Water Resources Systems

Modeling Techniques and Analysis

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Lecture No #29

Chance Constrained Linear Programming for Reservoir Operation and Design (1)

Good morning and welcome to this the lecture number 29 of the course Water Resource Systems, Modeling Techniques and Analysis. Over the last two lectures, we have been now discussing the stochastic optimization; essentially, in the last last two lectures, we have covered the basics of probability theory that will be requiring in developing the optimization model; and in the previous lecture, in the last lecture specifically, we dealt with the normal distribution, the lognormal distribution and the exponential distribution.

Now, these are not the only distributions that will be typically using; we also use gamma distribution, and when we are talking about extreme values, we may use compiles extreme value distribution and so on. But, the three distributions that I covered in the last lecture, will give you some idea about how to use the pdfs in the optimization models and that usage of the pdfs in the optimization models, we will be discussing today.

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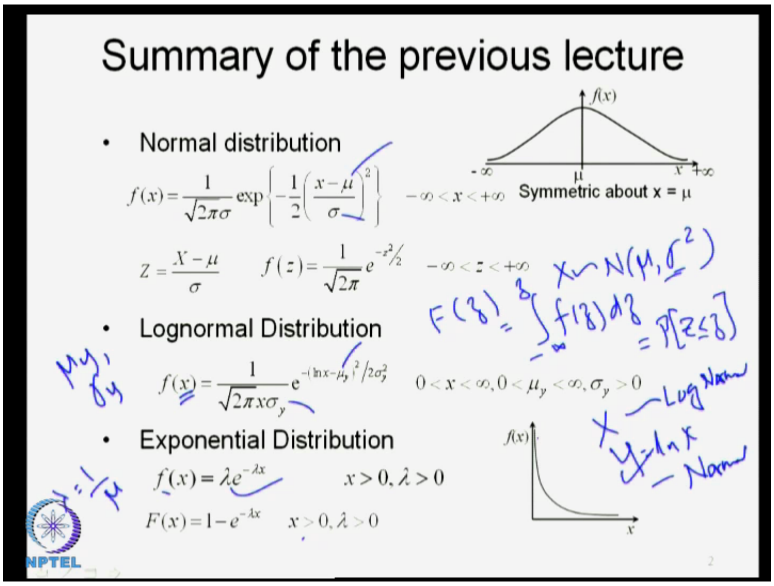

So, let us quickly recall what we did in the last lecture. We discuss the normal distribution; as I mentioned, normal distribution is the most commonly used distribution, especially in the optimization models and so on; because it is rather easy to handle the normal distribution, however we also use other distributions like lognormal distribution, exponential distribution, gamma distribution and so on. The normal distribution is the symmetric about x is equal to mu, and we define the standard normal distribution, by the standard normal variant by Z is equal to X minus mu over sigma, where x is the original random variable and x (Audio not clear from 02:04 to 02:07) the notation, x follows the normal distribution with mu comma sigma square, indicating that it has the parameters mu and sigma square. Normal distribution has two parameters mu and sigma; sigma is the standard deviation. So, we typically use sigma square, which is the variance.

And then once we convert it into a standard normal distributions, standard normal variant like this (Audio not clear from 02:38 to 02:40) z f (z), and then the capital F of z, which is integral of minus infinity (( )) f (z) d z; this gives (( )) of z being less than or equal to z, and (( )) f (z) is tabulated. So, we use the tabulated values of capital F (z) to obtain different probabilities associated with the variable z, and then transfer it into the variable x. (( )) seen a a few examples, of how to use the normal distribution tables for obtaining several probabilities associated with the random variable x.

Then we also discuss the lognormal distribution; recall that if x follows normal distribution, y is equal to ln (x) follows lognormal distribution. So, if we say x follows lognormal distribution, then y is equal to ln (x) follows normal distribution; and we have also seen, how to obtain the parameters of y, which follow normal distribution; that means, typically mu y and sigma y. Starting with the parameters on x that means, we have given mu x and sigma x, we can obtain mu y and sigma y. We have seen the expressions associated with this.

This is the pdf of lognormal distribution. So, this will have parameters mu y and sigma y defined on y is equal to ln of x; and then it will also be a function of x, if we are considering pdf. But, a more simpler way of… But a simpler way of handling with the lognormal distribution is to simply convert your sample data x into log (x) ln (x); and then deal with the ln (x), the sample data on ln (x), because ln (x) follows normal distribution, you can then compute the movements of mu and sigma on ln (x), on the sample containing ln (x); and then use the use the normal distribution, as we have done earlier.

Then we also looked at the exponential distribution, the exponential distribution gives the f (x) of exponential distribution is given by lambda e to the power minus lambda x, this has only one parameter lambda and lambda is estimated by 1 over mu. So, if you have a sample, you can estimate mu, which is the sample estimate of the mean; and then get lambda, and lambda is lambda defines the pdf. The cdf - f (x) is then given by 1 minus e to the power minus lambda x, for x greater than 0, lambda greater than 0; and we have also seen a few examples, one example typically, on how to use the f (x) to obtain different probabilities; and the exponential distribution looks like this; it is the exponential curve in fact, f (x) versus x.

That was just to introduce a few typical probability density functions and the associated cumulative distribution functions. Now, we will progress further, and see how we use the pdfs or and the cdfs in optimization problems. As I have been mentioning earlier in the deterministic optimization cases, we assume that the inflows, inflows are deterministic; in the sense that you specify a sequence of inflows, you assume that the same sequence keeps on repeating; for example, if you were solving the reservoir sizing problem with 1 year mean inflows mean monthly inflows; let us say june month, july month etcetera like this up to december… you up to may you have monthly inflows, and you solve the optimization problem, what it indicates is that the same sequence of inflows is likely to… Is remaining constant, it it in fact, remains constant in the future, that is what is the implication of the deterministic optimization.

However, in reality, it is not so, as you know the inflows are governed by rain fall, which is the random variable, and the soil moisture if you are looking at, it is the random variable, evaporation if you are looking at, it is the random variable and so on. And we have seen in the last two classes, what we mean by random variable; how we address the problems associated with the random variable, given the pdf, how we access the probabilities associated with the random variables and so on. So, this information that we have we have seen in the last two classes. We will now, try to synthesize with the optimization problems.

And the first level of optimization that I will introduce is the chance constrained linear programming problem. So, we have earlier seen the linear programming problem, in which the constraints were deterministic; in the sense that there was no probability associated with any of the constraints. The movement we introduce any random variable into an optimization problem, any constraint that contains that particular random variable or a function of that particular random variable in any constraint becomes a probabilistic constraint. You can you can no longer write it as a deterministic constraint, because one of the variables in that constraint is the random variable.

So, you have to write it in terms of the probabilities associated with that particular random variable, and that is what leads to chance constrained linear programming, if we are look looking at the linear programming problems. So, we will start with the chance constrained linear programming; remember, this is an explicit stochastic optimization technique; explicit, because as I mentioned in one of the earlier classes, it considers the probabilistic behavior of the random variable, explicitly in the optimization, in terms of taking into account, the probability distributions of the particular random variable into the optimization directly. Contrast this with the implicit stochastic optimization, where the optimization model remains deterministic, and you run it for several sequences.

So, we are now, dealing with the class of optimization problems called as the explicit stochastic optimization; and the first model that we will consider is the chance constrained linear programming problems. And specifically, we deal with the chance constrained linear programming for reservoir design and operation; and we denote this as an abbreviation by CCLP - Chance Constrained Linear Programming. Now, the chance constrained is also called as reliability constrained, for the reasons that I will mention just now. So, it is also called as reliability constrained linear programming problem and we will take the reservoir design or reservoir operation problem for demonstrating, how the CCLP models are formulated.

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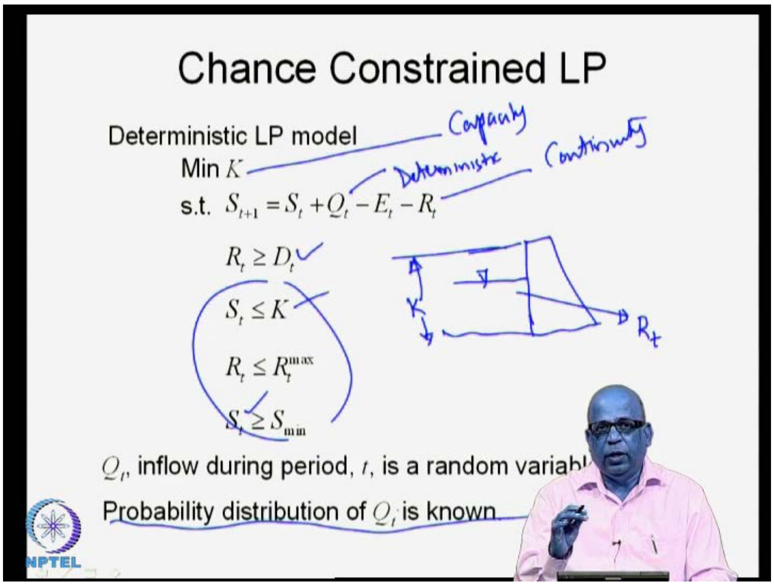

We will start with the deterministic LP model that we had seen earlier. Recall that we formulated the deterministic model like this. We are looking for the minimum storage capacity, this K is the storage capacity, to satisfy a certain demand pattern. So, we are saying R t greater than or equal to D t, that means every we are specifying the demands, and you want to meet those particular demands always, and if possible, you want to supply more water than the demand. So, R t greater than or equal to D t, indicates that the release should be atleast equal to the demand.

Then we have considered S, this is the storage continuity - continuity constraints; S t plus 1 is equal to S t plus Q t minus E t minus R t, simple mass balance; and this is the capacity of the reservoir, which means if we are talking about storage is, this is the capacity, in terms of volume; and these are the capacity constraints; this R t is… S t must be always less than or equal to K, and R t must be less than or equal to R t max (( )) maximum release that is possible, and S t must be greater than or equal to S t S minimum. You may specify a minimum storage, and then say that S t must be greater than or equal to S minimum.

Now, look at these constraints now; we assume the Q t to be deterministic in this. In fact, all the variables we assume to be Q deterministic, but Q t is the main driving variable, which is assume to be deterministic; and therefore, in this, if you also assume E t to be deterministic or negligible, this constraint becomes deterministic, and all of these constraints are in fact deterministic. Now, in this now, if you want to introduce randomness; that means, the uncertainty associated with a hydrologic variables Q t and E t, then what happens? You can no longer write any constraint, which has Q t explicitly or which has a function of Q t as a deterministic constraint. You can only write them as constraints containing probability distributions.

Let us see, how we incorporate that? So, what we are deriving at now, is starting with the deterministic optimization that is shown here. We want to now address uncertainty associated with the randomness in the hydrologic variables; and specifically, we will talk about the uncertainty associated with the inflow, reservoir flow Q t. For the time being, we will ignore evaporation; although, it can be readily incorporated indict into the optimization, but to understand how we incorporate randomness, for the time being, we will only assume that Q t is random, and we will ignore evaporation.

Further, we assume that them probability distribution of Q t is known. Although I say assume, this information can be obtained or this we can obtain the probability distribution fairly accurately, if you have the sample values on Q t, that is the reservoir inflow at a particular location is known, are measure for the last about 30 40 years, you have the data, from the data you can estimate, which probability distribution best suites that particular sequence of inflows. So, the probability distribution of Q t is known. Remember we are talking about Q t here, which means that, from one time period to another time period, it may have a different distribution; Q 1 may have some distribution, Q 2 may have distribution, Q 3 may have it is own distribution and so on. So, we are assuming that all the all the distributions of Q t are all known, for all t; I repeat that the distribution of Q t, for all t is known.

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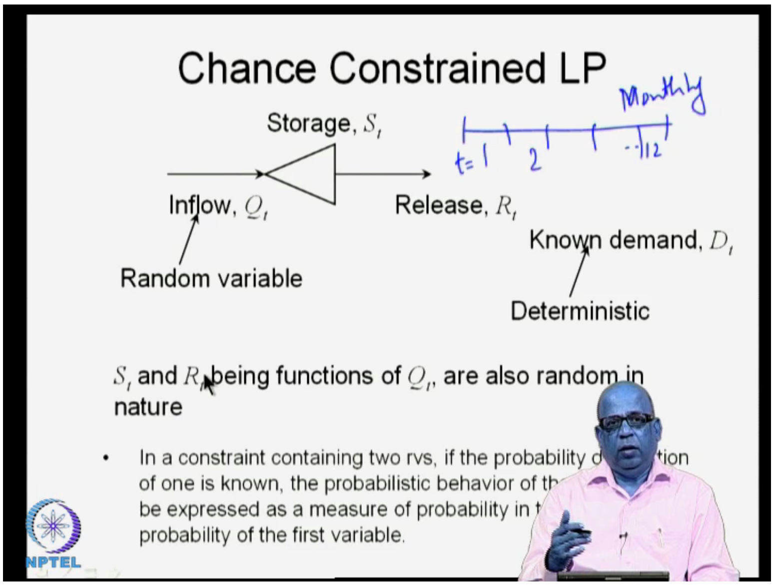

So, this is the problem that we are talking about now; we are looking at the minimum storage capacity; we want to meet a certain demand pattern, known demand, so demands remain deterministic. We are specifying a set of demands, let say D 1 is equal to 100 units, D 2 is equal to 100 units, D 3 may be 200 units and so on. These demands are fixed, and they are deterministic; there is no uncertainty associated with the demands. But the reservoir inflows, which control the storage and therefore the release are random variables.

So, we are considering the reservoir inflow Q t as the random variable; and we may be talking about a monthly operation, let say (( )), you may have 12 time periods for monthly operation; that means, we are saying now, Q 1 is a random variable, Q 2 is a random variable, Q 3 is a random variable etcetera. So, the inflow associated with these time periods is the random variable.

Now, look at these S t and R t, if you look at the continuity here, both S t and R t are ignoring evaporation for the time being, let us ignore evaporation; both S t and R t depend on Q t, and because Q t is the random variable, S t as well as R t become random variables. So, S t and R t both being functions of Q t, they become random variables; and therefore, S t plus 1 is also a random variable. So, in the same continuity equation now, Q t you know, the distribution half, that is the probability distribution of Q t is known. S t and R t are functions of Q t and therefore, they are also random variables. But you do not know the distributions of S t and R t.

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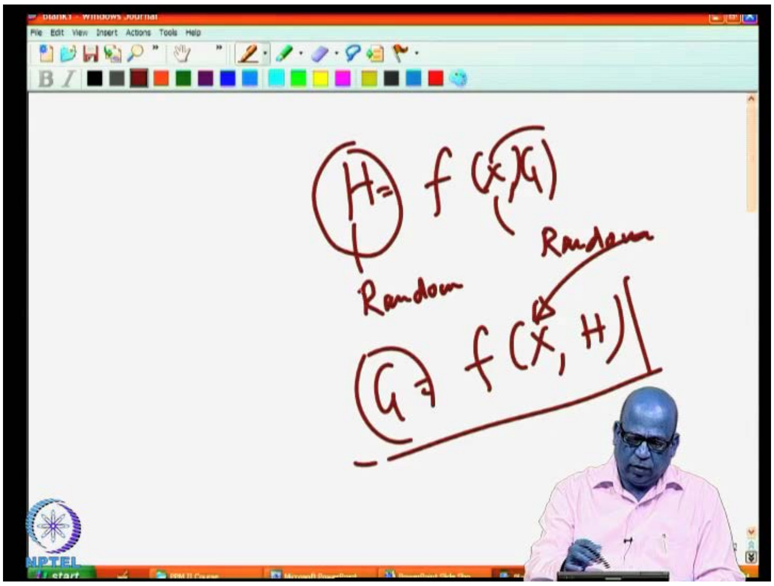

If you have a function of a random variable, let us say H is equal to f (x), where x is the random variable. Let me write it this way, this concept let us understand slightly carefully, that is, if you have a function of a random variable, H is equal to f (x), let us say; x is random and H is also… H therefore, becomes random. A function of a random variable is also a random variable. Now, if if you have only one function, that is h is directly related with x; then you can determine their distribution of H from the distribution of x. But typically, if you have H is equal to f (x) and you have another variable let say, G is also a function of x and H. If you have such a situation, and and in which you have only the distribution of x, then it becomes difficult for you to get the distribution of G and also H, here it may have G also here.

So, if you have only one random variable, whose distribution has to be determined based on another random variable of which it is a function of, it is possible and easy. Although, you can do some numerical methods etcetera, there are ways of obtaining these, but it is not simple, obtain the distributions in this particular case of both S t as well as R t as a as they are functions of Q t. So, you know the distribution only of Q t. It is difficult to obtain distributions of both S t and R t; and therefore, what we do is, we use what is called as a linear decision rule. Remember the problem, we are talking about is one of reservoir design, which has inbuilt into it, how we operate, inbuilt into it the policy of operation itself, because we are saying R t must be greater than or equal to D t, which means you want to meet the demands in every time period, at at least, the R t must be atleast equal to D t is what we are saying.

So, in a constraint containing two random variables, if the probability distribution of one is known, the probability probabilistic behavior of the second can be expressed as a measure of the probability in terms of the probability of the first variable; that means, if you have only R t and R t being a function of Q t, because Q t distribution is known, you can transfer the distribution to R t, whereas if a constraint contains both S t and R t and both of them are functions of Q t, then it is difficult for you to get the distributions of S t and R t. To overcome this difficulty, we introduce what is called as a linear decision rule. We will see, what we mean by that.

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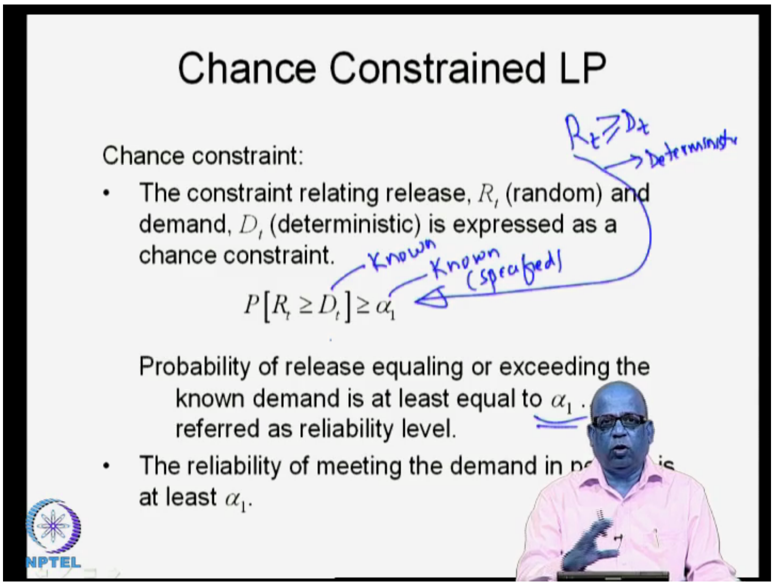

Now, before that, we will start understanding how we write the chance constraints or the reliability constraints. You had to begin with, you had the deterministic constraint R t greater than are equal to D t; this is the deterministic constraint. (No audio from 21:16 to 21:21) We could write this earlier as in this particular form, because R t was deterministic, because Q t was deterministic and therefore, S t was deterministic, everything was deterministic and therefore, you could write these in this form.

The moment you introduce any random variable in the optimization problem, and any variable, which is the function of that particular random variable also becomes random variable; and therefore, R t is now, a random variable, because it is the function of Q t, which is the inflow; and therefore, you can no longer write this in this particular form, you must be able to write this as a probabilistic constraint, because you cannot be 100 percent sure that this constraint will be met always, because there is a uncertainty associated with the inflows, and that uncertainty is also transfer to the releases; and therefore, this constraint we write it in this form.

Probability of R t being greater than or equal to D t is greater than or equal to alpha. So, what we are saying is, instead of stating R t greater than or equal to D t, we state that the probability of R t being greater than or equal to D t must be at least equal to alpha 1. I may specify this alpha 1, to be let us say, 80 percent, 90 percent and so on. So, what we will say is, instead of saying R t must greater than or equal to D t, we will say that at least 90 percent of the time, R t must be greater than or equal to D t, at least 70 percent of the time R t must be greater than or equal to D t. So, we are stating now, in terms of the probabilities associated with the releases R t. So, that is the way we will state.

Here, we may specify this, so we may say, this is known or specified and R t is (( )) decision variable, D t is known, it is deterministic; (( )) a (( )) random variable and therefore, this becomes a probabilistic constraint, in terms of the (( )) R t. Now, the the statement reads like this, probability of release equaling or exceeding the known demand is at least equal to alpha 1, and the alpha 1 here is referred as the reliability. The reliability with which the demand can be met, is denoted as alpha one here in this particular case.

So, what we are stating is that the demands in period t must be met with at least a reliability of alpha 1; that is why we are putting a greater than or equal to sign here; and that is the implication of this. So, the reliability of meeting the demand in period t is at least alpha 1. Now, this alpha 1 can be a constant for all the time periods, we you may say that at least 70 percent of the time, the demand should be met in period t and that 70 percent you may hold constant; or alpha 1 can also be a function of time period t, in which case, you will write alpha 1 t, instead of writing simply alpha 1. But we will start with the simple case and understand how we handle this kind of chance constraint. So, this constraint that we have written now; from the deterministic constraint R t greater than or equal to D t; we have converted into probability of R t being greater than or equal to D t must be greater than or equal to alpha 1. This becomes a reliability constraint or a chance constraint or a probabilistic constraint, all three are synonymous. So, we have converted the deterministic constraint into a chance constraint now.

Similarly, we had other constraints in the deterministic optimization S t less than or equal to K, R t less than or equal to s t R t max and S t greater than or equal to S s min or S t min. Because Q t is random, R t is random, because Q t is random, S t is random and therefore, any of these constraints, you cannot state in deterministic forms like this, because there is the random variable associated with this; S t is the function of Q t therefore, S t is the random variable; R t is the function of Q t therefore, R t is the random variable; and therefore, you cannot state these constraints in deterministic forms. So, much the same way as we stated the constraint R t greater than or equal to D t in the reliability constraint form.

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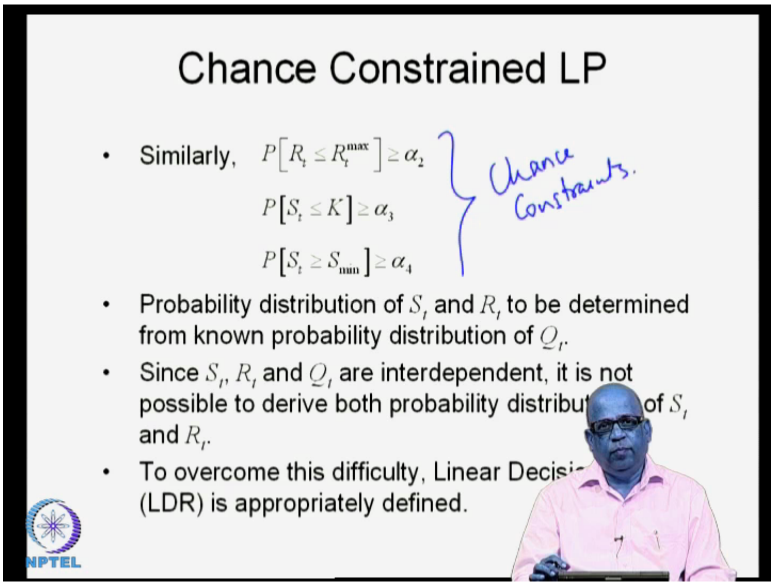

We also state the other constraints for example, R t less than or equal to R t max, which is the constraint on the maximum release, because R t is the random variable, we write this as probability of R t less than or equal to R t max must be greater than or equal to alpha 2. We use another reliability level for this; similarly, probability of S t being less than or equal to K greater than or equal to alpha 3. Probability of S t being greater than or equal to S min greater than or equal to alpha 4. It is likely that the new students will get confused with the implications of this; so, just let us understand what this particular constraint says, there is the physical limitation physical limit up to which the storage can go, S t must be less than or equal to K; however, because S t is the random variable, you cannot state it in deterministic form as S t less than or equal to K.

So, storage must be less than or equal to K is a condition, which we have to incorporate through the continuity relationship etcetera, addressed using the probability distributions of Q t; however, when you are saying, S t less than or equal to K, if S t is the random variable, you cannot simply say state it as a deterministic constraint, you have to state it as a probabilistic constraint; and therefore, associated with any random variable, remember, if a a constraint contains any… if any term of the constraint… of a constraint contains the random variable, then that constraint cannot be stated as a deterministic constraint, it has to be stated as the probabilistic constraint, which means S t less than or equal to K, we are saying that by and large S t must be less than or equal to K, which means about 90 percent, 80 percent and so on.

So, we associate the probability of S t being less than or equal to K, and then we state, remember see here, all the right hand sides are greater than or equal to, irrespective of what are your constraints, which means, we are saying that R t must be less than or equal to R t max with the probability of alpha 2, with the minimum probability of alpha 2. Similarly, S t must be less than or equal to K with minimum probability of alpha 3, S t must be greater than or equal to S min with the minimum probability of alpha 4. So, these are stated also as reliability constraints or chance constraints.

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Now, we will see, how to handle all these chance constraints in the LP problems. Now, the probability distribution of S t and R t, we do not know; all we know is the probability distribution of Q t. Let us say that the Q t from your sample data etcetera, you can assume that the Q t follows normal distribution in time period t, with varying parameters across time periods t; that is Q 1 follow the normal distribution with certain parameter, Q 2 follow the normal distribution with certain other parameters and so on. So, Q 1, Q2, Q 3 etcetera up to Q 12, you have estimated the probability distributions, and therefore, you know, the probability distributions of Q t.

However, S t and R t, which are functions of Q t, you do not know the probability distributions, and it is not straight forward as I just mentioned to determine the probability distributions of both Q t and R t; both of which will appear in a particular constraint in the continuity constraint. And since S t, R t and Q t are interdependent, it is not possible to derive probability distributions of both S t and R t; and to overcome this difficulty, we use, what is called as the linear decision rule. We will understand the linear decision rule, although this is I I must alert you that it is slightly a dated concept now, we have much more regress and much more sophisticated ways of addressing the chance constraints; however, as the background the students must know, how the chance constraints are typically handled in a linear programming problem. We use the linear decision rule; linear decision rule you must remember always is, an approximate way of handling the chance constraints.

We are actually, if I may say so, taking a back door entry in into handling the chance constraints. What we are doing there is in principle, what was the function of two random variables; we are converting into a function of one random variable, by somehow treating the other random variable as a deterministic variable; that means both S t and R t are random; however, we will device some intelligent way, by which we will say that, although S t and R t are both random variables; we will treat only R t as random variable; and do something with S t, so that we it becomes deterministic. Then we do not have to deal with two random variable, we deal with only one random variable as the function of Q t, whose distribution is known. So essentially, this is the principle of LDR; that is, we are reducing the number of random variables, which are functions of Q t, which is the random variable. Now, the justification we give for this is, we will see that I will come to the explanation.

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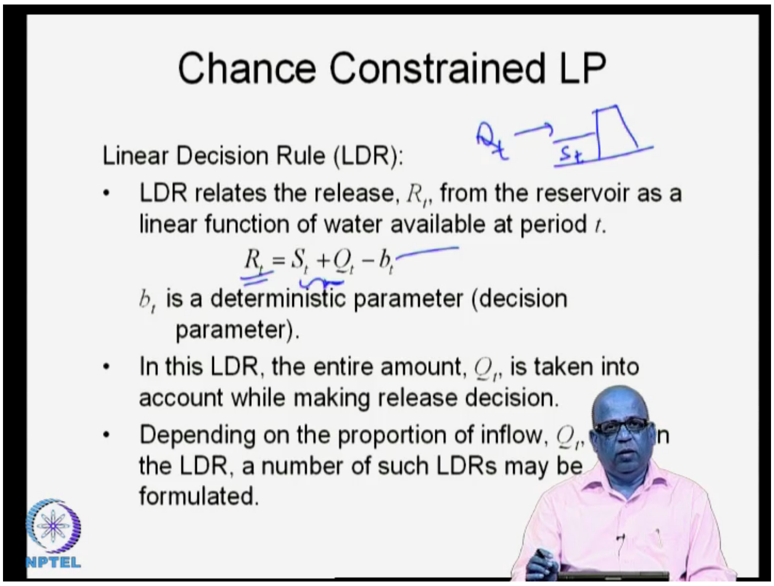

Now, look at the linear decision rule; we say this is linear decision rule, because we are relating R t as a linear function of the total water available in time period t. So, in a reservoir in time period t, you had S t, which is the storage; and Q t, which is the flow; and therefore, the total water available to make the release R t is S t plus Q t. Now, out as the total water available, I will now release only S t plus Q t minus b t. So, this is the water available, I will release S t plus Q t minus b t and b t is a deterministic parameter; that means we will say that b 1, b 2, b 3 etcetera have to be determined, but they do not have uncertainty associated with it, they remain the same across a across years; that means, b 1, b 2 etcetera up to b 12, if we are talking about the monthly operation will remain constant and they are deterministic.

So, what we are saying is, I will relate the random variable R t as S t plus Q t, which is which are both random variables, S t is the random variable, Q t is the random variable, minus b t, all I am saying is (( )) water S t plus Q t, I will release only S t plus Q t minus b t, which makes perfect sense, because there is so much amount of water available, and you will deduct b t, and then the remaining amount, you will release you will use it as release, and b t is the deterministic parameter, you can take it as greater than or equal to 0. We will see various forms of LDRs, in which b t need not be greater than or equal to 0, but b t typically can be a in this particular form of the LDR, you can take b t to be greater than or equal to 0; because we do this and this is the linear decision rule; that means that we apriori decide, a priori decide that my release in time period t will be a linear function of what is available. So, I will take S t plus Q t minus b t as release.

Now, in this LDR now, the entire amount Q t is taken to account for is taken into account for making the release decision; that is what we are saying is, we are making the release during time period t and we are taking into account the initial storage, which is known; the entire amount of Q t, we are accounting for in taking that is you know, R t; however, we may develop a general linear decision rule, in which we will account for only part of Q t, not the entire Q t. So, you may say that R t will be equal to S t plus some fraction beta Q t minus b t; that means I will consider 70 percent of Q t, 80 percent of Q t, while making my release decision. So, this is the idea that we use.

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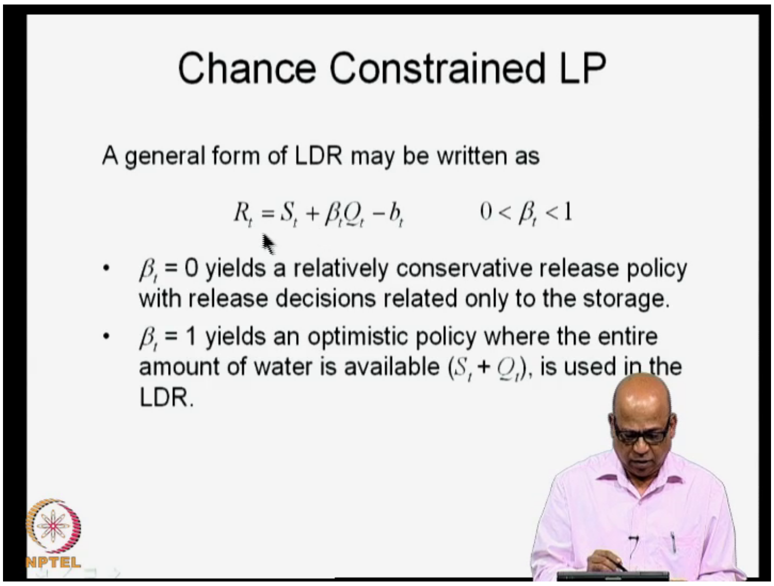

So in general, we can state the LDR is follows; R t is equal to S t plus beta t Q t minus beta minus b t (( )) equal to 1, this is for all t. So, we may have beta 1, beta 2 etcetera. So, you may use different proportions depending on your requirement, beta 1 (( )) from beta 2 etcetera, which means (( )) this, in this constraint is that out of the total (( )) is coming, I will take only 70 percent to account for the water availability in deciding the release, and the release is still a linear function of the S t as well as Q t, because b t is the, beta t is known and b t is the deterministic parameter.

Now, in this, if you put beta is equal to 0; that is beta t is equal to 0, what does it mean? It means that you are making the release based on only water available (( )) for the inflow at all; that is the conservative way of accounting for it, because you are only looking at, what is available; do not even look at what is likely to come during this time period, and then simply make the decision based on what is available in the storage. It is a conservative way of making decision; whereas, as the other extreme of beta is equal to 1, it indicates that you are accounting for the entire flow Q t in the time period while taking the decision on R t. So, that yields an optimistic policy; and in between you can have several policies.

So, the linear decision rule essentially, is introduced to make sure that you do not have to deal with two random variables, you have to deal with only one random variable as the function of Q t, whose distribution is known. How that you have linear decision rule simplifies the optimization model? So that we can solve it, obviously you know, when we are looking at the chance constraint like this; you cannot put this chance constraints directly into linear programming problem; unless you convert them in into deterministic forms, and then use the deterministic forms in the chance constrained. For example, you cannot directly put probability of R t greater than or equal to D t greater than or equal to 0.7 or 0.8 and so on, into ALP problem, you need to start looking at the distributions of them, and then use the distributions and make sure that in the linear programming problem, you are you are posing these constraints, you are stating these constraints in a linear form.

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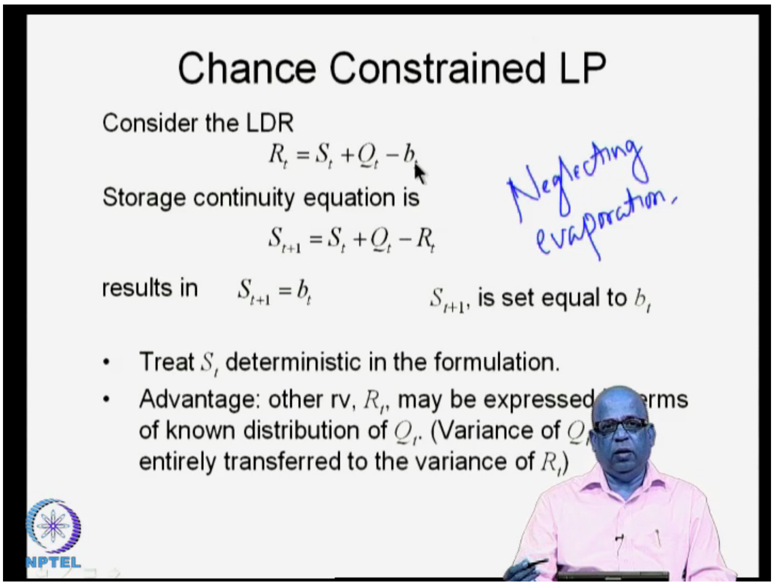

Now, that is what we will do now; that means, we have a chance constrained of the form (( )) probability of R t being greater than or equal to D t is greater than or equal to alpha 1; that means the reliability which which the demands are met in period t must be at least equal to alpha 1 is what we are stating. So, we will take a LDR here, LDR is the linear decision rule. So, we are saying R t is equal to S t plus Q t minus b t.

Now, we have the storage continuity equation; remember in all of these (( )) neglecting the evaporation. (No Audio from 39:20 to 39:31) So, what we will do is, we will consider the storage continuity equation S t plus 1 is equal to S t plus Q t minus R t this is the storage continuity. Now, when you put R t (( )) t in this. So, we are saying S t plus

Q t minus S t minus (( )) b t, this result in S t plus 1 is equal to b t; simply (( )) S t plus 1 is equal to b t.

Now, b t is a deterministic parameter and therefore, what did we do through this, we simply said that the S t plus 1 which is in fact, a random variable, z equal to a deterministic parameter and this is how we avoid… S t avoid treating S t as a random variable we are simply saying that S t plus 1 is equal to b t, which is the deterministic parameter unknown deterministic parameter, but it is deterministic still. So, the effect of using this LDR is to treat S t as deterministic in the formulation, that is the major achievement that we did through the linear linear decision rule.

So, the advantage of this is that the other random variable R t may be expressed in terms of the known distribution of Q t; Q t is the inflow and it is distribution is known. So, in effect what we are saying is that all the uncertainty in Q t is transferred only to R t and not to S t. This is the way we address the problem associated with dealing more than 1 random variable; obviously, it is not a elegant way, it is not a straight forward way, it is a sort of a devious way, if I may use the one in which we are we are saying that one of the random variable I will set it as deterministic variable; however, this simplifies the calculations a lot and it also addresses the uncertainty associated with the reservoir inflows. So, we get S t plus 1 is equal to b t the as a result of your linear decision rule. Remember now, we have already included the storage continuity equation, in deriving S t plus 1 (( )) is equal to b t. So, the movement I put S t plus 1 is equal to b t, we have included the reservoir continuity equation with this particular linear decision rule.

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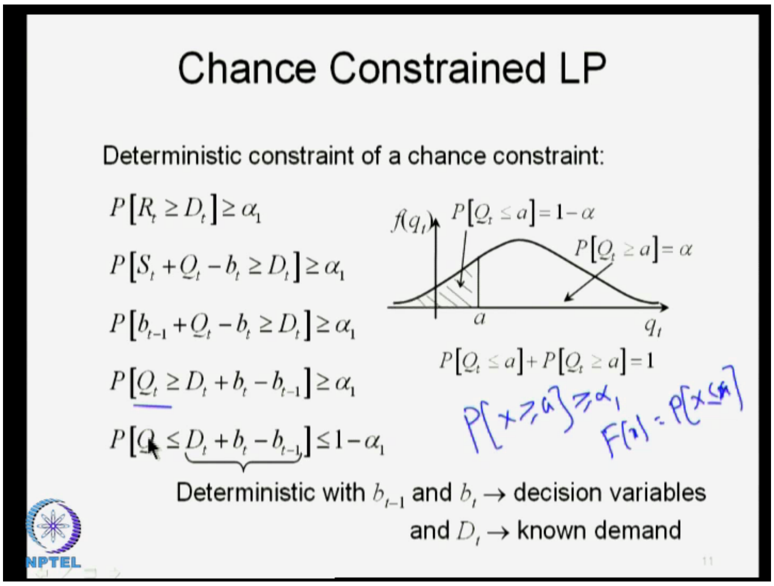

So, we will now, look at how the optimization model appears to be. We formulated the chance constraint probability probabilistic constraint as probability of R t being greater than or equal to D t greater than or equal to alpha 1. To use in the linear programming problem linear programming algorithm, we need to state this constraint in a deterministic form. We will see, how the deterministic equivalent of the chance constraint is formulated. So, this is probability of R t being greater than or equal to d t greater than or equal to alpha 1. Now, what is R t? R t is from this S t plus Q t minus b t. So, I will put for R t, S t plus Q t minus b t greater than or equal to d t greater than or equal to alpha 1.

Now, Q t is the inflow, which is the random variable, whose probability distribution is known. So, Q 1, Q 2, Q 3 etcetera all these random variables, we have already estimated the probability distributions and S t we have said, it is a deterministic variable; remember here, we got S t plus 1 is equal to b t. So, I can write S t as b t minus 1. So, the idea of doing this is, that I will express everything in terms of only one random variable. So, that is what we are doing now. So, S t is b t minus 1, because S t plus 1 is equal to b t I will write S t as b t minus 1 Q t a written as it is, b t as b t greater than or equal to D t greater than or equal to alpha 1.

And this one, I will write with Q t taking on the left hand side of the inequality, Q t greater than or equal to D t plus b t minus b t minus 1. So, I have taken Q t on the left hand side greater than or equal to alpha 1. Now, this is of the form probability of x being greater than or equal to a greater than or equal to alpha 1, so, this is of this form; but what is our f of x capital f of x, remember capital f of x is probability of x being less than or equal to a less than or equal to x, and therefore, always I would like to express the probabilities in terms of the random variable being less than or equal to some deterministic quantity.

Look at this probability of Q t being greater than or equal to a; Q t is the random variable. Let us say, this is a and this area is probability of Q t being greater than or equal to a, and this area is probability of Q t being less than or equal to a this will be 1 minus alpha if it is equal to alpha. So, if this area is alpha, you look at this one; you said this has to be greater than or equal to alpha 1, then that should be less this will be less than or equal to 1 minus alpha. So, when you change the inequalities inside, this inequality changes with 1 minus alpha 1; that means, you are pushing this towards this side and therefore, this becomes less than or equal to 1 minus alpha 1.

So, this will write it as probability of Q t being less than or equal to D t plus b t minus b t minus 1 less than or equal to 1 minus alpha 1. You have change the inequality inside therefore, this change with 1 minus alpha 1. Now, look at these quantities D t is known, b t is deterministic, b t minus 1 is deterministic; therefore, this right hand side is deterministic and Q t is the random variable. So, if we know the distribution of Q t, we we can express this in terms of f of Q t, because you are saying that Q t is less than or equal to some deterministic quantity. So, that is what we will do.

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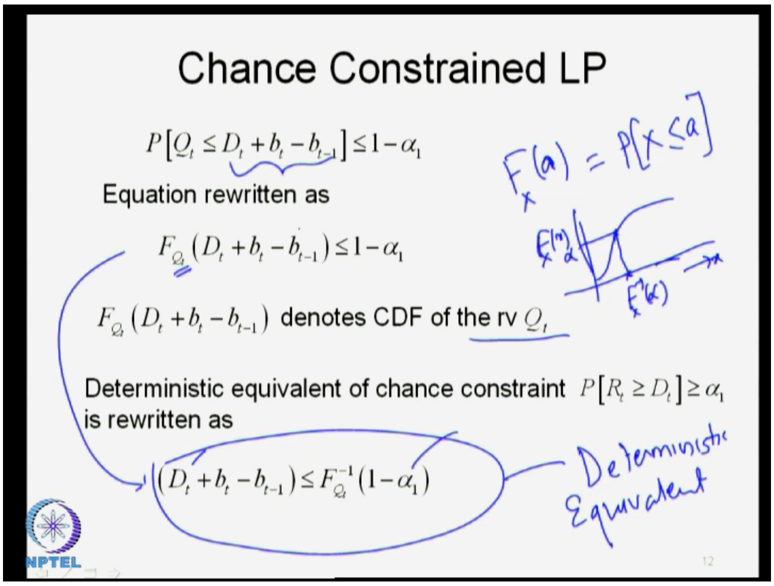

So, this term here inside the brackets, we write as, again I will repeat that probability of Q t being less than or equal to D t plus b t minus b t minus 1 is less than or equal to 1 minus alpha 1. So, for completeness sake, again you F of a F x of a, we write it as probability of x being less than or equal to a. We are saying, the cumulative distribution function of the random variable x, at x is equal to a is equal to probability of x being less than or equal to a. So, using that we write this as F Q t, which is the random variable of the deterministic quantity here, d D t plus b t minus b t minus 1 is less than or equal to 1 minus alpha 1.

Now, this is the CDF of Q t. So, F Q t (( )) is the CDF of Q t. We will understand another simple notation here; let us say this is your F of x. So, for a given level of alpha, we can write this as F inverse of alpha, and if we want to denote the random variable, we can write this as F x. So, corresponding to alpha the value of x on this x axis, we will be written as F inverse of that particular value alpha. So, that is what we do here. So, this is the deterministic equivalent, we are saying that F Q t of D t plus b t minus b t minus 1 must be less than or equal to 1 minus alpha 1, we write this as D t plus b t minus b t minus 1 less than or equal to F Q t inverse like here, F Q t inverse of 1 minus alpha 1.

Now, this becomes the deterministic equivalent. Now, look at this you have specified alpha let us say alpha 1 is 0.7, 0.8 etcetera. So, you does specified alpha and the CDF of Q t is known and therefore, you should be able to get this value, like I as mentioned here the moment you set alpha, because the CDF is known you will get F inverse corresponding to that alpha. So, this value will be known. This is known and these two are the decision variables. These two are the deterministic, they are decision variables therefore, this entire constraint here, becomes a deterministic constraint. In fact, we call it as deterministic equivalent of the chance constraint that we started with, that is of this chance constraint.

So, essentially what we did is, we started with the probabilistic constraint namely, probability of R t being greater than or equal to D t, use the linear decision rule, by using the linear decision rule, we could achieve setting one of the random variables to be a deterministic random variable, that is S t plus 1 is equal to b t; and then we went to the probabilistic constraint, and express the probabilistic constraint in a deterministic form by using the information on its distribution. So, the distribution is known, and you have set a particular value of alpha therefore, this value you can determine from the distribution, and D t is known b t and b t minus 1, they become decision variables. So, what what was the earlier decision variables R t, S t etcetera; have been now converted in terms of b t and b t minus 1; like this each of the constraints that we stated in probabilistic form, we can convert it into deterministic equivalent.

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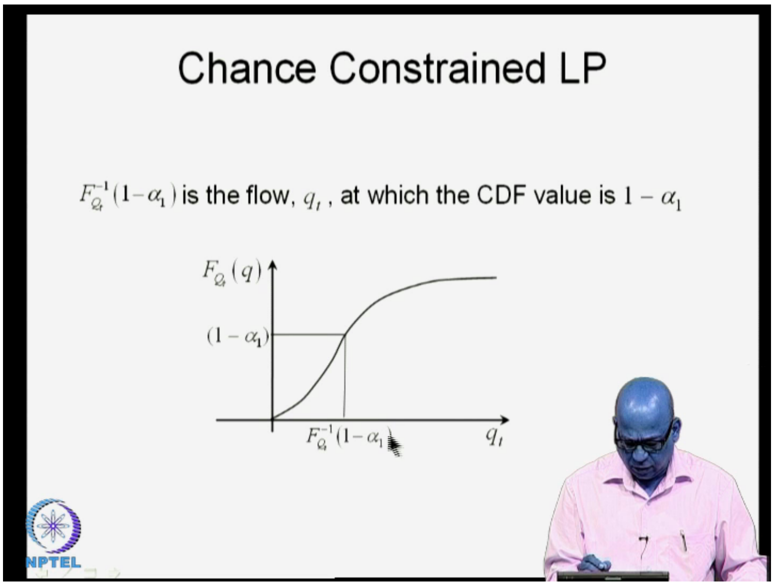

This figure in fact, shows what I mean by F Q t inverse 1 minus alpha. So, this is the distribution of Q t, which is the random variable and small q t the value that it takes; now, this if you look at this, for any specified value of 1 minus alpha, you go to the curve and come to the x axis, that indicates the value F Q t inverse 1 minus alpha 1, which been In fact, a value of flow, if you you are dealing with Q as a flow. So, this is how you can determine this quantity.

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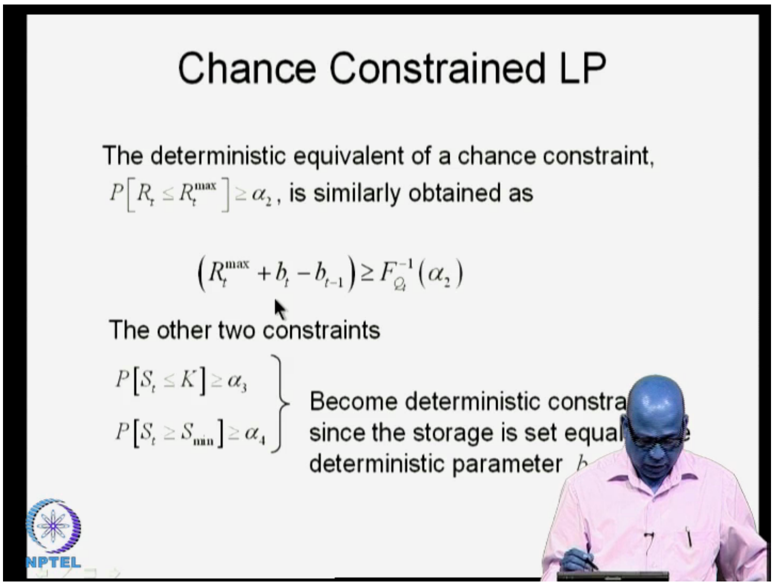

Now, let us say you take probability of R t being less than or equal to R t max is greater than or equal to alpha 2. So, this is another chance constrained; now, we want to write this chance constraint also in the form of a deterministic constraint, it turns out to be like this, but for a practice, let us do this now; that is probability of R t being less than or equal to R t max is greater than or equal to alpha 2, we will start with.

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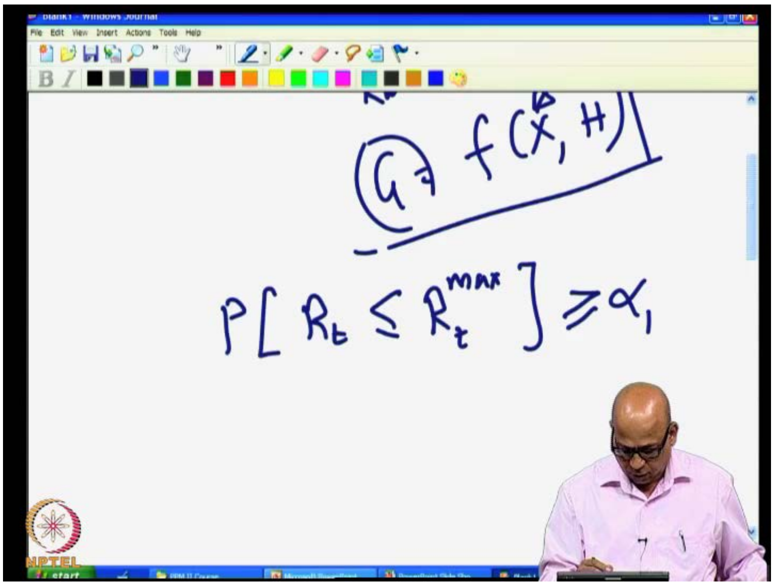

So, we will say probability of R t being greater being less than or equal to R t max is greater than or equal to alpha 1; (No Audio from 52:27 to 52:39) alpha 2 is what we have used, let us do it with respect to alpha 2; and then we will use this expression here, the same LDR, R t is equal to S t plus Q t minus b t, we will use this LDR and see, how we convert the second constraint, in terms of the deterministic equivalent.

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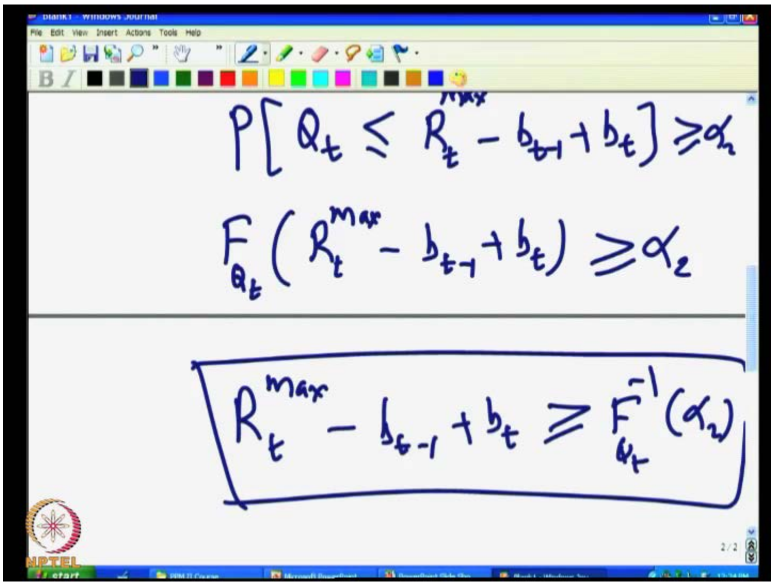

This we have said is greater than or equal to alpha 2, so I will put it as alpha 2 here. Remember, in converting any probabilistic constraint into a deterministic constraint, you should write the constraints, in terms of the random variable, whose distribution is known. So, in in this particular problem, we know the distribution of Q t. So, to convert any probabilistic constraint, you must be able to write those that particular constraint in terms of the random variable Q t. So, we will use the linear decision rule, we said this is S t plus Q t minus b t, this is the linear decision rule is less than or equal to R t max, which is greater than or equal to alpha 2.

Now, what is S t? We have put S t plus 1 is equal to b t so, S t will be equal to b t minus 1 deterministic plus Q t will written as it is, because it is distribution is known minus b t less than or equal to R t max greater than or equal to alpha 2. Now, this is deterministic, this is deterministic this is known therefore, I will written only the Q t on the left hand side Q t less than or equal to R t max minus b t minus 1 plus b t greater than or equal to alpha 2, what is this you can identify this as F of Q t of this particular value. So, I will write this as F Q t of R t max minus b t minus 1 plus b t greater than or equal to alpha 2 and this we will write as R t max minus b t minus 1 plus b t greater than or equal to F Q t inverse alpha 2.

So, this is the deterministic equivalent of the constraint R t greater than or equal to (( )) R t being greater than or equal to R t max must be greater than or equal to alpha 2. So, this is how we use we convert the probabilistic constraints into the associated deterministic constraints. (No audio from 55:48 to 55:55) Now, we will see the other constraint similarly, we now, wrote the second constraint also as deterministic constraint.

Look at the other two constraints now, S t less than or equal to K and S t greater than or equal to S min. We have already by using the linear decision rule. We have set that S t is the deterministic; because we are saying S t is equal to b t minus 1. So, we have set the storage as a deterministic variable already and therefore, these two constraints can be state away written as S t being less than or equal to K, then need not be written as probabilistic constraints, because S t is the deterministic variable.

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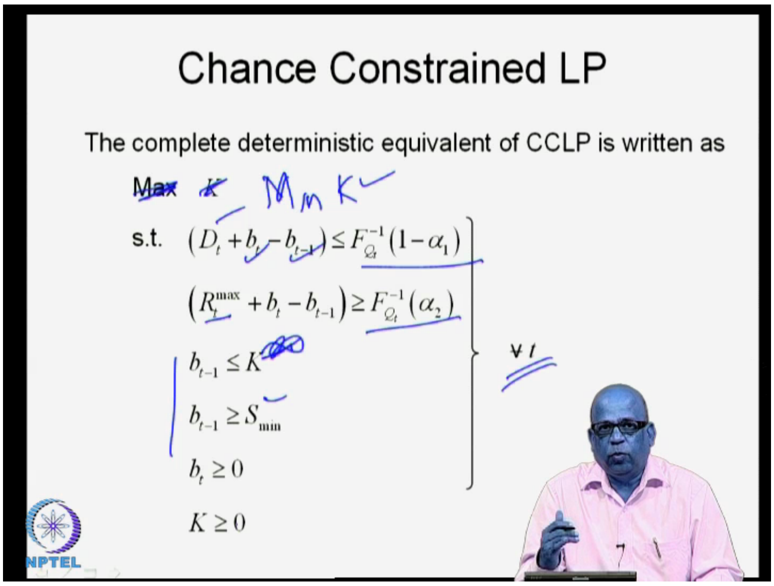

And then, we write the complete form of the deterministic equivalent, we write as minimize K, this is not maximize, this is minimize k, subject to D t plus b t minus 1, this is the first deterministic equivalent we got, this is the second deterministic equivalent you got, and then we are writing S t less than or equal to K as b t minus 1 less than or equal to K, b t minus 1 is the greater than or equal to S min b t is greater than 0 K is greater than 0.

This we have to write for all t, look at this now; what are the decision variables? k is the decision variable you do not know, what is the value of k; b t is the decision variable, b t minus 1 is the decision variable, this value is known, because alpha 1 you have fixed, this value is known, this is also known, b t are decision variables this is known, because alpha 2 is fixed, and these are decision variables; and this is known, this is this is to be determined. So, this can be solved using any linear programming algorithm.

So essentially, what we did through this exercise now is that by treating Q t as random and by using the distribution of Q t probability distribution of Q t, we reformulated the problem of the determination of a storage as a stochastic optimization problem. Any constraint that contains a random variable becomes a probabilistic constraint, and we state these probabilistic constraints, and then determine the associated deterministic equivalents of the probabilistic constraints.

For determining the deterministic equivalent equivalents of the constraints, we have use the linear decision rule, which actually converts one of the random variables into a deterministic variable. So, typically we use R t is equal to S t plus Q t minus b t in the example that I showed, and then by using the linear decision rule, we can convert the probabilistic constraints into deterministic constraints, and then we can use the linear programming algorithm to solve this.

Now, in the particular way of a handling this probabilistic constraint that I just showed, the b t and b t minus 1, the b t is will become decision variables along with the capacity K, and once b ts are fixed your storage is are fixed, because S t plus 1 is equal to b t and then R t can be determined and so on, using the continuity equation. So, we will continue this discussion and solve an example, using the chance constrained linear programming problem in the next class, thank you for your attention.