Water Resource Systems

Modeling Techniques and Analysis

Prof .P .P. Mujumdar

Department of Civil Engineering

Indian Institute of Science, Bangalore

Lecture No # 06

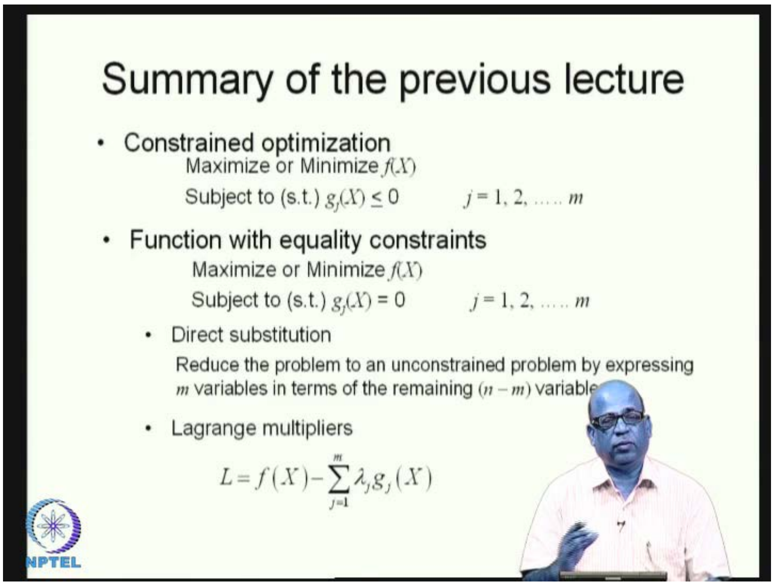
Constrained Optimization (2)

Good morning, and welcome to this the lecture number 6, of the course, Water Resources Systems - Modeling Techniques and Analysis. We are introducing the systems techniques now, which mean essentially, we are talking about optimization techniques. To begin with, subsequently we will go to the simulation and other stochastic optimization and so on; so right now, we are talking about deterministic optimization; we started with optimization of a function of single variable without any constraints. So, we specifically dealt with unconstraint optimization as we started with function of single variables, extended them to extended the optimization to function with multiple variables without any constraints, and in the previous lecture, we started introducing constraints to the optimization problem.

So, we are now entering into domain of constraint optimization, and in the previous lecture, we introduced a general form of the optimization problem where you have minimize or maximize of f of x, where x is a vertex of decision variable x 1, x 2 etcetera x n, and subject to the conditions g j of x is less than or equal to 0, where g j of x are set the set of constraints j is equal to 1, 2 etcetera up to n, number of constraints; so we are dealing with n variables and n constraints. In general m is less than or equal to n; in this general class of optimization, we started with equality constraints that means g j of x is equal to 0.

So, we are we dealt with the previous lecture, maximizations of a functions subject to equality constraints, and typically we introduced two different methods of dealing with equality constraints; one is the direct substitution where you will express n minus m number of variables in terms of the remaining m variables; I am sorry; I will repeat we express n variables in terms of remaining n minus m variables and convert the constrained optimization problem into unconstrained optimization problem with n minus m variables; and then solve using any of the technique that we have dealt with earlier. This is called as direct substitution method and the direct substitution method is useful when we you have less number of constraints and the substitution is fairly straight forward; if you have a highly complex non-linear constraints or if you have a large number of variables then this kind of direct substitution will not in general be feasible. Then we introduce the Lagrange multiplier method where associated with each of the equality constraints, we add a variable called as Lagrange multiplier. So, g j of x equal to 0 is the equality constraint that we are talking about associated with each constraint g j of x, for each j we associate a Lagrange multiplier named as lambda j, denoted as lambda j and then we formulated the necessary conditions and the sufficiency conditions; we continue the discussion now with Lagrange multipliers.

(Refer Slide Time: 04:07)

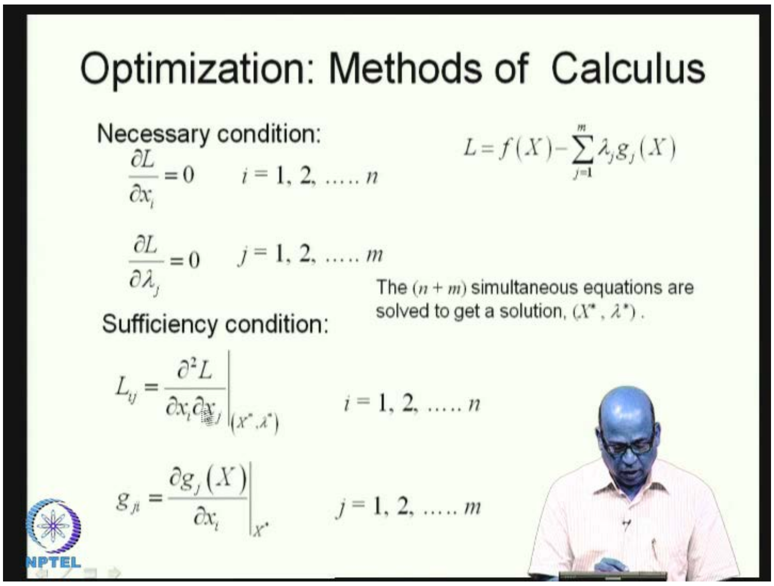


So, this is a summary of the previous lecture; we introduce the constrained optimization where we are talking about minimization or maximization of function f of X, subject to g j of X equal to 0; this is the general form of optimization problem and we dealt with function with equality constraints that is optimization of function subject to equality constraints. So, the problem statement will be maximize or minimize f of X subject to g j of x equal to 0 and you have m number of such constraints, and X remember is a vector of n variables X 1,X 2 etcetera X n. So, there are n numbers of variables, m number of constraints; and in the direct substitution, we used m variables and express these m variables in terms of the remaining n minus m variables and therefore, converted the original problems of n variable with m constraints to a new problem of n minus m variables without any constraints.

So, essentially in the direct substitution, we reduce the constrained optimization problem with two a unconstrained problem and in the process, we are also reducing the number of variables, because we have been able to express the m variables in terms of the remaining n minus m variables; however, this is useful only in very simple cases and in most engineering problems where the problem size are is huge, and the nature of the constraints is quite complex, and of an non-linear constraint, you have often non-linear constraints, then the direct substitution method is not elegant. We introduce the Lagrange multipliers where associated with each of these constraints g j of x is equal to 0, we insert an another addition variable lambda j, we introduce another addition variable lambda j.

So, in the direct substitution method, you reduce the number of variables, whereas in the Lagrange variable method, you increase the number of variables. So, instead of the original n number of variables, you now have n plus m number of variables, the n original decision variables X 1, X 2, X n, and the additional Lagrange multipliers lambda 1, lambda 2 etcetera lambda m; and then we formulate the Lagrange L is equal to f of X minus summation j is equal to 1 to m lambda j g j of X, where g j of X is the original constraints, the left hand side of the constraints. Now some places we use plus lambda j g of X, it is does not matter, because the sine of lambda j in the case will be difference. So, the Lagrange function will be f of x plus summation j is equal to m lambda g j of x in some text books; it does not matter whether we use minus or plus here; starting with this Lagrange functions, we introduce the necessary and the sufficiency conditions, we will continue the discussion from where we left last time; so I will repeat the necessary conditions and the sufficiency conditions here just. So, that the notation and the way we use the differentials is all correctly understood.

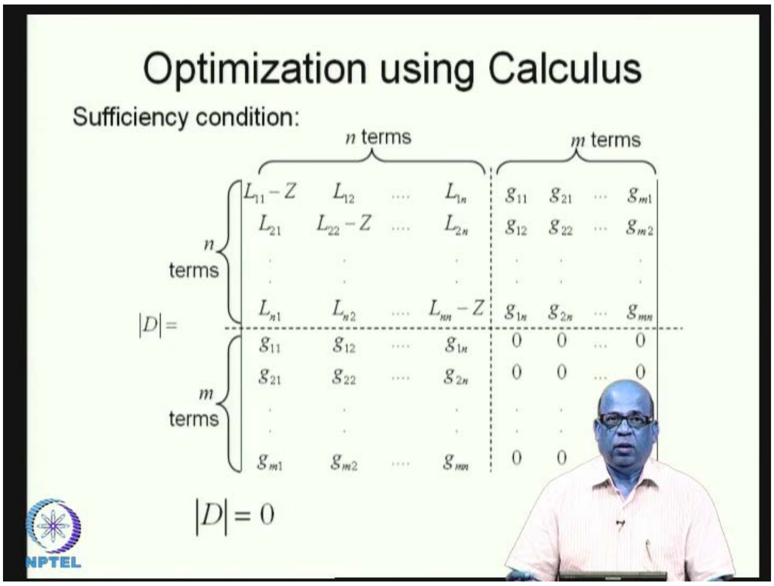
(Refer Slide Time: 07:44)



So, the necessary condition for this is that we look at the Lagrange function, this is the original functions f of X minus j is equal to 1 to m summation lambda j g j of X; this is the Lagrange functions; for the Lagrange function to have a local minimum or a local maximum at X is equal to X star; the necessary conditions are that the first order differential of the Lagrange functions with respect to each of its arguments namely X 1, X 2 etcetera X n and lambda 1, lambda 2 etcetera lambda m; the first differentials with number of each of these arguments must be equal to 0. So, d l by d x i is equal to 0, for i is equal to 1 to n, number of variables and d l by d lambda j is equal to 0, for j is equal to 1 to m.

So, these will form constitute n plus m number of equations; n equations here and m equations here. So, we use these equations and solve for n plus m number of variable, what are the n plus m number of variables X 1, X 2 etcetera X n and lambda 1, lambda 2 etc lambda m; so n plus m number of variables, so we solve for x star and lambda star; then we go to the sufficiency conditions. For the sufficiency condition, we consider the second order derivatives of the Lagrange function L, and the first order derivatives of the constraint functions g j of X is equal to 0. So, we look at the first order derivatives of the constraints and the second order derivatives of a Lagrange - Lagrange functions. So, we define L i j is equal to d square L by d x i d x j which is the second order derivative of the Lagrange functions with respect to the variables x i and x j, this is defined for all i is equal to 1 to n, then g j i which is the differential of j eth function j eth constraint with respect to i eth variable. So, we define g j i as d g j by d x i, which means the derivative of j eth constraint with i eth variable, and this we defined for all js.

(Refer Slide Time: 10:32)



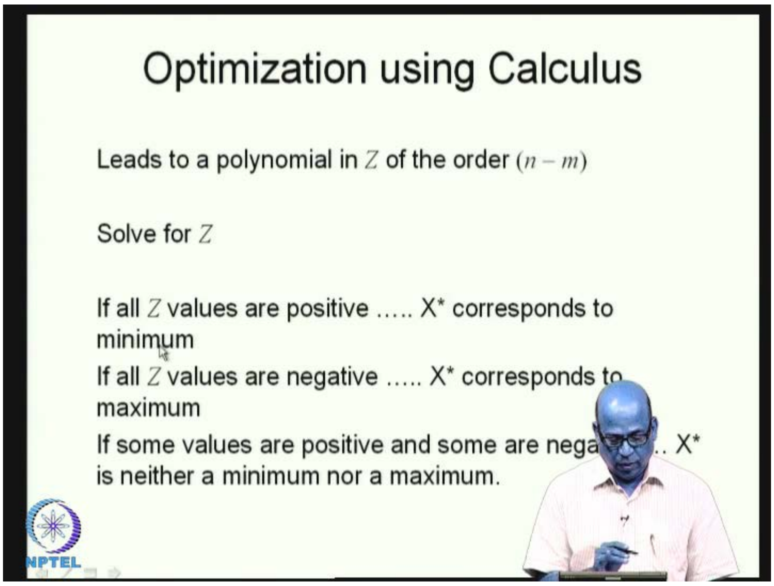
So, once we have these derivatives, we formulate the determinant D as we start with L 11 minus Z L 12 etcetera L 1n, which is here X 1 etcetera X n, X 1 etcetera X n; So, this is d square l by d x square d square l by d x1 d x2 etcetera d square l by d x1 d x n, and then we go to L 21 etcetera L 2n L n1 etcetera L nn minus Z. So, all the diagonal elements we introduces minus Z here. So, this is n by n terms, this we formulate based on the second order derivative L ij, then we introduce next m terms here; this is the first partial derivative of the first constraint with respect to the first variable; second constraints with respect to the first variable etcetera, m eth constraint with respect to the first variable; then this is a partial derivatives of the first constraint with respect to the second variable, second constraints with respect to the second variable, etcetera m eth constraint with respect to second variable and so on. So, we get m eth constraint with respect to n eth variable.

So, this will be m terms here and n terms here; so this is a m by n component; we transpose this and put it here; so, we have n terms in this direction. So, we will have n terms in this directions and m terms is in the directions; so we transpose this and put g 11 g 12 etcetera g 1n, then g 21 g 22 etcetera g 2n in this direction; and similarly, we g m1 g m2 etcetera g mn. So, this m by n terms we put it as n by m terms here. So, I left with the last column, the last corner which has n by m terms this n by m terms is substituted by 0. There is a proof available for why this has to be a sufficient conditions, sufficient conditions for optimal, will not go into the proof, we will only use the final results of the proof, and then we will see, how we apply the sufficient condition for actual problem.

So, this a way we formulate the determinant D, and we evaluate the determinant D at the stationary point X star, lambda star; remember we have determined X star, lambda star by solving the n plus m equations, and we evaluate the determinant D at the point X star lambda star; and then equate the determinant D which is evaluate the X star lambda star to 0. When we do that, you get a polynomial in Z, remember only Z is the unknown here, all other derivatives here are known, and this are all of course 0. So, we get a polynomial of order n minus m in Z; we solved this for Z and look at all the roots that will get for Z; if all roots of Z are positive that is greater than are equal to 0, then the point x star lambda star corresponds to a local minimum; positive always correspond to local minimum; if all the roots of the Z is negative, then the point X star lambda star corresponds to maximum; if some of them are positive, some of them are negative, then the point X star lambda star neither corresponds to a maximum nor to a minimum, that is a sufficiency conditions.

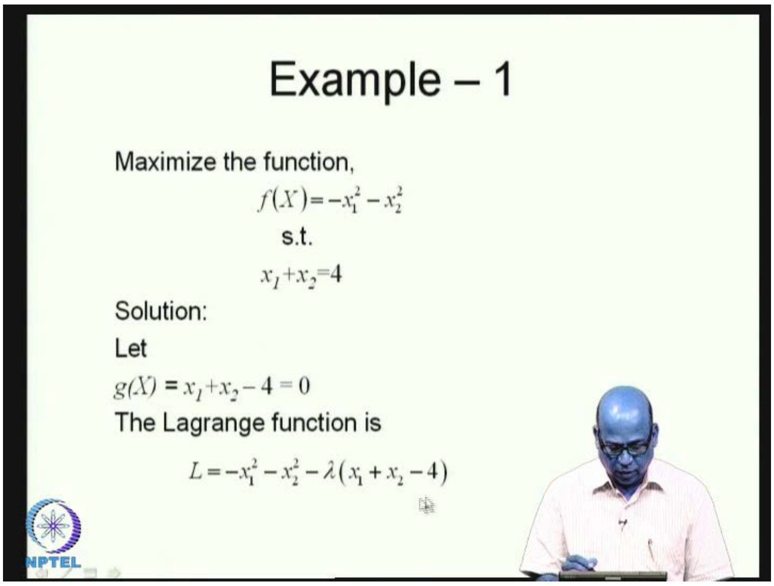
So, I repeat, we are dealing with function f optimization of a function f of x, subject to constraints g j of x is equal to 0, we introduce the Lagrange multipliers lambda j one associated with each of the constraint g j of x, we formulate the necessary conditions differentiating the Lagrange with respect to each formulating of the arguments namely X 1 X 2 etc X n and lambda 1 lambda 2 etcetera lambda n thus formulating n plus m equations, we solve the n plus m equations to obtain the solutions X 1, X 2 etcetera X n and lambda 1, lambda 2 etc lambda m, then we go to the sufficient conditions, we formulate the determinate D and explain evaluated at X star lambda star equate determinant D is equal to 0 to 0 and then you obtain a polynomial of order n minus m in Z, solve the polynomial to obtain roots of Z; if all the roots Z are positive, then the stationary point X star lambda star correspondent to the minimum local minimum; if all roots are negative, then the points X star lambda star correspondent to a local maximum if some of them are positive some of them are negative, then the point X star lambda star neither correspondent to the maximum nor to the minimum. So, this the method of lag range multiplier and in fact, the Lagrange multiplier have some physical significant in terms of ability to explain the sensitivity of the solutions and form which will discuss slightly later, when we are dealing with the linear programming; but right now, we just remainder that lambda which are the Lagrange multiple in fact, have a physical significance.

(Refer Slide Time: 16:46)



We will take, this is what I just explain; so, if all the Z values are positive, then X star correspondent to a minimum; if all the results are negative, then X star of correspond to maximum; and then some of them are positive, some of them are negative, it is neither the minimum nor the maximum.

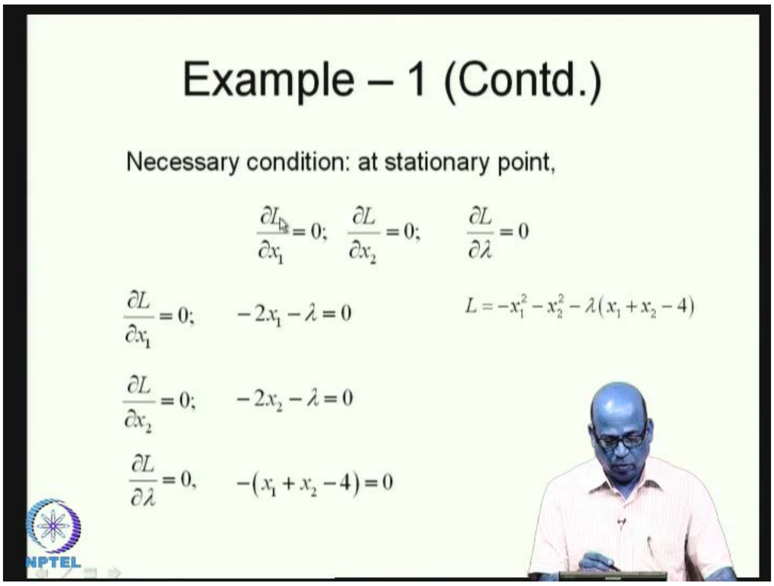
(Refer Slide Time: 17:07)



We will take an example now; let see we are looking at the maximization of the function f of X is equal to minus x 1 square minus x 2 square, subject to x 1 plus x 2 is equal to 4. In fact, the problem statement can be just optimize that means maximize or minimize subject to a equality constraint x 1 plus x 2 minus 4 is equal to 0. In fact, the problem statement meant have been just optimization; that means, maximize or minimize that to a equality constraint x 1 plus x 2 is equal to 4, because we do not know apriori, whether the stationary point that we can get after applying the necessary condition, in fact leads to the maximization or maximization of the functions or minimization of the functions. So, we just want to explore the optimal values relative maximum, relative minimum of these particular functions. So, this has one constraint, this has two variable x 1 and x 2 are the two variable, x 1 plus x 2 is equal to 4 is the constraint.

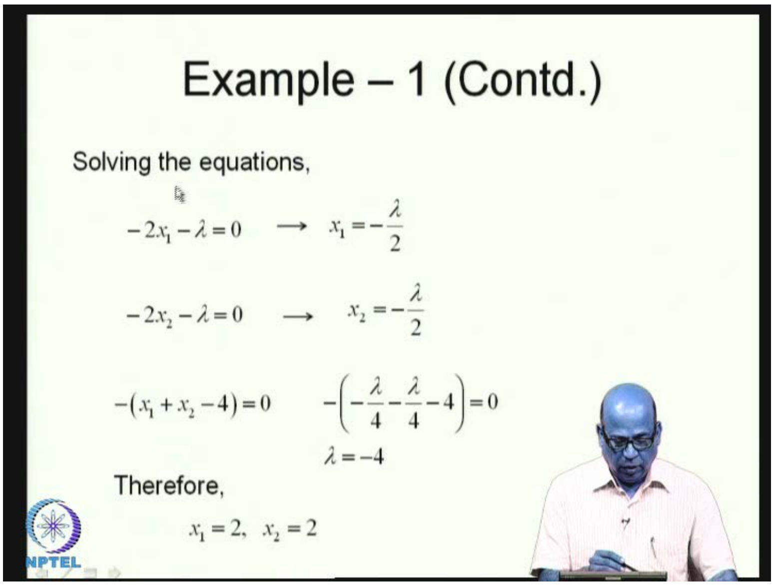
First we express the constraint in the form g j of x is equal to 0, which means in the right hand side we make it 0. So, we express this as g of X, because there is a only one function, one constraint I write it as g of X is equal to x 1 plus x 2 minus 4 is equal to 0. So, we writing g of x is equal to 0; and then formulating the Lagrange, Lagrange is f of X, which is the original function minus lambda, in this case there is only one constraints therefore I will use only one lambda and therefore I get rid of the subscript; otherwise I would put lambda 1, lambda 2 etcetera associated with constraint 1, 2 and so on; because there is only one constraint, I use lambda into g of X which is a x 1 plus x 2 minus 4, so this is x 1 plus x 2 minus 4; this is a way we formulating the Lagrange, once we get the Lagrange function, we are apply necessary conditions.

(Refer Slide Time: 19:00)



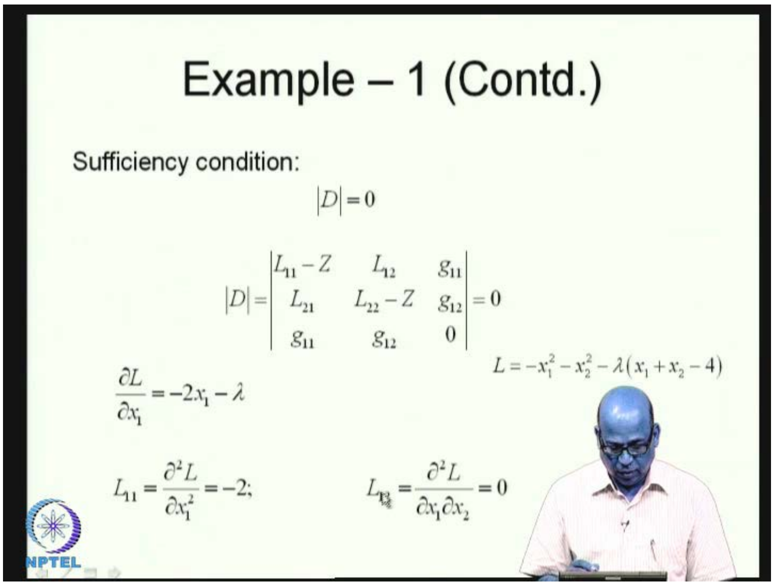
So, at the stationary point, we want the necessary condition is d L by d x 1 is equal to 0, d l by d x 2 is equal to 0 and d L by d lambda is equal to 0; there are two original value variable and one Lagrange multiplier therefore, we take three partial derivatives of the Lagrange functions, and the Lagrange function L is minus x 1 square minus x 2 square minus lambda x 1 plus x 2 minus 4. So, d L by d x 1 is 0 that leads to minus 2 x 1 minus lambda is equal to 0; d L by d x 2 is 0 that leads to minus 2 x 2 minus lambda is equal to 0 and d L by d lambda is 0 that leads to the original constraint x 1 plus x 2 minus 4 is equal to 0 we are differentiating with respect to lambda.

(Refer Slide Time: 19:58)



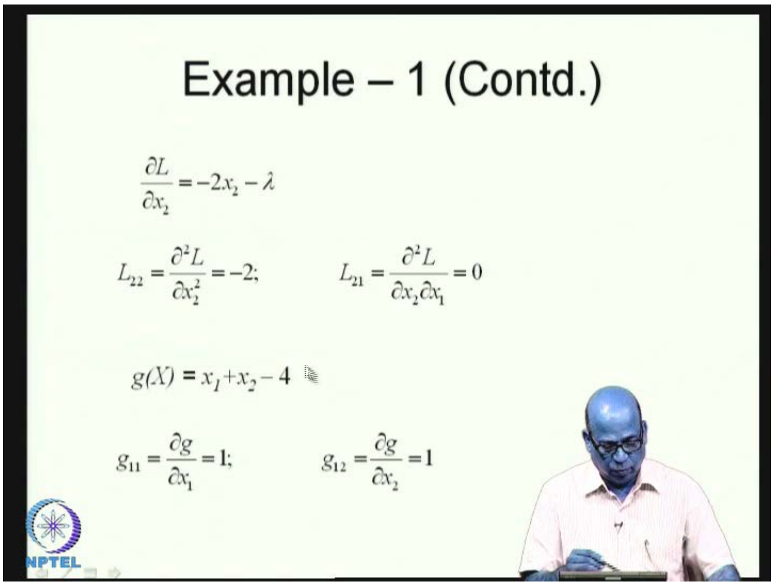
Then, we solve these equations to minus 2x 1 minus lambda is equal to 0, this is the equation minus 2x 2 minus lambda is equal to 0 and x 1 plus x 2 minus 4 is equal to 0; so we solve this equations to get x 1 is equal to minus lambda by 2 here and x 2 is equal to minus of lambda by 2 here and the third one is x 1 minus x 2 plus 4 is equal to 0 or minus of x 1 plus x 2 minus 4 is equal to 0; you get from this, substituting x 1 is equal to minus lambda 2, x 2 is equal to minus lambda by 2; you get lambda is equal to minus 4; and therefore, x 1 is equal to 2, and x 2 is equal to 2 is the solution.

(Refer Slide Time: 20:50)



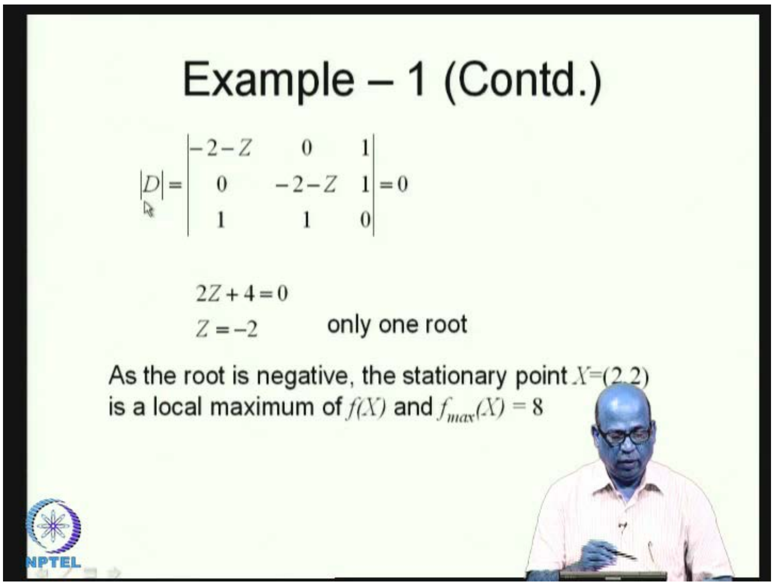
We then go to the sufficiency conditions, for which we need to formulate the determinant D, now the determinant D is in this particular case L 11, there are two variables and one constraint. So, L l1 L 12 L 21 L 22 this is a second order derivative of the Lagrange function with respect to the first variable that is d square L by d x 1 square this is d square L by d x 1 d x 2 and so on; this is g 11 is first order derivative of the function g 1of x with respect to the first variable and first constraint with respect to second variable, like this we formulate the determinant and equate it to 0; so d L by d x 1 is as we obtain earlier minus 2 x 1 minus lambda therefore, d square L by d x 1 square will be minus 2 and d square L by d x 1 d x 2 will be minus 2 this with respect to x 2, that is 0.

(Refer Slide Time: 22:06)



So, similarly d L by d x 2 you know, minus 2x 2 minus lambda. So, d square L by d x square will be minus 2 and d square L by d x 2 d x 1 as before will be 0, and you have the constraint g of X x 1 plus x 2 minus 4, g of X is equal to x 1 plus x 2 minus 4, this we differentiate with respect to each of the variable. So, d g by d x 1 is equal to 1 and d g by d x 2 will be equal to 1 also.

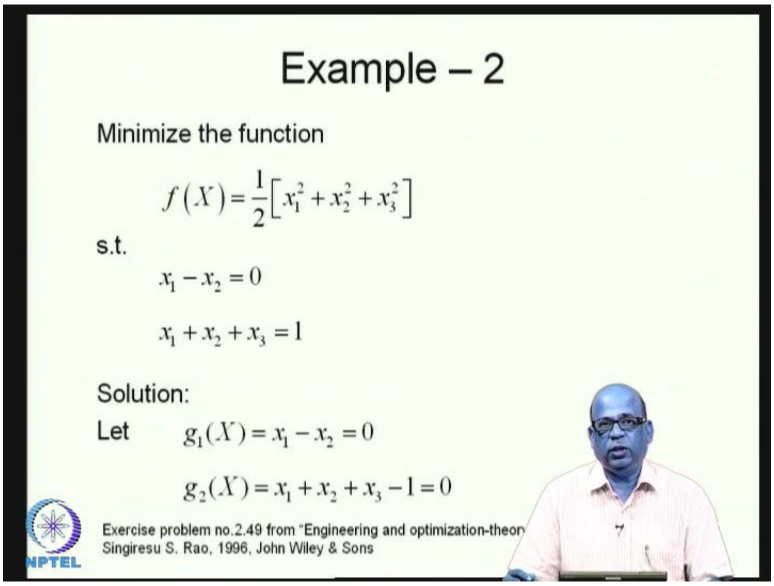
(Refer Slide Time: 22:45)



So, with this now, we formulate the constraint, we formulate the determinant I am sorry determinant D as L 11, look at this L 11 minus Z , L 12 g 11 and L 11 is minus 2; so, minus 2 minus Z 0 1 and 0 minus 2 minus Z 1 1 1 0. So, this is the last n by m terms which is 1 by 1 term here, which is 0. So, when we get the determinant and equate it to 0, you get 2 Z plus 4 is equal to 0, you know how to get the determinant and therefore, you get as solution Z is equal to minus 2; in this case, you got a polynomial of the order n minus m, which is 2 minus 1 which is 1; so, you get a polynomial of order 1 in Z and therefore, in this particular case, you get only one root which is Z is equal to minus 2, and Z is equal to minus 2 is the negative root and therefore, the point X star lambda star that you obtain corresponds to a maximum.

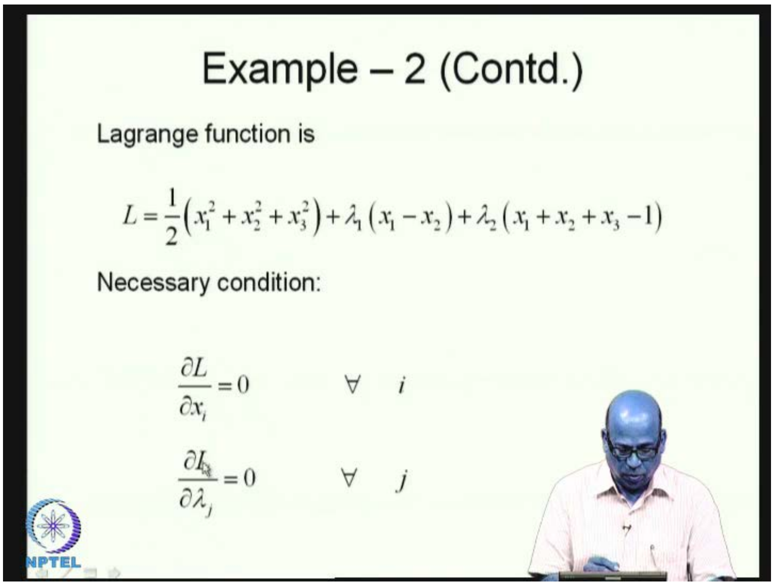
So as the root is negative the stationary point X (2, 2), which is x 1 is equal to 2 x 2 is equal to 2; what are these values? These are the values that we obtain by solving the necessary conditions. So, x 1 is equal to 2, x 2 is equal to 2. So, this is the stationary points. So, this stationary points x (2, 2), x is equal to (2, 2), which is x 1 is equal to 2, x 2 is equal to 2. In fact, corresponds to a maximum value, local maximum, and then we substitute the value (2, 2) in your original functions which is minus x 1 square minus x 2 square and so here we put x 1 equal to 2 and x 2 is equal to 2, and you obtain the value as 8. So, the maximum value of the function is 8, subject to constraint that is the constraint that we consider. So, this how we use the Lagrange multipliers to obtain the local minima, local maxima of a function of multiple variables subject to a set of equality constraint, Lagrange multipliers the way we have introduced is applicable only to equality constraints.

(Refer Slide Time: 25:16)



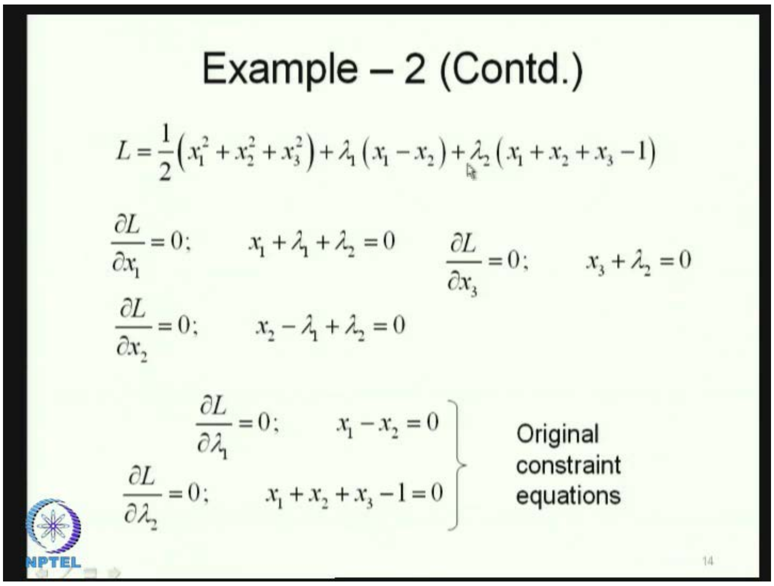
Let us look at one more examples; let say you take f of X, this is the exercise problem from the text book, not solved exercise problems, so I am solving that example now; f of X is equal to half of x 1 square plus x 2 square plus x 3 square, subject to two constraint x 1 minus x 2 is equal to 0, and x 1 plus x 2 plus x 3 is equal to 1. So, these are the two constraints; there are three variables and two constraints. So, first as I said, make the constraints of the form g j of X is equal to 0, so make the right hand side is equal to 0 for the constraint. So, x 1 minus x 2 is 0 is the first constraint, x1 plus x 2 plus x 3 minus 1 is the second constraint. So, this is the first step we just express the constraints in the form g j of X is equal to 0.

(Refer Slide Time: 26:13)



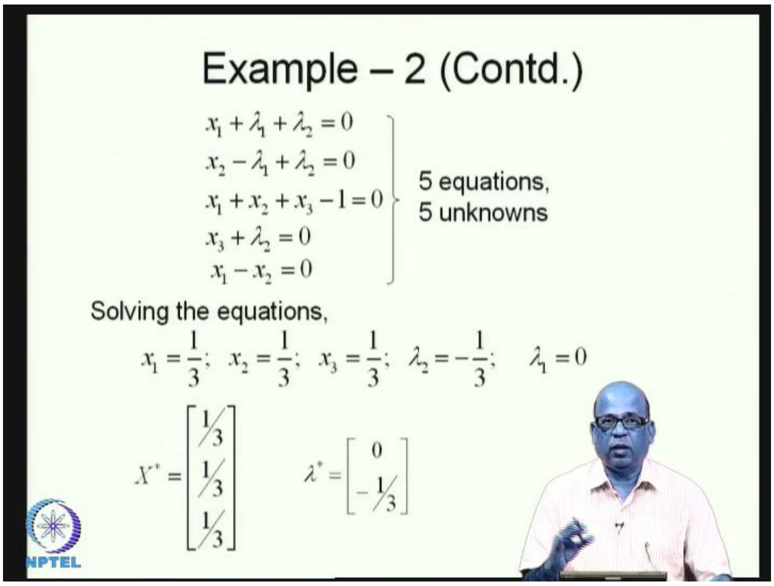
Then we formulate the Lagrange function, we will take plus lambdas now, instead of minus lambda we took last time. So, we formulate the Lagrange functions by introducing one lambda associate with each of the constraints. So, we have two constraints and we put lambda 1 lambda 2 as the two constraints. So, this is the original functions f of X plus lambda 1 in to g 1 of X plus lambda 2 in to g 2 of X; so, this how we formulate the Lagrange functions. Then we apply the necessary conditions by taking the first order derivatives of the Lagrange functions with respect to each of the arguments, the arguments are x 1, x 2, x 3 and lambda 1 and lambda 2 so we get five equations so, d L by d x i is equal to 0 it will get three equations and d L by d lambda j is equal to 0, which will give two equations associated with the two lambdas.

(Refer Slide Time: 27:17)



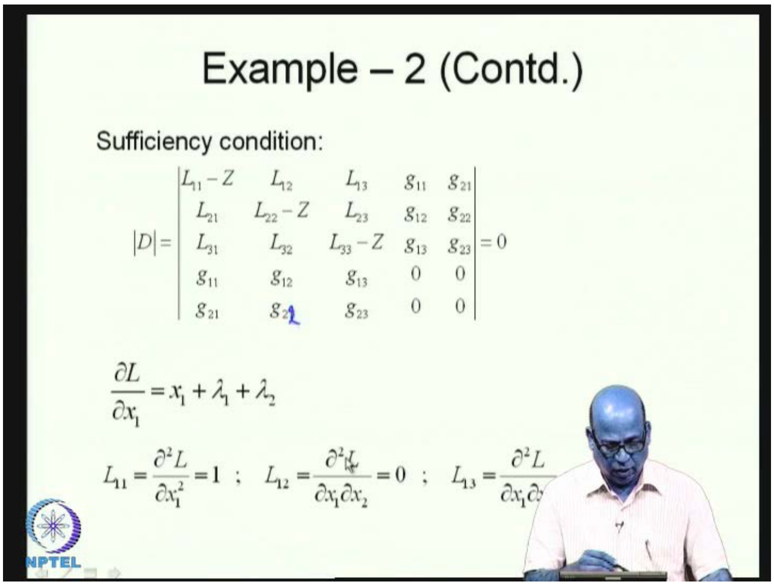
And this is the L, so, d L by d x 1 is equal to 0, gives the x1 here plus lambda 1 plus lambda 2 is equal to 0, I am differentiating with respect to x 1; similarly differentiate with respect to x 2, you get 2 x 2 here divided by 2 so x 2; minus lambda 1 plus lambda 2 equal to 0. So, this is the differential of L with respect to x 2; then we respect to x 3, 2 x 3 by 2, so you will get x 3, there is no x 3 term here, and plus lambda 2 is equal to 0. So, we get three equations here; 1, 2 and 3; then we also differentiate with respect to lambdas lambda 1 and lambda 2 to get the original constraints remember when we differentiate the Lagrange with respect to Lagrange multipliers lambda 1 and lambda 2, all were doing is reproduce the original constraints. So, d L by d lambda is equal to 0 will give me x 1 minus x 2 is equal to 0, and x 1 plus d L by d lambda 2 is equal to 0 will give me x 1 plus x 2 plus x 3 minus 1 equal to 0, these are in fact the original set of constraints; so you got five equations now and there are five unknowns x 1, x 2, x 3 and lambda1and lambda 2; there are five unknowns; we solve this equations simultaneously and get the values of lambda 1, lambda 2.

(Refer Slide Time: 28:51)



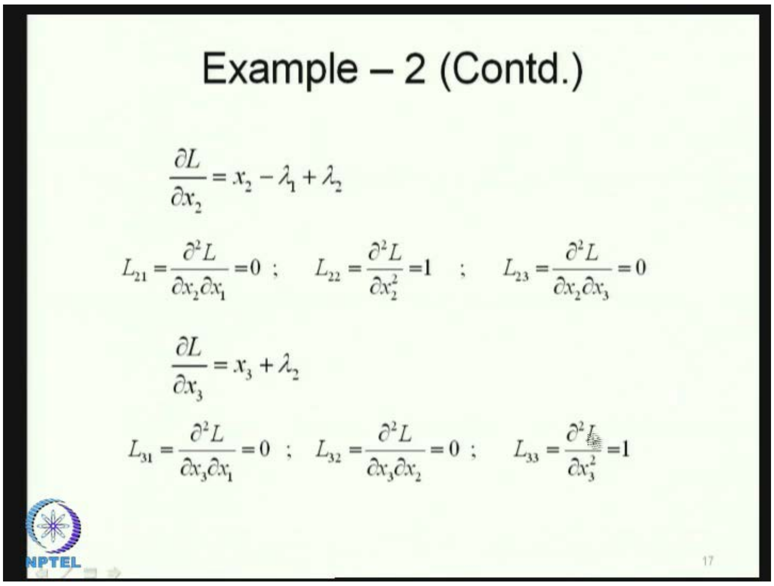
So, we reproduce this equations x 1 plus lambda 1 plus lambda 2 equal to 0; look at this, so these equations I am reproducing So, these are the five equations, in five unknowns, we solve these equations, I leave the solutions to you, the solutions comes to x 1 is equal to 1by 3, x 2 equal to 1 by 3, x 3 equal 1 by 3 and lambda 2 is equal to minus 1 by 3, lambda 1 equal to 0. So, this is the set of solutions; so we get X star as 1by 3 1 by 3 1 by 3 and lambda star is to be lambda 1 lambda 2 is 0 and minus 1 by 3. So, this how we get the stationary points where the necessary conditions are satisfied; now we need to examine whether the stationary points that we obtained is in fact, corresponds to a maximum or a minimum value for which we have to look at the sufficient conditions. So, in the sufficiency conditions, we deal with the second order derivatives of the Lagrange functions with respect to the variables x 1, x 2 etc x n and the first order derivatives of the constraints, constraint functions with respect to each of this variable.

(Refer Slide Time: 30:21)



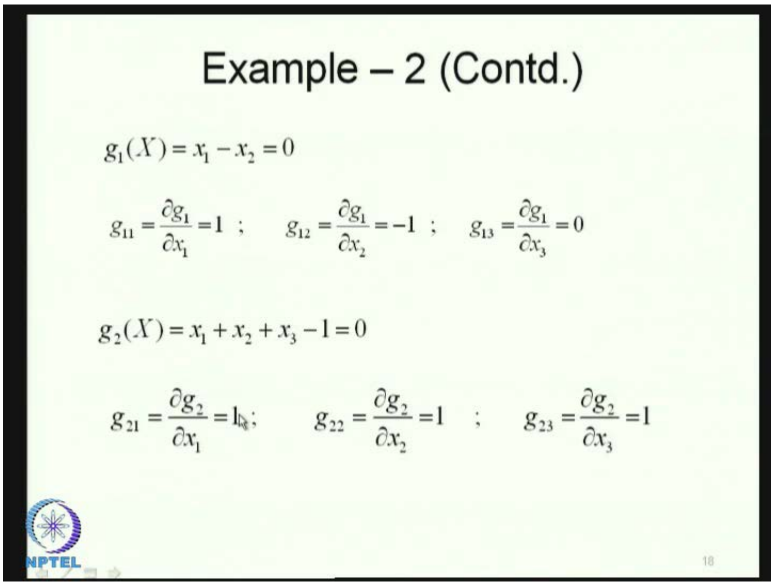
So, we formulate the determinant D as L 11 minus Z L 12 L 13 there are three variables x 1 x 2 and x 3; similarly, we have three constraints and two variables; I am sorry we have three variables and two constraints. So, this is the first constraint partial derivatives of the first constraint with respect to the first variable; second constraint with respect to first variable; first constraint with respect to first variable; second constraint with respect to second variable; first constraint with respect to third variable; second constraint with respect to third variable; and then it is transpose of its g 11 to g 23 and g 21 to g 22 and g 23; this is g 22 here and g 23; and this determinant we equate to 0, and then solve for it. So, we get the values now L 11 is d square L by d x 1 square d L by d x 1 we know x 1 plus lambda 1 plus lambda 2; we take the second order derivatives with respect to x 1. So, you get 1 here similarly d square L by d x 1 d x 2 becomes 0, then d square L by d x 1 d x 3 becomes 0.

(Refer Slide Time: 31:48)



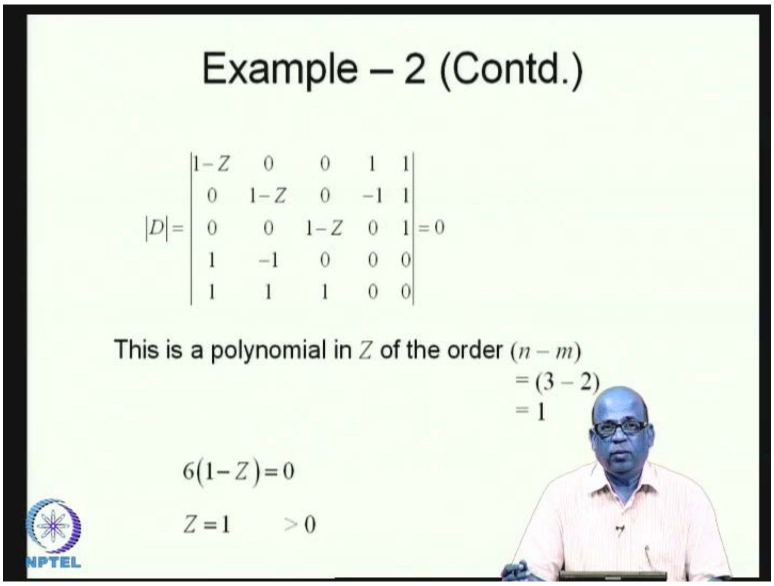
Similarly, d square L by d x 2 d x 1 is 0; similarly, d square L by d x 2 square is 1 d square L by d x 2 d x 3 is equal to 0; then we go to the third variable d L by d x 3 is x 3 plus lambda 2; then d square L by d x 3 d x 1 is 0; d square L by d x 3 d x 2 is 0; d square L by d x 3 square is equals to 1.

(Refer Slide Time: 32:22)



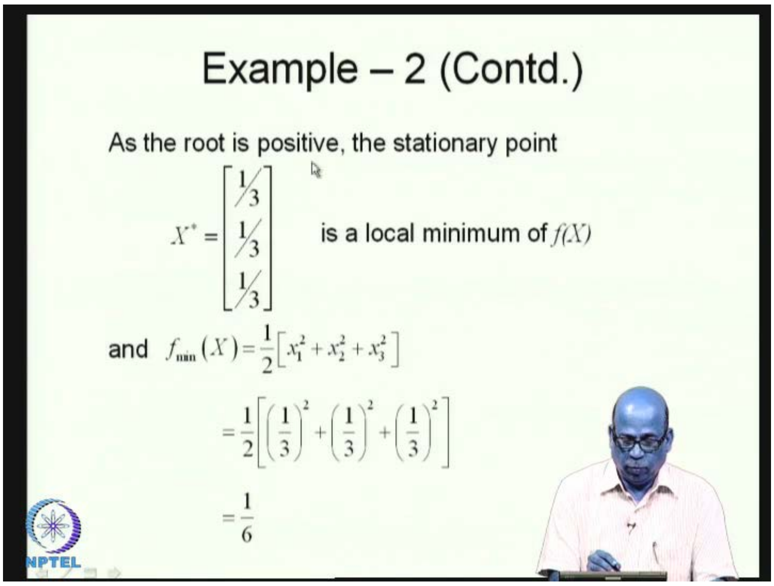
Similarly, we look out the constraints g1 of X is x 1 minus x 2 is 0 so g 11, which is the first constraint with respect to first variable d g 1 by d x 1 is equal to 1, first constraint with respect to second variable d g 1 by d x 2 is equal to minus 1 first constraint with respect to third variable d g 1 by d x 3, which is equal to 0; then the second constraint is x 1 plus x 2 plus x 3 minus 1 is equal to 0. So, we take the second constraint with respect to first variable, partial derivative of the second constraint with respect to second variable which is 1, second constraints with respect to second variable which is 1and second constraints with respect to third variable which is 1. So, we have obtained the second order derivatives of a Lagrange functions and first order derivative of the constraint functions.

(Refer Slide Time: 33:16)



So, let us formulate the determinant D, so determinant D is one minus Z; we are using this values now; this 1 0 0, so using this values now I put the constraints as the determinant 1 minus Z 0 0 1 1 0 1 minus Z 0 minus 1 1; I am using these values now; minus 1 is here, and 0 is here. So, we formulate the determinant D as written here; equate the determinant to 0; then you get a polynomial of the order of n minus m; n is the number of variables in this case it is 3, m is the number of constraints in this case it is 2. So, we can use any of the classical method to solving the determinant or you can use the matlab directly to solve this for Z, so you get Z is equal to 1 in this particular case. So, you have a polynomial of order n and then therefore, one root is possible here namely Z is equal to 1; this is a positive value so, we have only one root which is positive and therefore, what should it correspond to? Positive values use in the sufficiency conditions always corresponds to a minimum value and therefore the point X star lambda star in fact, corresponds to a relative minimum.

(Refer Slide Time: 34:45)

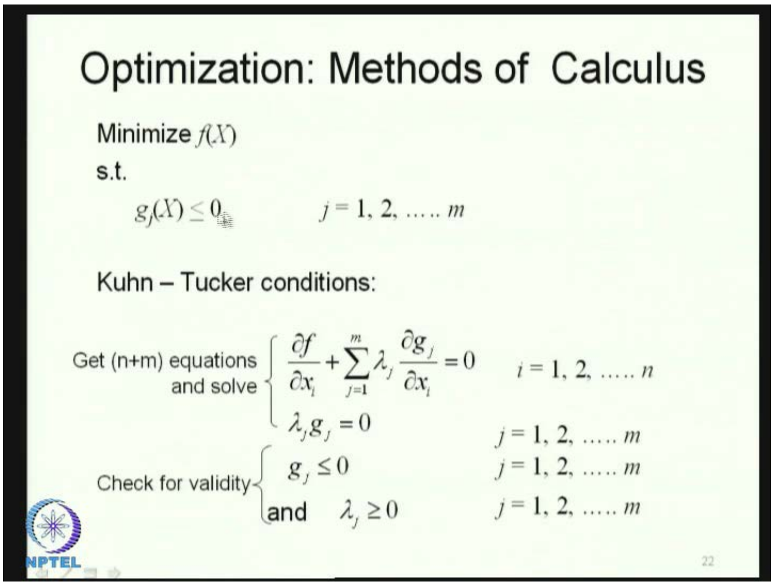


So, as the root is positive, the stationary points which is 1by 3 1by 3 1 by 3, which is our X star is our local minimum of f of X, and then we substitute these values in our original functions and get the f minimum as 1by 6. So, this is how we use the methods of Lagrange multipliers; when we are optimizing a functions of f of X of n number of variables x 1, x 2 etcetera x n, subject to equality constraints g j of X is equal to 0, j is equal to 1, 2 etcetera m, m numbers of equality constraints; and now in most of the engineering decision making problems, engineering optimizations problems we do not has a luxury of having equality constraints and therefore, we do not use Lagrange multiplier method directly; we will be in general dealing with a general problem statement a general problem of the type maximize or minimize f of X subject to g j of X less than are equal to 0, there are inequality constraints.

So, the most general formulation of the optimizations problems maximize or minimize f of X subject to g j of X less than or equal to 0 and in fact, g j of X can be greater than or equal to 0 also, depending on how the problem is formulated; for this general statement of the problem, there is no single algorithm which takes place of various specific cases that arise out of this general statement of problem for example, some of the constraints can be equality constraints, some of the constraints can be less than or equal to, some may be greater than or equal to, some of them may be non-linear constraints, objective functions can be non-linear, objective functions can be convex, the set of the constraints can be convex, can be concave and so on.

So, there are larger number of cases that arrives and there is no single algorithm or single method by which this general statement, general problem can be solved; however, all optimizations problem problems stated in that general form must satisfy certain necessary conditions for them to have an optimal value at a particular point - at a particular stationary points indeed all optimization problems that are formally stated must satisfy the necessary conditions and these are called as Kuhn-tucker conditions are abbreviated as K-T conditions and in fact, more generally they are called as K-T conditions after the mathematicians you have formulated these conditions any optimizations problem must in fact, satisfy this conditions KKT conditions or K-T conditions will check to conditions as K-T conditions as Kuhn-Tucker conditions.

(Refer Slide Time: 38:19)



So, let us see what are the Kuhn tucker conditions? Now the Kuhn tucker conditions can be stated for general optimizations problem. So, we state this problem for minimize f of X subject to g j of X less than or equal to 0. So, this is the conditions that…This is the optimization problem that we have stated; we are looking at the minimize, we are looking at the minimization functions and we stating the conditions in the form g j of X less than or equal to 0 for all j is equal to 1, 2 etc m; for this combination of the objective functions and the constraints, the Kuhn-tucker conditions can be stated as follows;

d f by d x i, which is the first partial derivative of original objective functions with respect to the i eth variable d f by d x i plus summation j is equal to 1 to m, you have to m number of constraints lambda j d g j by d x I, which means the j eth constraints with respect to i eth constraints; remember we are writing this conditions, for i is equal to 1, 2 etcetera n. So, for a given i, first partial functions of the objective derivatives f with respect to i eth variable x i plus summation j is equal to 1to m lambda j d g j that is the j eth constraints differential of the j eth constraints with respect to i eth variable by writing this constraints is equals to 0. So, this is the first set of conditions d f by d x i plus j is equal to 1 to m plus lambda j d g j by d x i is equal to 0, and this condition we write for all i 1, 2 etcetera n, so you get n number of such conditions; then the second set of conditions are simply lambda j g j is equals to 0. So, you have the original constraints g j of X less than or equal to 0, multiply with respect to multiply with lambda j which are in fact, a Lagrange multiplier so we will see that in linear programming, lambda j g j is equals to 0 and this we write it for all the conditions and all the constraints j is equal to 1, 2 etcetera m. So, you get m equations here; so, get n conditions here and m conditions here.

Then we have the original constraints g j of X less than or equal to 0 and this combination which is minimization of f of X and subject to g j of X less than or equal lambda must be greater than or equal to 0. So, you have j is equal to 1 to m. So, you get n equation here, m equations here, to get m conditions here n conditions here. So, you have n plus 3 m conditions, n plus 3 m equations; how many variables you have? You have n plus m variables namely, the first the original n decision variables namely x 1, x 2 etcetera x n. So, you are writing this for n different variables, one conditions associated with each of the variables and the lambda j lambda 1, lambda 2 etcetera lambda n. So, you have n plus m variables.

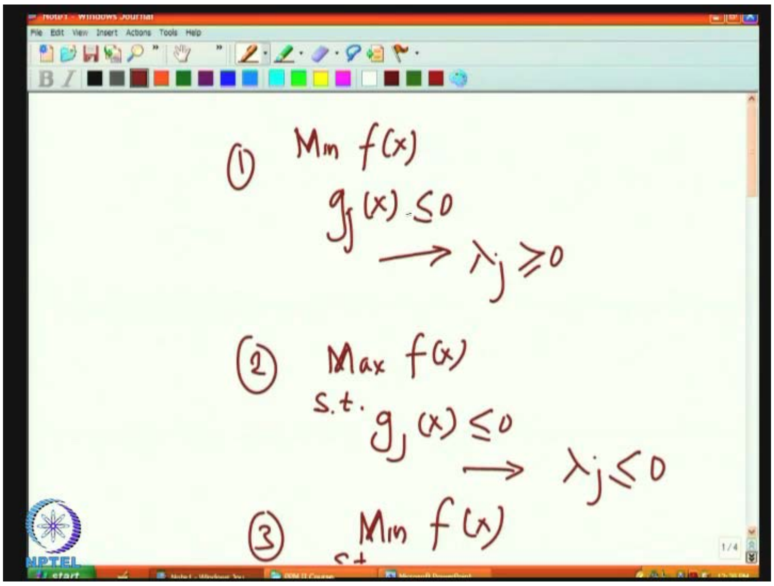
So, the way we apply these Kuhn-tucker conditions is, we use the first n plus m conditions, these sets of equations and solve for the n plus m variables namely, x 1 plus x 2 I am sorry x 1, x 2 etcetera x n and lambda 1, lambda 2 etcetera lambda m. So, we solve for the n plus m equations for n plus m variables using the n plus m equations then we use the other conditions namely, g j less than or equal to 0 and lambda j greater than or equal to 0 to verify which among this n plus m which on the various sets of n plus m values that you have to obtain in fact, to satisfy this conditions also. So, we solve using n plus m equations conditions, we solve them n plus m conditions variables and check other two conditions check whether there this solutions, we have to satisfy the other two conditions.

So, this is how the Kuhn-tucker conditions are stated and used for optimizations problem; remember again that we lambda j greater than or equal to 0 depends on this particular conditions minimize f of X subject g j of X; so when you have this particular type of combinations the lambda j will greater than or equal to 0 g j of X g j less than or equal to 0, just depends on its original conditions, original constraints if g j of X greater than or equal to 0, here to be x greater than or equal to 0.

The sign of the lambda j however will depend on what kind of combinations will have here, minimize subject to less than or equal to, maximize subject to less than or equal to, minimize subject to greater than or equal to, maximize subject to greater than or equal to, so depending of this the sign of lambda j will change; these conditions is simply reproductions of the original conditions and this will remain the same and these two conditions will always remain the same irrespective of the combinations, here we taking the first order derivative with respect to each of the n variable, and this term will simply respect to of the j eth constructs partial derivatives of the j eth constructs with respect to i eth variable multiplied by lambda j Lagrange multiplier of the particular constraint; remember lambda j is associate with the constraint g j.

So, these conditions remains the same irrespective of the combinations m this conditions is simply lambda j g j is equal to0, this conditions is same irrespective of combination here, this conditions is simply reproduction of the original set of constraints only this conditions which is lambda j whether it is greater than or equal to 0 or less than or equal to 0 depends on the particular combinations. So, I have stated that this here for minimize f of X to g j of X equal to 0 for that lambda j are greater than or equal to 0, I will now state this also for other combinations let us say that you have a combinations I write all the combinations.

(Refer Slide Time: 45:52)



So, if you have minimize f of X subject to g j of X less than or equal to 0, this is the problem that I stated; for this problem, you have lambda j greater than or equal to 0; if you change one of them, let say from minimizations I make it maximizations, but written the other conditions the same; then lambda j will reverse the sign; if we change both of them then lambda j it will be greater than or equal to 0. I repeat, here we talking about minimize f of X subject to g j of X less than or equal to 0; if I change one of them from minimize I make it maximization; however, I written g j of X still less than or equal to 0, then lambda j becomes negative; if I change one of them, lambda j will reverse the sign; if I change both of them that will make maximize f of X subject to g j of X is greater than or equal to 0, I change both of them then lambda j will still remain the same.

So, we will write them more formally, I change the first one that is the first form, in the second form I write maximize f of X subject to g j of X less than or equal to 0. So, the constraint I have not changed, but the objective functions I have changed from minimize I write to I written as maximize; in this case lambda j will less than or equal to 0; now the third one, I will retain the objectives functions as same, minimize f of X, but I will change the constraints that is subject to g of X instead of less than or equal to, I will make it greater than or equal to, because I change one of them, your lambda j in this case will be greater than or equal to 0; I am sorry with respect to this I am talking now minimize f of X g j of X is less than or equal to 0, lambda j is greater than or equal to 0; if we change either the objective functions or the nature of the constraints then lambda j will reverse the sign then therefore, in this case the lambda j will be less than or equal to

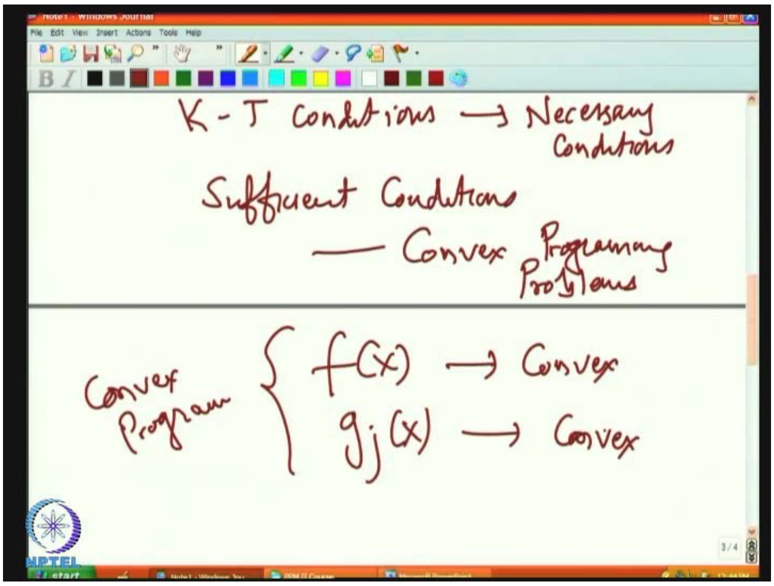
0, if we change both of them namely, I make the minimization functions in to maximizations, but the objective functions f of X and I also change the constraints from original less than or equal to make it less than or equal to g j of X, I make it greater than or equal to, because the original sign lambda j is remain namely lambda j greater than or equal to 0.

So, this will indicate the Kuhn-tucker conditions for all possible combinations objective functions minimizations constraints less than or equal to 0, then lambda j are greater than or equal to 0; objective functions maximizations constraints less than or equal to 0 to lambda j is will be negative; objective functions minimizations constraints greater than or equal to 0 then lambda j must be negative functions; objective functions maximizations greater than or equal to 0, then lambda j must be greater than or equal to 0. So, just remember one of them that is this particular form you remember minimize f of x subject to g j of x is less than or equal to 0, then lambda j must be greater than or equal to 0; remember one of them, and then if you change one of these that is either the objective functions or nature of this constraints; if you change one this then lambda j will reverse the sign; if you change both of them lambda j will not reverse the sign will be same as greater than or equal to 0.

So, this how the Kuhn-tucker conditions will followed relatively then of course,, the remaining conditions will be same as I just indicate that the remaining conditions will still remain unaltered that means these two conditions will remain the same, this conditions is depends on type of constraints that you have for example, the j eth conditions can be greater than or equal to 0, then we put j eth constraints as greater than or equal to 0 and sign of this will depends on the nature of the objective functions and particular type of constraints whether it has less than or equal to 0 or greater than or equal to 0. So, this how we formulate the Kuhn-tucker condition conditions for any optimizations problem, and any even optimizations problem can be expressed by minimize f of X subject to g j of X less than or equal to 0 or variance of that, that is maximize f of X subject to g j of X less than or equal to 0 maximization f of X subject to g j of X less than or equal to 0 and so on. So for any given general optimizations problem, it is possible for as to write down the Kuhn-tucker conditions and the Kuhn tucker conditions are the necessary conditions for the functions f of X to have a minimum f of X values subject to this conditions this constraints g j of X less than or equal to 0.

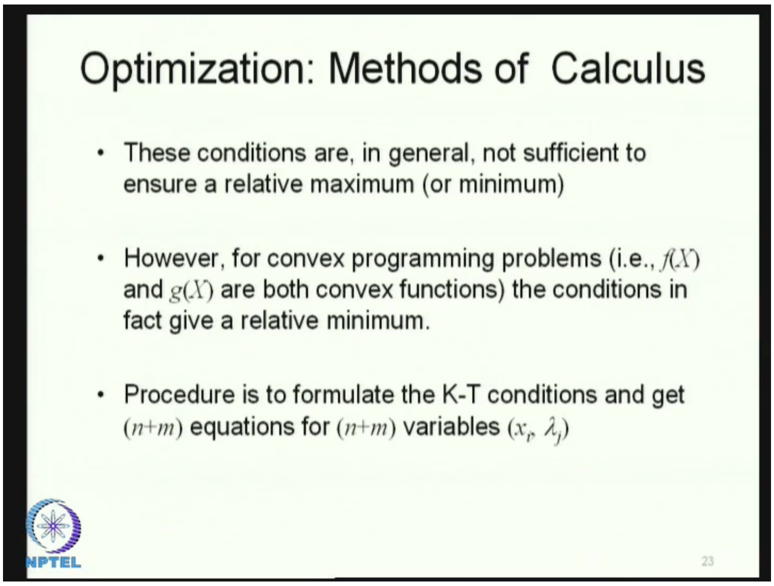
There are certain important task remember for the Kuhn-tucker conditions one is that any optimizations problem must satisfies the Kuhn-tucker conditions the second important thing is the Kuhn-tucker conditions are necessary conditions; there are not sufficient conditions that is conditions that we just had look at, we just are necessary conditions for a local optimum there not sufficient conditions, in general for any general problems there not sufficient conditions; however, there are certain types of problems called as convex programming problems where these conditions that also be sufficient conditions.

(Refer Slide Time: 52:45)



So, K-T conditions are necessary conditions and any optimizations problem must satisfy the necessary conditions; they are also sufficient conditions for what are called as convex programming problems; that is you are looking for the minimization of a function that is f of X is the convex functions; your objective function is a convex functions, and all your constraint functions g j of X, they are all convex functions, and you are looking for minimization of a functions such problems are called as convex programming problems functions. So, these are this is the specific place of a general optimizations problems that you guide with. So, if you have f of X as convex and d j of X also convex, all the constraints are also convex functions, then the Kuhn-tucker functions that will stated are both necessary as well as sufficient conditions. So, this is what you must remember.

(Refer Slide Time: 54:17)



So, Kuhn - tucker conditions are both necessary as well as sufficient, if we are dealing with convex programming problems; and how do we use this, we state the Kuhn-tucker conditions depending upon the nature of the optimal functions, whether it is minimizations or maximizations of and nature of the constraints g j of X less than or equal to 0 or greater than or equal to 0; we use the first n plus m conditions to solve for the n plus m variables, and then check the other two conditions namely g j of X less than or equal to 0 and lambda j d j is equal to 0, to verify whether I am sorry d j of X less than or equal to 0 and lambda j greater than or equal to 0, to verify whether the n plus m variable to obtain in fact, to satisfy that sets of conditions. So, we use this to solve for n plus m variables, and then verify whether the solutions that you have obtain in fact, satisfy these conditions; you may have, you may in general have multiple number of solutions and some of those solutions may satisfy all the conditions, some of the solutions may not satisfy some conditions and therefore, we use only one conditions among the among the solutions that you obtained; only those solutions that you obtain for all the conditions are in fact, chosen or they in fact, satisfy these necessary conditions and they lead to local minimum or local maximum, again these are necessary conditions and not the sufficiency conditions.

So, we will examine through an example, how this Kuhn - tucker conditions are applied we will do that in next lecture. So, essentially in today lecture, we started of with equality constraints there is an optimizations of functions, subject to equality constraints; and in the previous lecture we had dealt with the method of direct solutions; in today lecture we introduces the Lagrange multiplier method where associated with each constraints, we introduces a variable called as Lagrange multipliers. So, you have m number of constraints for have m number of Lagrange multipliers; we then formulate a Lagrange functions and formulate the necessary conditions for Lagrange functions, the basic premises and the principal here is that the Lagrange functions is so formulated that if you formulate the lag range functions itself is optimize.

So, associate with each of the formulate Lagrange functions the necessary conditions obtained by differentiate the Lagrange functions with respect to each of the arguments that is x 1, x 2, etcetera x n and lambda 1, lambda 2, etcetera lambda m; then we formulate the sufficiency conditions we wrote down the sufficiency conditions we did not formulate we just using the sufficient conditions through the determinant D; I just explain to formulate the determinant D based on the second order derivatives of the Lagrange functions with respect to each of the original variables with x 1, x 2 etcetera x n; and the first order derivative of each of its constraints d g j by d x i. So, the j eth constraints derivatives with respect to the i eth variable

And then using these we formulate the determinant D, equate it to 0; it gives the polynomial of order n minus m in the variable Z, if all the variables if all the roots of this polynomials are positive then the stationary points corresponds to minimum; if all of them are negative if corresponds to maximum, then we went on to formulate the state Kuhn-tucker conditions for any given optimizations problems the Kuhn-tucker conditions are necessary conditions then no sufficient conditions any optimizations problem must satisfy the Kuhn-tucker conditions for any given general optimizations problems the Kuhn-tucker conditions are necessary conditions and not sufficient conditions any optimizations problem must satisfies the Kuhn-tucker conditions; in cases where the objective functions f of X is convex, as well as constraints are all convex functions then Kuhn-tucker conditions are also sufficient conditions. So, we continue the discussion in the next class by starting of with the example problem, thank you for your attentions.