Water Resources Systems
Modeling Techniques and Analysis
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Lecture No. # 03
Optimization: Functions of a single variable

Good morning and welcome to this lecture number 3, of the course, Water Resources Systems - Modeling Techniques and Analysis. In the previous lecture, we essentially studied the definition of a system and then looked at various types of systems.

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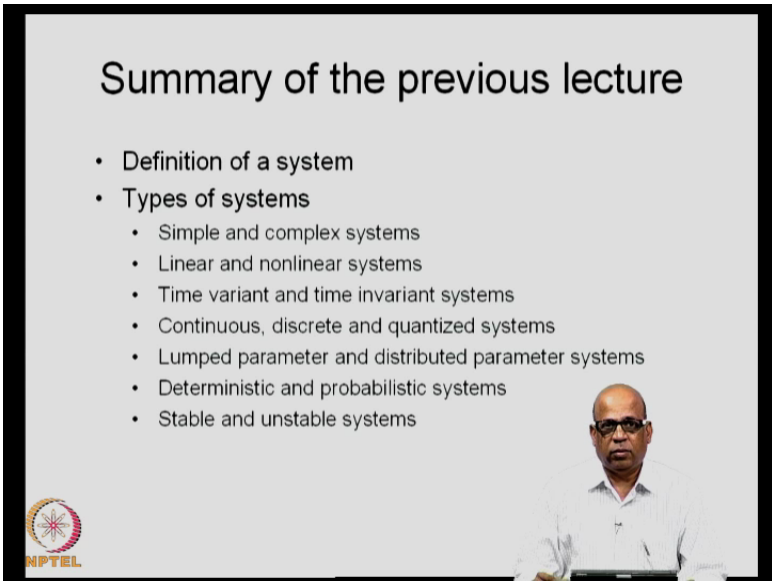

Typically, we saw what is meant by a simple system, what is a complex system, then linear and non-linear systems, where principle of proportionality is valid or not, then time variant and invariant systems. In the time invariant systems essentially the properties will remain the same across time and in the time variant systems several of the parameters will be changing with respect to the time. Then,, we also saw continuous, discrete and quantized systems. In many of the water resources system, although the processes are continuous we may approximate them to be discrete and therefore, treat the systems as discrete systems and in the in the quantized systems, we may have time step during weight is sudden changed in the system occurs and that is what we call as quantized systems, then we have lumped systems and distributed parameter systems.

In the lumped systems, we lump the various parameters in the sense that the spatial variation of the parameters and the processes are not considered in the lumped systems. Whereas, in the distributed systems we do consider the spatial variation of the parameters and the processes we have the deterministic and probabilistic systems. In the deterministic systems the same input always provides the same output, where as in the probabilistic systems there is the probability associated with an output for a given input, then they have the stable and unstable system. In the stable systems if the input is bounded then the output is also bounded, where as in the unstable system the systems become unstable if the input becomes input is you know you know it is bounded, but the output may not be necessarily bounded. So, the output of the system in that sense becomes unstable. So, these are types of system that we studied in the previous lecture and towards the end of the lecture, I introduced a simple problem which we wanted to convert into a mathematical problem. The physical problem statement we wanted to convert into a mathematical problem. We will start with that now and that is quickly recapsulate what the problem was.

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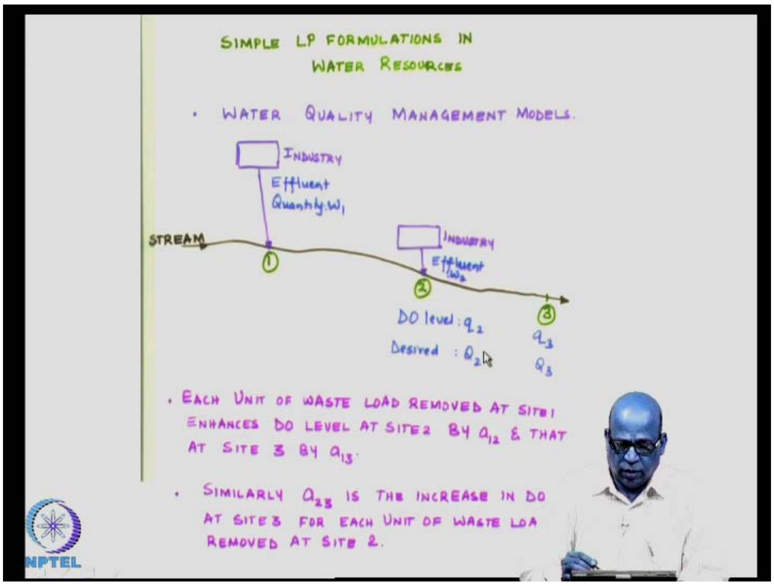

It is a simple problem of water quality where we are interested in maintaining water quality at these 2 check points, point number 2 and point number 3. This is the stream which receives a effluence from 2 industries, industry 1 here and industry 2 here with the effluent quantity of w1 and effluent quantity of w2 at location 2. If you do not provide any treatment at this location, the DO level was q2 at location number 2 and q3 at location number 3. By providing a treatment appropriate treatment at the industry number 1 and at the industry number 2 we would like to maintain the dissolved oxygen level at a desired level of Q2 at location number 2 and at Q3 at location number 3. Now the dissolved location at any oxygen at any location will depend on the amount of stream flow that is coming in which is the flow discharge flow quantity etcetera and it also depends on a large number of climatic factors for example, the temperature, the relative humidity, the (( )) coefficient and so on. So, there are large number of processes that govern the transport of the load BOD into its translation into the dissolved oxygen at a given point. Let us assume that all of these processes have been taken into account and we have 1 parameter which gives the amount of DO dissolved oxygen that can be enhanced at a particular location by providing a unit treatment at the upstream location.

Let us say that we we treat 1 unit of this effluent quantity w1, then the dissolved oxygen at these location will be enhanced by certain amount. Now that is what we will define as the parameters a12 and a13 what I meant by that is each unit of waste load removed at site 1 here enhances the DO level at site 2 at this location by a12, which means 1 unit of waste load you remove you will get an enhancement of a12 at location 2 and a13 at location 3. Similarly, at 2 you remove 1 unit then it enhances the DO at 3 by a23; obviously, it does not affect the location here and nor it does not affect the location upstream of that. They will only affect the treatment, that only affect the dissolved oxygen, downstream of that particular location. So, a23 is the increasing DO level at site 3 for each unit of waste load removed at site 2. So, this is the physical problem and then we have to now convert using this information, we have to now convert it into a mathematical problem.

First let us take to look at what is that we want to achieve. What we want to achieve in this problem is that we would like to make the treatment levels at industry 1 and industry 2, the sum of the treatment levels or sum of the cost of the treatment levels a minimum and achieve the desired DO level q2 at 2 and q3 at 3. So, from q2 we want to announce it to at least q2 at location 2 and from small q3 we want to achieve a desire level of q3 at site 3, but providing the treatments at industry 1 and industry 2, but by providing a minimum treatment; minimum sum of the treatment or minimum cost of the total treatment. So, this is the problem. So, let us look at the conditions that we want to put first. what is the condition? That we want to provide the treatment such that the dissolved oxygen level q2 is at end at this location. So, how do we write this?

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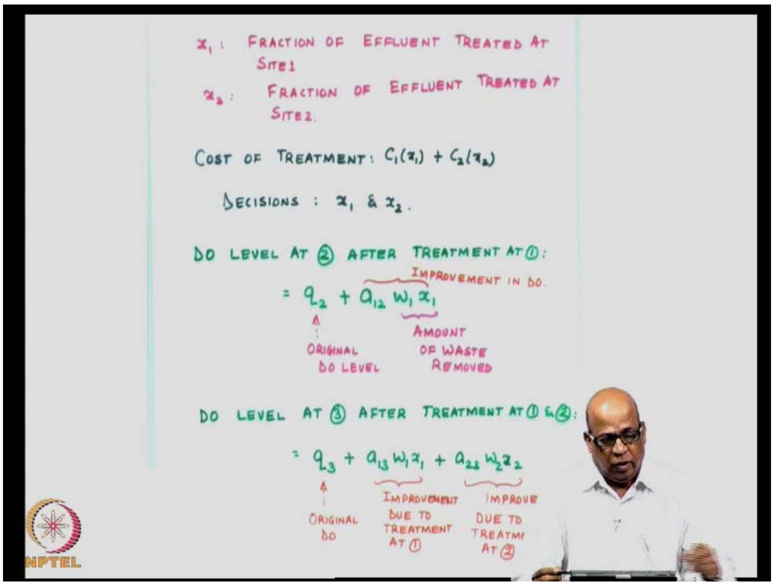

Let us say that x1 is the fraction of effluent treated at site 1 which means w1 was the load that was coming at site 1, x1 fraction has been removed. So, what is the total amount removed w1 into x1. The total amount that is removed, the total amount of waste that is removed is w1 into x1, which means if you remove 10 percent it will be 10 percent of w1 and so on. Similarly x2 is the fraction of effluent treated at site 2. we will take the cost as c1(x1) and c2(x2); that means, if I want to remove x1 fraction at w1 at the industry 1 the cost associated with that is c1(x1) and if you want to remove x2 fraction at site 2 the cost associated with that is c2(x2) and we want to make the distance on x1 and x2 such that the total cost is a minimum, then what is the condition that we want to achieve? We just look at what are the definition of a12 for every unit amount of effluent that is removed from site number 1 you will get an enhancement in the DO of a12 at site number 2 that is the definition of a12. So, you had a load of w1 and you removed a x1 fraction of that.

So, the total amount of waste removed is w1 into x1. So, this is the total amount of waste removed at site number 1. For every unit of waste removed at site number 1 you will get an enhancement of a12 into that amount at site number 2. So, the total improvement in DO dissolved oxygen at site number 2 is a12 w1 into x1. This is the improvement and what was the original DO? Original DO was q2 small q2. So, q2 plus a12 into w1 into x1 will be the total dissolved oxygen at site number 2, after a treatment of x1 has been given at site number 1 and this dissolved oxygen at site number 2 must be atleast equal to what we desired at that point it is Q2, we will write it presently. Similarly we look at site number 3; dissolved oxygen at site number 3 what is the existing before the treatment is q3 and what we desire is Q3.

So, starting with q3 you have provided a treatment of x1 at site number 1, which means the total amount of wastage removed is w1 into x1. For every unit of waste that you remove at w1 you will get an enhancement of a13 at site number 3 and therefore, the total improvement of DO at site number 3 corresponding to the treatment x1 at site number 1 will be a13 into w1 into x1. So, a13 into w1 into x1 is the improvement at site number 3 corresponding to a treatment at site number 1. Similarly, a23 into w2 x2 is the improvement in the dissolved oxygen at site number 3 for a treatment of x2 at site number 2 and the original DO was q3. So, q3 plus the improvement due to treatment at 1 plus the improvement due to treatment at 2, this will be the total dissolved oxygen after the treatment. So, we got the total dissolved oxygen after the treatment at site number 1 and site number 2 where the treatments are given and we got the dissolved oxygen level at site number 2 and site number 3 corresponding to these treatments. Now this total dissolved oxygen that you get after the treatment must be atleast equal to the desired dissolved oxygen which is Q2.

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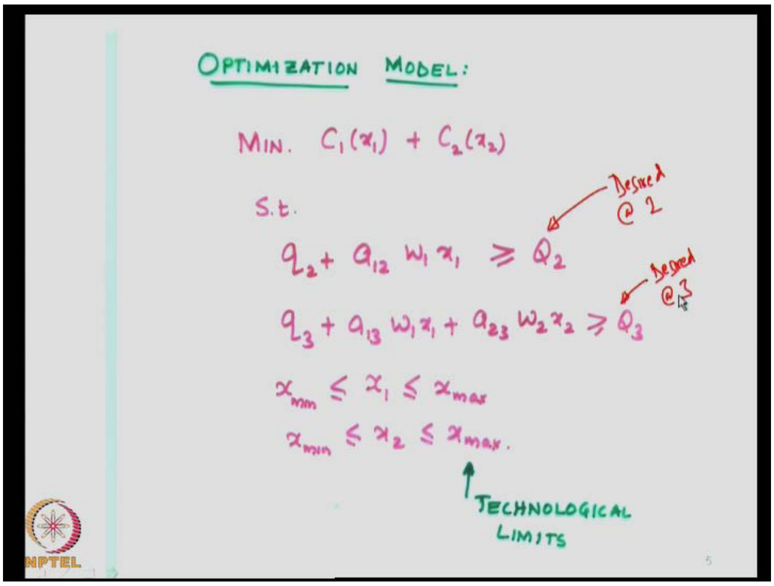

So, that is what we write, this is the dissolved oxygen at site number 2 after treatment x1 has been provided at site number 1. So, this is q2 plus a12 w1 x1 and this should be at least equal to Q2. Now Q2 is the desired DO level. So this is the desired level at site number 2. Similarly, this is desired at site number 3. So, these are the 2 major constraints that will be talking about. These constraints arise from our management condition that we want to achieve a minimum desired level of Q2 here and Q3 at site number 3 then there may be some technological limits. For example, you may not be able to say that you treat 100 percent of the effluent it may not be technologically feasible. Therefore, you may want to put an upper limit of the treatment levels. So, x1 should be less than or equal to an upper limit, x2 should be less than or equal to an upper limit. These can be different for the 2 sites then we may want to put a minimum level, typically we put a 35 percent of a treatment level has a minimum treatment level that is mandatory.

So, depending on the problems you may fix x min and x max. So, you are making decision on x1 and x2 such that they vary in this range and such that there cost is the minimum, total cost is the minimum and such that they satisfy these 2 conditions. So, this is how the original physically stated problem has been converted into a mathematical problem. That are a few things that you must note here. What are the actual decisions that you are making? The decisions that will be interested in or what is the optimal value of x1 and what is the optimal value of x2. Look at all the other parameters are they known. q2 is known which is the original dissolved oxygen level. a12 which is the parameter which indicates the improvement in the water quality at site number 2 corresponding to a unit treatment at site number 1 is known or can be determine and therefore, for as for as this problem is consider, we will take it as a constant. Similarly a13 is the constant, a23 is the constant. w1 is the data what is the amount of effluent that is coming in this is known, w2 is the data what is the amount of water water effluent that is a waste load that is coming in that is known, c1 which is the cost coefficient that is known, c2 cost coefficient that is known. So, the only (( )) and similarly the desired levels Q2 and Q3 are also known and these technological limits we are fixing, similarly the minimum levels we are fixing. So, all of this other variables in this problem are all known. We are making decision on x1 and x2. So, x1 and x2 are only unknown in this particular problem.

So, this is how we converted a physically stated problem into a mathematical statement and this is an optimization problem and we will see what is an optimization problem, what is the simulation problem etcetera presented in today's class, but essentially what you must understand is that a physically stated problem has been converted into a mathematical statement of a problem. These are this is called as the mathematical formulation of the optimization problem and so, in most of the situations in in all most all the situations, we understand the problem physically and states the problem in linguistic terms first and then convert them convert these problems into mathematical statements such as this and this becomes the mathematical formulation of the problem starting with such a mathematical formulation we will provide a data for example, what are the values of c1,what are the values of q2, Q2, a13, a12, a23 etcetera we provide the data and there we should be able to solve for what is the optimal value of x1 and x2. How to solve for is the different question we will study during the course of this particular subject.

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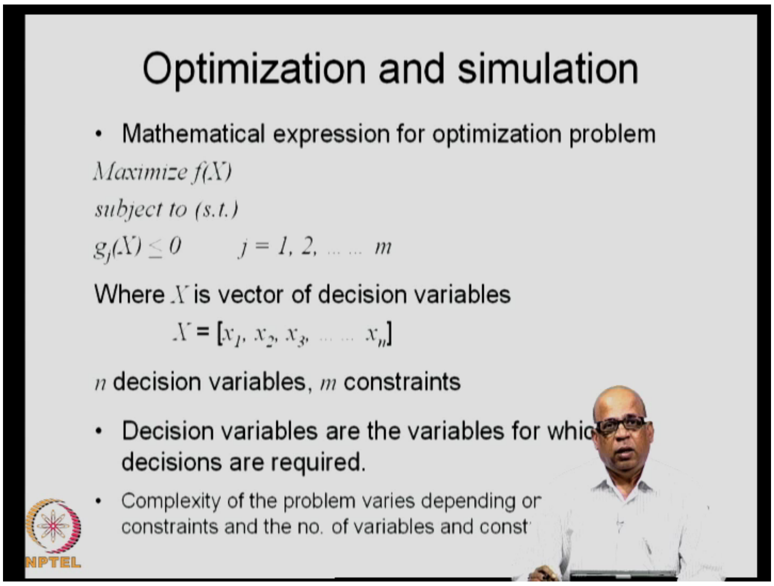

So, this is just a broad idea of problem formulations. So, as long as you understand the physical statements as the problem, you must be able to convert that into a mathematical statement. we will just go through some broad basics of what is optimization and what is simulation in optimization, we were looking for optimal values of a particular function by optimal I mean it may be a minimum value or a maximum value like say that you have to looking at in the previous problem that I just discussed you may be looking at minimization of the cost or in certain situations, you may be looking at maximization of the flood control benefit or maximization of the hydro power or minimization of the water pollution in some sites d o (( )) let say d o (( )) you would like to be minimize has have to location and soon. So, that is always a physical quantity which you may want to minimize or maximize. So, you have a function which needs to be optimized, then you may have certain conditions that you want to satisfy while we are optimizing them, you want to optimizing the function that we just define you may want to satisfy certain conditions. These conditions will define what are called as the constraints.

So, you have a function which has to be optimize and you have a set of conditions that need to be satisfied that are called as a constraints. So, in the optimization let us have a look at it we may say maximize f of X now I say maximize you can also talk about minimization problem. So, in this example I am saying it is maximization problem. So, maximize f of X subject to g j of X less than or equal to 0. Now this X is the vector of decision variables. So, I may write X is equal to a vector of x1, x2, x3 etcetera xn, n decision variables and you may have m constraints. So, g j of X less than or equal to 0, j is equal to 1,2 etcetera m.

So, these are m number of constraints in the previous example. For example, you just look at it this is your f of X. X consist of 2 variables x1 and x2, these are the decision variables and these are the 2 constraints or if into this also is constraints. These are the 4 constraints 1, 2, 3,4. I will say if I include with the purpose because the many of the situations the bonds need not be included included as the explicit constrains, but; however, we will take the math constraints for that I am being. So, 1,2,3,4 4 constraints,2 variables and this is the objective function. So, this is what we are relating, that is the objective function, there are decision variables, there are n n number of constrains.

So, this is how typically an optimization problem is stated, you have an objective function which has to be either minimized or maximized, there are a set of decision variables in our case that decision variables where x1 and x2 the treatment at industry number 1 and the treatment at industry number 2, then you have a set of constrains, n number of constrains. Now the decision variables are the variables for which decisions are required very simple; that means how much is the treatment at industry number 1, how much is the treatment at industry number 2, these were the 2 decisions. In other situation, if you are talking about reservoir operation problem, you may be talking about the decision has what is the amount of release during the month June, what is the amount of release during the month July and so on. In water allocation problems you may talk about allocation to hydro power, allocation to irrigation, allocation to flood control as 3 different decision variables. So, the decision variables are the variables on which we would like to make decisions.

Now in this simple statement of problem, this is the general statement of the problem, as you can see in the complexity of the problem increases or changes with respect to the nature of a function itself it may be highly non-linear of a function or it may be a simple linear function and so on. So, the complexity of the problem varies depends on the depending on the nature of the function that you may also have a set of quite complex constraints, there may be highly non-linear in nature, there may be interactive in nature and so on. So, that also defines the complexity of the problem and also the number of variables and the number of constraints and so on all of this will determine the size and also the complexity of the problem. So, this is what we doing optimization. There is the function which we call it as objective function we want to minimize or maximize, there are set of conditions and then there are a set of decision variables on which we want to like to make a decision.

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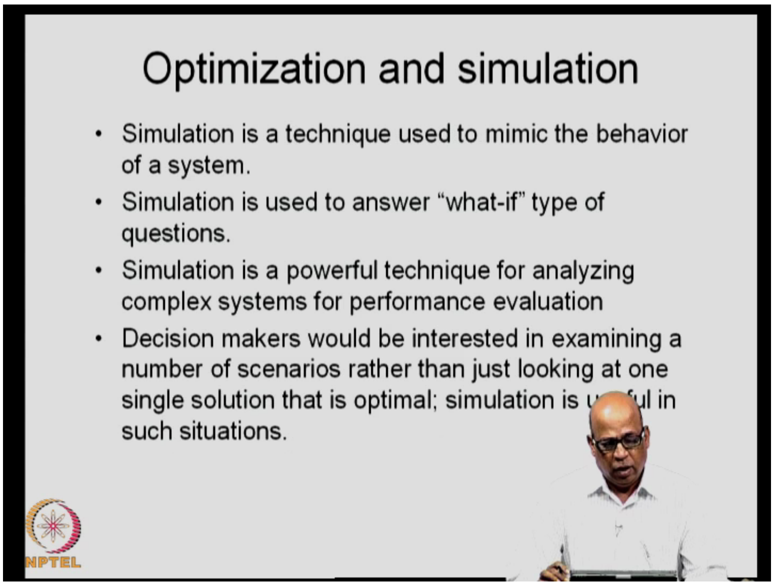

Simulation has against optimization will not provide you with one solution, one optimal solution. Generally we use simulation when we want to look at several alternatives. Let us say that you would have a large river basin and then let me let me draw that in explain what do what do you mean by simulation.

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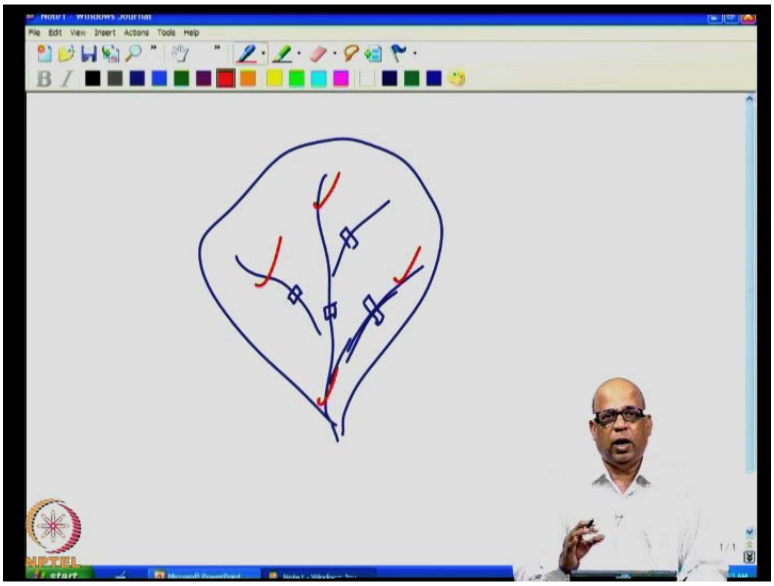

Let us say that you have river basin like this and then you would like to develop this river basin .The question is that you may be interested in answering is, if I put a reservoir here and a reservoir here and a reservoir here and a reservoir here is the benefit that are likely to get the maximum or should I be putting the reservoir at this point, at some other point here, at some other point here, some other point here etcetera. So, among virtually infinite number of possibilities you would like to explore which combination is better in some sense. So, you are actually screening alternative, there are large number of alternatives and you want to screen which among them is the best. For example, do I need to have 20 reservoirs here or only 15 is alright or 13 is alright, 12 is alright and if 12 is alright at which which location you should have or I may not want the reservoirs in big dams, through big dams I may just want to use the water through lift irrigation. So, some lift irrigation somewhere, some ground water somewhere, some reservoir somewhere and so on.

So, there are large number of alternatives that you would like to explore. In such situations it is easier handle and feasible to go with simulation rather than optimization. So, the simulation essentially helps us with checking alternatives and in simulation we mimic the behavior of a system. For example, we put those dams and then run the flow through those things and then see how the system actually behaves in practice. So, the physical behavior of the system is mimicked or imitated mathematically in simulation .we also use simulation to answer “what- if” type of questions. For example, we do not actually put the dam we say, what if you put the reservoir here, what happens to the downstream stream flows or what happens to hydro power generation, what how much of irrigation can be met and so on. So, there was the simulation is typically used for answering “what-if” type of questions and as I just said large water resource system which of the extremely complex in terms of their interactions among various components and so on. So, when you have large complex systems, simulation is a handy tool to providing how well the system performs. So, in terms of the performance evaluation of the system, simulation is a handy tool and as I just said that number of scenarios exist and therefore, you want to do the screening of alternatives screening of scenarios. So, rather than looking at just one optimal solution you may be interest in a range of solutions from which we would like to pick up. In such situations, simulation becomes very useful.

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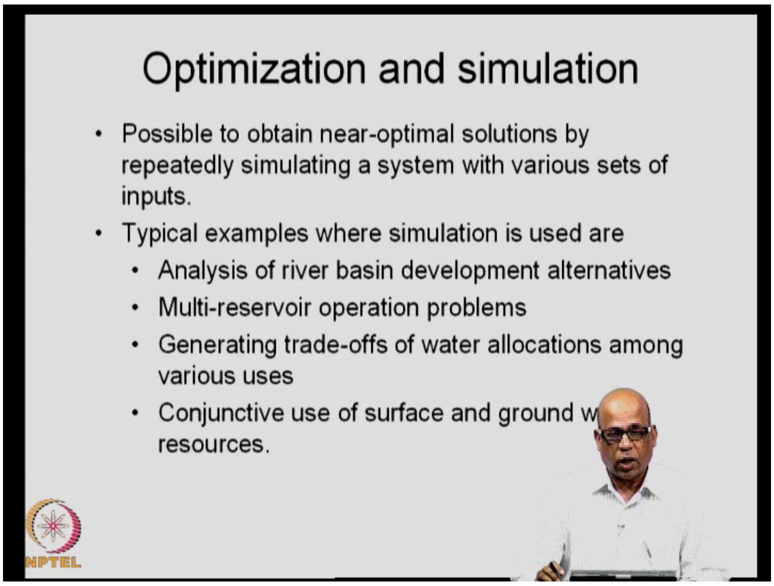

Through simulation, let us say that you run the simulation many times with different types of inputs and So on. So, while you may not be formally doing a optimization, but it is possible for you to get near-optimal solutions through simulation and in water resource systems, typically we use simulation in several situations. For example, analysis of river basin development alternatives, just now I I give the example you may want to put lift irrigation schemes, you may want to put reservoirs, you may want to put ground water utilization points and you may want to have a conjunctives at certain points, you may have recycling of water at certain points and so on. So, the large river basins you want to develop with several alternatives. In such situations you may want to use simulation. Then large Multi-reservoir optimization Multi-reservoir operation problems although they can be formally optimized because the systems are large and systems are complex, it is better to go with simulation first to get an insight into the behavior of the system itself first and then go with the formal optimization. Then generation of alternatives are trade-offs of water allocations; that means, between hydro power and flood control, between hydro power and irrigation, between one city with another city and so on. So, water allocations among various users you want to provide trade- offs, if I put more water here what happens to how does the other one suffer and so on.

So, you you want to develop trade-offs between among water allocation water users. Then conjunctive use of surface and ground water, how much of surface water you would like to use, how much of ground water you would like to use across time periods in a year, across time periods in a decades and so on, such that the system become sustainable, such questions become best answer through simulation. So, we just saw the difference between optimization and simulation, typically we use optimization and simulation both in most water resource problems and there are situations where we use both optimization as well as simulation in a in a sort of interactive manner we simulate the system and then run through an optimization get an optimal value, look at the performance of the system, rerun the simulation and so on. So, simulation optimization combination methods are also available.

So, now onwards we will for the, for some time will focus on optimization. It is a formal method of obtaining optimal values of a function. So, we start with the simplest methods and then proceed on to more complex problems of optimization. What do you mean by optimization? Let us say that you have a function f of x, we will start with the simplest function of function of a single variable f of x. Now f of x we want to get either a maximum or a minima that is a optimization problem, you may have a range over which x varies or over which f of x is defined or over which we are interested in. So, while the function may be defined in the entire rate, your interest may be only between certain values of x, x1 and x2. Within this rate you would like to check where the minimum of the function occurs, where the maximum of the function occurs and this is the optimization problem. There are classical ways of getting this optimal these optimal values and we will start with that that is optimization with methods of calculus.

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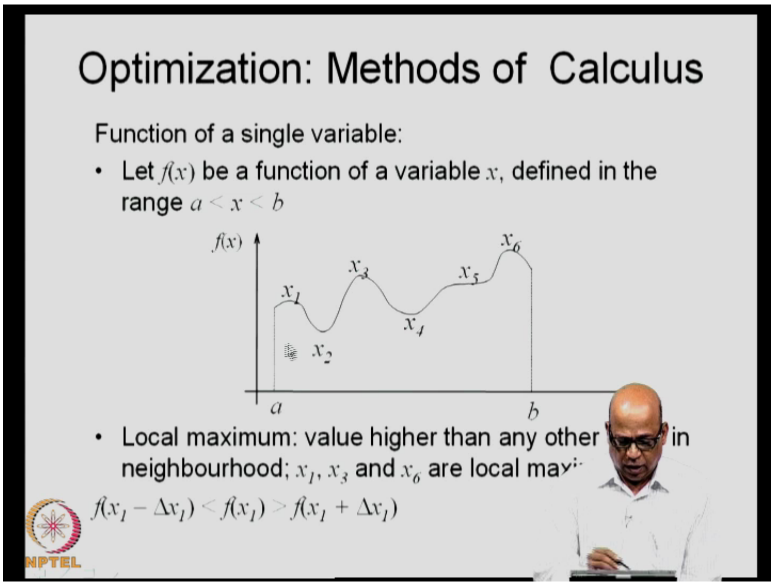

We will start with the simplest of the functions that is function of a single variable. Let us say f of x is the function of a single variable x defined in the range x between a and b, this is x is equal to a and this is x is equal to b and the function nature is something like this. So, between a and b, this is the behavior of the function. The function attains certain value certain maximum value here and certain minimum value here, again picks up again attains a maximum value comes on which is the minimum value and so on like this. Now these peaks here x1, x3, x5, x6 etcetera where the maximum value has occurred. In fact, at x 5 maximum value may not have occurred that a change in the slope has occurred here, as well as the change in the slope 2 has occurred at x2 and x4 and so on. Now these are the points where the maximum has occurred x1, x3 and x6 are called as the local maximum values. So, there is the local maximum at this point, local maximum at this point, local maximum at this point and so on .

At the local maximum what is the happening here? Let us say look at x1, the value of the function is higher than any other value in the neighbourhood. So, if you look goes slightly to the left, the function value decreases, slightly to the right the function value decreases. So, f of x1 is greater than f of x1 minus delta x1 which is slightly to the left, also it is greater than f of x1 plus delta x1 slightly to the right. So, at the local maximum the functional value will be more than it is the more than the function values at its neighbourhood that is how that the local maximum is obtained. Similarly at the local minimum, you look at x2 here. The function value will be smaller than its than the function values at the neighbourhood. Similarly, here the function value at x4 will be smaller than the function values at the neighbourhood.

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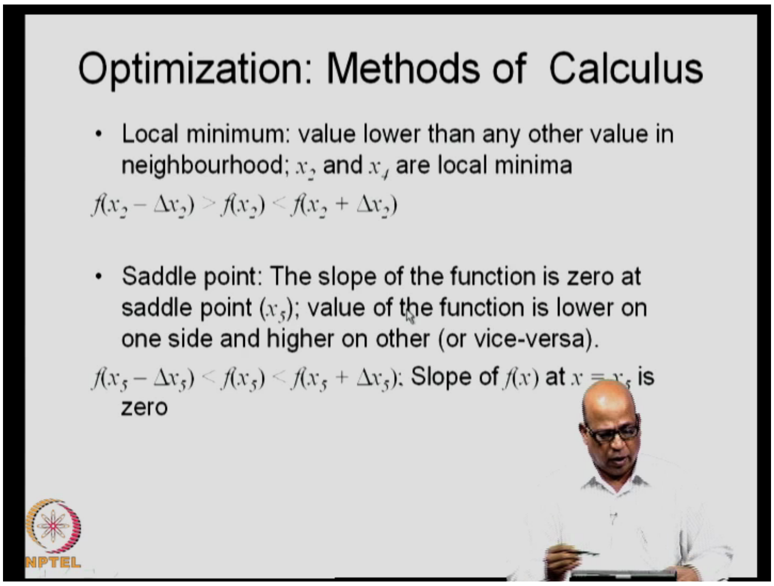

So, we write that as f of x2 is less than f of x2 minus delta x2 and also it is less than f of x2 plus delta x2 which means you you go slightly to the left or slightly to the right, the function value will be more at those points. Now you look at the point x5. alright one more interesting feature is whenever we have a local maximum or local minimum, what is happening? The slope was in this direction when we came from point a, slope kept on reducing and then slope becomes zero at this point. Similarly, at x2 this slope becomes zero, at x3 slope becomes zero, x4 slope is zero, x6 slope is zero. So, at local maxima and local minima, the slope is zero. Slope of a function that is dy by dx or f of f dash of x will be zero. The first derivative of the function will be zero at his location .Look at what is happening at x5, the slope is zero here; however, it is neither a maximum nor a minimum in the sense that in the immediate neighborhood of x5, the function value has not changed or the function value is not less on either side making x5 is the maximum or it is neither more on either side making x5 the minimum.

So, at x5, although the slope is zero the function value is either a maximum nor a minimum either a local maximum nor a local minimum. such points are called as saddle point. So, at the saddle point, the slope of the function is zero, but the value of the function is lower on one side and higher on the other side and vice versa. So, at f of x5, you can see f of x5 is less than x5 minus delta x5 and is also less than x5 plus the function value at x5 plus delta x5. So, we saw two points now two major observations. one is that at local minimum and local maximum the slope changes and the slope becomes zero and then we have what is called as the saddle point where even if the slope is zero it does not correspond to a minimum or a maximum.

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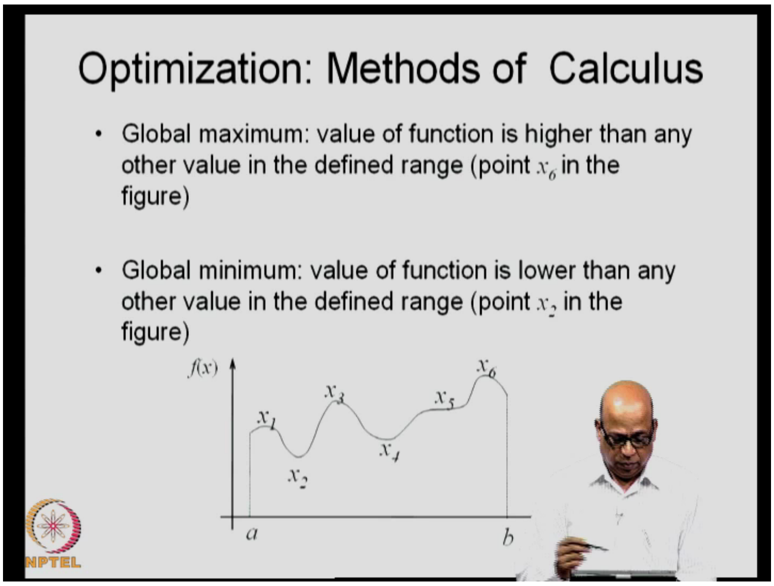

Now we got what are called as local maxima, these are also called as relative maxima. So, x1, x3, x6 are local maxima. The maximum among all such local maximum maxima is called as the global maxima; that means, in this range between a and b, x6 is the highest among all the local maxima namely x1, x3 and x6. Among all these 3, x6 is the highest and that is called as the global maxima. Similarly between x2 and x4, x2 is the lower among the 2 minima. So, x2 is the global minima. So, when we are talking about global maximum and global minimum, we are talking within a specified range; that means, within the range a and b, x2 is the global minimum and x6 is the global maxima within this range.

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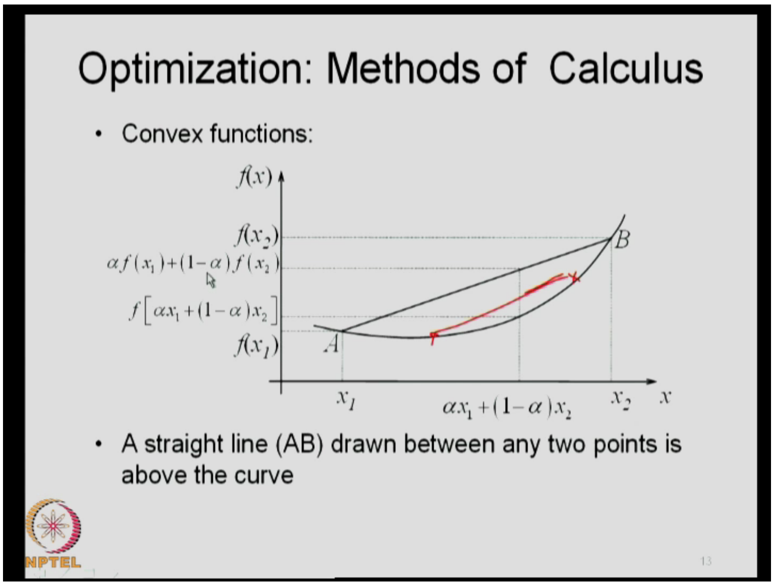

Now we will go in to an important aspect of optimization with calculus namely the concept of convex and concave functions. Remember we are talking about functions of a single variable and we have just defined what are called as local maxima and local minima and then the global maxima and global maximum and global minimum. Global maximum is the maximum among all the local maxima. If you have a function where there is exactly one minimum which means the local minimum that you obtain is also the global minimum. Let us look at a function. Let us say that you have a function like this here and you get a minimum point at this location and this is the range that you are talking about. In this range, the function is something like this, there is the only one minimum here and this minimum also corresponds to the global minimum.

So, the local minimum also happens to be global minimum. What is the feature of those function? Let us say that you draw, you join any two points on the curve. This is the function of curve. Let us say you join any two points, this is the straight-line you will be always about the curve. You take any two other points like this you want to join this point and this point this will still be above above the curve. So, you join any two points on the curve that will be always above the curve. This is the feature not the definition this is the feature of the (( )) feature of the function and such functions are called as convex functions where the local minimum is also equal to is also the same as the global minima. Let us say that there are two points a and b and you have joined and this point has to be always about the curve itself.

So, let us see what is the condition for this; that is x1 and x2.You take any one point in between that would be alpha x1 plus 1 minus alpha x2 where alpha is a scalar between zero and one. So, this is alpha x1 plus 1 minus alpha x2. What is the function value corresponding to x1? It is f of x1 at this location, the function value corresponding to x2 is f of x2. The straight line point corresponding to this location which is alpha x1 plus 1 minus alpha x2 is alpha f of x1 plus 1 minus alpha f of x2 that is on the straight-line. The function value corresponding to this is f of alpha x1 plus 1 minus alpha x2, that is the function value corresponding to this. So, what we are saying is this value which is on this straight line namely alpha f of x1 plus 1 minus alpha f of x2 must be always greater than f of alpha x1 plus 1 minus alpha x2. So, this point is always lower than this point. That is what we are call it as convex function. So, this are the convex function, the local minimum is also equal to a global minimum and the straight line that is joining any two points will always lie above the curve itself.

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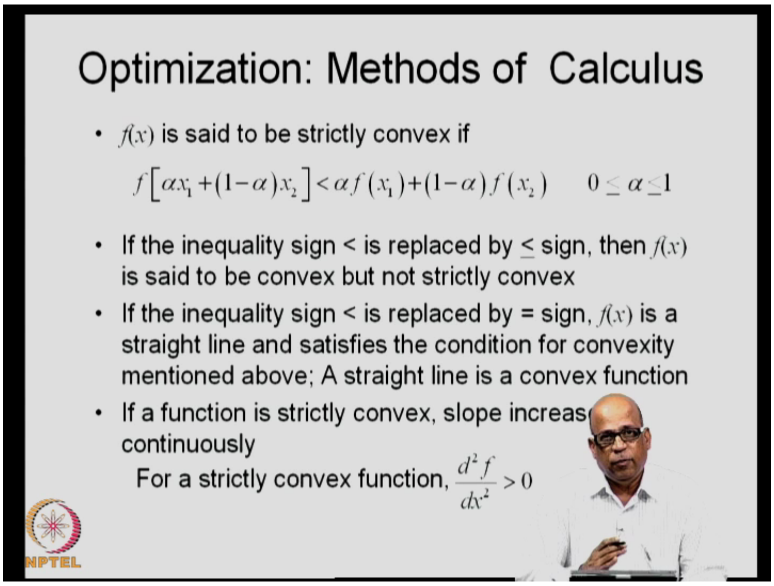

So, we use this condition and write this as f of alpha x1 plus 1 minus alpha x2 which is I am talking about this point here. The function value corresponding this point must be always lower than the corresponding point on the straight line joining any of these two points could be less than alpha f of x1 plus 1 minus alpha f of x2 for all values of alpha between 0 and 1. Now this is if it is less than then it is called as strictly convex, if it is less than or equal to than it is still convex function, but not strictly convex. However, I repeat this is the feature of a convex function then definition of a convex function is a strictly convex function. The second derivative d square f by dx square is greater than 0 for all values of x in that particular rate,for d square f by dx square of the second derivative of the function. If it is strictly positive in the range than the function is strictly convex in that particular range. If it is greater than or equal to zero that is d square f by dx square is greater than or equal to zero in the range that we are interested in then it is called as the convex function.

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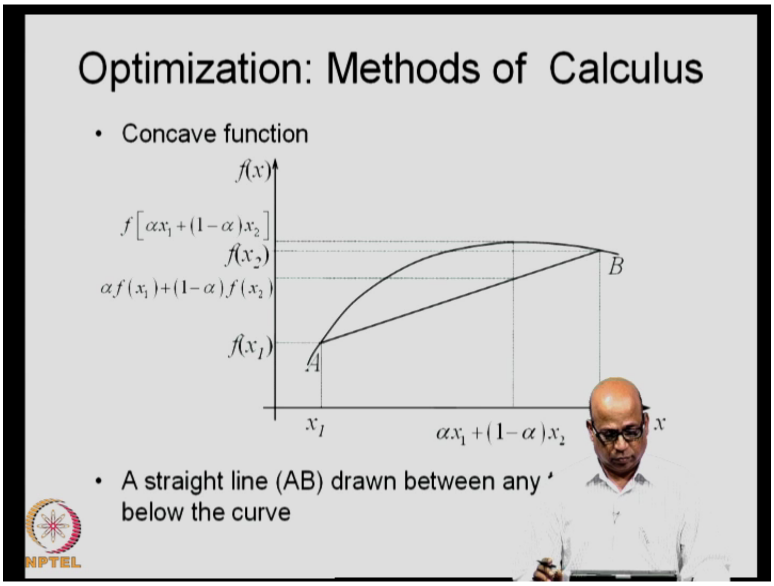

What we just do it with the convex functions the same thing is true of the convex concave functions. In the concave functions the local maxima; local maximum will correspond to global maxima. So, there is only one maximum value here and any line that is joining between a and b, any two points you join, any two points on the curve that line will be always below the curve itself and giving the same argument that will used just now for the concave functions the convex functions on sorry.

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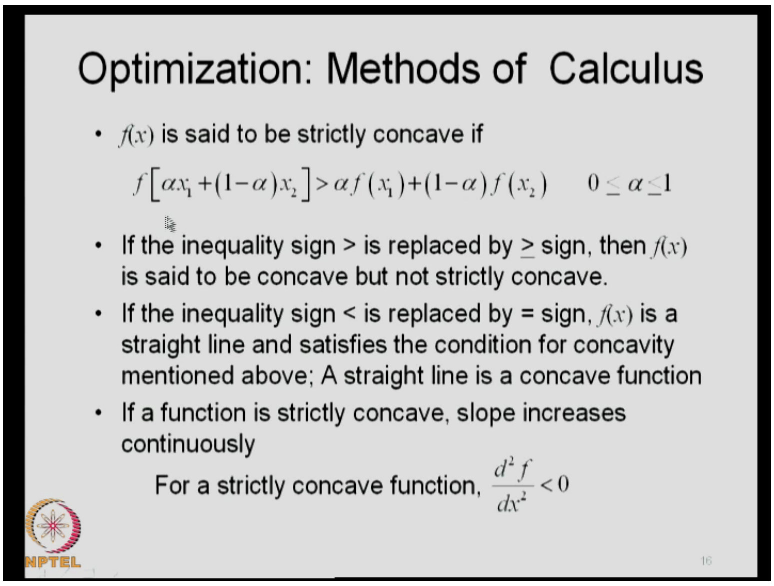

We can write this as f of alpha x1 plus 1 minus alpha x2 will be greater than alpha f of x1 plus 1 minus alpha f of x2 for all values of alpha between 0 and 1. For a strictly concave function, d square f by dx square is less than zero in that particular range. So, in the range if you have in in the particular range in of interest if you have d square f by dx square strictly negative then it is called as the strictly concave function where as if d square f by dx square is less than or equal to zero than it is called as strictly it is called as a concave function. Remember, a convex function corresponds to minima and concave function corresponds to maxima. So, the second derivative being negative corresponds to maximum values, second derivative being positive corresponds to minimum values. So, these are the important point set for must remembering this.

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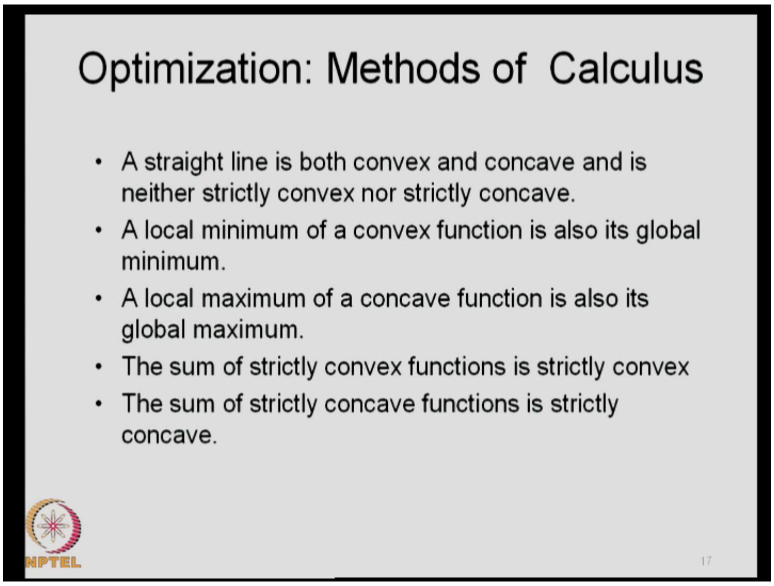

Now look at a straight line, let us say that y is equal to m x plus c a simple straight line equation. What happens to the second derivative? The second derivative is zero. So, the straight line is both a concave function as well as a convex function. Now the other two points at I have already mentioned is for the convex function there is only one maximum in the range that we are considering and that maximum also corresponds to the global maximum. So, local maximum is also the same as local global maxima in terms of in the concave functions. Similarly local minimum is also the same as global minimum in the case of convex functions. The sum of strictly convex functions is strictly convex. So, if you have a two strictly convex functions, the sum of them is also a strictly convex function. Similarly, sum of strictly concave functions is also strictly concave.

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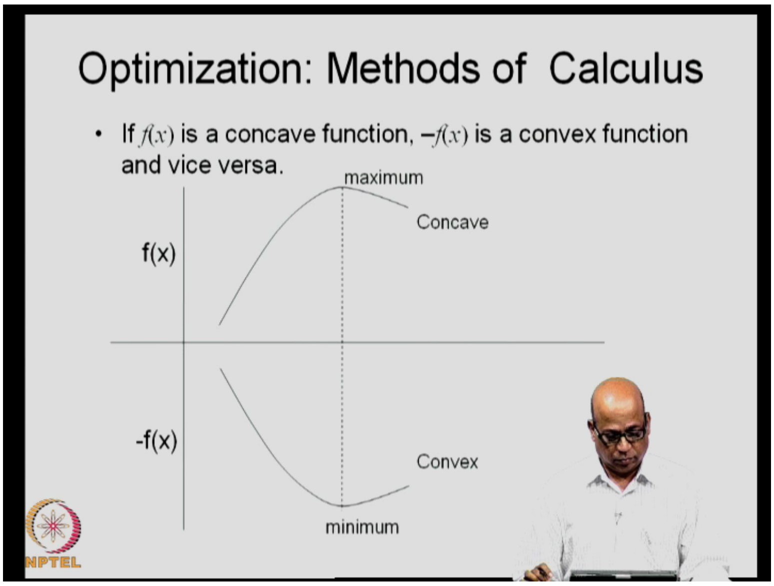

Let us say you have a concave function like this which corresponds to a global maximum and you take a mirror image of this or f x is plotted like this which is also in a concave function. You take minus f of x which plots like this this corresponds to a local minimum which is also equal to is same as global minimum. So, f of x is if it is a concave function minus f of x is a convex function and vice versa. In fact, maximization of f of x will mean the same as minimization of minus f of x both the optimum values correspond to the same point on the x axis.

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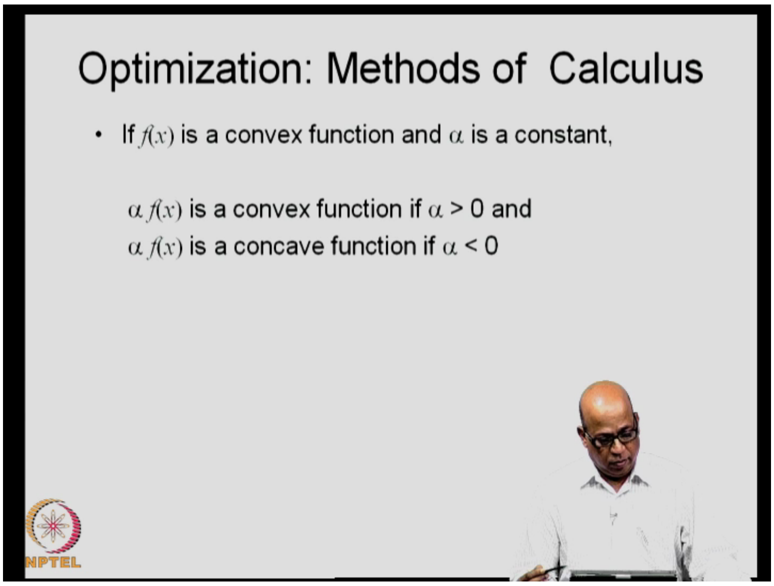

Another important point is if f of x is the convex function and alpha is any constant, then alpha f of x is the convex function if alpha is greater than zero and alpha f of x is the concave function if alpha is less than zero which means if you multiply it with the positive quantity then you still get a convex function, if you multiply with the negative quantity you will get a concave function and the same is true if it replaces with concave this would become concave and this becomes convex. So, these are the features of concave functions and convex functions. Why are we interested in the definitions of concave and convex functions? If you know that the function that you are optimizing is in fact a convex or concave function and if you are looking at the minimum value or the maximum value and then you have methods of isolating or identifying local minimum, you will be sure that the local minimum also corresponds to the global minimum in that range if it is in fact concave convex function and that is where the notations of convex functions then concave functions would be of use.

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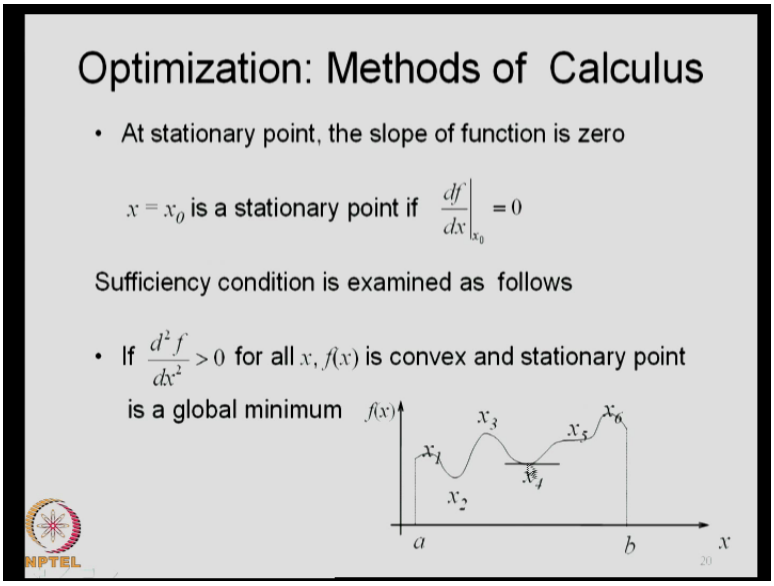

Now look at what is happening to our original function here. So, this is the function with which we started x1, x3, x6 were local maximum and x2 and x4 were local minima. At all these points, local maxima as well as local minima the slope is zero. So, if you draw a tangent here at this point, the slope will be zero. Similarly at x6 the slope is zero, x3 the slope is zero and so on. These points at which the slopes are zero are called as the stationary points. So, at stationary point the slope of a function is zero. So, we write that as df by dx is equal to zero, we are talking about functions of single variables and therefore, we write it as df by dx is zero at x not then x is equal to x not is the stationary point.

So, what we are looking at is in a particular rate we want to get the local maxima and the local minima. This is the optimization problem that we are looking at. So, the first condition we need to examine is whether the slope is zero at that particular point. This is called as the necessary condition. So, the necessary condition for a function to have an optimal value at a particular point is that the slope of the function at that particular point must be zero. So, d f by dx at x is equal to x not is zero. This is the necessary point necessary condition. So, in the necessary condition, you got slope is zero at this point, at this point, at this point and so on.

So, what is the sufficiency condition that. Now the sufficiency condition is if d square f by dx square is greater than zero if it is valid for all x in that range then what happens it becomes a convex function. So, if d square f by dx square zero for all x in that range we just saw that in the in that range if it is greater than zero then you have only one minimum and that also becomes the global maxima the global minima. So, if d square f by d x square is greater than zero for all x f of x is convex and the stationary point itself is the global minima.

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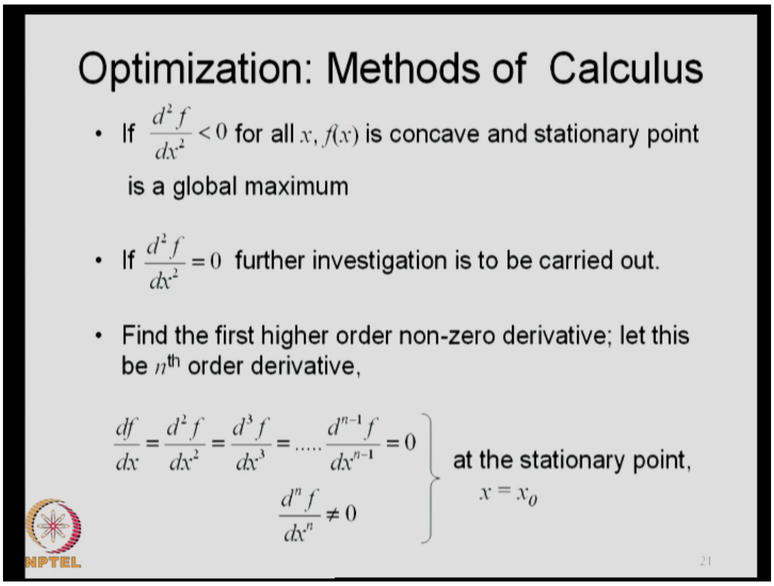

Similarly if d square f by dx square is less than zero for all x, f of x is concave and stationary point is also a global maxima. Now it may so happen that you may get d square f by dx square is equal to zero, then further investigation needs to be carried out. Let us just go through it again what we are interested in is that within a particular range between a and b, we want to examine whether the point x is equal to x not is the minimum or a maxima and in fact, whether it is the local minimum or a global minimum, local maximum or a global maxima. First we examine the slope of the function. So, slope of the function at x is equal to x not if it is zero; that means, that it may either correspond to minimum or correspond to maximum or perhaps may not correspond to either of them as it happens in that case of x5 here. So, if we have exactly one minimum; that means, if in this range the function for functions behaves as a convex function then we get that particular point change of slope and that itself corresponds to the global minimum. Similarly if in this range it behaves as the concave function, then the function the point corresponds to a local maximum which is also the global maxima; however, you may get different maxima and different minima like this. So, you may get slope is equal to zero, here slope is equal to zero here etcetera and the second derivative may be positive at some point may be negative at some points and so on.

So, if you look at the second derivative it may positive at certain of these points which are the stationary points. It may be negative at certain points which are the each corresponds to the local maxima. So, we may capture all of these local maxima and local minima by looking at the second derivative. Let us say that the second derivative also becomes zero at some of these stationary points. Remember these are the stationary points x1, x2, x3 etcetera, these are all x5, x6 they are all stationary points because the slope of the point at those locations, slope of those slope of the function at those locations is all is zero and therefore, they correspond to their stationary points. Now some of the stationary points may correspond to local maxima and some of them correspond to local minima and some of them may correspond to saddle points and so on. So, these are examined based on the second derivative. We will see how we use the second derivative as sufficiency condition and what happens if the second derivative also becomes zero. What we do in the cases where the second derivative also becomes zero? So, we will will extend this problem and then see how will use the second derivative as sufficiency condition. So, the necessary condition for a function to have an optimum value at a particular point is that the slope at that particular point of the function must be equal to zero.

So, in today's class, essentially we started with definition of what we mean by a local minima, local maxima, local minimum, local maximum and so on and then looked at how we identify a convex function and a concave function. Remember the definitions for a concave function is that d square f by d x square must be negative and d square x by d x square if it is positive then it is the concave function and the convex functions will give you the local will just go through that once. So, that refresh our memory correctly. For example, in the convex functions we always get the global minimum and in the concave functions we always get the global maxima. So, the the notion of convex functions and the concave functions are useful because if you know that a function that you are optimizing is either a convex function or a concave function then the local minima or local maxima that you get are in fact the global minima and global maxima.

The necessary condition for a function to have an optimal value at our given point is that the slope at that point must be zero or the first derivative of the function with respect to the variable must be equal to zero where treating functions of single variable. Then we go on to the second derivative to see if the stationary point is in fact a local minimum or a local maxima and then in the next class, we will also see what happens that the second derivative is also zero; that means, the first derivative is zero has the necessary condition the slope has changed, then we will look at the second derivative and decide whether it is a local minimum or local maximum depending on whether the second derivative is positive or negative. What happens if the second derivative is also zero? There we will continue this discussion in the next class. So, we are talking about optimization using methods of calculus of a function of a single variable now. We will continue our discussion in the next class. Thank you for our attention.

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