 0, we write it for j is equal 1 to 2, g j less than or equal to 0, you write for j is equal to 1 to 2, and lambda j less than or equal to 0; why less than or equal to 0 here, because you are objective function is maximization, and the g j of X are all less than or equal to 0. So, that is why lambda j must be less than or equal to 0. So, we state the Kuhn - tucker conditions, and then start applying the Kuhn - tucker conditions to examine the stationary points, to determine the stationary points for this optimization problem. So, f of X is given here, then g of X, as I mentioned in the last class we express the constraints as g of X less than or equal to 0. So, you take express g 1 of X as X 1 plus 2 X 2 minus 4 and g 2 of X as 2 X 1 plus X 2 minus 5; look at the first Kuhn - tucker condition, we write this for i is equal to 1. So, d f by d X 1 plus lambda 1 d g 1 by d X i for i is equal to 1 plus lambda 2 d g 2 by d X i is equal to 0; this we write for i is equal to 1 and i is equal to 2. So, you generate two equations corresponding to this; f of X is this.

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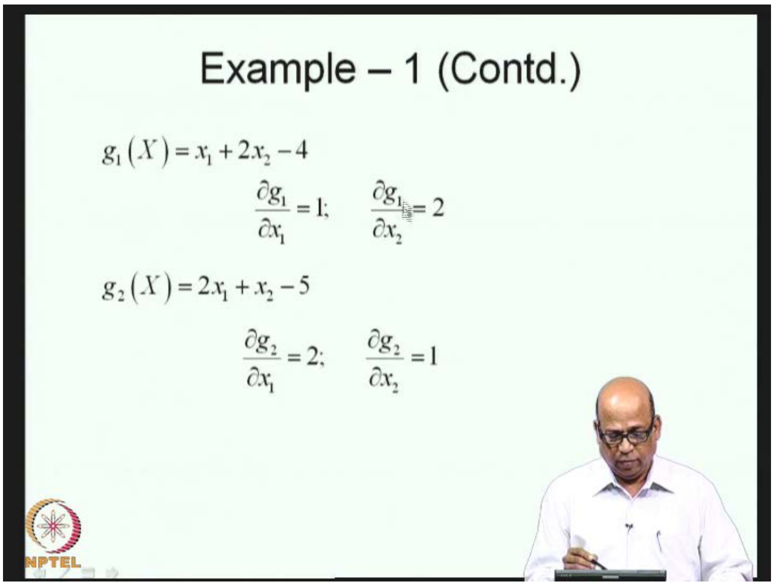

So, we will write that and d g of… d g 1 by d X 1 is 1 here and differentiating this with with respect to X 1; similarly d g 1 by d X 2 is the differential of the first constraint with respect to the second variable and that comes to 2; similarly you take the second constraint which is 2 X 1 plus X 2 minus 5, differential of the second constraint with respect to the first variable, second constraint with respect to the second variable. So, these are the values that you obtain.

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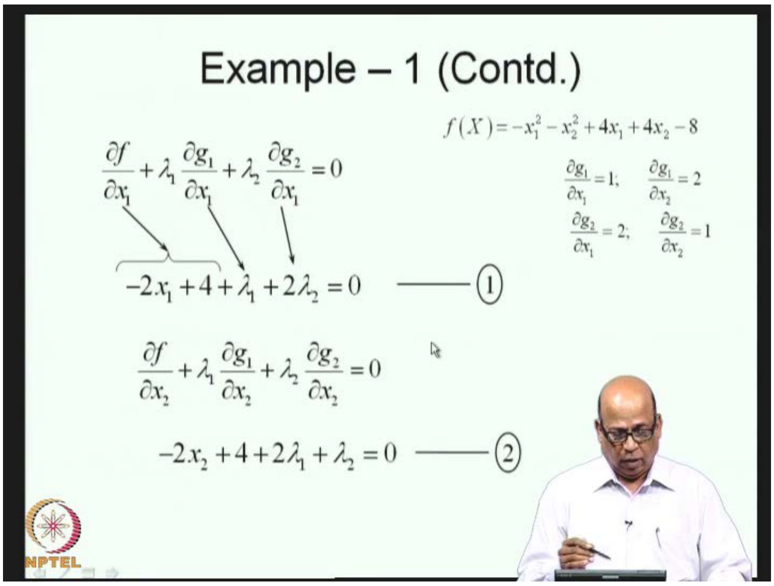

Now, we start applying the Kuhn - tucker conditions. So, the first condition with respect to i is equal to 1, d f by d X 1 plus lambda 1 d g 1 by d X 1 plus lambda 2 d g 2 by d X 2 is equal to 0. So, d f by d X 1 f of X is here therefore, d f by d X 1 will be minus 2 X 1 plus 4; I am differentiating with respect to X 1, then lambda 1 d g 1 by d X 1, d g 1 by d X 1 is the differential of the first constraint with respect to the variable X 1 that is 1; and therefore, this will be lambda 1 plus this is 2 therefore, this will be 2 lambda 2 equal to 0, this is a first condition; similarly with respect to the second variable that is i is equal to 2 now we will write and that equation will be minus 2 X 2 plus 4 plus 2 lambda 1 plus lambda 2 is equal to 0.

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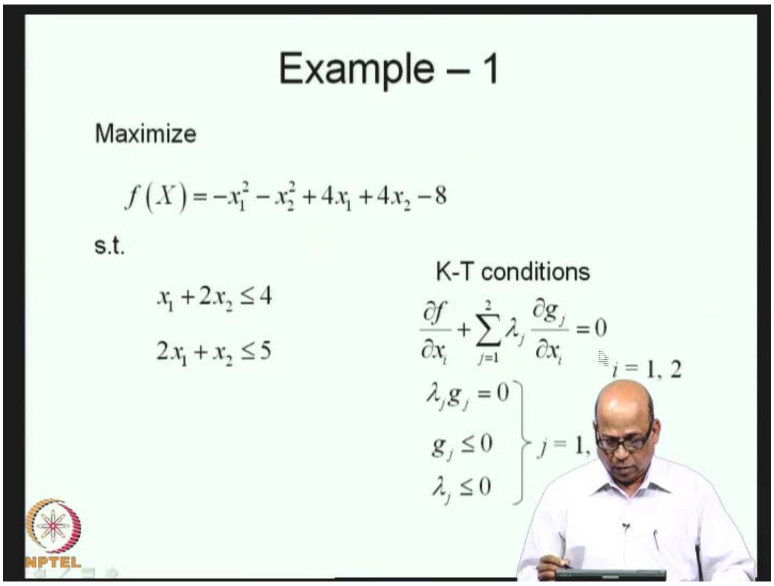

So, you get equations 1 and 2 corresponding to the first set of conditions namely, these condition d f by d X i plus lambda plus summation j is equal to 1 to 2 lambda j d g j d X i, this is written for i is equal to 1 to 2. So, you get two equations associated with the first set of conditions, and these are the two equations.

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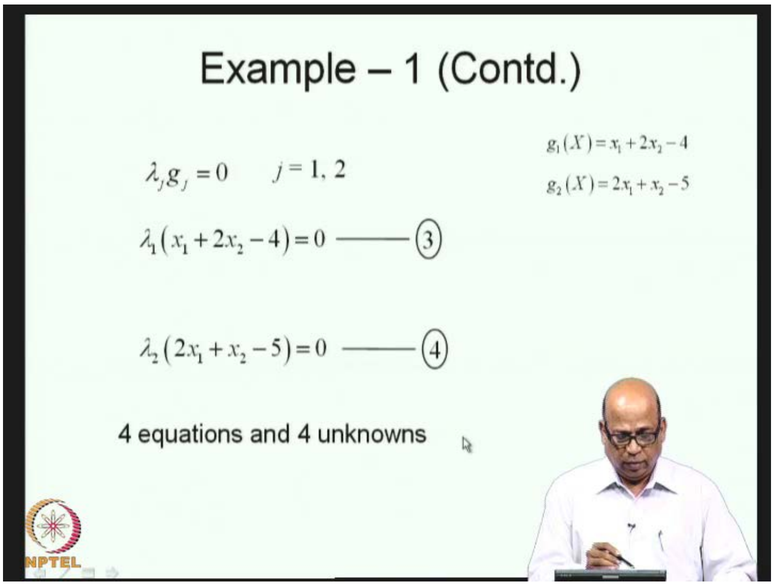

Then, you also look at the next set of conditions, namely lambda j g j is equal to 0 written for j is equal to 1 and 2. So, lambda 1 into g 1, this is g 1; g 1 of X is X 1 plus 2 X 2 minus 4 is equal to 0; similarly, lambda 2 into g 2 is equal to 0. So, you have four equations now 1, 2, 3 and 4; you have variables X 1, X 2, lambda 1 and lambda 2. So, you can solve all these four equations simultaneously to get X 1, X 2, lambda 1 and lambda 2. Let us say, we use equations 1 and 2 now; these two equations we use and then express lambda 1 and lambda 2 in terms of X 1 and X 2; these are fairly straight forward solutions, but let us go through it, because you get several combinations and each combination leading to certain set of… Certain sets of solutions, and then we need to examine the other conditions associated with each of these solutions.

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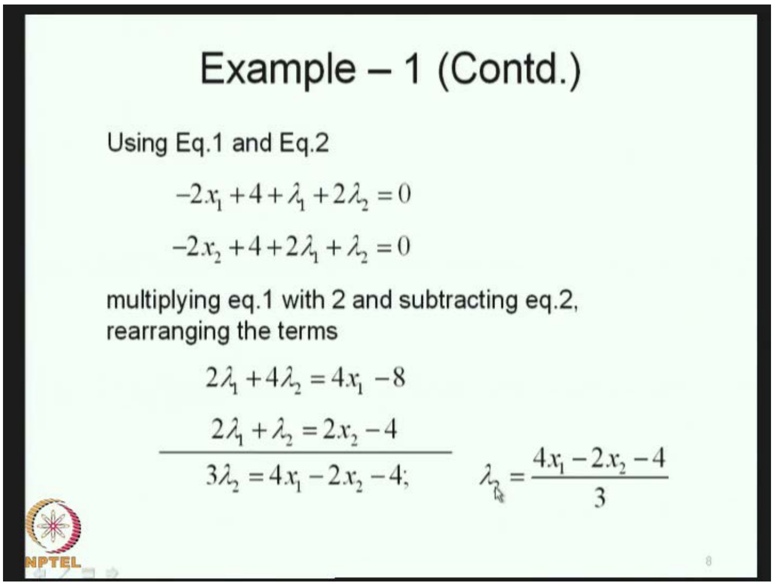

So, let us start with these two equations equation 1 and 2; from these we express lambda 1 and lambda 2 in terms of X 1 and X 2. So, lambda 2, I take this, these two equations, and then express lambda 2, I eliminate lambda 1 from here and express lambda 2 as 4 X 1 minus 2 X 2 minus 4 divided by 3.

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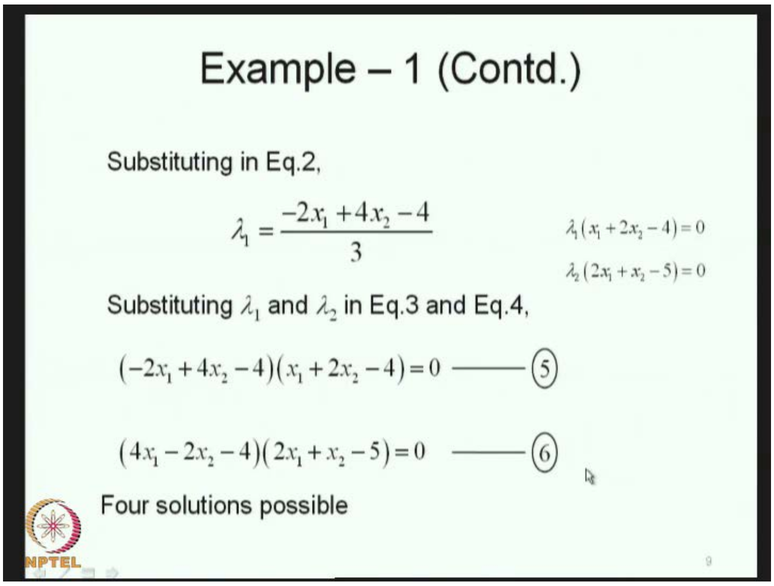

Similarly, I express lambda 1 using these two equations, and using the value of lambda 2 that I just obtained, as lambda 1 is equal to minus 2 X 1 plus 4 X 2 minus 4 divided by 3. Now you have equations 3 and 4, which are lambda j g j is equal to 0; we have now express lambda 1 and lambda 2 in terms of X 1 and X 2. So, we reformulate these equations 3 and 4 as equations 5 and 6. So, I am expressing lambda 1 here, and lambda 2 here in terms of X 1 and X 2, and these are g j of X is equal to 0; now for these sets of equations, what we then do is that we take this is to be 0 along with this to be 0, this to be 0 along with this to be 0, this to be 0 along with this to be 0, this to be 0 along with this to be 0; therefore, we formulate four different sets of equations to solve to obtain four solutions and associated with each of these solutions, we examine the conditions g j X less than or equal to 0 and lambda j less than or equal to 0. So, this is how we proceed.

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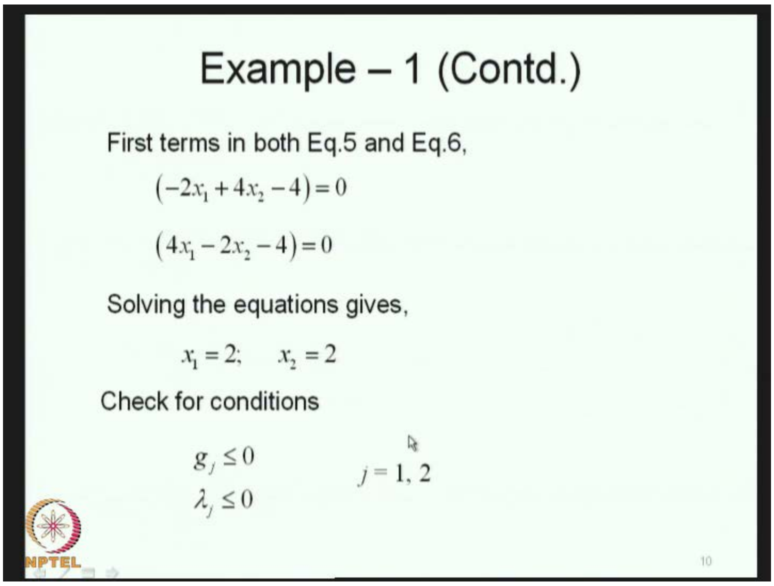

So, for the first terms in both equations that is 5 and 6, I take minus 2 X 1 plus 4 X 2 minus 4 is equal to 0; 4 X 1 minus 2 X 2 minus 4 is equal to 0; we obtain the solutions X 1 is equal to 0, I am sorry X 1 is equal to 2 and X 2 is equal to 2. Using these, we now examine whether the other two conditions are satisfied, namely g j of X is less than or equal to 0, which are the original constraints, and lambda j is less than or equal to 0 which is the requirement of the K - T conditions. So, we substitute X 1 is equal to 2 and X 2 is equal to 2, which is 1 of the 4 solutions that we need to obtain; for this solution, we examine the conditions g j is less than or equal to 0 and lambda j less than or equal to 0.

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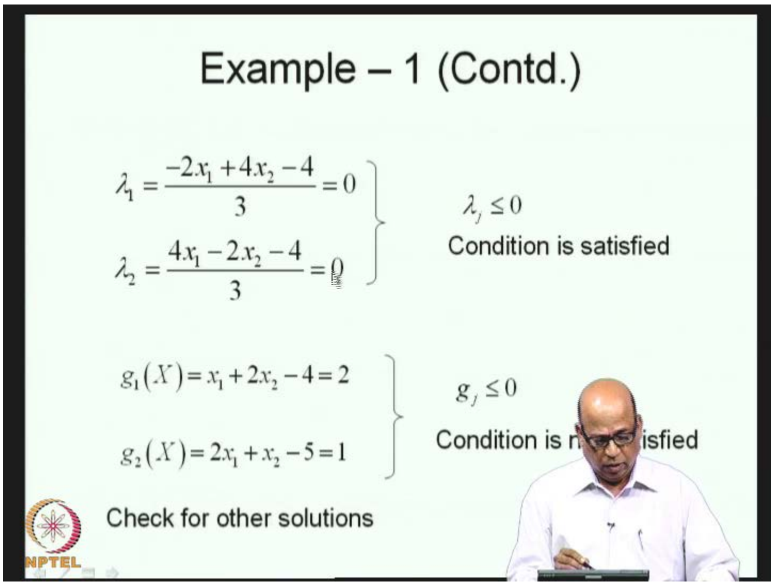

So, you put for lambda 1 you have the expression minus 2 X 1 plus 4 X 2 minus 4 divided by 3, and this turns out to be 0, and lambda 2 is equal to 4 X 1 minus 2 X 2 minus 4 divided by 3 this turns out to be 0; this is the expected, because these conditions that we have obtained here these are in fact, the lambda values, lambda 1 and lambda 2 values, and you are setting it to 0; and therefore, you get lambda 1 is equal to 0 lambda 2 is equal to 0; and therefore, lambda j less than or equal to 0 is in fact, satisfied. We then go to g j of X less than or equal to 0; when you substitute the values of X 1 and X 2 you get g 1 of X is equal to 2, and g 2 of X is equal to 1, which are not both less than or equal to 0, and therefore, this set of conditions is not satisfied by the point 2 comma 2, which you obtained as the first set of solutions. So, this condition is not satisfied, and then we move to other solutions. In fact, even if this set this point satisfies all the conditions, we still exhaust all the solutions, to examine how many stationary points we can obtain. So, this condition is not satisfied, therefore we proceed to the next set of solutions; how do we obtain the next set of solutions? We take the second terms of equations 5 and 6; these are the equations 5 and 6, we initially took these two sets, we take now these two sets, and then obtain the solutions. So, you take these two sets of equations, and obtain X 1 is equal to 8 by 5 and X 2 is equal to 6 by 5; with these now, we again examine whether g j X is less than or equal to 0 and lambda j is less than or equal to 0 are satisfied.

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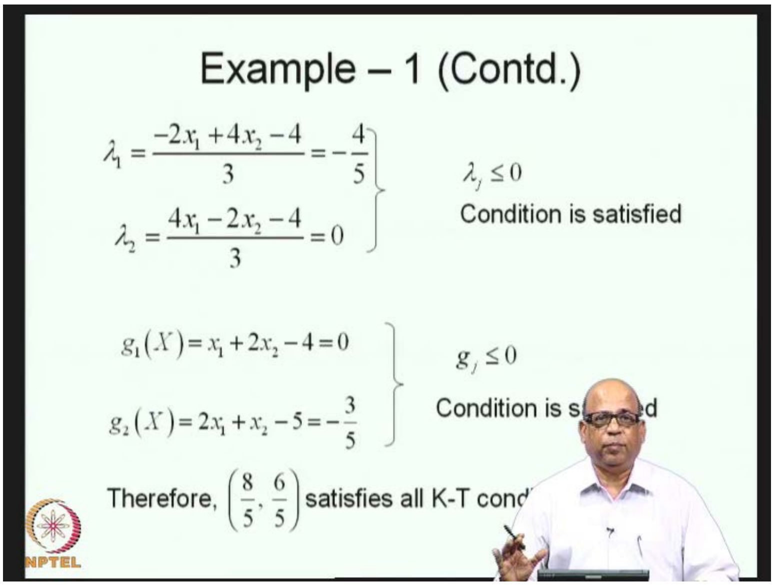

lambda j you get as minus 4 by 5 and lambda lambda 1 you get as minus 4 by 5 and lambda 2 you get it as 0, therefore this condition is satisfied; now you look at g 1 X and g 2 X, so g 1 X is 0, it turns out to be 0, with these X 1 and X 2; and g 2 of X turns out to be minus 3 by 5, and therefore this condition is also satisfied. So, the point 8 by 5, 6 by 5 satisfies all the conditions, all the 4 sets of K - T conditions are satisfied by these points, this point 8 by 5, 6 by 5; however, we have still not exhausted all the remaining solutions. So, even if we identify 1 point to satisfy 1 point as satisfying all the conditions, we still exhaust all the remaining solutions to capture all the stationary points.

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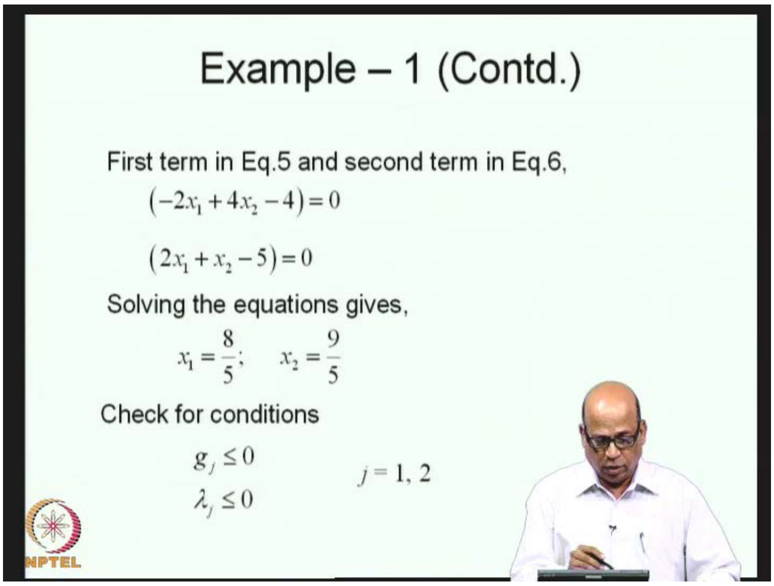

So, you move to the next solution where we take the first term in equation 5 and the second term in equation 6, and obtain the solutions as X 1 is equal to 8 by 5 and X 2 is equal to 9 by 5, and then again check for the conditions g j of X less than or equal to 0 and lambda j less than or equal to 0.

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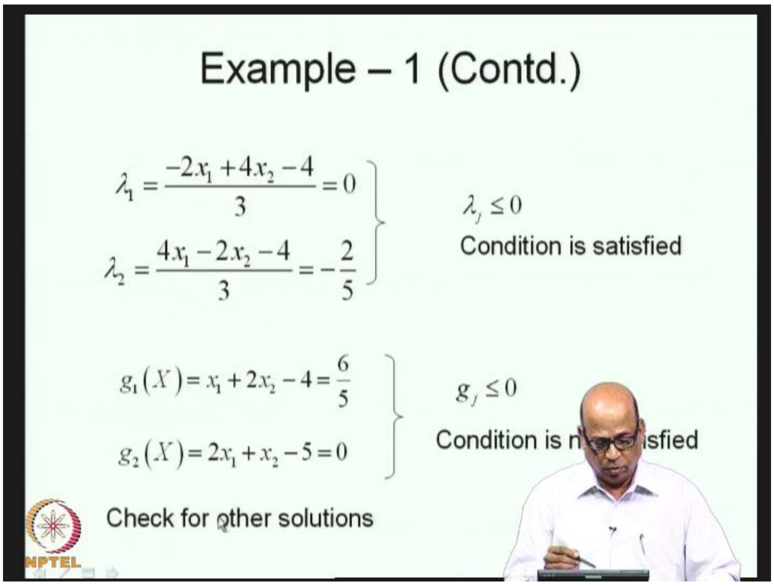

When we examine for lambdas, you get lambda 1 is equal to 0 here and lambda 2 is equal to minus 2 by 5, associated with X 1 is equal to 8 by 5 and X 2 is equal to 9 by 5; and therefore, this condition is satisfied; however, when we come to g 1 of X, you get g 1 of X is equal to 6 by 5, which does not satisfy the condition that g j of X must be less than or equal to 0; and therefore, this condition is not satisfied. Then we move on and take for other solutions, what are the other solutions? We take the second term in equation 5 and second term in equation 6, these are the equations 5 and 6; now we take this equal to 0 and this equal to 0, and formulate the equations, here we formulate the equations, and obtain X 1 is equal to 2 and X 2 is equal to 1, again we examine whether this point in fact, satisfies the other two conditions. We see that lambda j is less than equal to 0 is not satisfied whereas g j of X is less than or equal to 0 is satisfied. So, one of the conditions is violated, and therefore this point does not correspond to a stationary point.

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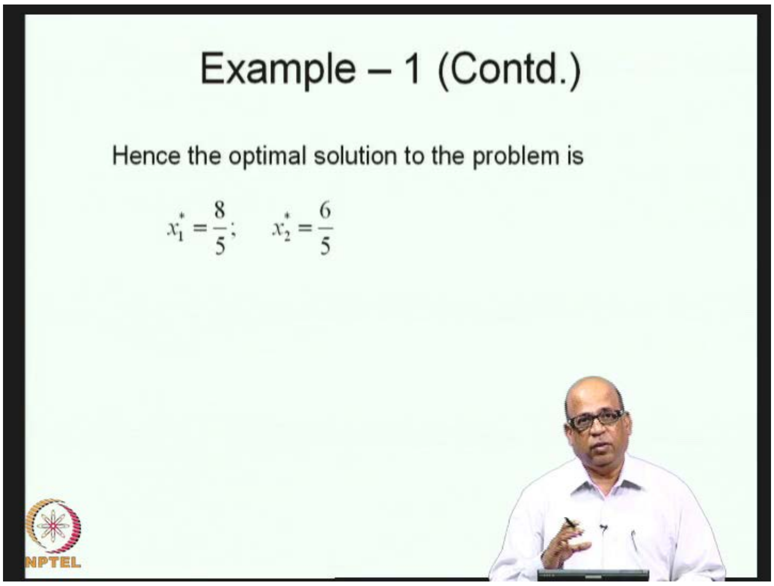

So, with that then, we have exhausted all the solutions; among all these four sets of possible solutions, we see that only one set of solutions satisfies all the conditions, and that point is X 1 star is 8 by 5 and X 2 star is 6 by 5, and that corresponds to the optimal solution. So, in general then, the Kuhn - tucker conditions, we apply to a general problem of maximization or minimization of f of X is subject to a set of inequality constraints. In fact, as I as I mentioned in the previous lecture, the Kuhn- tucker conditions are necessary conditions for any optimization problem. So, any optimization problem must satisfies the Kuhn - tucker conditions for the solution to be optimal; and Kuhn - tucker conditions are necessary conditions and or not in general sufficiency conditions.

In water resources systems problems however, we hardly ever come across such (( )) formulated optimization problems, where we can straight way apply the Kuhn - tucker conditions and obtain the stationary points and the obtain the optimal points and so on. So, we however, come across large number of large size problems where the number of variables as very large, typically of the order of thousands of variables, and the number of constraints that you obtain are also extremely large, because you imagine a large river basin, in which we are looking at several alternatives of placement of reservoirs, the size of the reservoirs subject to the hydraulic constraints and so on. So typically, you are looking at large number of variables and large number of constraints; and such simple applications of Kuhn - tucker conditions on a class room type of exercise will not be viable for large scale water resources systems problems.

We use algorithmic ways of solutions for such conditions, such large scale water resources problems, and the most important optimization technique that we use in water most commonly used optimization technique in water resources systems is the Liner Programming. So, starting with today's lecture and subsequent few lectures, I will focus on Liner Programming problems, just beat of history of Liner Programming, you know when we are dealing with large scale resource allocation problems, the Liner Programming is a very elegant and handy tool for optimization of such problems; in fact, the Liner Programming was conceived and formulated first in its general form by a US army engineer named Dantzig during the world war two; during the world war two, the problem of resource allocation was prominent especially, the military resources they have to allocate to various places, and the problem of Liner Programming was formulated by a US army engineer, air force engineer I am sorry named Dantzig; and in fact, he develops the simplest algorithm, subsequently for solution of the Liner Programming problems.

Liner Programming is so commonly adopted in all branches of engineering that it is said about the 25 percent of all the problems on computer or in fact, LP problems - Liner Programming problems. Let us see why and how the Liner Programming is applied to several problems, we have seen a general statement of optimization problems, where we have the objective function f of X, which is to be minimized or Maximized; which is the function of n decision variables X 1, X 2 etcetera X n; subject to g j of X less than or equal to 0, you have m conditions; if the objective function f of X is a linear function of the decision variables, if all the constraints g j of X are also linear functions of the decision variables, then such a problem is called as a Liner Programming problem. In fact, in the general form of the Liner Programming problem, we also add the condition that all the decision variables X 1, X 2 etcetera X n are all non negative; that means, X i is greater than or equal to 0. So, in general you have three conditions to be satisfied, namely that the objective function f of X is the linear function of the decision variables X 1, X 2 etcetera X n, all the constraints g j of X or all function, all linear functions of the decision variables X 1, X 2 etcetera X n; additionally, all the decision variables are all non negative, s j great than equal to 0 for all j; now these three conditions lead to the general form of Liner Programming problem

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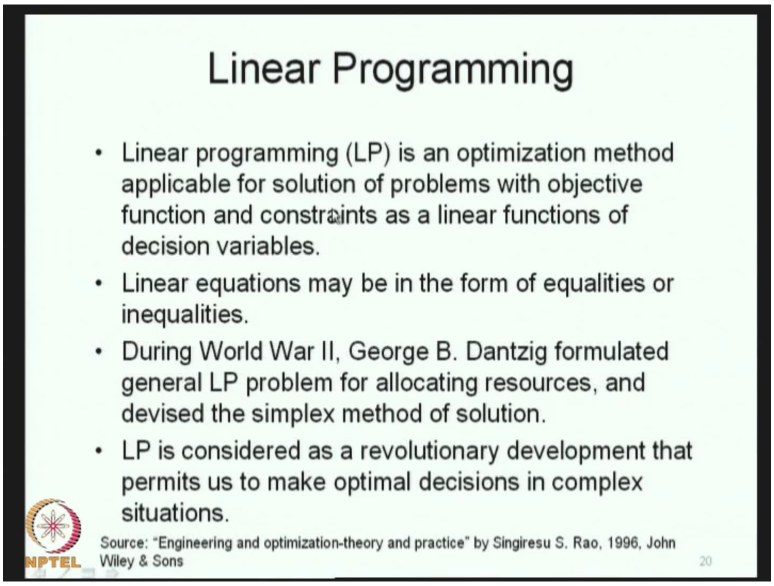

We will just see some of the features of the Liner Programming; as I just mentioned, it is the optimization method for solution of problems with objective function and constraints as a linear Functions of the decision variables. So, both the objective functions as well as the set of constraints are all linear functions; now this was formulated by Dantzig as I said, he formulated the general LP problem during the World War 2, and also devised the simple method of solution. In fact, in the field of operations research optimization etcetera the Liner Programming is considered as a revolutionary development that permits us to make optimal decisions in complex situations, and as we will see during the in the in the applications lectures, where we deal with large scale water resource systems problems, LP becomes of very handy and elegant tool for screening of alternatives, for obtaining optimal reservoir operating policy, for obtaining optimal hydro power generation, obtaining optimal flood control and so on. So, when we were dealing with large scale optimization problems, Liner Programming problem mainly because of the simplex revise simplex and the algorithm becomes an extremely handy tool for systems analysis.

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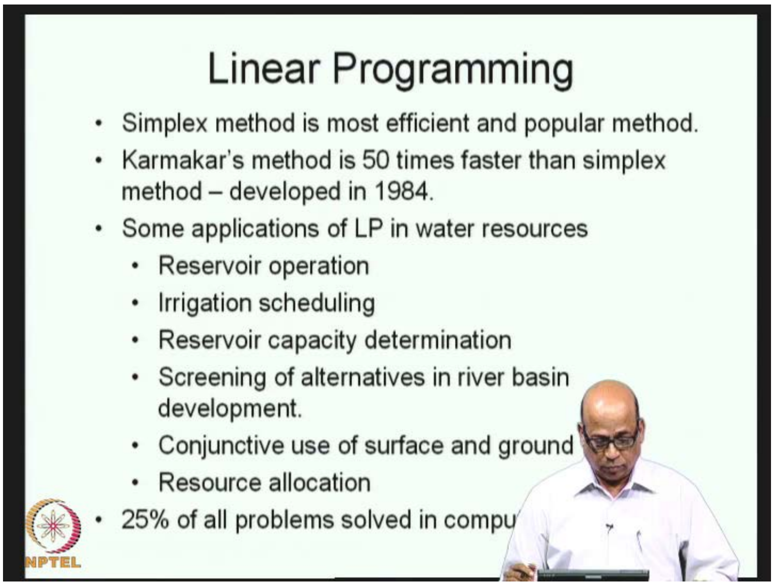

The simplex method as I mentioned, and as we will introduce in the subsequent courses, is the most elegant and efficient and therefore, the a very popular method of optimization. The Karmakar's algorithm, which is developed by Karmakar, an Indian, is 50 times faster than the revise simplex method; the general principle of the Karmakar's algorithm I explain, when I am talking about the revised simplex method. And in the field of water resource systems, a large number of applications exists today of the Liner Programming problem. In fact, any first cut systems analysis that we do in water resources, always almost always uses the Liner Programming. So, there are a large number of applications for example, you are talking about multi-reservoir operation, multiple reservoirs systems operations, look at systems of Narmada river basin, we where you have 14 or 15 major reservoirs which are to be operate in an integrated way. So, you are looking at optimization of reservoir operation of large systems, even when you are talking about a single reservoir optimization, you have multiple functions, water supply, irrigation, flood control, hydro power, down steam water control and so on. So, you have a single reservoir catering to multiple functions, and then you are talking about time periods of the order of 10 days 50 days within a year. So, you may have 36 time periods within a year 12 time periods within a year and so on.

So, you have sequential optimizations, monthly decisions to be taken over 12 months of a year, and therefore you may have associated with each set of constraints, you may have 12 number of constraints; and like this, you may have several constraints over correspond to mass balance, corresponding to the constraints due to the physical of the system and so on and so far. So, you will have large number of constraints associated with even single reservoir operation problems; so the size of the problem will be quite large. Then you may talk about irrigation scheduling where you are looking at allocation of a given amount of water, among different time periods, among different crops and so on. So, the irrigation scheduling problem is how much amount of water to be applied during a given time period, among several different competing crops; there is a irrigation scheduling problem; we apply Liner Programming for such problems. Then, you may have the reservoir capacitive determination, you would like to meet the demands form a river, there is the supply; so the supply will determine the constraints of water availability, and then you would like to have a given reservoir of minimum capacity to meet a set of demands; So, that is all reservoir capacity determination, I will talk about this problems, specifically when we go to the applications.

Then you have screening of alternatives in river basin development, let us say you have a large river basin, in which you you would like to build several reservoirs, and you would like to use the ground water at several locations. So, you are looking at where to build reservoirs, how much capacity of a reservoir to be build, how much of ground water to be used in conjunction with the surface water and so on. So, there are virtually infinite number of alternatives that are possible; so you would like to screen the alternatives. So, Liner Programming is a powerful technique as a screening technique. So, out of several such alternatives that are possible all of which will be the requirements - so functional requirements, which among this large number of alternatives or best alternatives. So, these are called as the screening problems.

So, for screening problems in water resources we often used Liner Programming; then we also talk about conjunctive use of surface and ground water, how much of surface water to be used how much of ground water to be used, such that the system becomes sustainable over a period of time, these will involve optimal decisions; let us say, 30 percent of ground water, 70 percent of surface water during certain time periods, then you start reducing the use of surface water, and then start using the ground water more. So, that you are able to meet to conditions, what are the conditions as I mentioned in the very first lecture that you do not want the ground water to come into the root zone by thus causing the water logging and you do not want the ground water to deplete so much, that it becomes unsustainable. So, you would like to maintain the ground water level within a certain conditions, yet at the same time, you are able to meet the requirements; both irrigation requirements as well as municipal and industrial requirements.

So, such problems are the conjunctive use problems, and typically in conjunctive use problem, because you have a ground water model, you have a surface water model and then two will be integrated together in an optimization framework; the size of the optimization problem will be really large; even for very small catchments, small basin etcetera, we deal with 30,000, 40,000 constraints, 30,000, 40,000 variables and so on. So, any water resource systems problem, when we want to address the problems in a meaningful manner, the size becomes a major issues; and whenever we are talking about large size optimization problems, Liner Programming is provides as a with a very tool for solution of such problems; then the resource allocation problems as I mentioned, it is not just the water resource, when we are talking about developing river basin, you may be talking about economic resource allocation, the labor resource allocation, the land resource allocation, water resource allocation and so on. So, when we are talking about allocation of various resources, the Liner Programming is… It is a very elegant and useful tool of optimization.

So as I mentioned, not only in water resource engineering, in most of the engineering optimization problems, Liner Programming is by far the most widely applied optimization technique for of the advantages that it has; however, the requirement that the objective function be linear, all the constraints be linear constraints of a decision variables, and all the variables themselves be non negative; often is not directly satisfied by most of the engineering problems, engineering optimization problem, in which case we adopt some linear irrigation techniques both on the objective function as well as the constraints, to convert the non-linear objective function and non-linear constraints into linear objective function and linear constraints; we will look at these techniques of linear irrigation subsequently; now what we will do is, we will start with a very simple Liner Programming problem and look at how we solve… Look at the basis for the solution of all the Liner Programming problems.

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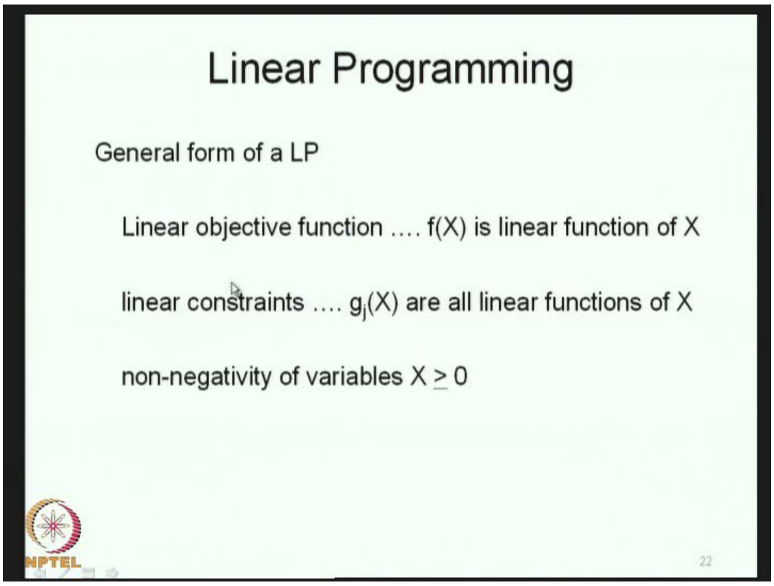

As I mentioned, you have in the Liner Programming, you have f X as a linear Function of X, X is the vector of Decision variables, X 1, X 2 etcetera X n, you have a number of decision variables. So, the objective function is a linear function of the decision variables; all the constraints or all linear functions of X, so you may write for example, 2 X 1 plus 3 X 2 less than or equal to 5, which is the function, which is the linear Function of the decision variables X 1 and X 2; whereas if you have 3 X 1 into X 2 plus 5 X 1 square of a 5 X 2 square etcetera, this is not only a linear function. So, you must have all the constraints also as linear functions, then in the general form the requirement is that all the variables be non negative.

In most of a water resources problems and in fact, in most of the engineering decision making problems, the decision variables that will be dealing with are all almost always physical variables for example, we may be talking about maximization or minimization of the reservoir capacity, maximization of flood control storage maximization of hydro power development and so on. So, the variables associated with these will be storage variable in flow variable the discharge variable and so on. So, these are in general non negative, we will be talking about mostly the physical variables, which are all in general positive variables or non negative variables; however, there may be situations where we may be talking about let us say temperature, which can also take negative values. So, there may be cases, where you may have you may have to deal with variables which are unrestricted in sign, we will see presently, how we deal with unrestricted sign, and also as I mentioned, you may have certain situations where your equations are not linear functions, they may be non-linear functions of the decision variables, in which case, we adopt what are called as pice-wise linearization techniques or linear linearization techniques to convert the non-linear functions in to linear functions for using Liner programming, similarly your objective function can be non-linear we convert that into linear Functions.

So, the requirement of the general form of LP is that the objective functions should be linear function constraints should all be a linear functions of the decision variables and all the decision variables be non negative; when you satisfy when your optimization problem satisfies all these three conditions, then we use the Liner Programming problem; let us look at the simplest form of solution which in fact, provides as with the motivation for the algorithmic way of solutions

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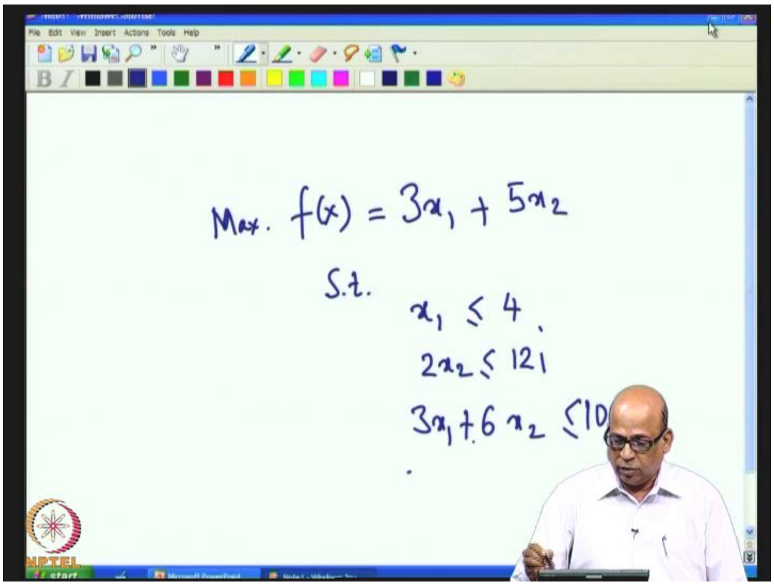

Let us say we start with a simple problem; let us say, we are looking at a function of just two variables 3 X 1 plus 5 X 2, this is our objective function f of X, and we may be interested in maximizing this, so we will write Maximize f of X is equal to 3 X 1 plus 5 X 2, this is the linear function of X 1 and X 2, X 1 and X 2 are the decision variables, then we may write a constraint subject to let us say, X 1 is less than or equal to 4; and 2 X 2 is less than or equal to 12 let us say, we write only these two conditions; just have a look at this function, we have positive coefficients here 3 X 1 and 5 X 2, and we are looking at maximization of this function. So, as I increase the value of X 1, my function value increases. So, I would like to have as high value for X 1 as possible, similarly because of this positive sign here as I increase the value of X 2, this term increases and therefore, the function value increases. So, I would like to have in this problem, as high a value of X 2 as possible; to what extent I can increase the value of X 1 that is governed by this constraint, X 1 should be less than or equal to 4; typically this may be a resource allocation problem where my resources are limited and therefore, I cannot increase X 1 beyond 4. So, I would like to increase X 1 to the greatest extent possible and therefore, I will make X 1 is equal to 4 and therefore, this term becomes 3 into 12; then 2 X 2 will be 12, which means X 2 is less than or equal to 6 therefore, I will increase X 2 up to 6; so, that I get the maximum value of f of X; this is the motivation; that means, we will look at what are the limits up to which we can increase the variables.

Now, these are by for the simplest conditions that I put, let us say that we put another condition that we may have 3 X 1 plus 6 X 2 may be less than or equal to I will put some 10 or (( )). So, I may have another condition which is the function of X 1 and X 2. So, I am putting another condition here; which will we at along with the other two conditions and then decide on what are the optimal values of X 1 and X 2; I am still looking at the maximum values of X 1 and… The maximum values 3 X 1 plus 5 X 2, and I keep on adding constraint like this, the problem becomes more and more complex; if had only this condition, then X 2 would have been infinite infinity; then I add this condition, then X 1 will go to 4, X 2 will go to 6, and I obtain the optimal value; then I add one more condition, then X 1 can need not go to 4 or X 1 X 2 need not go to full 12, because it will be governed by this condition; like this as I keep on adding conditions, the values that the X 1 and X 2 can assume become more and more restrictive.

So, these sets of conditions in fact, will determine whether a given value of X 1 and X 2 is in fact, feasible or not what we mean by that let us say, I have a solution X 1 is equal to 5, that does not satisfy this condition and therefore, X 1 is equal to 5 is not a feasible solution. So, we identify the particular region in terms of X 1 and X 2 in this particular case, in which all the conditions are satisfying, and that will determine what is called as the feasible solution or a feasible rate, feasible space; we will understand this better when we use only two variables and a set of constraints, and use the Graphical procedure to see what is happening.

Let us look such a simple problem now; so, I will now demonstrate the use of the Graphical procedure for solution of a Liner Programming problem, and because we are using Graphical solution, I will use only two variables. So, that the principle is understood correctly, and then we will lead this… From the Graphical solution, we will lead to more complex algorithmic approach, not more complex more regress in fact, algorithmic approach of simplex method.

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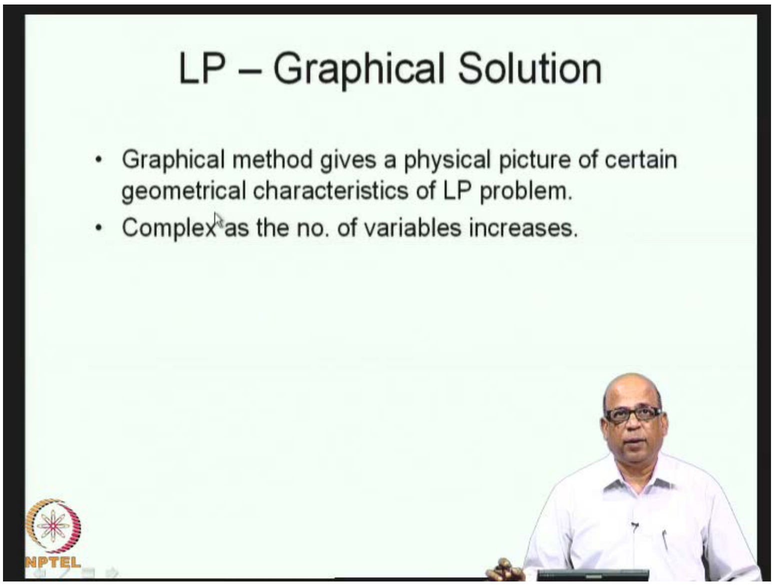

Let us look at a two variable problem with a set of constraints; so, in the Graphical method, we are looking at of a physical picture of the geometric characteristics of a LP problem; and this is best used only with two variables, in fact, in the class room type of examples we can use only with two variables, as the number of variables increases, the the Graphical picture becomes more and more complex and therefore (( )), it is becomes (( )), but however, to understand the basis of the solutions of for a Liner Programming we will use the Graphical method.

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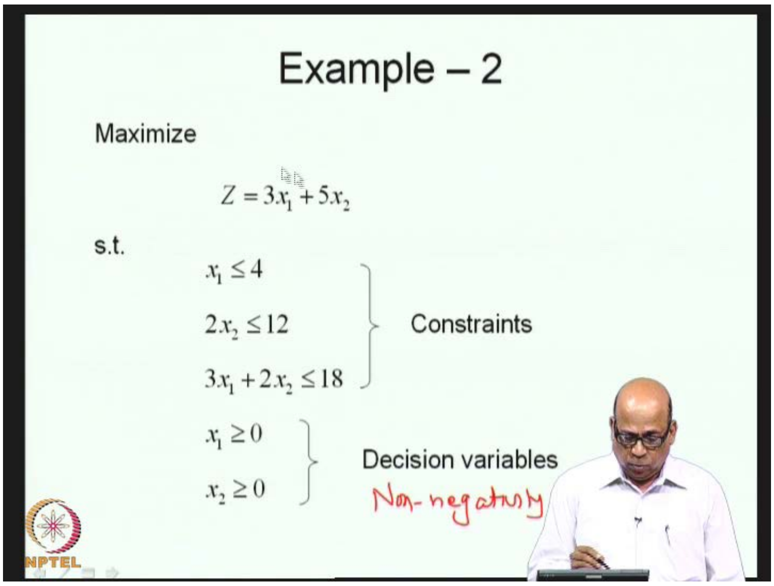

Let us look at a simple problem, we are looking at Maximize z is equal to 3 X 1 plus 5 X 2, in fact, in LP problems we use instead of a f of X, we use Z as the objective function, notation for objective function. So, you have such a simple function 3 X 1 plus 5 X 2 subject to X 1 is less than or equal to 4, 2 X 2 less than or equal to 12, 3 X 1 plus 2 X 2 is less than or equal to 18, X 1 is greater than or equal to 0 and X 2 is greater than or equal to 0; now these are the non negativity conditions; (No audio from 49:13 to 49:21) look at the objective function it is a linear function of the decision variables X 1 and X 2. So, the objective function is a linear function, these are the sets of constraints X 1 is less than equal to 4 is the linear function of X 1; 2 X 2 is less than or equal to 12 is a linear function of X 2, 3 X 1 plus 2 X 2 is less than or equal to 18 is a linear function of X 1 and X 2. So, all the constraints are linear Functions of the decision variables X 1 and X 2, and additionally you have the decision variables as non-negative. So, the Non-negativity conditions are stated as X 1 greater than equal to 0, X 2 greater than equal to 0; now to solve this by a Graphical method, what we notice is that because of the non negativity conditions, X 1 greater than or equal to 0, X 2 greater than or equal to 0, in the four quadrants that we have in the two dimensional space, the solution must lie in the first quadrant, because X 1 greater than or equal to 0, X 2 greater than or equal to 0.

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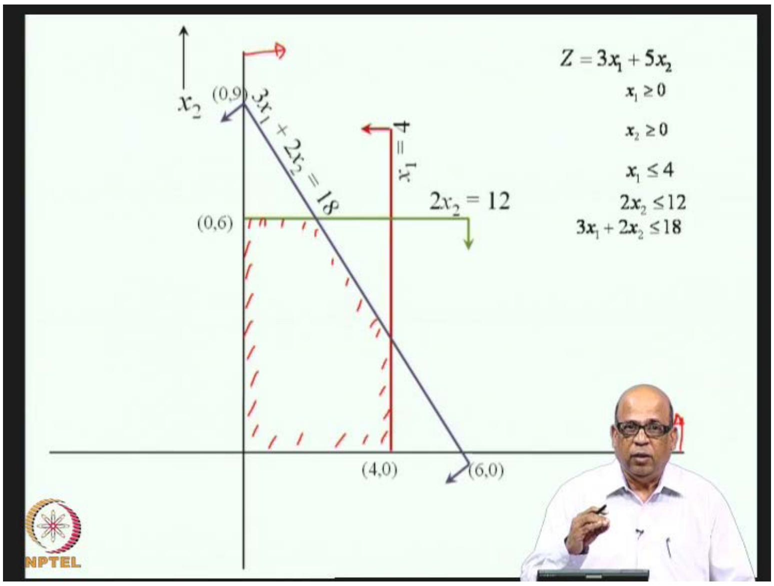

So, what we do is that we first draw the axis, this is the X 1 axis, this is the X 2 axis, and because X 1 is greater than or equal to 0, I am looking at this region X 1 greater than or equal to 0, and because X 2 is greater than or equal to 0, I am looking to the right of this axis; therefore, the solution lies in the first quadrant; then I look at the… And therefore, I mark the first quadrant. So, the solution has to lie on the first quadrant here, because of X 1 greater than or equal to 0 X 2 greater than equal to 0. Then, I look at the conditions, we have the condition X 1 less than or equal to 4; to mark this condition what I do is, I take X 1 is equal to 4, because I want to increase X 1, and I am putting bound of X 1 is equal to 4, X 1 cannot be greater than 4. So, X 1 cannot exceed beyond 4. So, I draw a line X 1 is equal to 4 which in fact, determines the bound on X 1, and because I want X 1 less than equal to 4, I look at look at left of this. So, any region within this will satisfy this condition namely X 1 is less than equal to 4.

Then I look at the second condition, which is 2 X 2 is less than equal to 12 or X 2 is equal to X 2 is equal to 6; so, this line is X 2 is equal to this, and I have to look below this, because X 1 cannot be more than 6; so, I am looking below this. Just look at what happen, the moment a considered these two conditions, I have a defined region here, in this region both these conditions are satisfied, any point outside of this region at least one of the condition is violated therefore, that becomes an infeasible point. So, by drawing these two conditions, these two constraints, I have identified a region, in which these two conditions along with the non-negativity conditions are satisfied.

Now we will look at third condition; third condition is 3 X 1 plus 2 X 2 is less than or equal to 18; now 3 X 1 plus 2 X 2 less than or equal to 18 is this line, how do I draw the line? I put X 1 is equal to 0 I get X 2 is equal to 9. So 0, 9 is one point; I put X 2 is equal to 0, I get X 1 is equal to 18, therefore 6, 0 is another point. So, I join these two points, by joining these two points I draw the line 3 X 1 plus 2 X 2 is equal to 18. What did I do, and because I want to this to be less than or equal 18, I am looking at this direction now. So, because of this condition now, I have now found another region where all the three conditions are satisfied. So, I mark this region now, this is the region, in which all the conditions are satisfied, all the three conditions are satisfied; and this in fact, becomes the feasible region of the problem.

Remember I have not looked at the objective function yet, I am only looking at the set of constraints, these are the non-negativity conditions, X 1 greater than equal to 0, X 2 greater than equal to 0, which makes my solution possible only in the first quadrant; then I looked at X 1 less than or equal to 4, by drawing the line X 1 is equal to 4, and then say that my solution should lie to the left of this line, then I drew 2 X 2 less than 12, by I drew 2 X 2 is equal to 12, which defines bound on this condition; I have to looked to the down to the below of this condition, then I draw the line 3 X 1 plus 2 X 2 less than or equal to 18, and then from this constraint. The intersecting region of all these constraints along with the non-condition is this region; this region is called as the feasible region. In this feasible region you take any point within the feasible region, it satisfies all the conditions and that is why it is called as the feasible region.

So, any point lying within or on the border of this region satisfies all the conditions, how many such points of there? There are infinitely many points, infinite number of points which will satisfy all these conditions, among this infinite number of points, we want that particular point which Maximizes this condition. So, we have still not looked at the objective function, we have simply identified the feasible region; the feasible region consists of infinitely many points, and among this infinitely many points, we want to choose or we want to identify, determine that particular point, which Maximizes the objective function 3 X 1 plus 5 X 2, how we do this, we will continued in the in the next lecture.

So, in today lecture essentially, we started with an example of use Kuhn - tucker conditions and Kuhn tucker conditions are the necessary conditions that any optimization problem has to satisfy. So, essentially what we do is we use first two sets of conditions to determine the n plus m variables which in fact, determine the stationary point, and then we examine corresponding to these stationary points, we examine whether the other two conditions are satisfied; in the example problem, we examine whether g j of X less than or equal to 0 and lambda j less than or equal to 0 are satisfied; we do this for all possible solutions that we obtain by by solving the first two sets of conditions.

So, in the example we had four sets of solutions, we do the examination for all the four sets of solutions, and out of the four sets in the example problem, we saw that only one of them satisfies all the conditions, in different types of problems you may have more number of solutions satisfying all the conditions; all of these will form the stationary points. Kuhn - tucker conditions are not sufficiency conditions, in general except for the problems of convex… Expect for the problems that are known as convex programming problems, in which the objective function as well as the constraints all of which are convex functions of the decision variables X 1, X 2 etcetera X n.

Then we went on to introduce the Liner Programming problem; in the Liner Programming problem, the objective function has to be a linear Function of the decision variables, all the constraints are all linear functions of the decision variables, and additionally in the general form of the Liner Programming that we will be using the decision variables are all non-negative. So, we typically write X j is greater than or equal to 0. So, when we satisfy when an optimization problem satisfies all these three conditions then we say it is a Liner Programming problem then we have just introduced the Graphical method of solution where we first draw the constraints the bounds determined by the constraints and identify the feasible region.

So, the feasible region is that region in the solution phase where any point within the region or on the boundary of the region satisfies all the constraints. So, the first step in the Liner Programming solution by Graphical method is to identify the feasible region. So, there are infinitely many such points, which satisfy, which all of which all of it is satisfy the constraints, set of constraints; among these infinitely many points, we have to identify one particular solution, which is in fact the optimal solution; we will see how to do this in the next class. Thank you for your attention.