Water Resources Systems

Modelling Techniques and Analysis

Prof. P.P. Mujumdar

Department of Civil Engineering

Indian Institute of Science, Bangalore

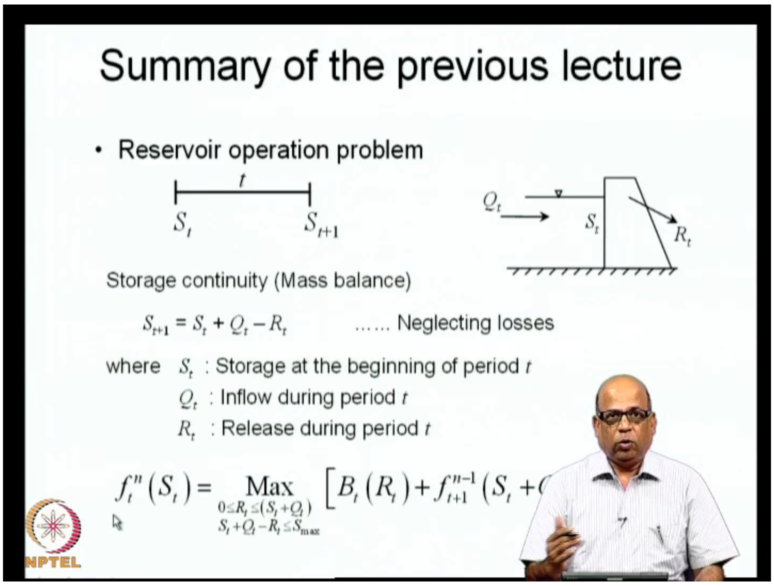
Lecture No. # 17

Dynamic Programming: Capacity expansion and shortest route problems

Good morning and welcome to this, the lecture number 17 of the course water resource systems modeling techniques and analysis.

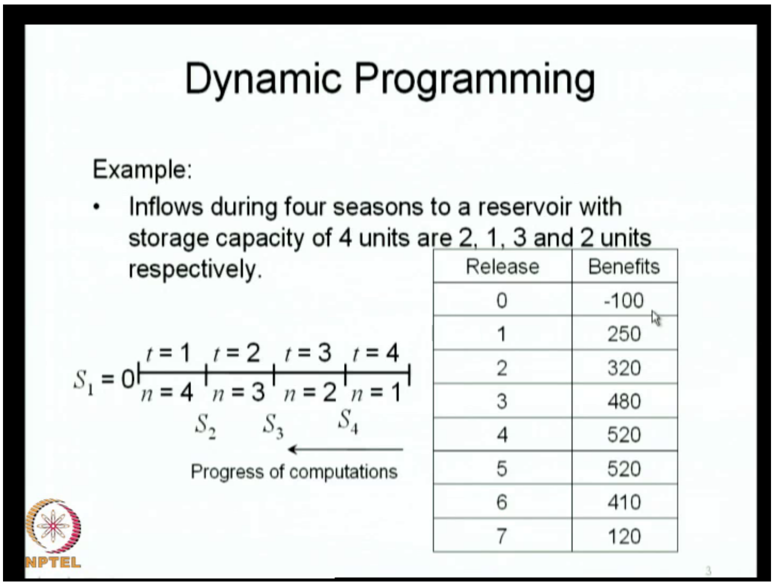
In the last class, I introduced the reservoir operation problem as a multi-stage decision problem and then, we formulated the dynamic programming structure for that particular problem.

(Refer Slide Time: 00:32)



For example, we are talking about the state transformation governed by the mass balance of the storage continuity in the time period t starting with storage S t. The storage S t plus 1 can be obtained by S t plus Q t minus R t neglecting the losses, and we also formulated the recursive relationship and we solved, we started solving a numerical example in which, the S 1 which is the storage at the beginning of the first time period is known to be 0, and the inflows during the four seasons are known, t is equal to 1 2 3 4 etcetera are known to be 2 1 3 and 2 units. The storage capacity is 4 units.

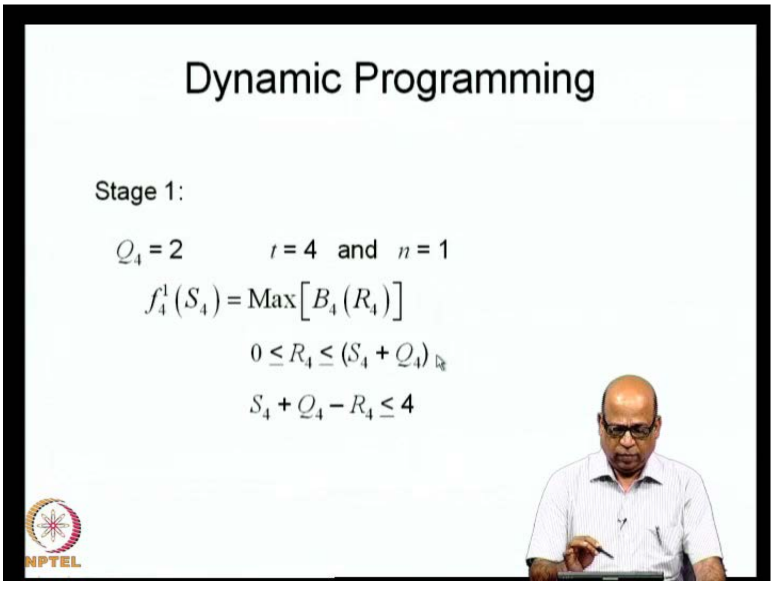
(Refer Slide Time: 01:03)



Then, you have the benefits associated with release. Now, these benefits as I keep mentioning need not be always economic benefits. They may be in terms of the hydropower that is generated, in terms of the crop field that is obtained, in terms of the flood control that you are able to achieve and so on.

So, these are the benefits which for these example, simple example we are assuming that they are the same for all the four seasons. In general, they can be different for all the four seasons and they will they may also depend on the storages.

(Refer Slide Time: 02:03)

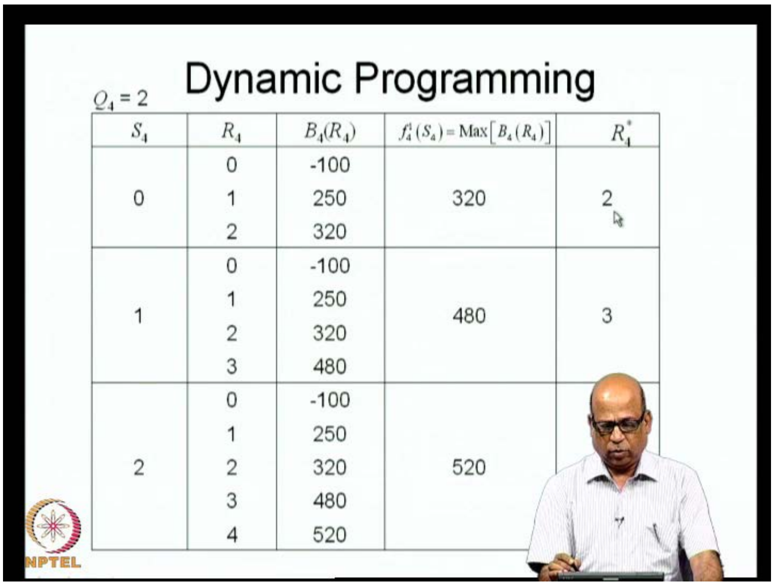


So for the stage number one, we solved. So, I will just rush through first stage calculation. Please refer to the previous lecture. So, in stage number one, what is the question we are asking?

If S 4 is known, S 4 is the storage at the beginning of the fourth time period, time period number 4 which corresponds to stage number 1. So, we write f 4 1 fourth time period, that is time period is equal to 4 and n is equal to 1, the stage number 1 because you do not have anything else to look beyond. It is simply maximize B 4 of R 4 and R 4 is a decision. Remember here that the storage plus inflow minus the release must be less than or equal to the total capacity available, which is 4 and the R 4 which is the decision that you are making will be taking on values between 0 and S 4 plus Q 4 because S 4 is the storage available plus Q 4 is the inflow that is coming there in that particular time period. Therefore, this is the total amount of water available. You can make releases from 0 to S 4 plus Q 4.

Now, this is what we saw in the previous lecture. I encourage you to go to the previous lecture, where I have discussed at length the formulation of the dynamic programming.

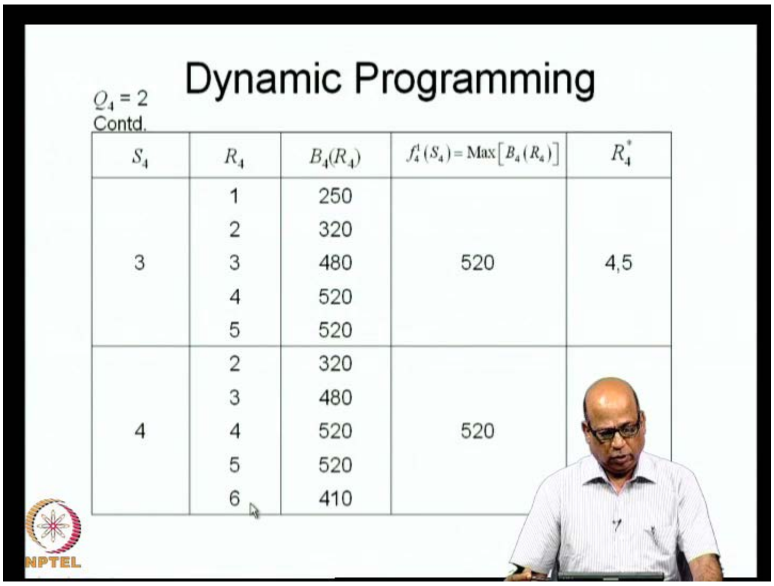
(Refer Slide Time: 03:25)



S_4	R_4	$B_4(R_4)$	$f_4^1(S_4) = \text{Max}[B_4(R_4)]$	R_4^*
0	0	-100	320	2
	1	250		
	2	320		
1	0	-100	480	3
	1	250		
	2	320		
	3	480		
2	0	-100	520	
	1	250		
	2	320		
	3	480		
	4	520		

Then, we also solve for stage number 1 in the previous lecture. Stage number 1 corresponds to time period 4, which means Q 4 which is given is 2. We ask the question, if S 4 is equal to 0, what should be my R 4? So, if S 4 is 0, R 4 can be 0, 1 or 2 because the total water available is 0 plus 2. So, you can either release 0, R 1, R 2 and so on. So, if you release 0, you get a benefit of minus 100, 1-250, 2-320 etcetera and you obtain the maximum of these, and that corresponds to R 4 is equal to 2. Like this, we keep going.

(Refer Slide Time: 04:08)

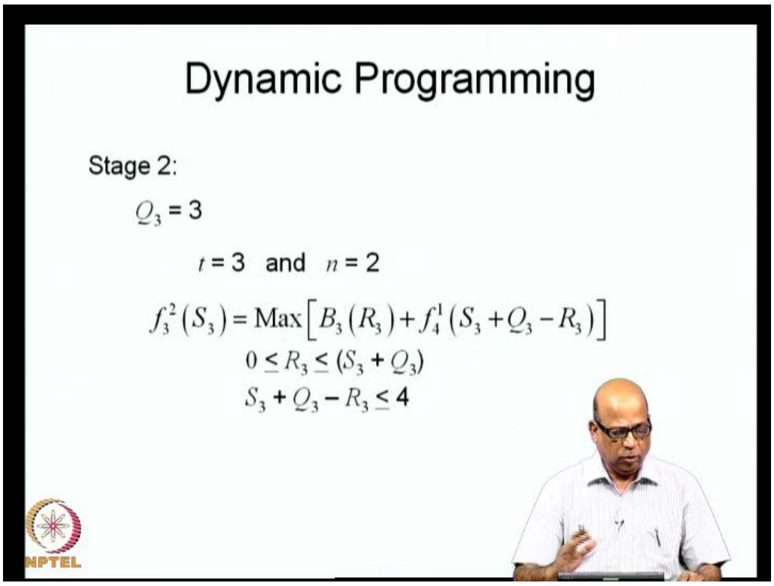


S_4	R_4	$B_4(R_4)$	$f_4^1(S_4) = \text{Max}[B_4(R_4)]$	R_4^*
3	1	250	520	4,5
	2	320		
	3	480		
	4	520		
	5	520		
4	2	320	520	
	3	480		
	4	520		
	5	520		
	6	410		

When you come to 3, you see that we start with R 4 is equal to 1 and not R 4 is equal to 0 because 3 plus 2 is 5, which is more than the storage capacity 4 and therefore, we start with R 4 is equal to 1. We go up to 5. So, R 4 can take on values up to 5.

Similarly, when I, when we come to X 4 is equal to 4, 4 plus 2 is 6 and therefore, minimum of 2 has to be released. Remember the release also includes the overflows, that is, the excess flow or the spill over and above the storage capacity is the overflow and that includes this. Therefore, for 4, we go up to 6.

(Refer Slide Time: 04:55)

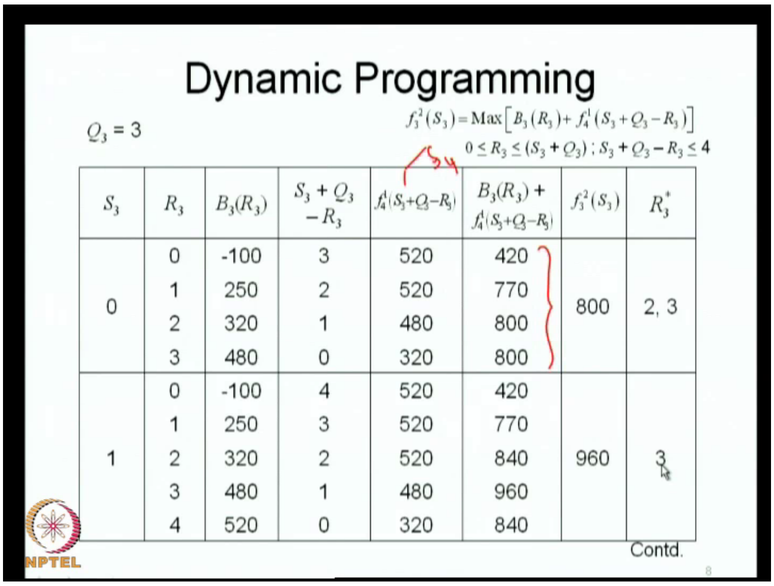


So, this is what we did in the previous lecture and then, we went to stage 2. In the stage 2, we are in t is equal to 3, period t is equal to 3 and stage is 2. We are asking the question, if S 3 is known, that means, if S 3 is equal to 0, 1, 2, 3 etcetera, what should be my R 3?

So, we are making a decision on R 3, such that the total benefit starting with this particular stage and all the previous stages included. In this particular case, there is only one previous stage. So, the previous stages, the total benefits should be maximum and R 3 again takes on values between 0 and S 3 plus Q 3, which is the total water available. Q 3 is known and S 3 is the particular S 3 value for which we are solving this particular recursive relationship. S 3 is such that, S 3 plus Q 3 minus R 3 should be less than or equal to the total capacity 4.

Now, this is where we stopped in the previous lecture. Now, we will continue to the example.

(Refer Slide Time: 06:10)



S_3	R_3	$B_3(R_3)$	$S_3 + Q_3 - R_3$	$f_4^1(S_3+Q_3-R_3)$	$B_3(R_3) + f_4^1(S_3+Q_3-R_3)$	$f_3^2(S_3)$	R_3^*
0	0	-100	3	520	420	800	2, 3
	1	250	2	520	770		
	2	320	1	480	800		
	3	480	0	320	800		
1	0	-100	4	520	420	960	3
	1	250	3	520	770		
	2	320	2	520	840		
	3	480	1	480	960		
	4	520	0	320	840		

So, we will solve the recursive relationship for various values of S 3 and obtain the corresponding R 3. The question we are asking at this stage is, if S 3 is equal to 0, what should be my R 3? If S 3 is equal to 1, what should be my R 3 and so on, such that the total benefits accrued up to that particular time period in the backward direction, including all the other time periods is maximum. So, let us say S 3 is equal to 0; R 3 can be either, 0, 1, 2 or 3. If R 3 is 0, remember in the last class I also told how to set up these columns. Just look at these terms here and then, set up the columns.

So, if R 3 is 0, then I get a benefit of minus 100 and if R 3 is 0, I will have the end of the periods storage as 3 because 0 plus 3 minus R 3, which is 0, 3 and corresponding to this, what is this? This is the storage at the beginning of fourth time period equal to 4 and for time period equal to 4, we have already solved and this defines S 3 plus Q 3 minus R 3 defines S 4, which is in fact the state variable for stage number 1. So, S 4 is my state variable for stage number 1 which we have already solved. So, corresponding to this value of S 4, we go back to the stage number 1 calculation and it says, 520 is the f for 1 corresponding to S 4 is equal to 3, so that we take it as 520 and these two together will give you total benefit of 420.

Similarly, if R 3 is 1 starting with S 3 is equal to 0, your benefit accruing B 3 R 3 is 250 because R 3 is 1, your S 3 plus Q 3 minus R 3 will be equal to 2, that is, 0 plus 3 minus 1, that is, equal to 2. Corresponding to 2, you go back to the previous table, and pick up the optimal objective function value, which is 520 and that is the 520 that you put here. 520 plus 250 is 770 and so on. That like this you calculate.

So, 420, 770, 800, 800 and out of these, you will pick the maximum value because your objective function is maximization. So, this is 800. So, this is what defines your F 3 to S 3, which means for a given S 3 in this particular case, S 3 is 0. For a given S 3, what is the total maximized objective function value up to that point is what we are answering here. So, this will be 800 and the R 3 star is the corresponding release, which results in this particular objective function value, that is, either 2 or 3. Both of them result in an objective function value of 800 and therefore, R 3 star is 2 and 3.

Similarly, you come to 1 now, that is, we are asking the question, if S 3 is equal to 1, what should be the optimal release? So, if S 3 is equal to 1, we have a total amount of water of 1 plus 3, 4. So, you can go up to 4, 0, 1, 2, 3, 4 and if you go with 0, 1, minus 0, 4 is left or 1 plus 3 minus 0, 4 is left. Then, if you go with 1, 1 plus 3 minus 1, 3 is left. 1 plus 3 minus 2, 2 is left and so on, so 4, 3, 2, 1, 0. This is the amount of water available for the next time period which is S 4 and corresponding to S 4, you would have solved earlier, pick up the corresponding objective function values from the previous stage computations and then, add these two. The immediate benefits at this particular time period plus the optimized benefits of all the time periods, until this particular time in this particular case, it is only one time period.

So, 520 minus 100 is 420, 520 plus 250 is 770 and so on. So, you will get values like this. Then, pick up the maximum of that. That will be 960, that corresponds to 3, which is R 3 star is 3, which defines R 3 star to be 3.

(Refer Slide Time: 11:10)

Contd. $Q_3 = 3$

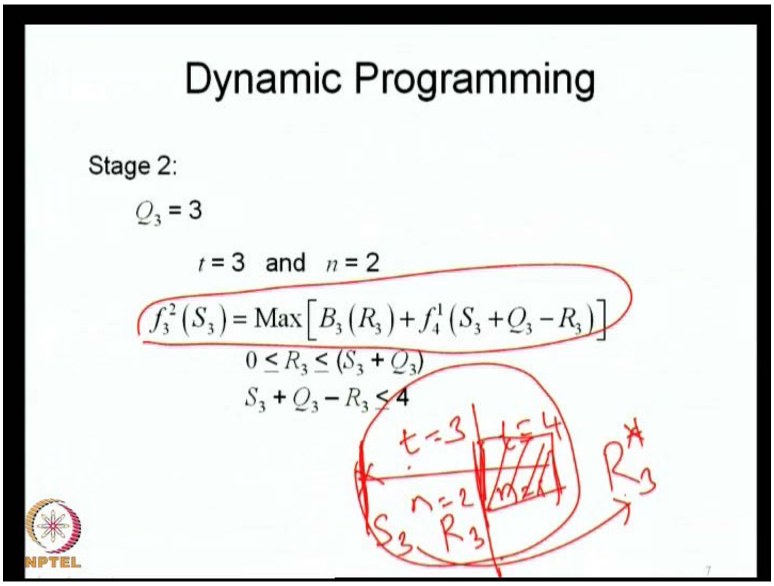
S_3	R_3	$B_3(R_3)$	$S_3 + Q_3 - R_3$	$f_4^1(S_3+Q_3-R_3)$	$B_3(R_3) + f_4^1(S_3+Q_3-R_3)$	$f_3^2(S_3)$	R_3^*
2	1	250	4	520	770	1000	3, 4
	2	320	3	520	840		
	3	480	2	520	1000		
	4	520	1	480	1000		
	5	520	0	320	840		
3	2	320	4	520	840	1040	4
	3	480	3	520	1000		
	4	520	2	520	1040		
	5	520	1	480	1000		
	6	410	0	320	730		
4	3	480	4	520	1000	1040	4, 5
	4	520	3	520	1040		
	5	520	2	520	1040		
	6	410	1	480	890		
	7	120	0	320	440		

Then, you go to 2, do the same calculations again. Remember, we are starting with a release of 1 and not as 0 earlier, as earlier because 2 plus 3, 5 is 1 unit above the capacity which is 4. Therefore, the minimum release that you have to make is 1. So, 2 plus 3, you go up to 5, 1, 2, 3, 4, 5. You can make a maximum release of 5. Do the same calculations as we did earlier and then, obtain R 3 star as 3 or 4 and the corresponding objective function value is 1000. Go to 3, 3 plus 3 is 6. So, a minimum release of 2 has to be made because the capacity is 4. Then, you can go up to 3 plus 3, 6, so 2, 3, 4, 5, 6. 2, 3, 4, 5, 6 corresponds to these releases.

Similarly, you obtain the next time period storage 3 plus 2, 3 plus 3 minus 2, that is, 4. 3 plus 3 minus 3, 3 and corresponding to this go to the previous table and pick up the maximum objective function value. These are the maximum objective function values and some of these you will get this, pick up the maximum and so on. Like this, you keep on continuing until S 3 is equal to 4. So, this completes the stage number 2 calculations. So, stage number 2 corresponded to period number 3.

Now, we will go to the next stage, stage number 3. So, remember in stage number 2, essentially what we did is, that we connected what we have already solved.

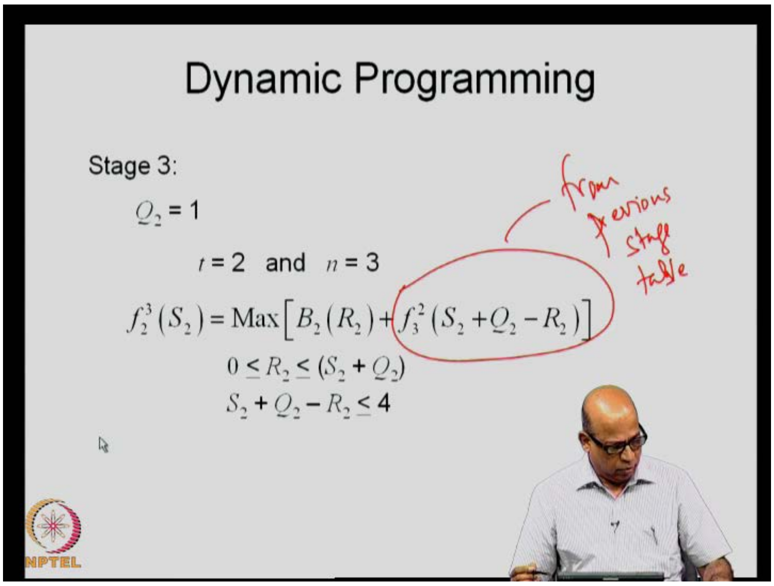
(Refer Slide Time: 12:51)



This is t is equal to 4 and t is equal to 3, n is equal to 1 and n is equal to 2 and stage number 2, we dealt with t is equal to 3. We were here. We said if the storage at the beginning of the time period 3 is known, then what should be my release R 3, such that the total objective function value including what we have already solved for stage number 1 is equal to 4? The total objective function value is maximized and this is where we are using the Bellmen's principle of optimality because we have already solved for various possible values of S 4.

We know that the moment you know S 4, we know what is the optimal objective function value and we get this S 4 from the storage transformation equation, namely S 3 plus Q 3 minus R 3 will define the storage at this particular time neglecting losses and the moment we know the storage, we know what the optimal objective function value for that is. Then, these two we add and get the optimal objective function value at stage 3, and we also obtain R 3 star corresponding to each of the S 3. So, this is what we did in stage 2.

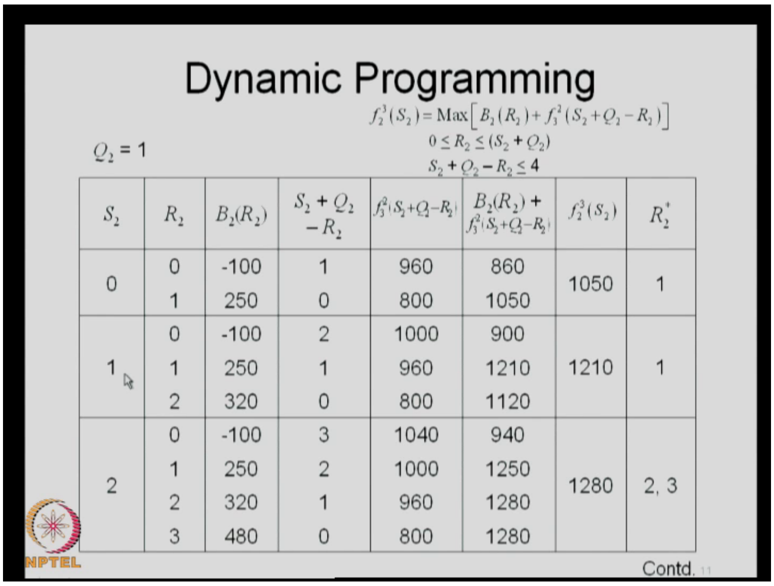
(Refer Slide Time: 14:26)



Similar computations we do for stage 3. Stage 3 will be much like stage 2 calculations. So, in stage 3, we are dealing with the time period t is equal to 2. So, t is equal to 2, n is equal to 3, that is stage number 3 and Q 2 is given, Q 2 is 1. So, we write the recursive relationship f 2 3. This is time period and this is the stage is equal to maximize D 2, R 2 which is the immediate returns from that particular time period, which is included now corresponding to a release of R 2 plus this term here is from the previous stage, from previous stage table. So, starting with S 2, making a release of R 2 for a given Q 2, we obtain S 2 plus Q 2 minus R 2 as the storage at the beginning of the next time period t is equal to 3 and associated with that, we have solved already. If we know the storage at the beginning of the time period 3, we know what is to be done, what is the particular optimal objective function value. This we have done in the previous stage calculation.

So, we pick up the objective, optimal objective function value corresponding to S 3 plus Q 3 minus that is S 2 plus Q 2 minus R 2, which defines S 3 from the previous stage calculation. That is what we do here. S 2 is that particular value of S 2 for which we are doing this calculation. R 2 is a search that we are making and Q 2 is known and it is constant and we are ignoring losses. So, that is what we do now again.

(Refer Slide Time: 16:34)



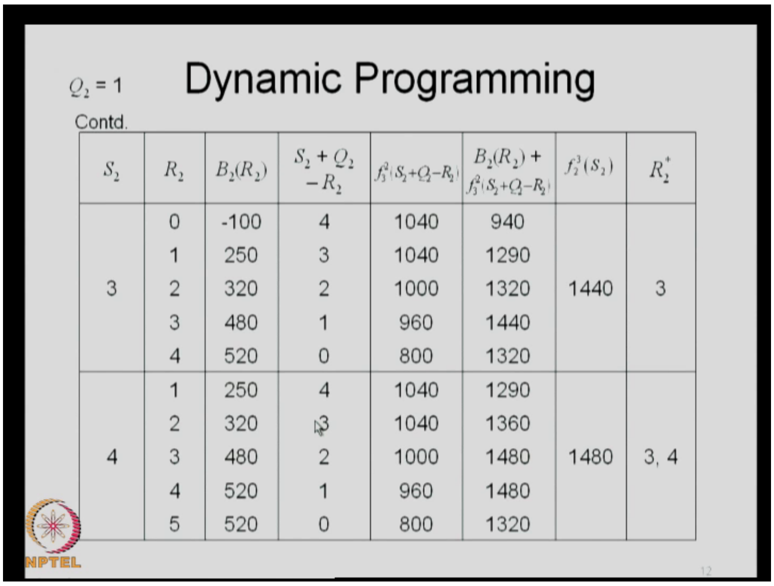
So, what we then do is, similar to what we did in the stage number 2. We set up the recursive relationship. This is f 2 3, that is time period 2, stage 3. S 2, which is the storage at the beginning of time period 2 is equal to maximize B 2. R 2 which is the return associated with a release of R 2 plus the optimized objective function value in the previous stage corresponding to the storage S 2 plus Q 2 minus R 2 and these two constraints are similar to what we did in the previous stage.

So, we start with S 2 is equal to 0, 0 plus 1. So, you can go up to 1. So, R 2 can be either 0 or 1. If R 2 is 0, the benefit is B 2 of R 2 is minus 100. If R 2 is 1, it is 250. If R 2 is 0, then 0 plus 1 minus 0 is 1. Corresponding to 1, you go to the previous table, corresponding to S 3 is equal to 1, you pick up the objective function value directly, 960. So, that is 960, so 960 minus 100 that is 860.

Similarly, if S 2 is equal to 0 and R 2 is equal to 1, you are left with 0 for the next stage because 0 plus 1 minus 1, that is 0. Go to the previous stage calculation corresponding to 0, it is 800. The objective function value is 800. Put that 800 here. 800 plus 250 that is 1050. In case of any doubts, you always refer to this equation. What we are doing is we are calculating for each possible value of R 2 for a given S 2, where calculating the terms B 2 R 2 plus this term.

So, this term we are picking up from the previous table, and we are calculating this term now, B 2 R 2 plus F 3 etcetera. Then, among all the possible values of this term here corresponding to a given S 2, we pick up the maximum value. So, that is what we get here as 1050 and R 2 star is the associated value of R 2, which results in this particular objective function value. In this particular case, it is 1. Therefore, we write 1. One more (( )) computation I will show you and then, you can verify all the other tables.

(Refer Slide Time: 19:12)



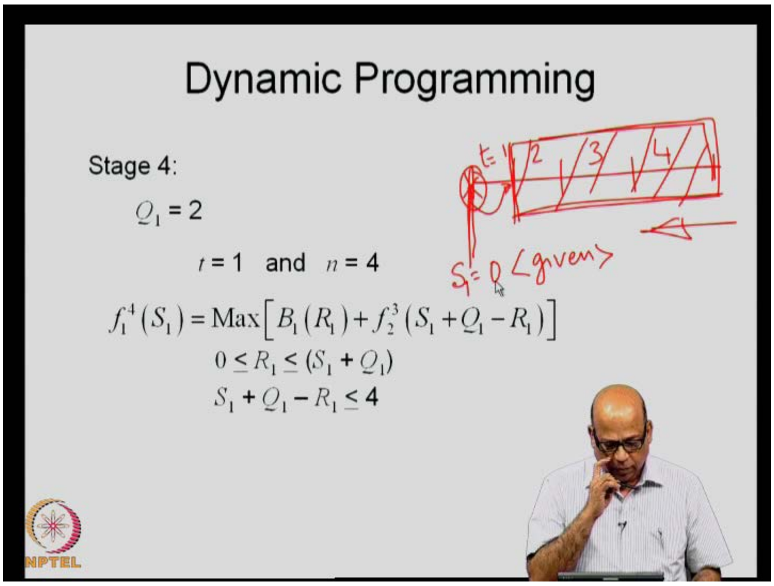
S_2	R_2	$B_2(R_2)$	$S_2 + Q_2 - R_2$	$f_3^2(S_2+Q_2-R_2)$	$B_2(R_2) + f_3^2(S_2+Q_2-R_2)$	$f_2^3(S_2)$	R_2^*
3	0	-100	4	1040	940	1440	3
	1	250	3	1040	1290		
	2	320	2	1000	1320		
	3	480	1	960	1440		
	4	520	0	800	1320		
4	1	250	4	1040	1290	1480	3, 4
	2	320	3	1040	1360		
	3	480	2	1000	1480		
	4	520	1	960	1480		
	5	520	0	800	1320		

So, this is what we do. Let us say that S 2 is equal to 3, you have Q 2 is equal to 1, so 3 plus 1 is 4. You are still within the capacity. Always check that the R 2 should be such that the end of the period storage that you get here should be a maximum of 4. So, 3 plus 1 minus 0 is 4. Corresponding to 4, you pick up from the previous table, the objective function value from the previous stage calculation, objective function value which comes to be 1040. So, you write 1040 here, and 1040 minus 100 is 940.

Similarly, let us look at another R 2 value 4 here. 3 plus 1 minus 4 that will result in value of 0 here and corresponding to 0, you go to the previous stage calculation and pick up the objective function value, optimal objective function 800. Remember all of these other values you do not worry. Simply look at the optimal objective function value corresponding to a given S 3 in this particular case. So, this is 800 and that is 800 that you have put somewhere here. Yeah here 800 you put. So, 800 plus 520 is 1320. So, you get 1440 and so on. Like this you complete for all possible values of S 2.

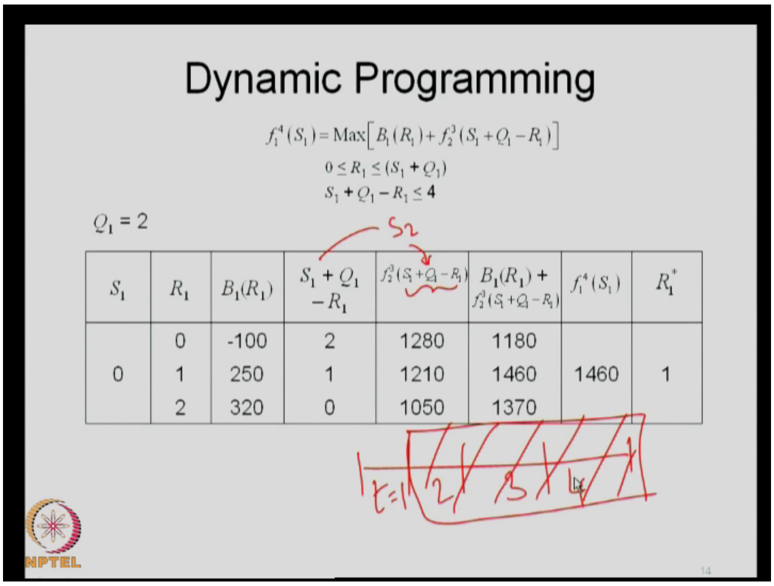
So, when you do the third stage calculation, what is the question that you answered? We answered the question, if at the beginning of the third stage that is stage number 3, which corresponds to period number 2. Now, if the storage at that beginning of the time period 2 is known, what should be my R 2, such that the total benefit inclusive of period number 1, period number 4, period number 3 and period number 2 is maximized? That is the question that you have to answer. We are proceeding in the backward direction and then, you are optimizing until that particular point where you stand which is at the beginning of time period 2. So, this is what we answered in this stage.

(Refer Slide Time: 21:43)



Now, we come to the fourth stage, that is, stage number 4. It is t 1 is equal to 1 now and n is equal to 4. We have to solve for values of S 1. What are the values of S 1? The problem statement is that you have fixed the storage at the beginning of time period 1. So, this is t is equal to 1, 2, 3, 4 and we proceeded in this direction and we have now reached this point. This point the storage is known. So, S 1 is equal to 0. This is given in the problem. So, you solve for S 1 is equal to 0 because that is given. So, S 1 will not take any other values unlike your S 2, S 3, S 4 etcetera. It will take on only one value, namely S 1 is equal to 0 and all other conditions remain the same. B 1 of R 1, which is immediate return from that particular time period 1, S 1 plus Q 1 minus R 1 will define the storage at the beginning of time period 2 here, and we have already solved for that particular stage. Essentially, what we have done is, at the time period 2, we would have solved in stage number 3, 1, 2, 3. We would have solved for all of these periods together. So, we just relate from this stage from the beginning of this time period, we relate with what we have already solved in the previous stage which included all other periods.

(Refer Slide Time: 23:48)



So, this is what we do in the stage number 4. So, Q 1 is given as 2 and we will solve this for S 1 is equal to 0 because S 1 is 0, S 1 is 0 in the problem. For all other stages, we solved for all possible values of the storage at the beginning of that particular time period, in that particular stage because we did not know the storage. We ask the question in other stages, if my storage at the beginning of this time period is so much, then what should be my optimal value, whereas when we come to time period 1, we know that the storage at the beginning of the time period is 0. So, we solve the problem for stage number 4, which corresponds to t is equal to 1, time period t is equal to 1 for the given value of storage. So, S 1 is 0 and Q 1 is 2. So, R 1 can go on up to 2. R 1 can take on values up to 2, so 0, 1, 2. If R 1 is 0, then these are the benefits you can pick up from the table for time period t is equal to 1. The benefits remain the same for all the time period.

So, it is minus 100, 250 and 320. If R 1 is 0 starting with S 1 is equal to 0, 0 plus 2 minus 0 which is 2. This defines the end of the period storage or the beginning of the period storage for next time period. This would be actually S 2. So, this would be S 2 and corresponding to S 2, we have already solved in the previous stage. So, corresponding to that, you would pick up the previous stage optimal value say, for example, S 1 plus Q 1 minus R 1 is equal to 2. Go to the previous stage calculation, look at S 2 is equal to 2, pick up the value which is 12, 18 and that is what we write here, 12, 18. Similarly, 0 plus 2 minus 1 is 1.

So, go to the previous stage value, look for 1 that is S 2 is equal to 1. The optimal objective function value is 1210. So, you write that 1210 here. Similarly, for 0, it is 1050 and you sum these up because that is sum, this and this 1280 minus 100 is 1180, 1210 plus 250 is 1460, 1050 plus 320 is 1370 and so on. Then, pick up the maximum that results from these, which means essentially what we are saying is given that S 1 is equal to 0, whether my R 1 should be 0 or R 1 should be 1 or R 1 should be 2 and this says that the maximum value resulting is 1460. Therefore, your R 1 should be 1.

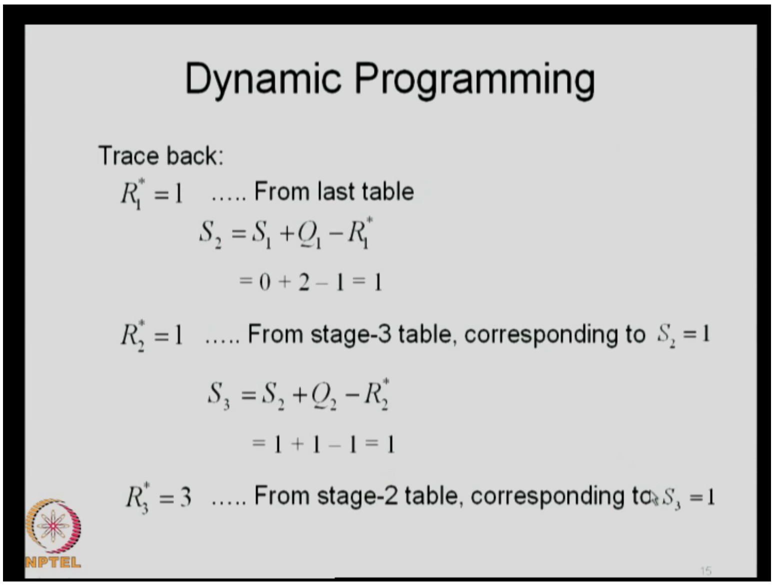
What is this maximum value? This is the maximum objective function value arising out of solution for all the four time periods; t is equal to 1, 2, 3, 4. All of these are included because you are relating it with the previous stage calculation, which included all these three periods optimal objective function value. So, this objective function value here, that is f 2 3 of S 1, Q 1, S 1 plus Q 1 minus R 1 has included all the other periods also up to that point and this where you are actually using the Bellmen's principle of optimality. That is what we are saying is starting with the known value of storage here, you proceed in an optimal way, optimal manner until the end of the horizon, until all the stages are completed. This is what we have done here.

Now, we have solved for the last stage which is t is equal to 1 or stage n is equal to 4. At this point now, we start tracing back the solution because you need the release is associated with the period number 1, 2, 3 and 4. The last stage calculations gave you the release for t is equal to 1, which is the time period 1. So, what it says because you are in S 1, you release 1. This is what it says, nothing else and it also says that your optimal objective function value until the end of the time horizon will be 1460.

Now, we have to trace back now. How do we trace back? In period t is equal to 1, you started with storage of 0 and you got an inflow of 2. So, 0 plus 2 minus 1 is the release that you made. So, 0 plus 2 minus 1, which is 1 will be the storage at the beginning of the next time period which is 2, t is equal to 2. With this storage, you go back to the previous stage calculation, which corresponds to period t is equal to 2 and for this particular storage that you obtain, you look up the corresponding R 1 star, R 2 star in that case and then, again do the mass balance. Look at the storage at the end of that period, which is t is equal to 2, a period t is equal to 2 and that defines the storage at the beginning of the time period t is equal to 3. Go to the time period 3 calculations which is the stage 2 calculation and look corresponding to that particular storage. You look up what is the optimal solution, apply that optimal solution for this particular time period, come to this time period, beginning of the time period, go to stage 1 calculation which corresponds to t is equal to 4, apply that particular release in this particular time period and thus, you get released during time period 1, 2, 3 and 4.

So, this is how we trace back the solution. After we obtain, after we solve the last time period, relate this with what we did in the water allocation problem. It is much similar. Simply arrive at the last stage calculations and then, keep applying the optimal decisions and in the forward direction now, you will obtain the trace back solution. Let us do that now. So, R 1 star is 1, which means 0 plus 2 minus 1.

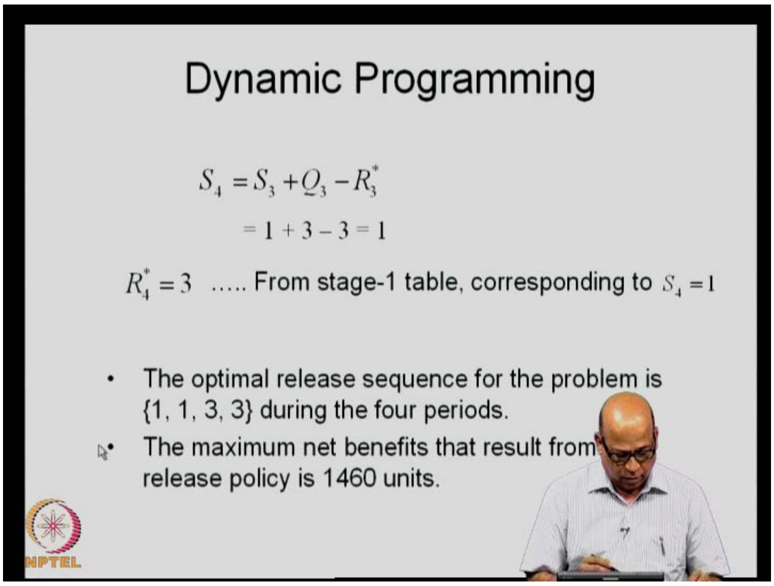
(Refer Slide Time: 30:25)



So, you will get S 2 is equal to S 1 plus Q 1 minus R 1 star. S 1 is 0, Q 1 is 2, R 1 star is 1. Therefore, you will get S 2 is equal to 1. Go to stage 3 calculation and look at what is the decision corresponding to S 2 is equal to 1.

So, you will go to stage 3 calculations and S 2 is equal to 1. It says R 1, R 2 star must be 1. So, will implement that, R 2 star is 1. So, that is S 2 which has 1, Q 2 which is given to be 1 and minus R 2 star, R 2 star is 1 from the table corresponding to the stage 3. So, 1 plus 1 minus 1 that is equal to 1. With this one now, you go to stage 2 calculations which means S 3 is equal to 1. So, look up for the table containing S 3 and S 3 is equal to 1. It says my R 3 star must be equal to 3.

(Refer Slide Time: 31:44)



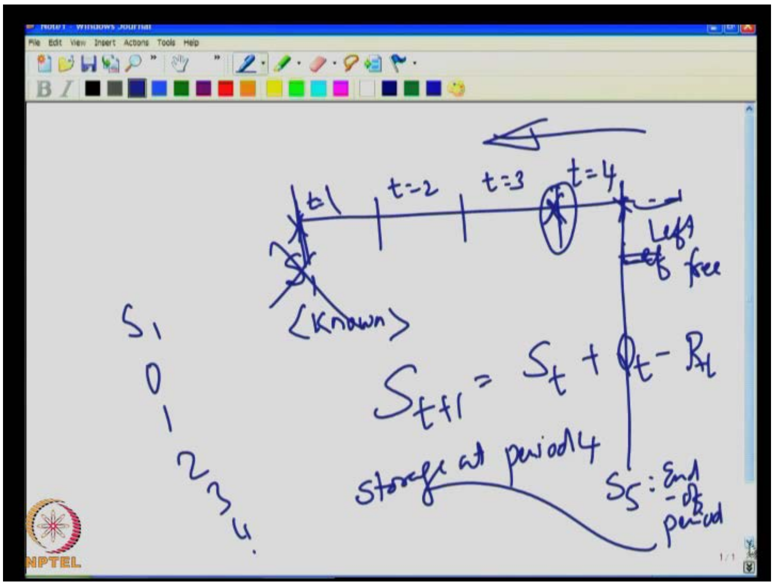
So, simply implement that R 3 star is equal to 3 and say S 4 is equal to S 3 plus Q 3 minus R 3 star, 1 plus 3 minus 3 which is equal to 1. This defines S 4, which is the storage at the beginning of the time period 4 which corresponds to stage number 1. Go to stage number 1, look for S 4 is equal to 1. So, stage number 1 we went and look for S 4 is equal to 1. It says R 4 star should be 3. So, R 4 star is 3.

So, this is how we trace back the solution after we have solved the problem completely. At the last stage you get the optimal decision and from that optimal decision, you trace back the solution. In all discrete dynamic programming problems of the type that we are talking about, this trace back will be necessary to obtain the optimal decision at various stages. So, the optimal release sequence for the problem is, you have 1 that is time period t, you had the solution 1, 1 3 3, which means that the release in the first time period is 1m second time period is 1, third time period is 3 and fourth time period is 3.

So, this is how we obtain 1, 1, 3 that is 1, 1, 3, 3. This is the sequence, optimal sequence of releases and the maximum net benefits resulting from this is 1460, 1460. This is the final objective function value. Remember you should not add now at this stage 1460 plus what you got here corresponding to that particular S 2 and so on. You should not add this objective functions again because you already added all of these. So, these terms, finally when you get this term, all of these objective function values, (( )) objective function values have been added and this is the final optimal objective function value. So, that is the objective function value that you get here. So, this is the classical reservoir operation problem when you have discrete states of storage and inflows are also taking on discrete values and release. Therefore, we will also take discrete values and we have ignored the losses here.

Now, this is the typical problem where we have specified the storage at the beginning of time period.

(Refer Slide Time: 34:37)



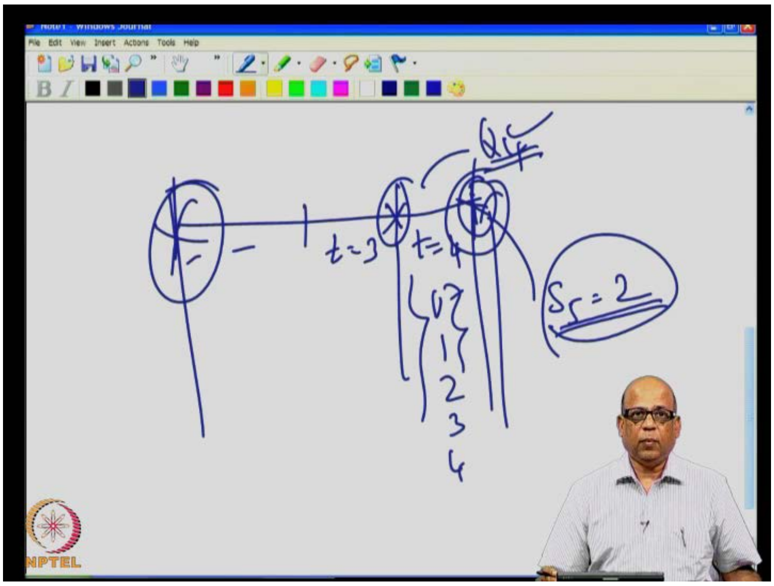
Let us say, what we did here is that the storage at certain point you must remember now, you had four periods, S 1 and the storage at the beginning of the time period. So, this was known in the particular case and this was left free, that is storage at the end of the fourth time period. Let us say this 1 2 3 and 4. So, the storage at the end of the fourth time period was left free and therefore, when you solved here for t is equal to 4, this was t is equal to 4, t is equal to 3, t is equal to 2 and 1. So, when we solve for t is equal to 4, what is that we did? We started with S. If S 4 is 0, S 4 is 1, S 4 is 2, etcetera like this we went far up to 4. Because this end was left free, we did not specify any storage at the end of the time period 4. Now, S 1 was set to be 0.

Similarly, S 1 could have been set to 1, set to 2, etcetera. So, these boundary conditions here we could have fixed based on some requirement. Let us say that we would have set S 1 should be 1 at this point. Now, the storage continuity equation e always relates the next time period storage with the previous time period storage. So, we are preceding the back ward direction here. Whatever the storage that we got here, we can relate it with the storage at the beginning of this time period using this and that is why, the backward recursive relationship is advantageous for this particular problem.

If you need to move forward, then you have to express S t. It becomes messy because you will be solving S t in terms of S t plus 1, whereas the continuity is in the forward direction. Therefore, for reservoir operation problems, the backward direction is advantageous. There is one more issue here. Now, we fixed S 1 to be 0. Let us say that we do not put any constraint on S 1. We simply leave S 1 also free, in which case you should solve this problem for S 1 taking on value 0, 1, 2, 3, 4 and so on. What is the question that we are asking there that you are saying that you do not want to put any constraints on what should be my initial storage. We leave everything free and you give us the optimal operating policy.

So, we solve for 0, 1, 2, 3 and 4 even for S 1 and then, we state if your S 1 is so much, then this is the policy, if S 2 is so much, this is the policy and so on. So, we specify the policy as the function of the storage at the beginning of the first time period. This is what we do typically in the applications. When we go to the applications, you will appreciate the problem, but this is the subtle point that you must remember. S 1 need not be fixed because you specify the policy. The policy can be specified as a function of the initial storage at the beginning of the first year. Then, the other thing is along with fixing S 1. We may also state S 5 which is the end of period storage at 4, at period 4. You understand this slightly clearly because in the application, these things become important.

(Refer Slide Time: 38:51)



Let us say, t is equal to 4, t is equal to 3 and so on. Like this we went, what we did was, we started the computation at this point, and this was stage number 1. So, we said the storage at the beginning of this time period is known or we solved it for various values 0, 1, 2, 3, 4 etcetera for the storage at the beginning of this time. We did not worry about what is the resulting storage at the end of that time period. In many situations we may fix this storage also, that is storage at the end of the time horizon must be at least so much. Let us say that you are talking about over the year storages in which case, you would like to maintain a minimum amount of water in the reservoir at the end of the year. So, we may state that this storage is also fixed, in which case what you will solve here? Your Q 4 is known. So, your S 5 which is the storage that results from this particular S 4 to this Q 4 will be fixed.

Let us say that we fix S 5 is equal to 2 or something, then the way you solve is in this example that I solved when the end of the year storage was left free. What did we do? We said S 4 can be either, 0, 1, 2, 3 and 4. When we fix this storage, end of the period storage, you choose only those particular values of S 4 for this given Q 4, that result in exactly S 5 is equal to 2. So, your S 4 and R 4 combination for this given Q 4 should be such that your combination will result exactly in that particular value of 2.

So, while you may have 0, 1, 2, 3, 4, your R 4 combi R 4 values must be such that, your result S 5 is equal to 2. Similarly, you may say that S 5 should be greater than or equal to

2. Then, you can have those particular values of S 4 and Q 4 and R 4. Q 4 is known of course, such that the resulting storage value will be greater than or equal to 2 and so on. So, you can fix several boundary conditions in the initial stage as well as in the final stage. You can fix these boundary conditions when you are looking for the combinations or S 4, R 4, S 3, R 3 and so on. You must make sure that you will use only those particular combinations which are feasible, in the sense that they meet the conditions at you have set for the problem.

So, in any given table you should always look at whether you are meeting all the conditions or not. In fact, we did this in our calculation because every time we are checking, whether it is the resulting storage is in fact below the capacity of the reservoir or not. That is how we pick our combinations, that is the release combinations along with a given S values. So, there are several subtleties you should start applying and then, you will realize keep all these subtle points in mind in reservoir operation.

So, we solved two problems in using dynamic programming. One was the water user allocation problem that is a known amount of water is allocated to a number of users. The second one is the reservoir operation problem which is the classical problem in water resources, where discrete values of storage is the concern. Now, in the remaining time of this particular lecture, I will introduce two more problems, but I will not discuss these problems in great numerical details as I did for the other two problems.

Now, that you know, now that you are aware that you need to define the state variable, you must identify what is the stage, then you must also identify what is the decision variable, what is the state transformation from one stage to another stage. Once we are able to write all of this, then the problem is quite simple. So, for any given problem, you must be able to identify these four requirements, what is the stage, what is the state variable, what is the state transformation from one stage to another stage and what is the decision variable.

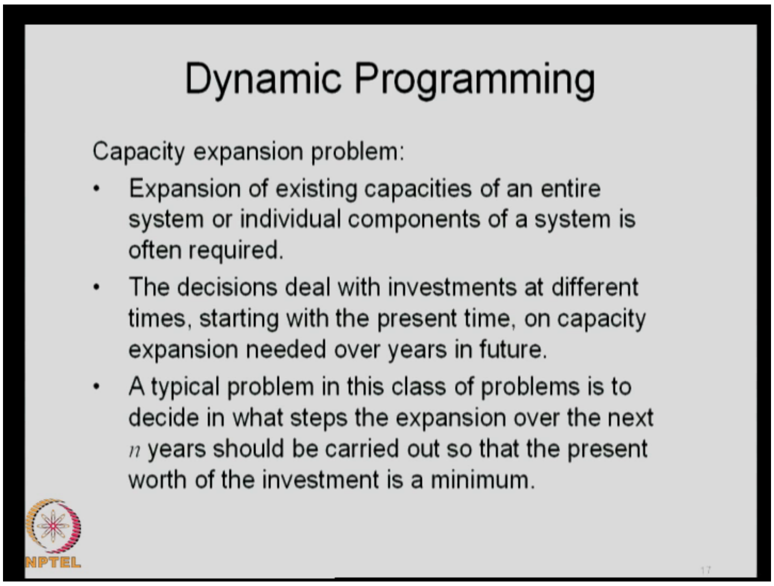
So, let us look at two more problems in the remaining time that I have today. One is another important problem in water resources, namely the capacity expansion. So, we will look at capacity expansion problem. What is this problem? You have large water resource systems in place, you are developing large water resource systems and these will have several components. For example, you may have a reservoir, you may have a canal network, you may have a hydropower plant, you may also have a (( )) scheme and so on. So, a water resource system when you are developing it, it involves developing several components and these components need to be added over a period of time. So, at the beginning of this time period t is equal to 0 when you are standing. Now, you need to make a decision on how do we expand the capacity, that is t is equal to 0, you have certain capacity. By capacity I mean, it need not only be reservoir capacity. If you have the reservoir, you may want to expand canal network over a period of let us say 20 years. You may want to expand the hydropower, you may want to expand the (( )) scheme and so on.

So, water resource systems by their very nature or large in size, large in space and therefore, they take significant time for complete development and it is important for us to assess how we are going to expand the capacity as time progresses. This is the problem of capacity expansion. So, we are asking the question starting at time period t is equal to 0, how do we plan for capacity expansion knowing that we need to invest certain amount of money at various times across. Let us say, you are looking at time horizon of 25 years. You have a given capacity at this particular time period. You are asking the question over the next 5 years, what is the capacity that I need to expand, from year number 5 to 10, what is the capacity that I need to put, 10 to 15 what is the capacity that I need to put and so on.

Knowing at the beginning itself that I need a certain capacity at the end of 5 years, a certain minimum capacity at the end of the 10 years, a certain minimum capacity at the end of 15 years and so on, so we specify a priory the minimum capacity that are required. Remember, this capacity you relate not just necessarily with the storage capacity. I keep repeating that it has to do with several components capacity. For example, you are talking about water supply systems stage 1 water supply enhancement, stage 2 enhancement and so on.

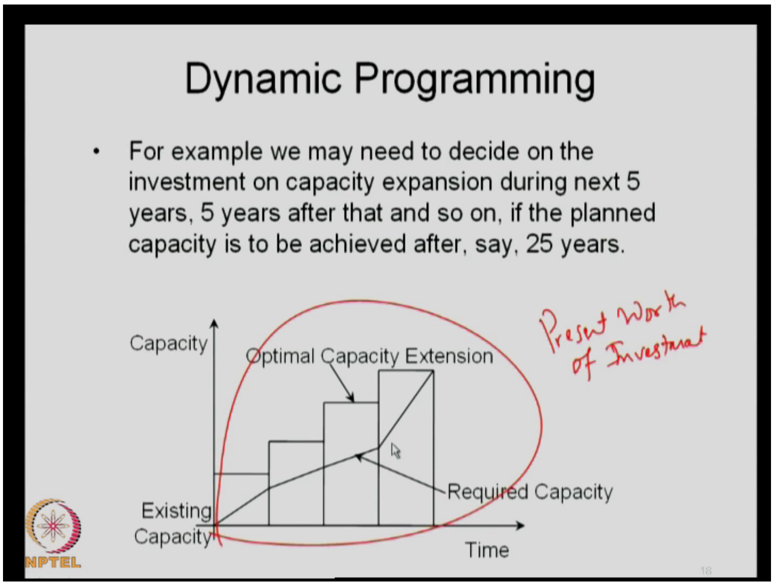
So, this capacity expansion can be various physical component expansions, all of which require significant investment and you are making the investment into future, but you are making the decision now. Therefore, you need to calculate the value of the investment that you make in future at the present time. The present worth of the investment that you are going to make in the future is what is important.

(Refer Slide Time: 46:37)



So, we will just look at the problem. You understand the problem correctly. I will also state the formulation, but will just rush through the solution itself.

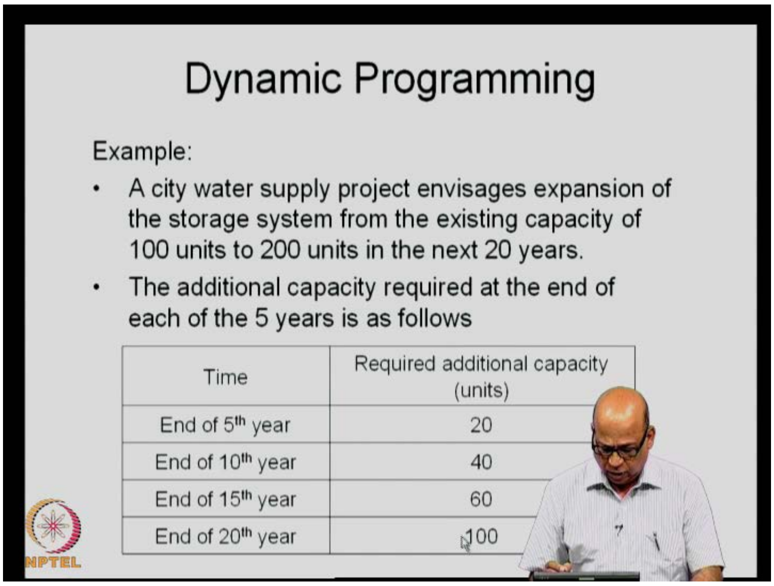
(Refer Slide Time: 46:42)



So, we will take this example. There is one existing capacity and you are specifying the minimum required capacity at each time horizon. Let us say that, this is 5 year period, this is 5 year period, 5 year and so on. So, from time t is equal to 0, this is 5, this would be 6 to 10, this would be 11 to 15, then 15 to 20 and so on. So, let us say that you want to expand the capacity in this particular manner. That means, you want to, you are saying that at the end of the first 5 years, my minimum required capacity is here, at the end of the second time period, my minimum required capacity is here and so on. At the end of this time horizon, I want to achieve the full capacity expansion and this is the point that I want to reach, whether we should reach along this line or along this line.

For example, I may provide a capacity much more than what is required here, much more than what is required here, much more than what is required here etcetera, such that I come here, such that the total investment that we make across all of these time period is the minimum and we are talking about the present worth of investment. So, we ask the question, given the existing capacity, what should be my expansion during the first 5 years, next time period, next 5 years, next 5 years and so on, such that the present worth of investment which we are in fact making in the future is a minimum? This is the problem.

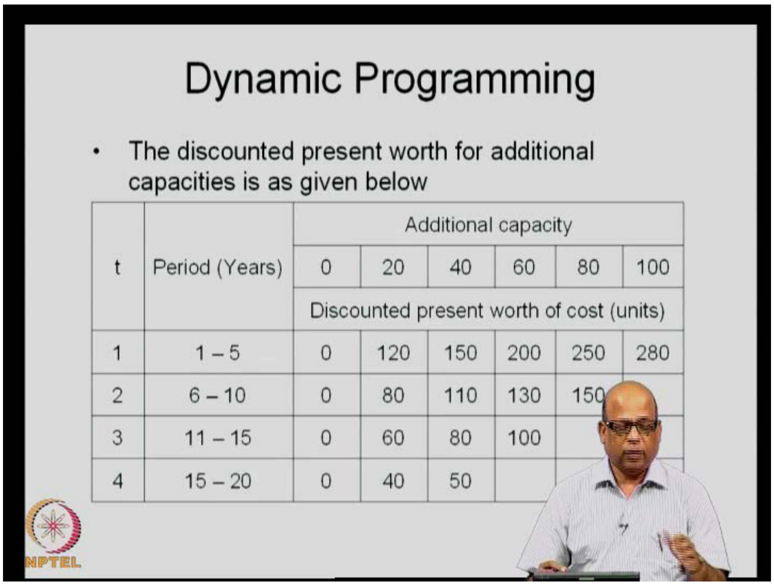
(Refer Slide Time: 48:34)



In this particular example, once you understand that statement of the problem, the example is straight forward. We want a minimum additional capacity. It has an existing capacity of 100 units and we want to reach a capacity of 200 units in the next 20 years. At the end of each of 5 years, we need a minimum capacity here, required capacity. That means, we are specifying these points now. So, the required capacity at end of the first 5 year, is twentieth, 20 units, which means starting with 100, we want 120, starting with

100, we want 140, starting with 100, we want 160 minimum, minimum capacity. At the end of the 20th year, we need a capacity of 200 units.

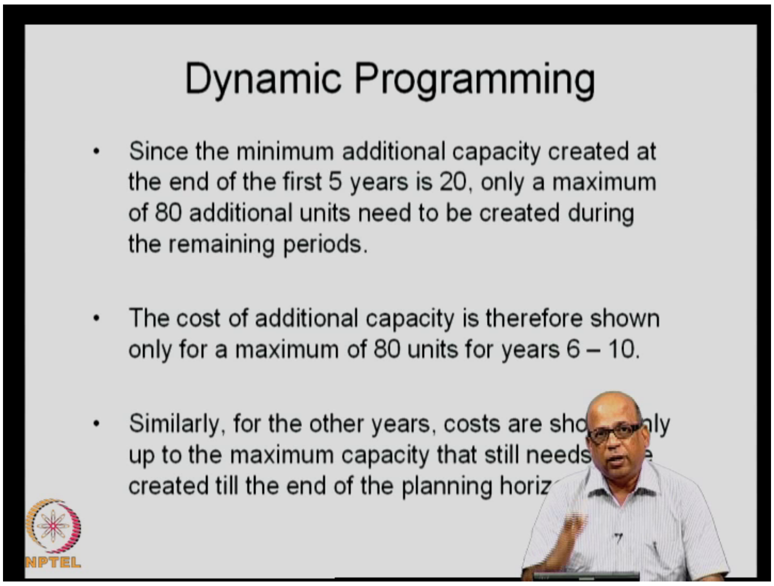
(Refer Slide Time: 49:25)



We have the present worth of the investments. Now, do not worry too much about the economic aspects of it. Simply assume that these values are the values that you will be investing in terms of their present worth, which means what? Although, you have invested for a capacity of 20 at the beginning of the sixth year, the value should be reckoned at this present time and like this. Therefore, you can see that if you invest for additional capacity of 40 for example, the amount that you may have to invest right now is 150, but if you invest at the end of 5 years which is at the beginning of 6 years, the same value will be less because we are talking about the present worth, which means the first 5 years you have not spent and you are going to spend it at the beginning of the next 5 years, at the beginning of the sixth year. Therefore, this value, this money is with you now. You have not spent it and therefore, this value will be lower and lower as you progress more into the future for the same amount of money.

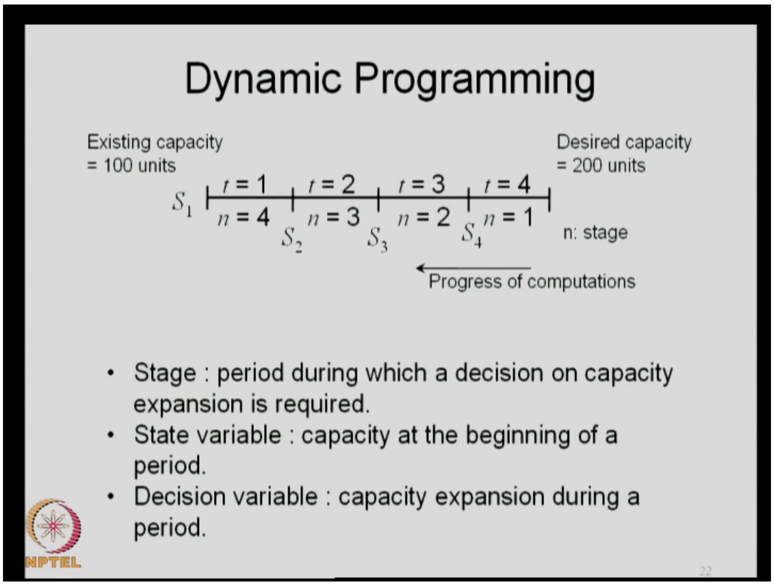
So, for the same capacity expansion as you progress in time, these values are smaller. Anyways, these are subtlety that you draw right now. Right now, we will focus on how we formulate this problem as a dynamic programming problem and then, solve it from stage to stage.

(Refer Slide Time: 50:56)



So, the question that we are asking now is, starting with a known capacity of 100 in stages of 5 years, how do we keep expanding the capacity, such that the minimum required capacity at the end of each of the 5 years which is specified is attained? Now, this is formulated as a dynamic programming problem.

(Refer Slide Time: 51:19)



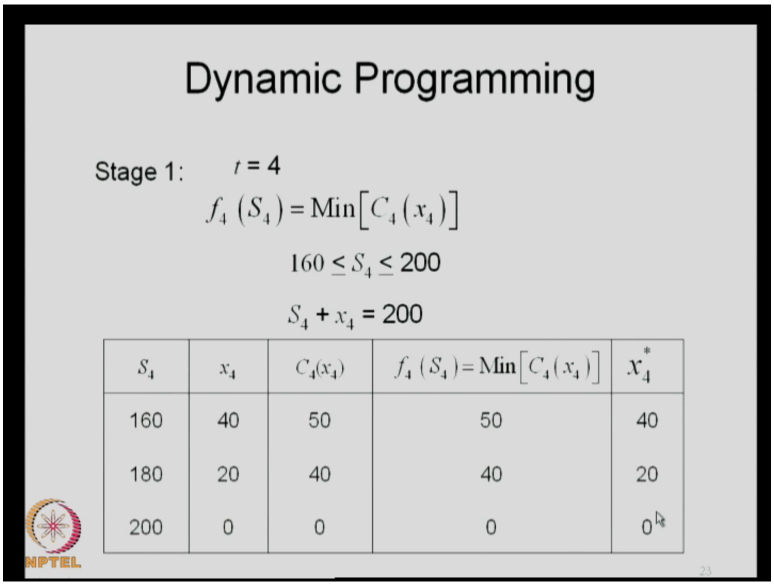
This is t is equal to 4, which means you have a 20 years horizon and each of this time horizon that I showed constitutes the stage here. So, this is stage number 4, stage number 3, stage number 2, stage number 1, t is equal to 1, t is equal to 4, t is equal to 2 and so on.

If you are proceeding in the backward direction, it will be n is equal to 1, stage is 1, stage 2, stage 3 and so on.

So, this is how we progress, n is equal to 1 and so on. So, stage is the period during which a decision on capacity expansion is required. Then, state variable is capacity at the beginning of the time period and decision variable is capacity expansion during the time period. So, these 3 definitions you must remember now. There is one more point here. When we specified these values here, you look at this. The minimum capacity that has to be expanded is specified from here 20, 40 and so on, so which means 100 and 120 and 140 etcetera. So, when we come to 6 to 10 years, the minimum capacity would have already been 120 and what you require is 200, maximum is 200. So, you go up to only 80 here.

Similarly, when you come to the next stage, you go only up to 60, next stage you go only up to 40 and so on. So, that is how the present worth additional capacity are known. When you are standing at the stage number 4, this is t is equal to 4, you do not know what the current capacity at this point is. So, you will ask the question if my current capacity is so much, what should be my decision for the next time period. This is the question that you will ask.

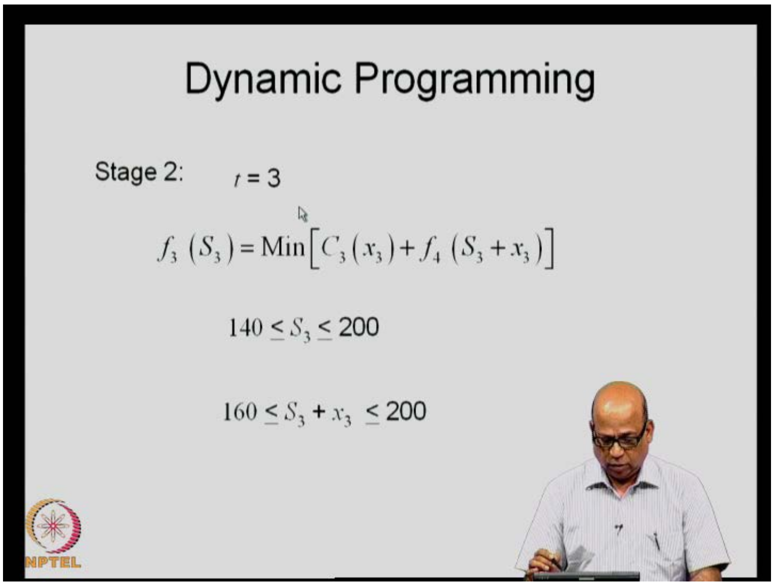
(Refer Slide Time: 53:25)



So, S 4 can start from 160 because the minimum required capacity at the end of stage 3 was 160. You see here, this was 60; you had 100, so minimum of 160 should be there. Therefore, S 4 cannot be anything less than 160. It has to be 160 and it can go up to 200.

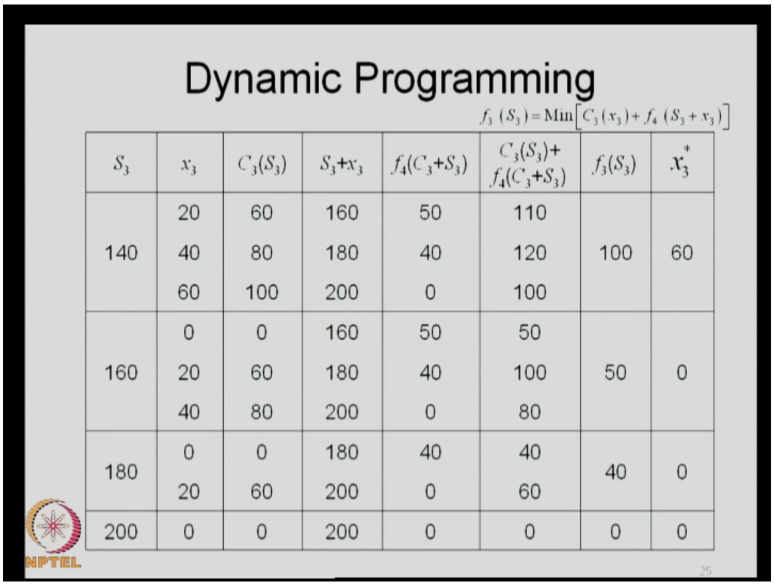
So, the question that we ask is, if S 4 is 160, if S 4 is 180, if S 4 is 200, what should be my S 4, such that the minimum cost is the cost associated that is minimum. Like this you go to next stage then.

(Refer Slide Time: 54:14)



Next stage, it goes on between 140 and 200 because 140, what would you achieved earlier and at the end of the time period, you should have 200 maximum. So, S 3 plus X 3 must be 160. What is X 3? X 3 is the decision that you make on the capacity expansion at time period t is equal to 3. So, this is what we do for stage number 2.

(Refer Slide Time: 54:43)

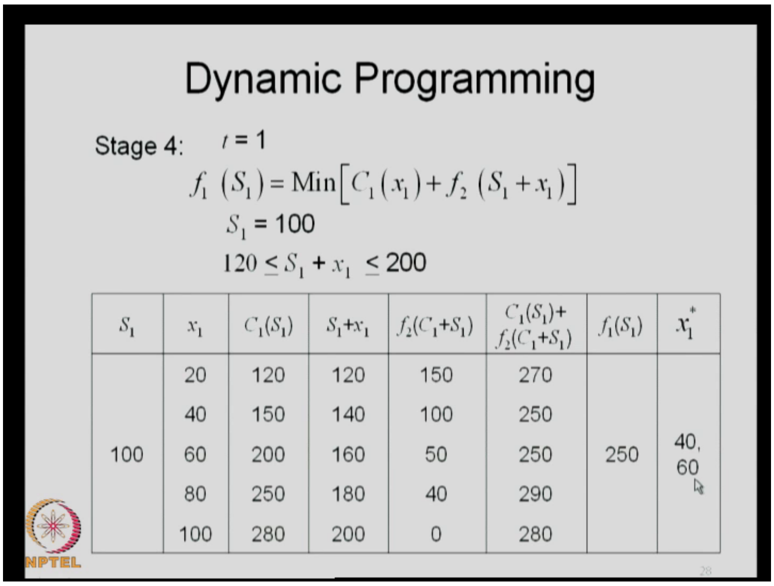


S_3	x_3	$C_3(S_3)$	S_3+x_3	$f_4(C_3+S_3)$	$C_3(S_3)+f_4(C_3+S_3)$	$f_3(S_3)$	x_3^*
140	20	60	160	50	110	100	60
	40	80	180	40	120		
	60	100	200	0	100		
160	0	0	160	50	50	50	0
	20	60	180	40	100		
	40	80	200	0	80		
180	0	0	180	40	40	40	0
	20	60	200	0	60		
200	0	0	200	0	0	0	0

Then, you go to much like the way we did the reservoir operation. In fact, this is very similar to the reservoir operation problem. You just verify all of these.

The question that we are asking is, if S 3 is known, what should be my X 3, if S 2 is known, what should be my X 2 and so on. So, much the same way as we did earlier, we keep on computing the cost for immediate expansion and the optimized cost for the state resulting from this S 3 and this X 3. In this particular case, it will be only from S 3 plus X 3 because you are adding the capacity S 3 and X 3. Like this you go to stage number 1.

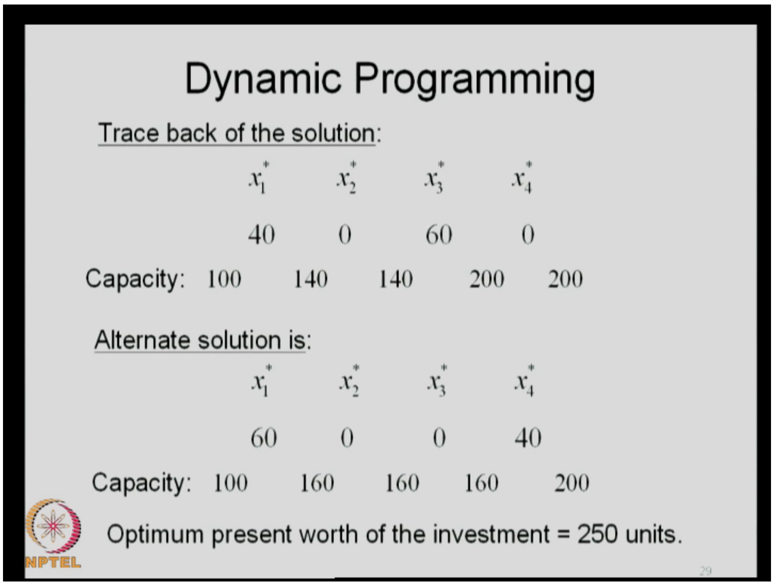
(Refer Slide Time: 55:26)



S_1	x_1	$C_1(S_1)$	S_1+x_1	$f_2(C_1+S_1)$	$C_1(S_1)+f_2(C_1+S_1)$	$f_1(S_1)$	x_1^*
100	20	120	120	150	270	250	40, 60
	40	150	140	100	250		
	60	200	160	50	250		
	80	250	180	40	290		
	100	280	200	0	280		

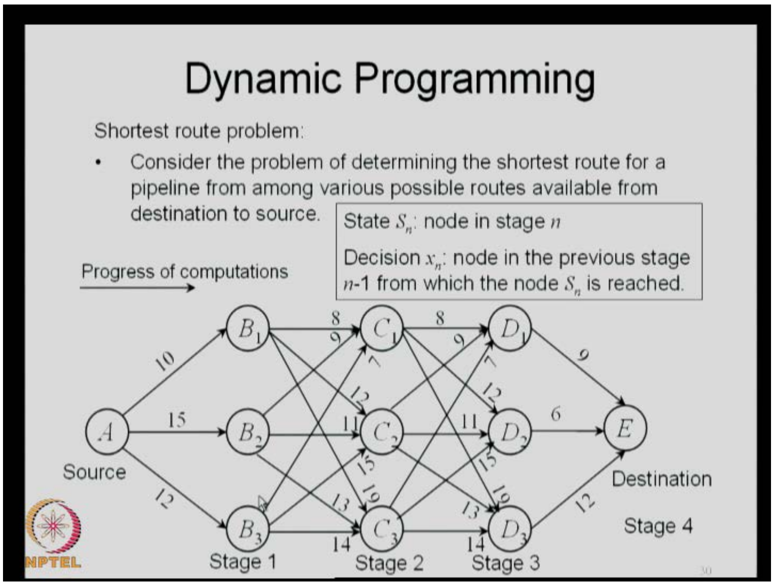
In stage number 1, it is given that S 1 is equal to 100. It is exactly same as the reservoir operation problem. The existing capacity at the beginning of the time period is known to be 100. So, you solve only for 100, you get the optimal objective function value as 250 and the associated values as 40 and 60. So, what does it say? It says that starting with 100; you go either to 40 or to 60.

(Refer Slide Time: 55:59)



Now, when you trace back the solution, you trace back for both 40 as well as 60. If you trace back for 40, this is what it says. Start with 100, expand 40, you get 140. For 140, expand 0, in the next stage you get 140. For 140, go expand 60, you get 200 and 200, 0, 200. So, finally, you achieve a capacity of 200, right at this point and the associated objective function value is 250. Corresponding to 60 because you can trace back for 40 and 60, corresponding to 60, the solution is 160. You end of with 160, 0 end up with 160, 0 end up with 160 because 160 was necessary at beginning of the fourth stage. Then, 160, 40 and 200 both of these lead to an optimal objective function value of 250. So, this is what we did in capacity expansion problem.

(Refer Slide Time: 56:59)



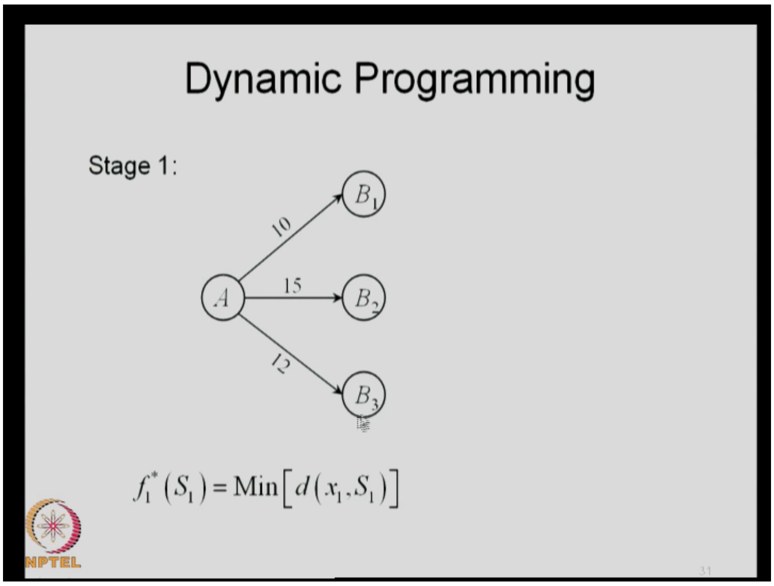
Now, there is a last problem that I would like you to understand. Many situations arise where you look at shortest route problems, typically in pipe network and so on. You may arrive at such a problem. I will just explain this problem and this problem solution is available. You can just look up this. From the source you want to go to destination and there are various paths available. From here you can go either to B 1, B 2 or B 3, from here you can go either to C 1, C 2 or C 3, from here you can go to D 1, D 2 or D 3 and finally, you reach the destination. So, we define for this problem the stage 1, the stage 2, the stage 3 as being here and being here and being here.

So, the question we ask is, we proceed in the forward direction. In this case, you can also proceed in the backward direction and I encourage you to look up, solve this problem also in the backward direction. We ask the question at stage 1, if I am in node B 1, from where I should have come, if I am in node B 2, from where I should have come, etcetera. Node stage number 1 computation is trivial, but when we are going to say stage 2, let us say if I am in a given stage, whether that is C 2. If I am in a given node C 2 in stage 2, whether I should come from B 1 or B 2 or B 3, such that the total distance covered form source up to this point is a minimum?

Similarly, if I am in D 3, stage 3, I ask the question whether I should come from C 1 or C 2 or C 3, such that the total distance covered form the source up to this point is a minimum. Like this for every stage, we keep on asking the question. In this stage, if I am in particular node, from where I should come from in the previous stage, such that the total distance covered from the source up to that point is a minimum. So, the state we decide as the node in stage n and the decision that you are making is node in the previous stage n minus 1, from where you have reach this particular node.

So, state variable is decided, stage decision variable is decided. What is state transformation? State transformation is simply the decision itself, that is you will define X n, that defines the state during the previous time period.

(Refer Slide Time: 59:51)



So, like this for example, stage 1, you look at these and minimize these and you will get, you are answering the question if S 1 is equal to B 1, then my X 1 star is A.

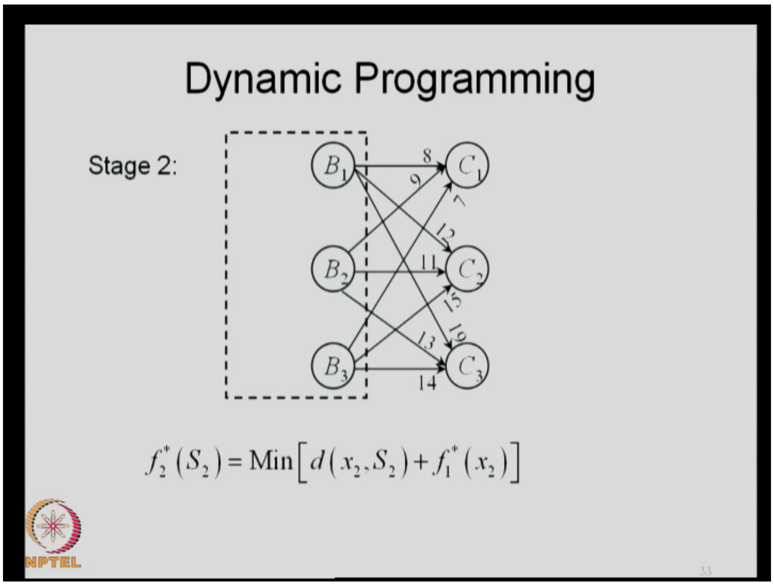
(Refer Slide Time: 59:56)

Dynamic Programming

S_1	x_1	$f_1^*(S_1) = \text{Min}\left[d(x_1, S_1)\right]$	x_1^*
B_1	10	10	A
B_2	15	15	A
B_3	12	12	A

So, we will continue this discussion in the next class, but I will not revisit this problem. The problem solution is available in the handout. It is much similar to the earlier two problems that we have covered.

(Refer Slide Time: 1:00:03)



I encourage you to solve this problem also in the backward direction. The problem solution that I am providing here is in the forward direction. You also continue in the backward direction and then, make sure that you get the same result. So, thank you for your attention. We will start with a next topic in the next lecture.