Water Resources Systems: Modeling Techniques and Analysis

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Module No. # 01

Lecture No. # 13

Good morning and welcome to this, the lecture numbers 13 of the course, Water Resource Systems: Modeling Techniques and Analysis. We have been covering the optimization techniques for the last few lectures and specifically we are dealing with the simplex algorithm in the linear programming problems.

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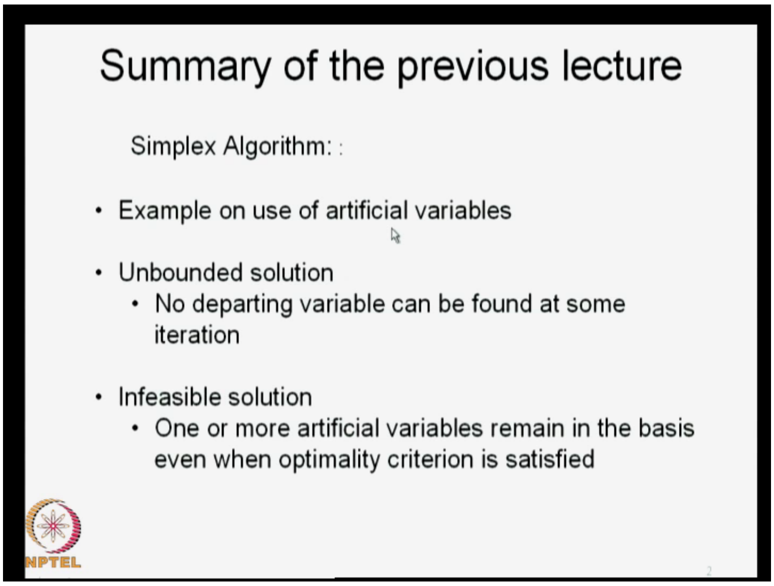

So, if you recall in the last lecture, we started with the example of use of artificial variables. When do we use the artificial variables? Whenever there are greater than or equal to or equal to constraints, equality constraints in the maximization problems, we introduced for the greater than or equal to constraints, we would have introduced a slack variables with a negative sign, and for equality constraint, we do not introduce any artificial variables and therefore, to start of the computations in simplex algorithm because we need an initial basic feasible solution, we introduce an artificial variables.

And then, we through an example of use of artificial variables in the previous lecture, recall that, we transform the Z row corresponding to each of the artificial variables and then use the transformed Z row in the simplex algorithm in the first iteration or the first tableau. If the solution is feasible, if the problem has a feasible solution, then the artificial variables that you have introduced to start the computations will quickly go out of the basis and then equate for the other original variables of the problems including the slack and surplus variables. Therefore, we introduce the artificial variables, penalize the objective function corresponding to the artificial variables, we ensure that if there is a feasible solution, the artificial variable exit the basis sooner than later.

Then we went to identify an unbounded solution, if you recall we identify an unbounded solution, when the optimality criterion is still not met, which means that the solution improvement is possible in the objective function, but we are unable to find a departing variable in the basis. We are able to find a entering variable which means that you can increase that particular variable value and therefore, increase the objective function value in a maximization problem, but corresponding to that you are unable to find a departing variable, which means what? The objective function value can be increased up to infinity without violating any of the constraints and therefore, we identify such problems as unbounded problems solution, the problems with unbounded solutions. Then we also identify an infeasible solution in the simplex algorithm, if your problem is feasible, we would definitely have one or more artificial variables sitting in the basis, when your optimality criterion is satisfied.

So, if the artificial variables have not departed from the basis and yet you have been able to meet the optimality criterion, then it means that the problem is infeasible. In general, if a problem is unbounded or infeasible, then the problem is ill formulated because, obviously if your problem is correctly formulated, if the physical problem is correctly formulated, then you cannot have unbounded solutions. Let us say that you are optimizing hydro power generation, maximize the hydro power generation and you get an unbounded solution. What does it mean? It means that you can generate infinite amount of power which is not possible. So, if your physical problem is correctly formulated as a mathematical problem, you should not get an unbounded solution. Similarly, if your problem is well formulated which means that, all the curve constraints are consistent with each other, all the constraints are correctly reflecting the physical features of the problem, then you in general, and you should not get an infeasible solution.

So, if the problem is correctly formulated, both unbounded solutions and infeasible solution should not exist. Now, we have been talking about the development of this algorithm and the procedure for solution and so on with very small size problems. For example, two variables, three variables, two or three constraints and so on so that the techniques are correctly understood by the students. However, any realistic engineering problems will have large number of variables, large number of constraints and so on. And therefore, identifying them in the simplex algorithm becomes important, identifying such situations of unboundedness, infeasibility of solutions, having multiple solutions, etcetera becomes important and we should be alert to such situations so that we can reexamine the problem formulations. Towards the end of the last lecture, I just started introducing the duality. So, we will continue the discussion today on dual problem.

So, as I mentioned, any LP problem can be treated as a primal problem and corresponding to every primal problem in a right a dual problem. The duality, it appears to be just a theoretical concept without having much of practicals in difficult, but like I said in the last lecture, towards the end of the last lecture, there is nothing more practical than a sound theory. So, let us understand theory of duality of LP and perhaps in today we will close the discussion on LP, although there there are large number of topics that can be covered in linear programming, but being a water resource systems course, we will quickly move on to applications.

So, this is just the theory of the linear programming and how we solve and so on, but for actual practical problems in water resources, we need to write programs using the techniques I have discussed now, or use commercially available or freely available softwares. In fact, mat lab will be, mat lab will have, mat lab does have certain routines or certain functions through which you can solve a linear as well as non-linear programming problems, which we will discuss when we are talking about the applications.

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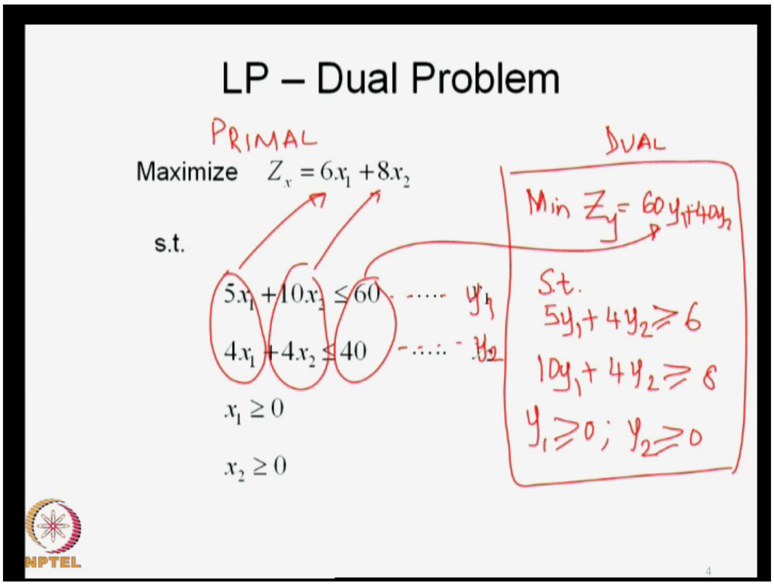

So, to introduce what is a dual problem, I will start with an example simple numerical example, simple LP example where we are talking about maximize Z x is equal to 6 x 1 plus 8 x 2 subject to 5 x 1 plus 10 x 2 less than or equal to 60, 4 x 1 plus 4 x 2 less than or equal to 40, x 1 and x 2 are non-negative. So, this is a problem which has objective function has maximization and constraints have less than or equal to pi, and both the variables are non-negative. Let us say for this problem, I will associate one variable y 1 here for the first constraint, and another variable y 2 for the second constraint, and then write another problem, where this objective function is maximization, I will write minimize.

So, objective Function, I write it as minimization and I will write it in terms of the new variables y 1 and y 2. Therefore, I will write this as minimize Z y, here it is Z x indicating that you are writing the problem with respect to the variable x, and writing the problem with respect to variable y. Therefore, I will write this as minimize Z y is equal to, I will write 60 y 1 plus 40 y 2, 40 into y 2. Then, I will write subject to I will just look at the x constraints here, subject to 5 y 1 plus 4 y 2 and then, I will write greater than or equal to 6. So, what I did, I looked at the x 1 variables here.

So, all of these are related with x 1 and write the constraints. So, corresponding to a variable, I am writing a constraint here. Similarly, I will write 10 y 1 plus 4 y 2, this one greater than or equal to 8, and I will write y 1 is greater than or equal to 0, y 2 greater than or equal to 0. Now, look at what we did in this case, we first looked at the nature of the objective function, the nature of the objective function was maximize, I wrote it as minimization when I am writing another problem, I am writing this has minimization, it was written in terms of the variables x, I am writing in terms of variable y. Therefore, I will write it as minimize Z y, the right hand side of the constraints of the original problem become the objective function coefficients of the dual problem.

So, I will write 60 into y 1 plus 40 into y 2. So, the right hand side constraints become the coefficients of the objective function, the right hand side values of the constraints become coefficients of an objective function, when I look at one variable at a time, corresponding to each of the variable, I write a constraint. So, what was a variable will lead to a constraint in this form. So, I look at x 1 now, the co-efficient of x 1 multiplied by the new variable, coefficient of x 1 in the next constraint multiplied by the new variable etcetera, will form the left hand side of the constraints, so 5 y 1 plus 4 y 2.

Then, because it is a minimization problem, I will write it as a greater than or equal to and the associated coefficient in the objective function becomes a right hand side of the constraint, so 5 y 1 plus 4 y 2 greater than or equal to 6. Similarly, 10 y 1 plus 4 y 2 greater than or equal to 8 and y 1 greater than or equal to 0, y 2 greater than equal to 0, you will see special cases of this, but for this problem, I am writing a problem like this in terms of y. Now, if you treat this original problem as the primal, this problem becomes dual. So, this is a dual problem for this particular problem, I will show now a more general way of expressing this.

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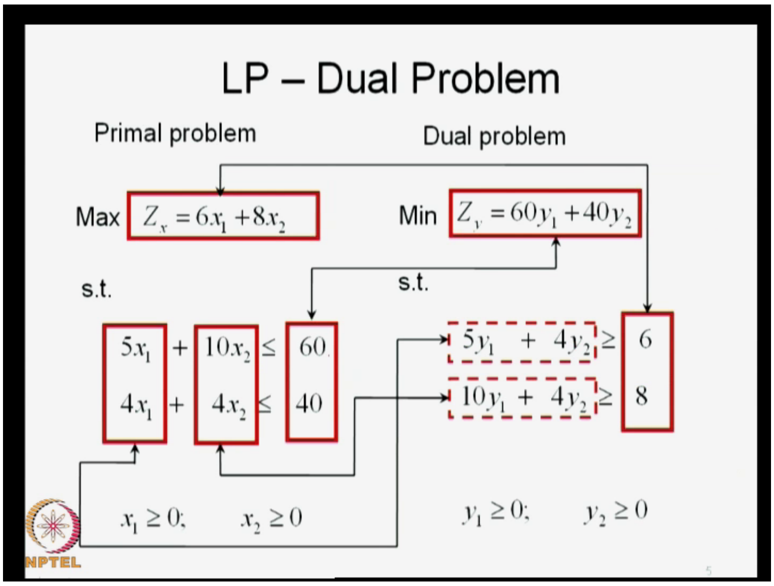

So, what we did here is that the primal problem was maximized Z x is equal to 6 x 1 plus 8 x 2, I write the dual problem objective function as minimize Z y and the coefficients of the objective function values are in fact, the right hand side values which are the b i values, if you recall b1 and b 2 of the original constraint, so 60 y 1 plus 40 y 2. So, this is how we write the objective function value, what was maximization becomes minimization. Then, we look at each of the variables, so I have to write the constraints of a dual problem now, constraints of the dual problem a variable in the primal problem corresponds to a constraint in the dual problem.

So, what we will do is, we will look at x 1 now, 5 x 1 and 4 x 1 are there corresponding to the two constraints, so we will write 5 y 1 plus 4 y 2. So, this is the constraint corresponding to the variable x 1 and the right hand side of the constraint is in fact, the objective function coefficient of this particular variable x 1, 6 and 8. So, 10 y 1 plus 4 y 2 and right hand side is 8, in maximization problem with less than or equal to constraints will have as it is dual, a minimization of the objective function with greater than or equal to constraints, and these are the right hand side values. So, this slide if you understand correctly, you will know how to formulate the dual.

So, we had a maximization problem with less than or equal to constraints, it becomes a minimization problems with greater than or equal to constraints, the right hand sides of the constraints are the original objective function coefficients, the coefficient of the objective functions are the original right hand side values of the constraints, the coefficient of the constraints themselves are the coefficients of the constraints of a particular variable. A variable will introduce a constraint, a variable of the primal problem will introduce a constraint, a variable of the dual problem will introduce a constraint in the dual problem, and y 1 greater than or equal to 0, y 2 greater than or equal to 0 in this particular case, because both this constraint are less than or equal to, I will come that later.

So, this is how we write the dual problem, a dual problem for a primal problem I am sorry there dual problem for a primal problem. So, it also does not matter which you call as primal, which you call as dual because dual of a dual is in fact, and a primal problem which we will see presently. So, what happened here, you had two constraints and two variables, you had two constraints and two variables associated with every every variable you have one constraint, and associated with every constraint you have one variable here, so we put y 1 y 2 and so on.

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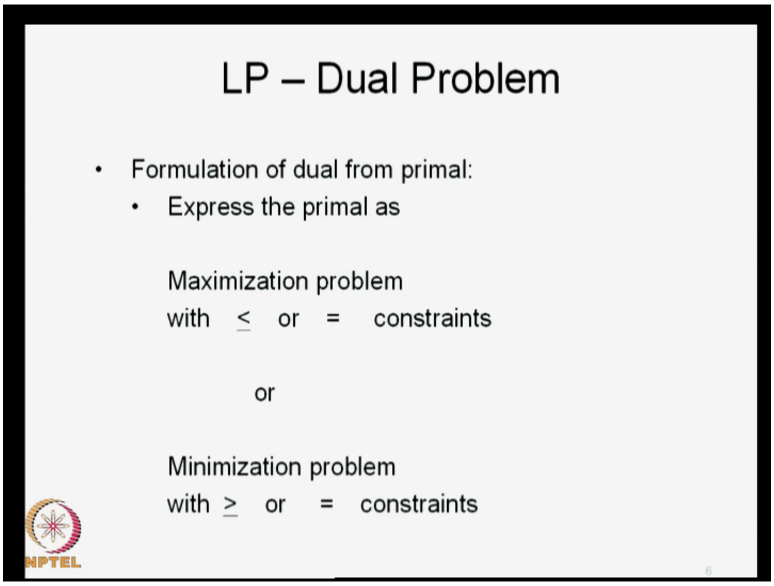

Let us look at how we formulate a dual problem of as a general procedure, any given LP problem, you express it as either a maximization problem with all less than or equal to or equal to constraints, or a minimization problem with all greater than or equal to or equal to constraint. So, these two are combined together; maximization with less than or equal to or equal to, minimization with greater than or equal to or equal to. So, you express it in any convenient form, any one of these two convenient forms, maximization with all less than or equal to or equal to, minimization with all greater than or equal to or equal to. As we discussed earlier, any problem can be converted into any of these forms.

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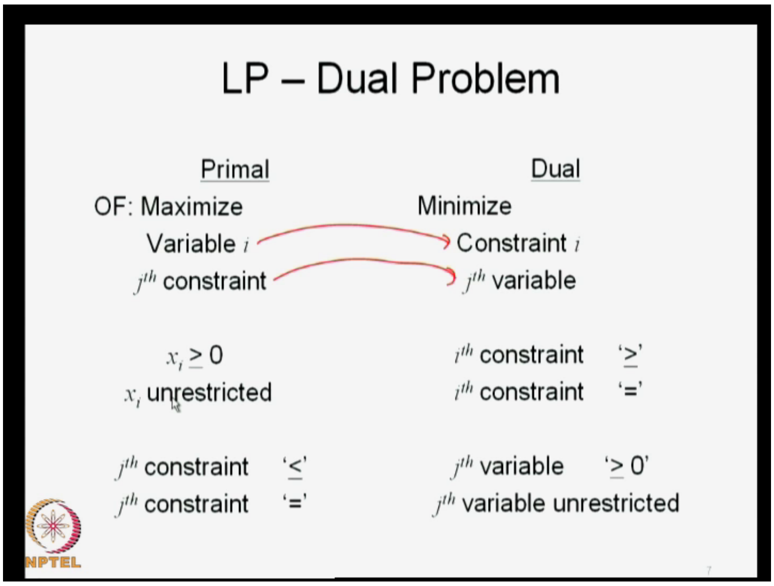

So, first we will express in any these two convenient forms. Once we do that, we write the objective function, if your objective function is of maximization type in the primal, it becomes minimization type in the dual, corresponding to the variable i as a primal, we defined the constraint i in the dual. So, if you have i is equal to 1, 2 etcetera 3; 1, 2 etcetera n variables, you will have the constraints 1, 2, 3 etcetera up to n variables in the primal will define n constraint in the dual, the j th constraint in the primal will correspond to the j th variable in the dual, so these are corresponding, these are corresponding and so on. If in a primal problem, you have an i th variable as greater than or equal to 0, you will have in the dual problem for this particular structure, you will have the dual problem, as in the dual problem i th constraint as greater than or equal to because we are operated function is minimization here.

So, this is one thing that you must remember, the i th variable is corresponding, i is corresponding to i th constraint here, if your i th variable is greater than or equal to then i th constraint will be greater than or equal to here, because of the minimization problem. Suppose, your i th variable is unrestricted in sign; that means, your sign take on either positive or negative values, in such a case, the i th constraint becomes equality constraint. So, as I said, we are starting with maximization objective function with all less than or equal to or equal to constraint in the primal, in this primal we have all less than or equal to or equal to constraint. Similarly, in the dual you will have all greater than or equal to or equal to constraint corresponding to this this particular structure of a primal problem.

So, if you are i th variable is unrestricted, then the i th constraint will be of equality type, if your j th constraint is less than or equal to in the primal problem maximization objective function, the j th variable in the dual will be greater than or equal to, if a constraint j th constraint is equality type, then the associated variable in the dual will be unrestricted inside. So, equality and unrestricted variables, equality constraints and unrestricted variables, unrestricted variables and equality constraints, so this is how you must remember, you must able to formulate the dual problem.

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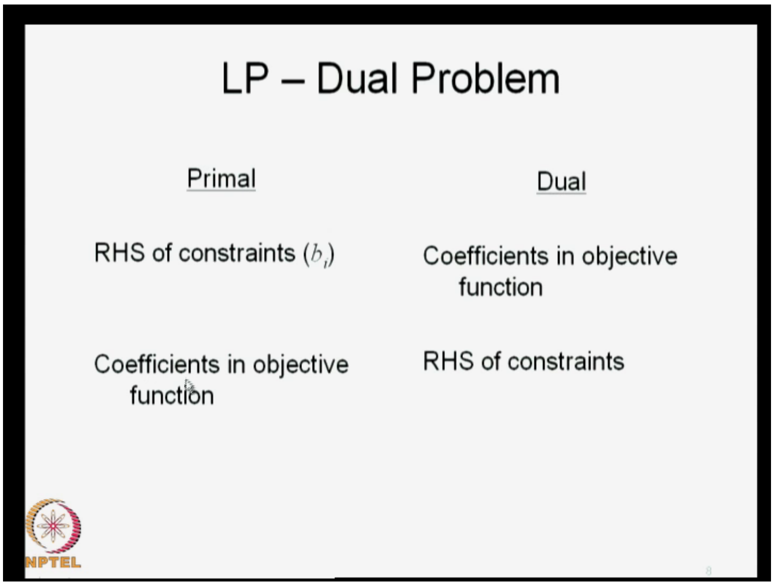

Then the R.H.S of a constraints, that is right hand side values of the i th constraint which is b i becomes the coefficient in the objective function, and the coefficient in the objective function of the primal become the R.H.S of the constraints. So, these are, this is how you relate the primal problem to the dual problem. It has in certain situations, writing dual has a great advantage than depending on whether you want you want to solve the problem with less number of constraints or more number of variables and so on. So, there are certain advantages of directly solving the dual problem, but there are also some very nice and interesting interpretations of the solutions of the dual dual problem, we will see that presently.

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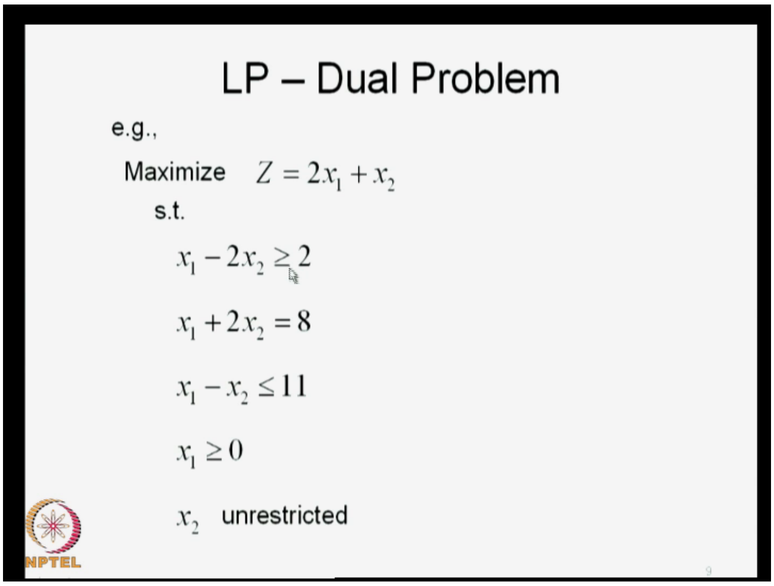

So, just to get used to this, we will solve an example again, let as write a dual problem for this particular problem, maximize Z is equal to 2 x 1 plus x 2 subject to x 1 minus 2 x 2 greater than or equal to 0, x 1 plus 2 x 2 equal to 8, x 1 minus x 2 lesser than or equal to 11, x 1 is greater than or equal to 0, and x 2 is unrestricted. So, you have all possible combinations here, there is a greater than or equal to constraint, there is an equality constraint, there is a less than or equal to constraint, one of the variable in non-negative or as the other variables is unrestricted in sign. So, let us write this problem as I said, given any LP problem, first you express an LP problem as a maximization with all less than or equal to or equal to constraint constraints or in minimization problem with all greater than or equal to or equal to constraints.

So, we will just for the demonstration what will do is, we will express the problem as a minimization problem, with all constraints are either greater than or equal to constraint, so equality constraints will return as they are. So, I will write this as minimization objective function, with all of these constraints as greater than or equal to, so this is greater than or equal to, so I do not do anything, this is lesser than or equal to, what do I do? I multiplied by minus 1. So, minus x 1 plus x 2 greater than or equal to 11, is how we write. So, the problem we rewrite as, minimize Z dash from Z and writing it as Z dash is equal to minus 2 x 1 minus x 2 and as taking the negative of Z, then the first constraint is x 1 minus 2 x 2 greater than or equal to 2, because I want the express all the constraints as greater than or equal to and equal to I have written this as as it is I written this as it is, x 1 minus 2 x 2 greater than or equal to 2.

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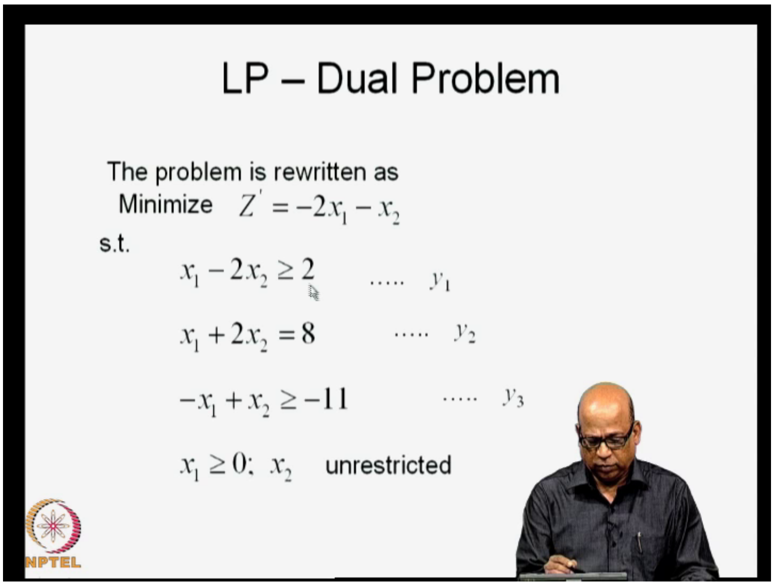

Then the second one, because it is a equality constraint, I do not do anything, I have written it as it is, x 1 plus 2 x 2 is equal to 8, the third constraint which has a less than or equal to constraint, I multiplied by minus 1 throughout and write it as minus x 1 plus x 2 greater than or equal to minus 11, x 1 is greater than or equal to 0, x 2 is unrestricted in sign. So, this is how I rewrite the original problem with the objective function as minimization problem, and all the constraints here are greater than or equal to or equal to. Remember, we are doing this only to write the dual problem, we are not interested in solving it as it, just we want to write a dual problem associated with this.

So, what is it that we do? First, we associate corresponding to each of the constraint we associate a variable y 1 y 2 y 3 and so on. So, there are three constraints, we write a variable y 1 associated with the first constraint, y 2 associated with the second, y 3 with the third, and then it is minimization with all greater than or equal to. So, what will be the objective function? Objective function will be maximization, all the constraint will be less than or equal to or equal to constraints and variables will be either greater than or equal to 0 or unrestricted in sign, we will write this now.

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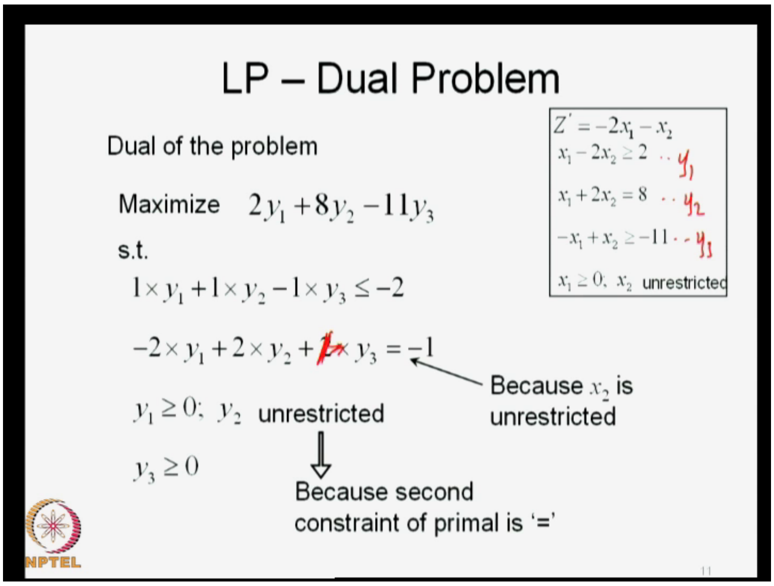

So, dual of a problem, this is a primal problem now, minimization is the problem, primal problem, the dual problem will be look here, what we are doing, this is the original problem. So, I am writing this as y 1 and y 2 and y 3, these are the variables that I am associating with each of the constraints. So, what would be my objective function value? Because, it was minimization objective function, the dual problem will have objective function as maximization, then 2 into y 1 2 into y 1 plus 8 into y 2 minus 11 into y 3. So, this becomes the objective function, subject to how do we write constraints, it pick up one variable, here the constraint in the dual problem, a constraint in the dual problem will be associated with a variable in the primal problem, so this is a primal problem.

So, I look at x 1 first and associate with the corresponding dual variable problem, so x 1 as the coefficient of 11 into y 1 plus 1 into y 2 minus 1 into y 3. So, 1 into y 1 plus 1 into y 2 minus 1 into y 3 less than or equal to minus 2, minus 2 which is a coefficient of x 1 in the primal problem, for in the objective function of the primal problem y less than or equal to because it is a maximization problem. So, we are expressing the dual problem as maximize with all less than or equal to or equal to constraints.

So, this is how you wrote the first constraint, remember the first constraint corresponds to the first variable, first constraint of the dual problem corresponds to the first variable in the primal problem, namely x 1. Similarly, we write it for x 2, so minus 2 into y 1 plus 2 into y 2 ((plus 2)) plus 1 here into y 3, let me just take this, yes this has to be let me look at this, minus 2 into y 1 plus 2 into y 2 plus 1 into y 3, this should be plus 1 y 3 is equal to minus 1. Why do I write is equal to now? We are writing a constraint corresponding to the variable x 2, the x 2 variable is unrestricted in sign therefore, the constraint that is defined for the variable x 2 in the primal will become a equality constraint, and equal to the coefficient in the coefficient of x 2, that the particular variable in the objective function value which is minus 1. Then, we look at y 1 which is associated with the constraint number 1 here, y 1 is greater than or equal to 0, because in your primal problem which is a minimization with all greater than or equal to 0, all greater than or equal to constraints, the constraint corresponding to y 1 is in fact, greater than or equal to, therefore y1 will be greater than or equal to 0 here.

The constraint corresponding to y 2 is equality constraint, therefore y 2 becomes unrestricted inside, then constraint corresponding to y 3 is greater than or equal to, therefore y 3 is greater than or equal to 0. So, because x 2 is unrestricted, you got a equality sign and because the second constraint of primal is equality, you got y 2 as unrestricted. So, this is how you form the primal problem, you should able to write a dual problem. So, this has all the features that are likely to appear in a primal-dual situation.

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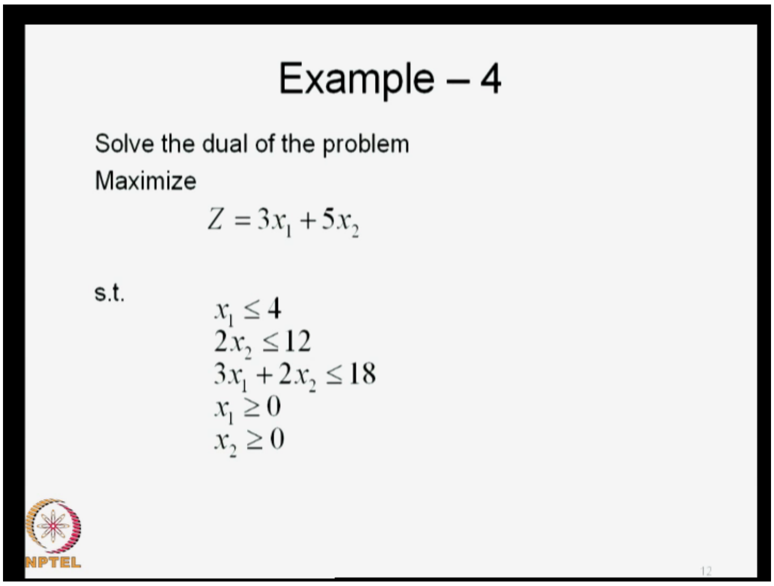

So, this is how we write the dual problem associated with particular primal problem. Now, what we will do is, we take the same original LP problem which we solve using using graphical method, also we introduce a simple method in fact, using this simple example and so on. So, we will take the same example and formulate the dual of that, solve the dual of that problem and then compare the primal problem solution, solution for the primal problem as well as the solution, we compare the solutions of the primal problem with the solution of the dual problem.

So, this is maximize Z is equal to 3 x 1 plus 5 x 2, x 1 less than or equal to 4, 2 x 2 less than or equal to 12, 3 x 1 plus 2 x 2 less than or equal to 18 and both these are non negative, we will write. So, we have the primal problem in the form that you need, that is the objective function is maximization and all the constraints are of the type less than or equal to, you may also have equal to constraint. So, either maximization with all less than or equal to or equal to constraints or minimization problem with all greater than or equal to or equal to constraints.

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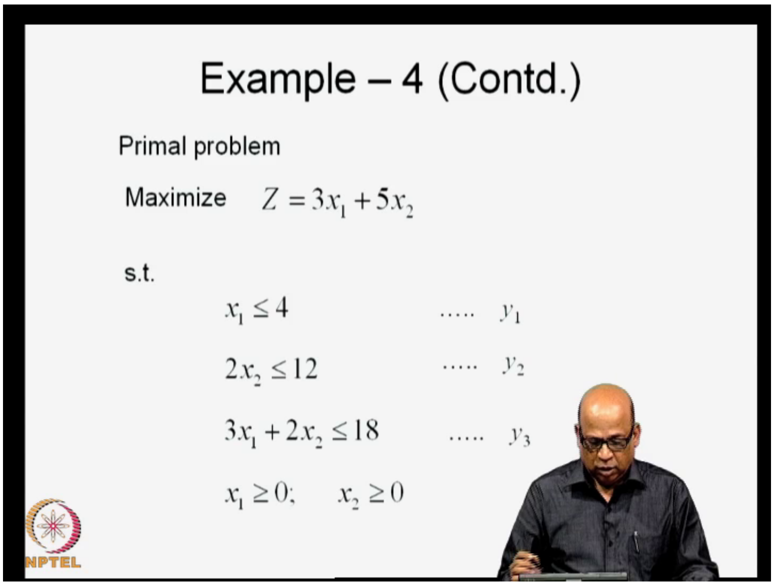

So, this is a primal problem, we associate the variables y 1, y 2, y 3 etcetera correspond to each of the constraints and rewrite this problem as a dual problem.

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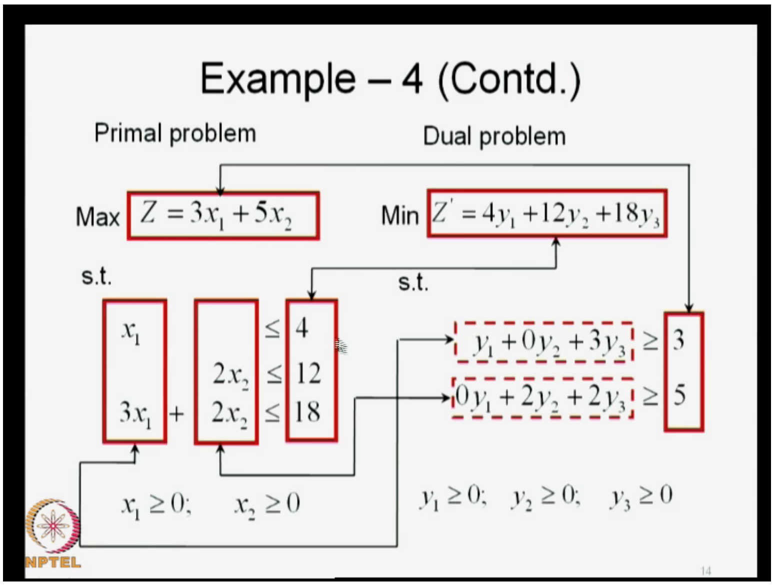

So, the dual problem you can verify this here, x 1 is less than or equal to 6. So, whatever maximization becomes minimization here and we are associating 4 y 1. So, this would be 4 into y 1 plus 12 in to y 2 plus 18 into y 3, subject to we write the constraints corresponding to the variable x 1, so 1 into y 1 plus 0 into y 2 plus 3 into y 3 greater than or equal to this 3 here, because you are writing with respect to x 1. Then, we write associated with x 2, 0 into y 1 plus 2 into y 2 plus 2 into y 3 is greater than or equal to 5, this 5 here, and y 1 greater than or equal to 0, y 2 greater than or equal to 0, y 3 greater than or equal to 0, because all these constraints here are of less than or equal to type. If there was a equality type, let us say this was equal to constraint, then y 3 would have become unrestricted.

So, this is how you write the dual problem, we will use the simplex method and solve this particular problem now. Let us say, I want to solve this problem in the procedure that we have introduced, what do we do? We first express the problem in the standard form and in our standard form, that you in the standard form that we have used our objective function in maximization type. Once you write the dual and you want to solve the dual problem, you forget about the primal problem, you just focus on this problem. So, this is problem that you need to solve, also notice an interesting feature here, you had a two variable problem which has become a three variable problem. Now, you had three constraints, now we have only two constraints, in general, having a lesser number of constraints will be advantageous in the simplex method.

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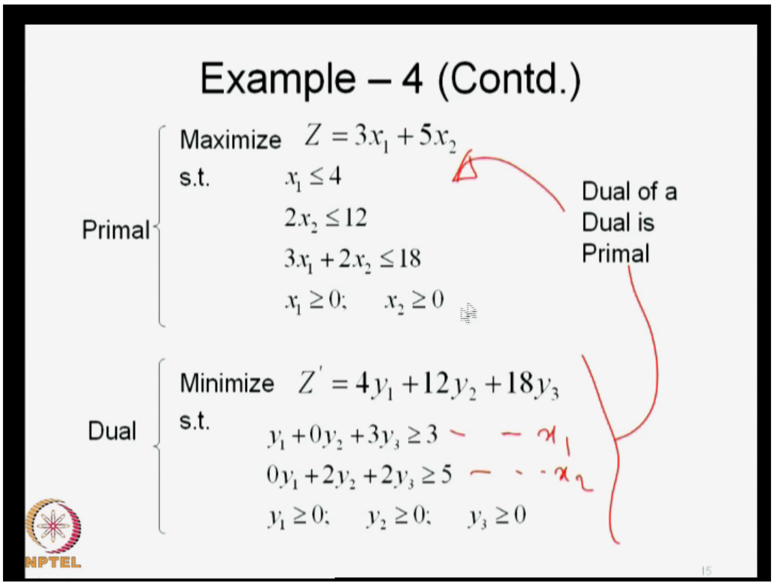

So, let us solve this particular problem now. So, we will just summarize this, for example, maximize 3 x 1 plus 5 x 2 subject to all these constraints, this was your original original problem which is a primal problem, but we converted into the dual problem which transfers to be this particular problem, minimize Z dash is equal to etcetera. Now, if you take the dual of this, you will get the primal, you can just verify that, so what I will do is, associated with these now, I will write variables x 1 and x 2.

So, my objective function is here, it is minimization with all greater than or equal to. Therefore, the dual of this will have the objective function as maximization 3 x 1, maximize 3 x 1 plus 5 x 2 which is the same as this, subject to we looked at the variable y 1 here, subject to 1 into x 1 plus 0 into x 2 less than or equal to 4, that is x 1 less than or equal to 4, like that you can just verify that, the dual of the dual is the primal. So, it does not matter whether you solve the dual or you solve the primal, which we will see, the solution will see but in general, which one you call primal? Which one you call dual? It is of no consequence, because dual of the dual is in fact, the primal. alright. So, you will focus on the solution of this now, solution of the dual, you have already solve you already solved the primal problem are here.

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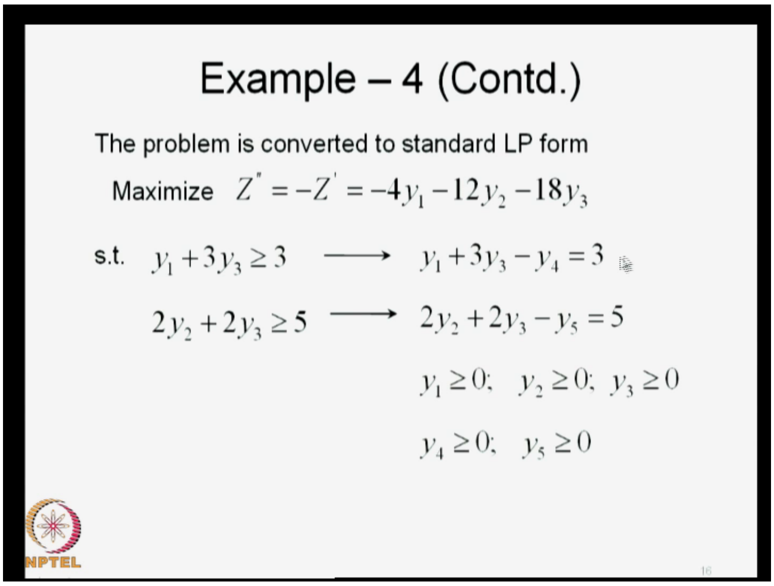

Let us see what happens to the solution of the dual, to solve the dual we will adopt our procedure, our standard procedure where we express the objective function as maximization and all constraints as equality constraints. We are now going on to simplex, solve the LP problem with the simplex method. So, you just revise, review what we did for simplex method are here. So, we write maximization now, now because we are solving this, where we had written as Z dash to distinguish between Z and Z dash, I write another variable now, maximize Z double dash which will be equal to minus Z dash because this is an objective function we are looking at.

So, minus Z dash which is equal to minus 4 y 1 minus 12 y 2 minus 18 y 3, I am just rewriting this, from the minimization objective function, I am writing this as maximization objective function, therefore this to be minus 4 y 1 etcetera. The first constraint is y 1 plus 3 y 3 greater than or equal to 3, for the simplex algorithm, I need an equality constraint, therefore because it is a greater than or equal to constraint, I will write a slack variable detect the slack variable. Therefore, y1 plus 3 y 3 minus y 4 and the right hand side I make it as equal to 3, then 2 y 2 plus 2 y 3 minus y 5 is equal to 5 and all of these are non-negative.

So, this is the way we start our simplex method. Now, because we had greater than or equal to and therefore, we have negative variables here your I am sorry negative, I repeat that the constraints will have slack variables with negative sign and therefore, there is no initial basic feasible solution possible, to ensure that you have a initial basic feasible solution, we have to associate artificial variables corresponding to the greater than or equal to constraints here.

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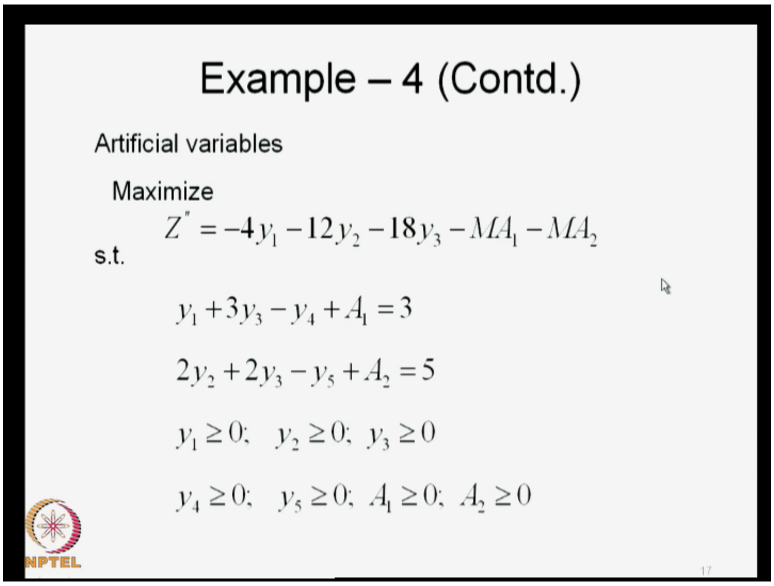

So, we associate an artificial variable A 1 here, artificial variable A 2 here, and then you penalize the artificial variables in the objective function. So, this is of just what we get earlier, except that this is the first time I am demonstrating the problem which has more than one artificial variable. So, you have two artificial variables, one associated with each of the constraints. So, we start of the computations with these artificial variables, what do we do? We transform this Z row with one constraint at a time containing the artificial variable. So, first we take this constraint along with Z row and transform the Z row and on the transform Z row, we use this constraint to retransform it. So, we have to do the transformation twice, this is how we do, so this is the Z row. Remember, we wrote it as an equation, so Z double dash plus 4 y 1 plus 12 y 2 and so on.

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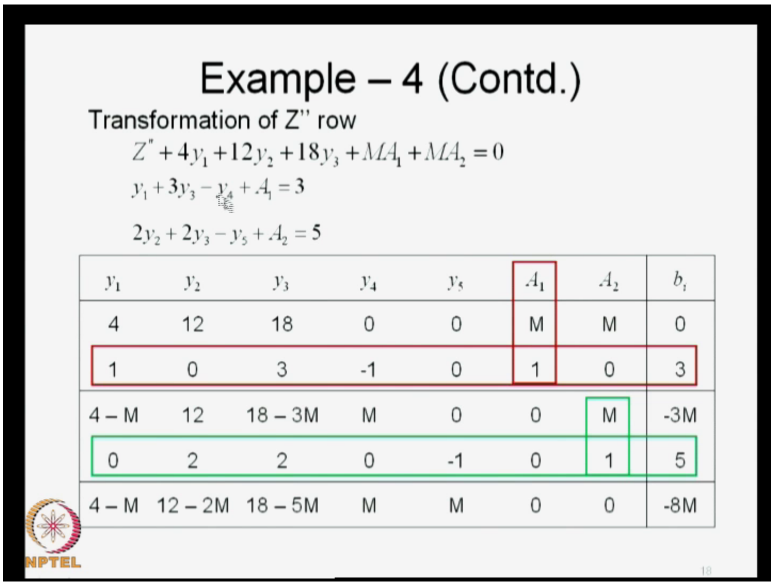

So, I write this as equal to 0, and then we take the first constraint containing the artificial variable and transform the Z row. So, this is Z row 4 y 1 plus 12 y 2 plus 18 y 3 and so on is equal to 0, this is a Z row and this first row containing the artificial variable 1 y1 plus 0 y 2 plus 3 y 3 etcetera is equal to 3, and we do the transformation are referred to the previous lecture on how we do this. So, we get the first transformation of Z, 4 minus M, 12, 18 minus 3 M etcetera minus 3 M. Now, on this transform Z row, we use the second constraint containing the artificial variable and redo the transformation. Remember, if you had several constraints, but only these two constraints had the artificial variables, let say you had five constraints, but only these two had artificial variables, do the transformation only with these constraints which have artificial variables, if you do not have artificial variables do not do that transformation with those particular constraints. So, in this particular case, both the constraints have artificial variables. So, we will do the constraints, we will do the transformation with both these constraints.

So, finally, you get this as the transform row, transform Z row 4 minus M 12, 12 minus 2 M and so on. We use this transformed Z in the first simplex tableau. So, we write the iteration 1 and this is the transform Z row and our initial basis is got by putting y 1 y 2 y 3 y 4 etcetera as 0. So, we get y 1 y 2 y 3 and y 4 as 0, y 5 also as 0. So, you will have A 1 is equal to 3, A 2 is equal to 5 is the initial basis feasible solution.

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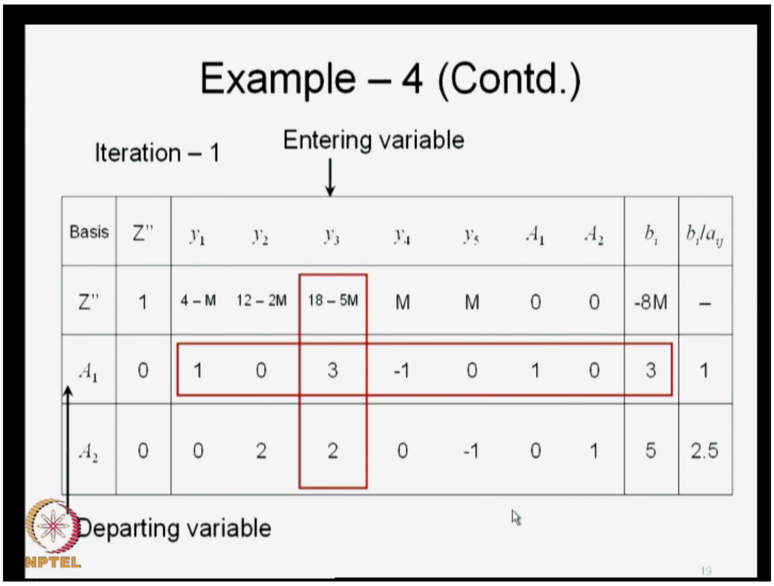

So, that is how we start. So, we do this as we have done earlier, this is the Z row and then you identify the entering variable, how do you identify the entering variable? The highest negative coefficient, remember M is a very large number and therefore, minus 5 M dictates which is the which is the variable having the highest negative coefficient in the Z row, so this becomes the entering variable. So, identify the entering variable, corresponding to that you get the b i by a i j, which ever has the lowest ratio will become the departing variable, in this case A 1 becomes the departing variable.

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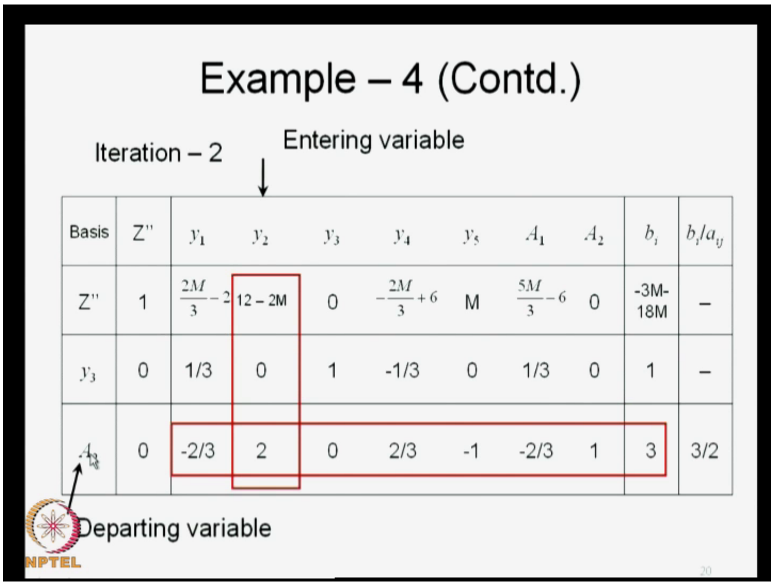

So, the next basis will be y 3 and A 2, so that is what we write, y 3 and A 2. Again, look at the Z coefficients, this is still not optimal because there is one there are two of the coefficients which are which are negative and therefore, you choose which ever is among the two, which over is having higher negative coefficients and it turns out to be y 2. So, y 2 becomes an entering variable, A 2 becomes the departing variable, so the next basis will consist of y 3 and y 2. So, that is how we write y 3 and y 2. Now, you look at the Z row, all these coefficients are non-negative and therefore, this becomes optimal solution and Z dash is equal to minus 36, when you when you do all these transformation etcetera, you will get Z dash is equal to minus 36, y 3 is equal to 1, y 2 is equal to 3 by 2.

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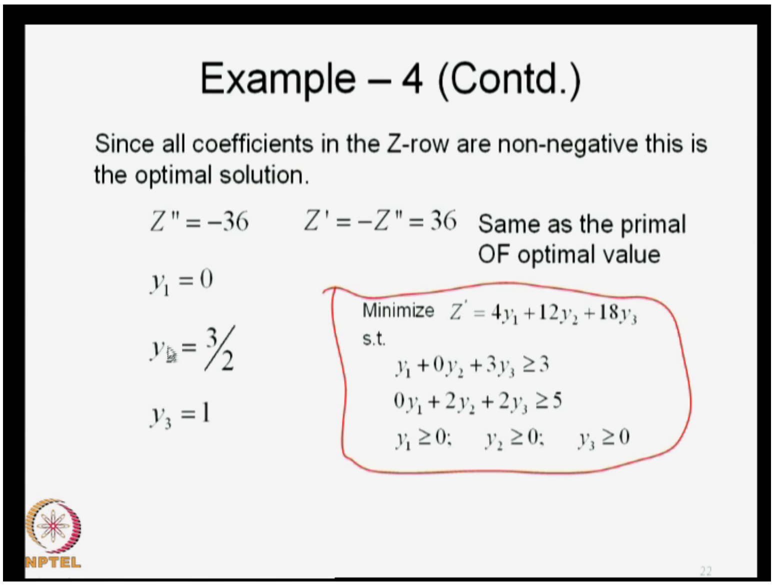

Now, what is Z dash? Z dash Z double dash in fact, this is Z double dash is equal to minus 36, Z double dash is equal to minus 36 you were looking for maximization. So, this is the maximized value and Z dash is equal to minus Z double dash is equal to 36 which is you are looking at minimized Z dash, this is also your original problem. So, actually you are solving this problem, you have solved this problem and therefore, the Z dash value that you get is plus 36, this optimal objective function value here is the same as the primal objective function value when the optimal solution had reached.

So, this is the same value, this is the first conclusion that you draw when you solve the dual of the problem, the objective function value will be the same as the objective function value, actually the optimal objective function value of the primal solution. Then you got the solution y 1 is 0 because it is not in the basis, y 2 is 3 by 2 and y 3 is 1. So, y1 is 0, y 2 is 3 by 2 and y 3 is 1.

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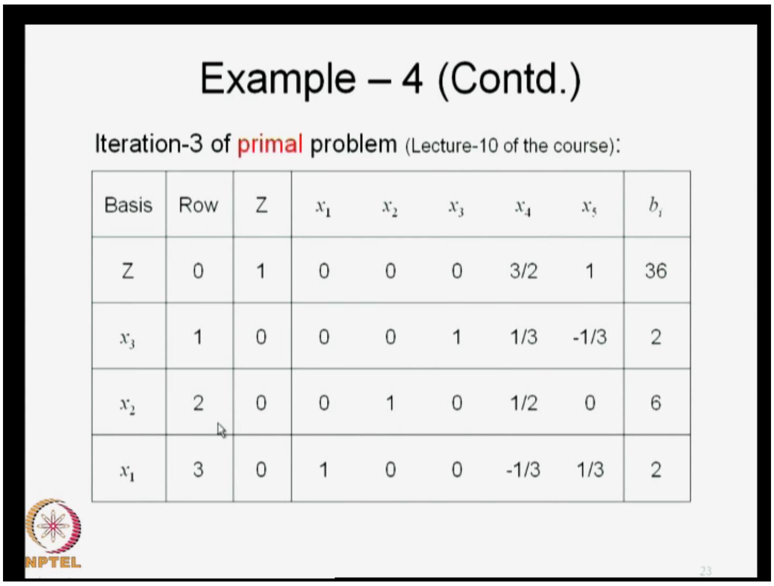

Basis	Row	Z	x_1	x_2	x_3	x_4	x_5	b_i
Z	0	1	0	0	0	3/2	1	36
x_3	1	0	0	0	1	1/3	-1/3	2
x_2	2	0	0	1	0	1/2	0	6
x_1	3	0	1	0	0	-1/3	1/3	2

Let us look at the last tabulae that you obtained for the primal solution, this was covered in the lecture ten, and you can just refer to that. So, this was the last tabulae of the primal solution, look at these coefficients here, x 4 and x 5. In fact, you see this problem statement of the problem, our problem is this, the primal problem is this. So, you have x 1 and x 2 as the original variables, you would have associated slack variables here, x 3 x 4 and x 5 as the slack variables associated with each of these constraints. The coefficients in the optimal tabulae of the Z row corresponding to the slack variables x 3, x 4 and x 5 are 0, 3 by 2 and 1 which are the same solution as y 1 is equal to 0, y 2 is equal to 3 by 2 and y 3 is equal to 1. What is slack variable x 3? x 3 was associated with the constraint and the dual variable associated with that constraint was y 1.

So, the coefficient in the Z row of the final tableau of the primal solution which is 0 gives me the solution for the dual variable y 1 is equal to 0. Similarly, x 4 which was a slack variable associated with the particular constraint and the dual variable associated with that that constraint is y 2 in the final tableau corresponding to the Z row. In the final tableau, the coefficient of x 4 in the Z row will give me the solution for y 2. So, y 2 is equal to 3 by 2 and y 3 is equal to 1, similarly which means what? The primal tableau contains in it, the solution for the dual, because Z is equal to 36, the same optimal solution and y 1 is equal to 0, y 2 is equal to 3 by 2 and y 4 y 3 is equal to 1, this is how we get the solution. So, just look at the Z row, you will be able to get the solution for dual problem.

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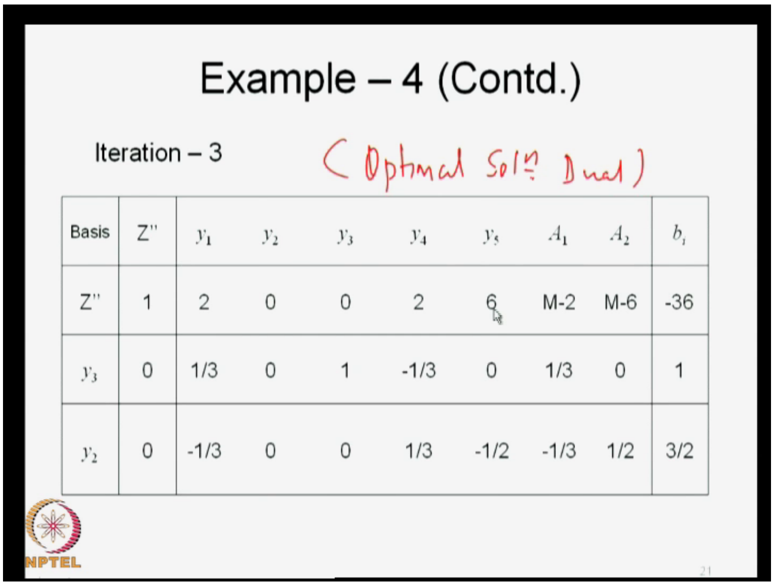

Basis	Z''	y_1	y_2	y_3	y_4	y_5	A_1	A_2	b_i
Z''	1	2	0	0	2	6	M-2	M-6	-36
y_3	0	1/3	0	1	-1/3	0	1/3	0	1
y_2	0	-1/3	0	0	1/3	-1/2	-1/3	1/2	3/2

Now, if you look at the last tabulae of the solution, the dual problem, now this is the optimal solution for the dual problem. Now, here you had introduced y 4 and y 5 as the the surplus variables, the coefficient corresponding to the surplus variables in the final tabulae in the Z row are 2 and 6 and this corresponds to the variables x 1 and x 2 of the primal problem.

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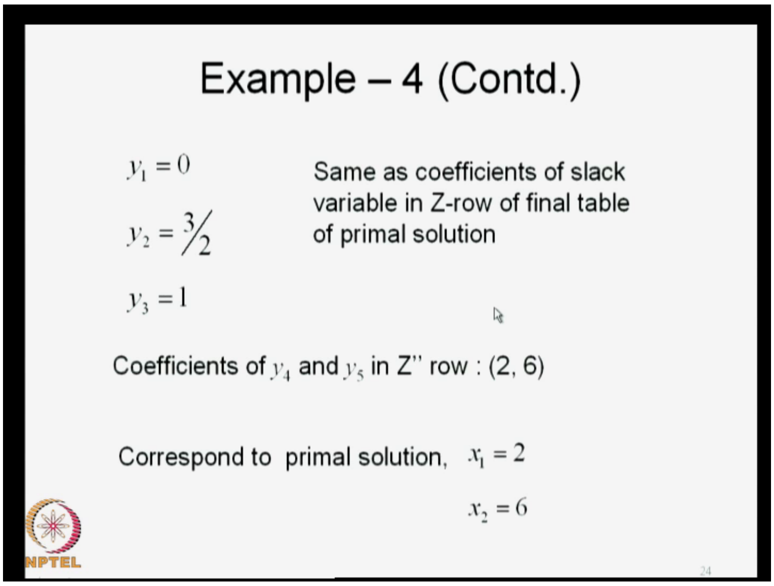

So, the solution 2 by 2 and 6 correspond to correspond to the primal solution, x 1 is equal to 2 and x 2 is equal to 6. So, this is how you can read the solution of primal problem from the dual or the dual problem from the primal, which means what? You formulate the problem, either the primal or the dual, one of them you solve, you should be able to get the solution of the other, complete complete solution, the objective function value and all the associated variable values, you will be able to read it from the final simplex tabulae of the problem that you have solved, now there are certain interesting features.

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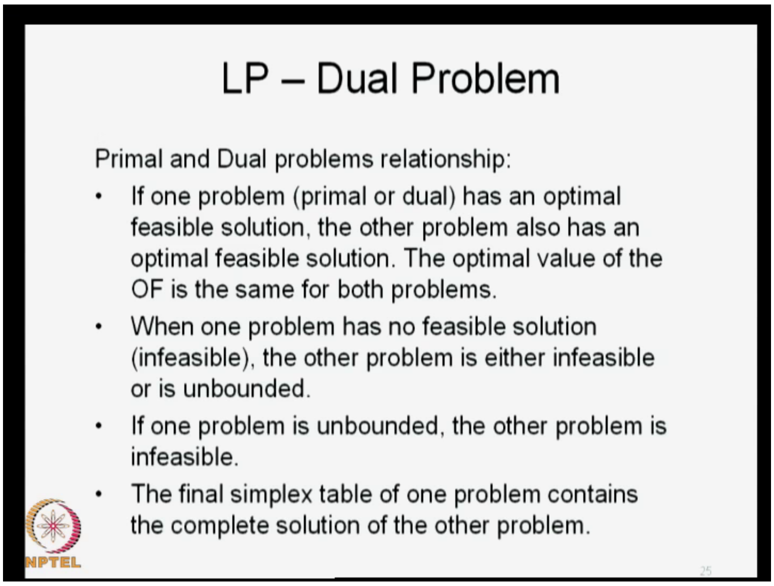

We will just run through them in terms of the relationship between the primal problem and the dual problem if either the primal problem or the dual problem has an optimal feasible solution the other one also has an optimal feasible solution let say that one problem has no feasible solution; that means, you are solving primal and you come a crosses situation that it is a infeasible problem if the problem that you have solve is infeasible than the other problem is either infeasible or unbounded.

So, primal if it is infeasible the dual may be either infeasible or unbounded if the problem that you have solved is unbounded than the other problem is always infeasible. So, if the primal problem let say is what you have solved unbounded in the intermediate iterations you really you do not identify a departing variable therefore, you say that it is unbounded than the other problem that will be dual in this case is infeasible then as we just for the final simplex tabulae of the problem whether it is a primal or the dual gives the complete solution for the other problem, you get the objective function values as well as you get the complete solution for the variable.

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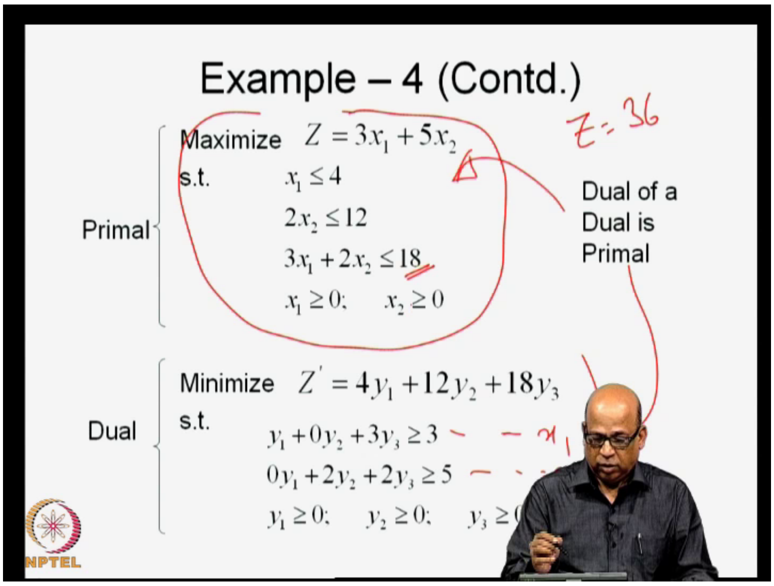

Now, we will just quickly go through one small topic, where we are looking at let us say, we are talking about this particular problem, this part you understand correctly, let say you are looking at this problem now. And you got the optimal solution Z as 36 and if you recall one of the constraints, at least one of the constant was binding or was at tight constraint, in the sense that when you put x 1 is equal to 2 and x 2 is equal to 6, this becomes equal to 18 and therefore, this becomes a tight constraint. Even when it is not tight, we would like to address the following question, what happens to Z value if I slightly change the right hand side value of one of the constraints or many of constraints? So, if you change the right hand side of the constraints, what happens to Z value, this we do it through what is called as sensitivity analysis and the dual solution that we obtain; that means, from the primal or otherwise you obtain the solution for the dual, the solution of the dual will give us a way or means of answering this question without having to resolve the problem; that means, let say that, from 18 I make it 17, or 18 I make it 19, I do not had to resolve this problem, simply use the solution for the dual and then, from the dual we should be able to get it. I will not go into theory of this, but will just go through how we do this.

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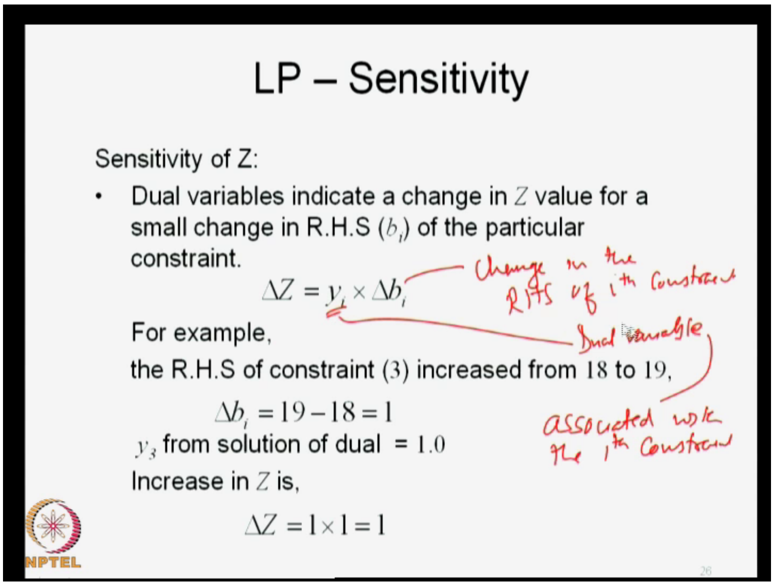

So, this is called as the sensitivity analysis, this is one of the sensitivity analysis that we can do, I will also list out the remaining, the other type of sensitivity analysis. So, the first type of sensitivity that we do is, if we change the coefficients that is, the right hand side of the constraints, what will be the change in the objective function value? Now, this type of sensitivity analysis is extremely useful in practical situations where I keep on repeating the sizes of the LP problems are very large. And you will be interested in answering the questions such as if my resource availability for example, is lower than what I used in the solution. Let us say, my resource availability was 18 units, I make it 16 units, then what happens to the objection function value? How much would I lose on the objective function value, or if I increase my resource availability from 18 to 20, what happens to the objective function value? It was 36 now, does it go to 40, and does it go to 50 and so on. You do not have to resolve the problem, you simply use the solution of the dual, that is the dual variables, with that dual variables associated with those particular constraints you should be able to answer this question.

So, this is what is called as sensitivity; that means, how sensitive is my solution to a slight change in the right hand side values of the constraints, I keep saying slight change because these analysis are valid for a certain range of changes, which will not at the time bother about this. So, the first level of sensitivity we get it by delta Z which is a change in the objective function value corresponding to a change in the i th constraint. So, this is the change in the R.H.S of i th constraint and this is the dual variable associated with the i th constraint, incidentally this variable we have associated with the particular constraint and we obtain the solution from the primal solution and these are called as the dual variables. So, the solution of the dual will give us the values of the dual variables. So, we know from the primal solution, you know Y i, corresponding to that particular constraint and the delta Z is change in the objective function value associated with the small change in right hand side values.

So, let say the R.H.S of constraint 3 which was 18, your original constraint we had this constraint 3. So, that associated dual variables will be y 1 here, y 2 here and y 3 here and from 18, I would like to make it to 19 which means delta b i here, delta b 3 in this case will be 1. So, I will change it by one unit. So, my delta b i will be 19 minus 18, it is increased from 18 to 19 and the associated dual variable y 3 is 1 how do where did you get this? y 3 is one associated with the third constraint which has a slack variable of x 5. So, this is the solution for y 3, y 3 is equal to 1. So, I will use that and the increase in delta increase in Z therefore, would be delta Z is equal to this is y i and this is delta b i therefore, delta Z would be equal to 1 which means what my new value of the objective function will be the original value of the objective function which was 36, the optimal value plus delta Z which is 1 and therefore, you will get 37.

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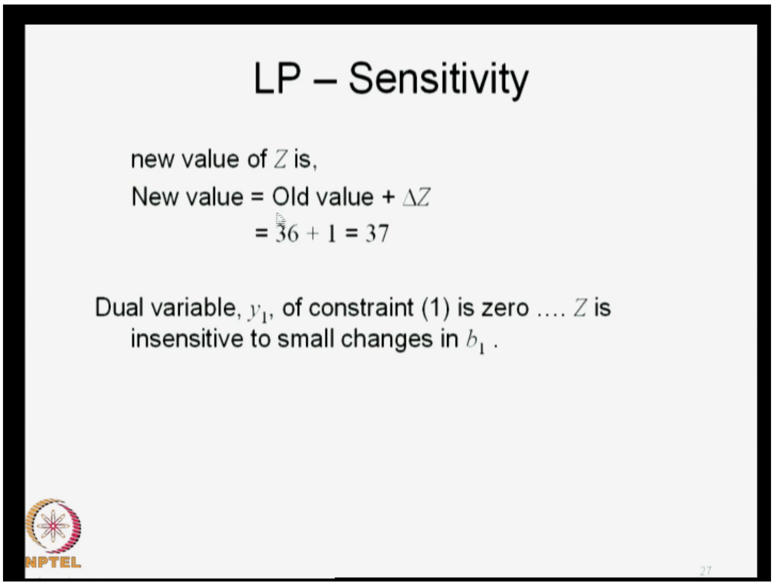

This means that, if I increase the third constraint the right hand side constraint by 1 unit, I will get an enhanced value of objective function of 1 unit. Now, this is the type of sensitivity that we do as the first level of sensitivity just to check what happens if I am if my resource availability changes. In this case, we increase the resource availability from 18 to 19 assuming that resource constraint, we increase the resource availability from 18 to 19 and saw that we got a increase in the Z value of one unit, let say you had decrease it by point 5. Then what will happen? Because the dual variable is one associated with that constraint, you will get a corresponding decrease of 0.5 in the objective function. Look at this, the dual variable y 1 of constraint 1 is 0, in that particular example, if you have a dual variable value as 0 in the final solution, it means that the change in the right hand side values or objective function value will be insensitive to change in the right hand side value of that particular constraint, because the corresponding dual variable is 0.

So, this is how we use the solution of the primal problem to identify the solution of dual problem, which gives us the values of the dual variables. Remember, dual variables are associated with constraints and when we are looking at sensitivity of the constraints, sensitivity of the objective function value to the change in the right hand side values of the constraints we use, we make use of the dual variables, values of the dual variables, and get the associated change in the objective function value. As you can see, we did not had to resolve the problem and this becomes very advantageous and handy, I will quickly list.

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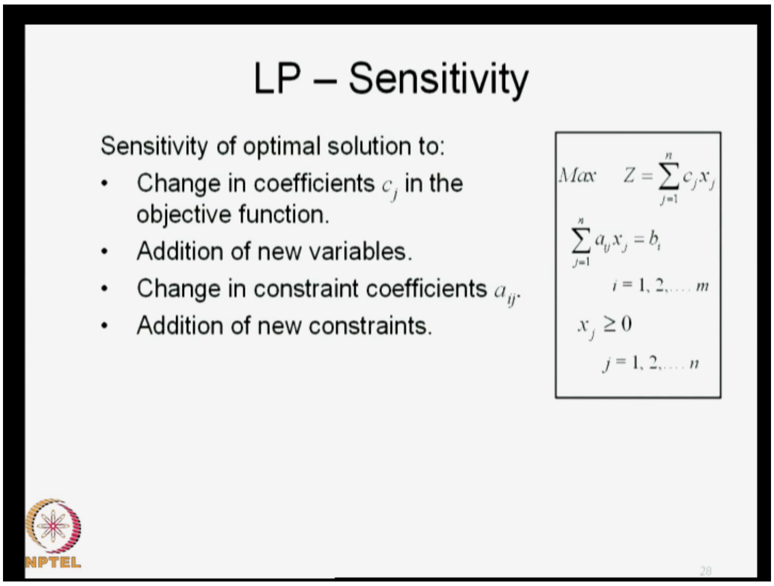

Apart from this kind of sensitivity, we also do sensitivity for example, this is the problem you have objective function coefficients, you have right hand side values, you have several number of constraints, you have several number of variables etcetera, and you have solved this particular example. With this example now, once you solve we should able to tell without resolving the problem, we should able to tell, what is the sensitivity of the optimal solution to change in coefficient c j? Let us say that my you are talking about maximization, therefore this may be benefit coefficient in terms of let us say, hydro power generation in terms of irrigational potential generated in terms of benefits Z prove and so on. So, if I change this, what would be the changed solution? Then I may want to add new variables, if I add new variables what happens to the optimal solution? Then I may want to change the coefficient in the constraint themselves, then what happens to the optimal solution? I may want to add new constraints. Let us say, your physical problem, you have already solved with the simplified version of the formulation, then you may want to add a few more constraints and you want to examine what happens to the optimal solution, all of these constitute the sensitivity analysis and we need not resolve the problem for getting the answers to these questions.

So, we will continue the discussion in the next class. So, essentially what we did today is introduce the concept of duality, we saw that the dual of the dual problem is in fact, the primal problem, we saw how to formulate the dual problem, we also saw that when you solve the solve one of the problem, see that the primal or the dual problem, the solution for the other is also included in the final optimal tableau. And in fact, the dual variables that you get from the solution of the primal problem can be used as can be used to answer questions such as, what is the sensitivity of the objective function to a small change in the right hand side values of the constraints? So, we will continue these discussions hopefully in the next class, next lecture; I will conclude the discussion on linear programming and will go on to the dynamic programming in the next class, thank you for your attention.