Water Resources Systems

Prof. P.P.Mujumdar

Department of Civil Engineering

Indian Institute of Science, Bangalore

Module No # 02

Lecture No # 12

Linear Programming: Unbounded and infeasible problems

Good morning and welcome to this the lecture number 12 of the course Water Resource Systems Modeling Techniques and Analysis. We have been talking about optimization techniques in this part of a course and specifically during the last few lecture, I have been dealing with the Linear Programming Problem.

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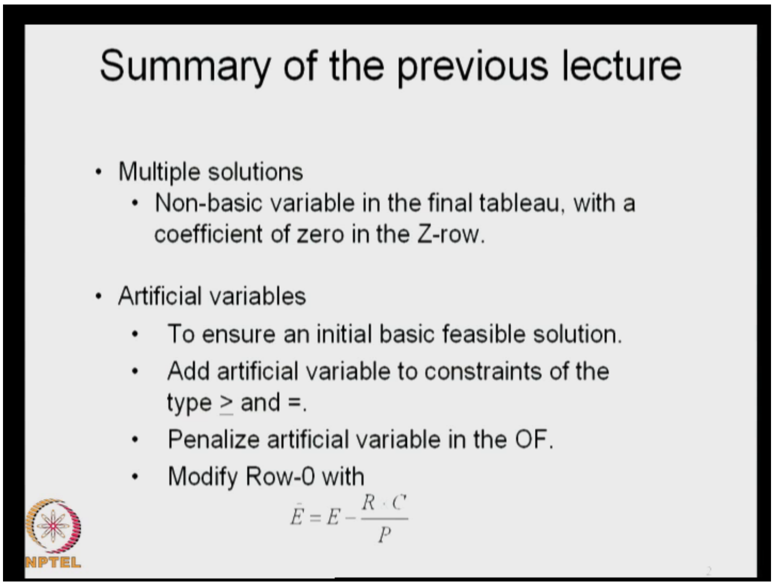

And in the last two lectures, we have been talking about the simplex method and specifically in the previous lecture, we have dealt with the multiple solutions, how do we identify the existence of multiple solutions in a linear programming through, when we are solving it through simplex method. If we recall, if the there is a non basic variable in the final tableau, which has a coefficient of 0 in the row 0 or in the Z-row, it indicates that that particular non basic variable can be brought into the basis, without changing the objective function value and therefore, you can generate one more set of solutions.

And as you can recall, if can generate two sets of solutions, you can generate infinitely infinitely many number of solutions by taking the linear combinations of the two solutions, that is, we write x star is equal to alpha x 1 plus 1 minus alpha x 2, where alpha is a constant between 0 and 1. And, that is how you generate infinitely many number of solutions. And as I have been telling, the existence of multiple solutions to a optimization problem is always advantageous for the decision maker, because your optimal value of a objective function remains the same across all these solutions, but you have an enormous flexibility to choose one solution from the other.

And therefore, you have a handful in fact, more than a handful of solutions possible all of which lead to leading to the same objective function value and therefore, you should be able to generate, identify and generate multiple solutions in the linear programming problem. Then, we went onto introduce the artificial variables, the concept of the artificial variables and then how we handle the artificial variables is what we were just talking about towards the end of the lecture.

We introduce the artificial variables essentially to make sure, that you have an initial basic feasible solution, to start of the competitions, to start of the simplex algorithm. As long as you have less than or equal to type of constraints, because our standard form is maximization objective function, for the maximization objective function, if all your constraints are less than or equal to type, corresponding to each of the constraints, you would have added one slack variable, which comes with the positive sign there, that is plus x 3 is less than or equal to etcetera, you you get.

And therefore, when you use the original decision variables as non basic variables, you would have got an initial basic feasible solution. However, when you have greater than or equal to or equal to constraints, as you choose your original decision variables as non basic variables, you will not get an initial basic feasible solution, because the surplus variable in the case of a greater than or equal to constraint comes with a negative sign there.

Therefore, you will not get a initial basic feasible solution, for the express purpose of getting an initial basic feasible solution to start of the computations, we introduce the artificial variables, in cases where you have greater than or equal to or equal to constraints associated with each of the greater than or equal to and the equal to constraints, you introduce one artificial variable each.

So, the artificial variables are essentially added to constraints of the type greater than or equal to and equal to constraints, because these were not existent in the original problem, you would like the artificial variables to assume a value of 0 in the final solution. And therefore, you would penalize the objective function for any non zero value of the artificial variables and artificial variables are non negative, therefore the only only desirable value for the artificial variables will be 0.

So, you would like the artificial variables to assume a value of 0 in the objective function, in the final optimal objective function value and therefore, you penalize the artificial variables, you I am sorry you penalize the objective function, such that any non zero value of the artificial variable will pull down the objective function value to a great extent, in the case of a maximization problem. And therefore, there is no incentive in fact, there is a penalty associated with any non zero value of artificial variables, this we do by the Big M method. And then, the way to use the Big M method, we will continue the discussion that we started in a previous class.

You take this Z-row and then, you take the row in which the artificial variable has been introduced and then transform the Z-row using our way of expressing, it in the canonical form as we have discussed earlier, transform the Z-row and then use this transform Z-row in your computations in the simplex term. And this terms are, you do one by one associated with each of the rows containing the artificial variables, so you take the Z-row take one of the rows containing the artificial variables, transform the Z-row. On this transform Z-row, you take another row containing another artificial row variable and then retransform this again and so on. So, associated with each of the rows containing artificial variables, you do this transformation one at a time.

And finally, when you exhaust all the artificial variables, all the rows containing the artificial variables that transform row, that finite transform row is what enters as the Z-row in your simplex tableau. I will demonstrate this with a simple example to make it further clear. So, will let us take this example maximize Z is equal to 3 x 1 plus 5 x 2, this is the same problem, that we have been solving using the graphical method and the first problem, that we solved is in simplex method and so on, except that, we will make a slight change in one of the constraints, x 1 less than or equal to 4, 2 x 2 less than or equal to 12, 3 x 1 plus 2 x 2 is equal to 18, recall that this constraint we had earlier 3 x 1 plus 2 x 2 less than or equal to 18.

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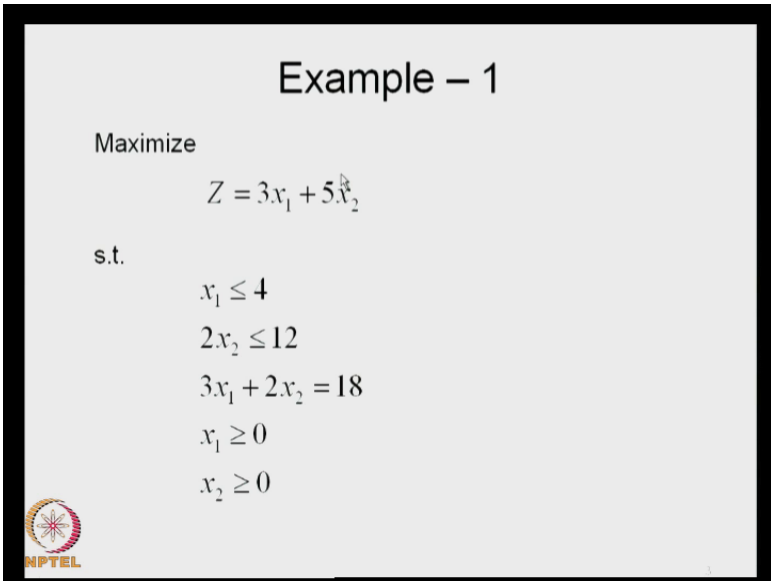

So, instead of less than or equal to, I would make it as equal to constraint equal to 18 and x 1 and x 2 are non negative. Because, this is an equal to constraint, if you put x 1 and x 2 as 0, let us say that, you would have put a slack variable here x 1 plus x 3 is equal to 4, 2 x 2 plus x 4 is equal to 12 and because, this is already an equality constraint you cannot add any slack or surplus variables, therefore you will written 3 x 1 plus 2 x 2 is equal to 18, because of this equality constraint, if you put x 1 and x 2 is equal to 0 what happens, these becomes infeasible, this constraint cannot be satisfied.

And therefore, that is no initial basic feasible solution possible. Therefore, to ensure the a initial basic feasible solution, we add a artificial variable here, so we will write this as 3 x 1 plus 2 x 2 plus a 1 is equal to 18. Just to start of the computations, so this is what we do, your original constraint your objective function is 3 x 1 plus 5 x 2, I will write this as 3 x 1 plus 5 x 2 plus 0 x 3 plus 0 x 4 minus M into A 1, where I am adding a artificial variable here, 3 x 1 plus 2 x 2 plus A 1 is equal to 18, I am adding a artificial variable here.

Now, because we add an artificial variable, we introduce a penalty in the objective function and this penalty we will add as minus M into A 1 for the Z expression. So, in the Z expression, I will write it as minus M into A 1, where m is a large number and therefore, any non zero value of A 1 will pull down the value of Z, that is idea and x 3 and x 4 here are the slack variables associated with these two constraints.

Now, there is one row containing the artificial variable. So, what we will do is, we will take the Z-row will express this as a row Z minus 3 x 1 minus 5 x 2 plus M A 1, that becomes your zeroth row or Z-row and take it along with this constraint this row and transform the Z value Z-row using our earlier method. So, I will write this as Z minus 3 x 1 minus 5 x 2 plus M A 1 is equal to 0, so this is the constrain that, I am rewriting is equal to 0 and the row containing the artificial variable is 3 x 1 plus 2 x 2 plus A 1 is equal to 18. So, we will write this in our canonical form, we will write this as minus 3 x 1 minus 5 x 2 0 x 3 0 x 4 A into A 1 is equal to 0.

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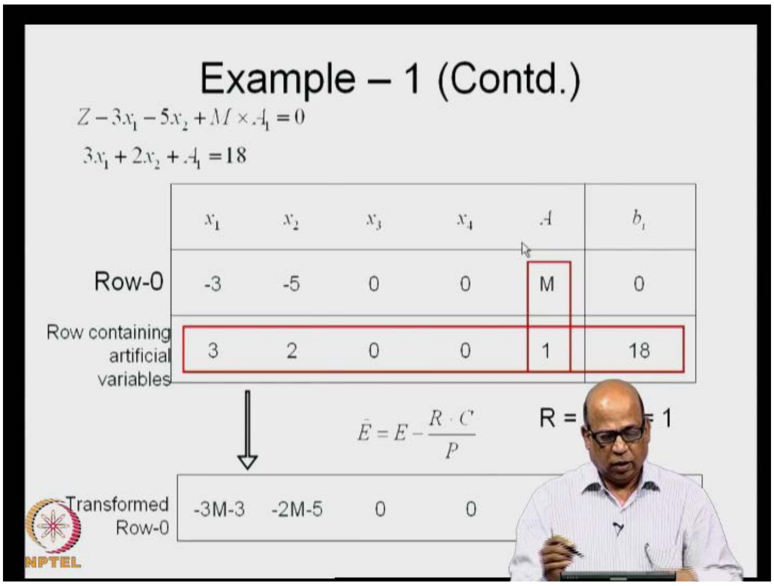

So, this is the row 0 is what we wrote. Then the row containing artificial variable 3 x 1 plus 2 x 2 plus 0 x 3 plus 0 x 4 plus 1 into A 1, this can be A 1 will write this as A 1 here, is equal to the right hand side, that is 18 here. Now, we will choose this as the pivotal column and this as the pivotal row and transform the Z-row, say for example, I want to transform the right hand side, how do I do 0 minus M into 18 divided by 1, so minus 18 M. Let us say, I want to transform this because, there is 0 that remains as 0, this because there is as 0 that remains 0, let us say you transform this, minus 5 minus 2 M by 1, so minus 2 Minus 5.

Let us say, I want to transform this minus 3 minus M into 3 divided by 1, that is R into C by P, R is the element in the row of that particular element, which is M, C is the element in the column of that particular element, which is 3, P is 1, which is the pivotal the pivotal element. So, this is how you transform the Z-row minus 3 M minus 3 minus 2 M minus 5 0 0 0 minus 18 M, now this is what we used in our simplex algorithm first table.

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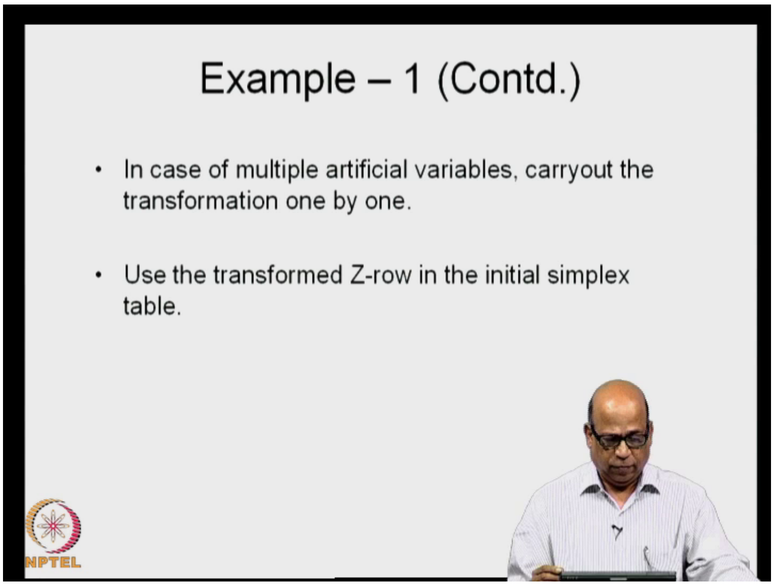

So, the transformed row is what we will use, in the first tableau of the simplex. As I mentioned, if you had more number of constraints of the type greater than or equal to or equal to, you would have more number of artificial variables, in such a case you do this transformation one at a time; that means, we took this row now, you got a transform row, then on this transform row, you take another row containing the artificial variables and then redo the transformation and so on. So, until you exhaust all the rows containing artificial variable.

And then we use the transformed Z-row or transformed row 0 in the first simplex tableau, this is what we will write now, so this is the transform Z-row minus 3 M minus 3 minus, 2 M minus 5, 0, 0, 0, minus 18 M, so this is what we write here, minus 3 M minus 3, minus 2 M minus 5, 0, 0, 0, minus 18 M 18 M and all other rows, you write as there.

So, this is row 1, row 1 is this, x 1 plus x 3 is equal to 4, row 2 is 2 x 2 plus x 4 is equal to 12, row 3 is 3 x 1 plus 2 x 2 is equal to 18, that is plus A 1 is what we will write. So, in the row 1, you have x 1 plus x 3 and 0 A 1 is equal to 4, in row 2 you have 0 x 1 2 x 2 0 x 3 x 4 0 A 1 is equal to 12 and in the last row, you have the artificial variable, you will write 3 x 1 plus 2 x 2 plus 0 x 3 plus 0 x 4 plus A 1 is equal to 18.

So, this is your first tableau now. Now, because these are first time you are coming across this m in the simplex tableau, I will explain bit more in detail remember always that M is a very large number and therefore, when you are comparing M with any other number the M magnitude, magnitude of m will be much much larger compared to any other values, that you have here. So, between minus 3 M and minus 2 M minus 3 M is a larger negative value, let us say that this was minus 3 M minus 5 only then you take this as a larger number.

Let us say, this was minus 3 M minus 3 and this was minus 2 M minus 50000, let us say even then this becomes larger negative value. So, M will override all other dimensions, even if you had minus 5 into 10 to the power 6 let us say, but M is so large compared to 10 to the power 6 that minus 3 M will be still larger negative value compared to minus 2 M minus 5 into 10 to the power 6 or whatever.

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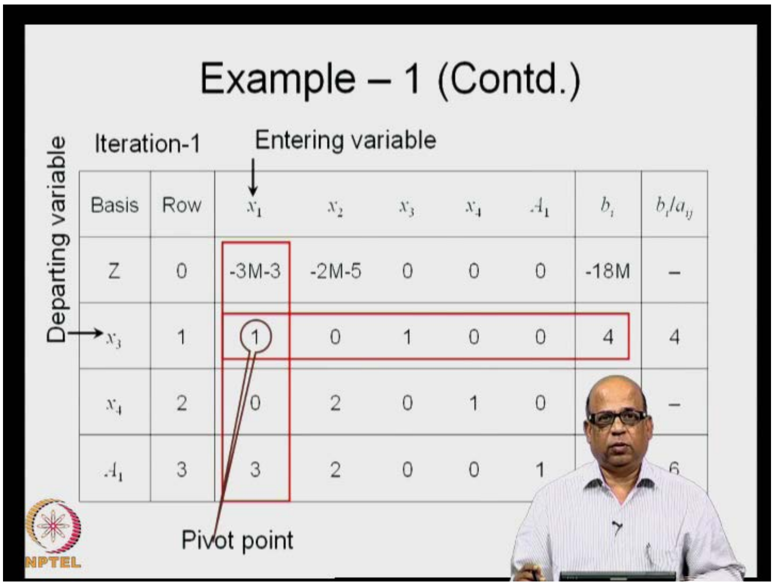

So, this is what you must keep in mind, the coefficient of M will determine the relative magnitude, because M itself is very large here and therefore, minus 3 M minus 3 is a larger negative number compared to minus 2 M minus 5. And therefore, this becomes the entering variable, that is x 1 becomes the entering variable and then you do all your calculations and see that, you can calculate your b i by a i j, only for these two which turn out to be 4 and 6 corresponding to the minimum b i by a i j value, you identify the departing variable in this case, it becomes x 3 and therefore, x 1 enters in place of x 3 you write it as x 1 x 4 and A 1.

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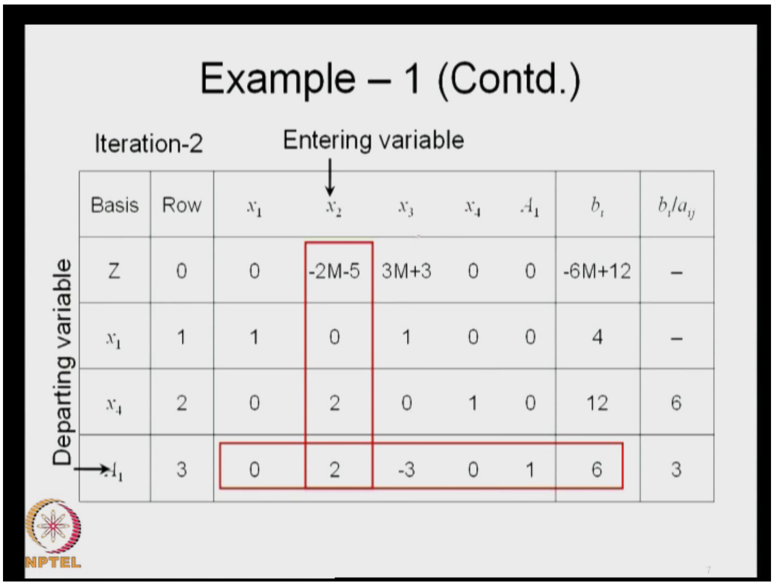

Basis	Row	x_1	x_2	x_3	x_4	A_1	b_i	b_i/a_{ij}
Z	0	0	-2M-5	3M+3	0	0	-6M+12	–
x_1	1	1	0	1	0	0	4	–
x_4	2	0	2	0	1	0	12	6
A_1	3	0	2	-3	0	1	6	3

That is what you write for the second iteration, so in the second iteration your basis would be x 1 x 4 and A 1, recall what you do then, you have identified the pivotal row, first you transform the pivotal row by dividing it by pivot, because in this case pivot is 1, this row remains the same, so that row remains the same. Then, all other elements you transform using our expression, the new number is equal to old number minus the element in the pivotal row corresponding to the element E, element in the pivotal column corresponding to the element E and divided by the pivotal itself, so that is what you do to transform everything.

Let us say that, this number we want to transform, so minus 18 M if you want to transform, how this will get transform, minus 18 M minus this number minus 3 M minus 3 into 4 divided by the pivot element, the pivot element is 1. So, this becomes minus 18 M, this will be plus 3, 12 M here, let me rewrite that, I am showing the transformation.

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So, this will be minus 18 M this is into 4, so this would be minus of minus, that is minus 12 M minus 3, for this if you write this will be minus 12 M, so it will be let me rewrite this, where doing transformation for this number now, so this will be minus 18 M minus 4 into minus 3 M minus 3 divided by 1, so this expression is right, minus 18 M minus 4 into minus 3 M minus 3. So, that comes out to 12 plus minus 6 M, so you get minus 6 M here minus 18 M plus 12 M, so 6 M minus 12, so this will be minus 6 M plus 12. Like that, you do all the transformations let us say, you want to transform this, now 0 how do we do 0 minus 1 into minus 3 M minus 3 divided by 1, you get 3 M plus 3.

Let us say, you want to transform another number x 3 is already in the departing variable, so let us transform x 3, so 0 minus 1 into minus 3 M minus 3 divided by 1, that comes 3 M plus 3 and so on. So, you would do the transformation using exactly the same procedure that we did, except keeping always in mind that M is a large number and when you are doing the transformation like this. You just keep it as M you do not change the number, so you just keep it as a constant m and then do the transformation. So, this is the transform table now, now you ask the question whether this is the optimal solution, it is not an optimal solution, because there is a negative coefficient here in the Z-row. Remember here, if you had let us say minus 2 M plus 50000, even then it becomes a negative number, because M is very large.

So, minus 2 M minus 5 is a negative number here and therefore, this solution is not optimal and because there is only one negative number, that variable the variable corresponding to that negative number will enter the basis, so this is the entering variable. Once you identify the entering variable calculate b i by A i j, if it is non zero, non negative and then we will choose that particular variable, which has a minimum b i by A i j among all these variables for which we could calculate and that goes as goes out of the basis as departing variable, so it makes way for the entering variable x 2.

So, you will have your next basis as x 1 x 4 and x 2 and then you do the transformation again of the pivotal row and so on. So, you get to the next iteration, so x 1 x 4 x 2 is your new basis, again do the calculation, you will see that there is still one more negative variable minus 9 by 2 and therefore, this becomes a entering variable and then x 4 becomes a departing variable, your basis then becomes a x 1 x 3 and x 2 in the next row you your next calculation, again redo the calculation redo the transformation and so on and go to the next tableau.

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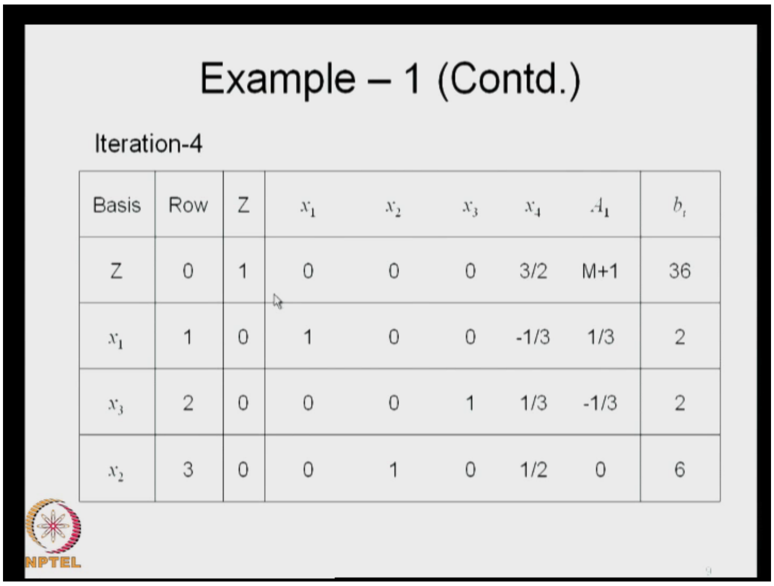

Example – 1 (Contd.)

Iteration-4

Basis	Row	Z	x_1	x_2	x_3	x_4	A_1	b_i
Z	0	1	0	0	0	3/2	M+1	36
x_1	1	0	1	0	0	-1/3	1/3	2
x_3	2	0	0	0	1	1/3	-1/3	2
x_2	3	0	0	1	0	1/2	0	6

When you go to the iteration number 4, when you ask the question, is the solution optimal and examine the coefficients in the Z-row of all the variables including the artificial variable, you will see that, all these coefficients are non negative and therefore, the solution cannot be further improved and therefore, this becomes the optimal solution. What is optimal solution? The solution is read as Z is equal to 36, x 1 is equal to 2, x 3 is equal to 2, x 2 is equal to 6, so this is the solution.

Now, there are some important aspects that you must remember, you introduce the artificial variable A 1, which was not existent in the problem, much like you introduce x 3 and x 4 for example, these were slack variables, x 3 and x 4 can take non zero values in the final solution, because essentially you introduce the slack variables to make the constraint as equality constraint. Therefore, x 3 can be non zero in the final solution; however, even has to be 0 in the final solution, because you have introduced a penalty corresponding to A 1, if there is any A 1, which is non zero in the optimal solution, if there is any artificial variable, which is non zero in the final solution, it means that the penalty has, penalty comes into play in the optimal Z value.

We will see the implication of that, when we proceed further, if there is the artificial variable, which has a non zero value, when the optimality criteria is satisfied let see, what it means presently. So, the solution here is Z is equal to 36, x 1 is equal to 2, x 3 is equal to 2 and x 2 is equal to 6, recall this, when we solve this problem earlier with a less than or equal to constraint 3 x 1 plus 2 x 2 was less than or equal to 18 in the earlier problem that we solved.

And that, thing we made it equal to just to demonstrate the artificial variable, when we made this equal to still get the same solution namely x 1 is equal to 2 and x 2 is equal to 6, if you forget about other variables x 1 is equal to 2 and x 2 is equal to 6 and Z is equal to 36 was in fact, the solution that you had obtained earlier, which means what if you look at these constraint 3 x 1 plus 2 x 2 less than or equal to 18 and you put x 1 is equal to 2 here and x 2 is equal to 6 here the left hand side becomes equal to 18.

Such constraints, which are stretch to the full extent, you were saying 3 x 1 plus 2 x 2 must be less than or equal to 18, but in the optimal solution, it becomes equal to 18 such constraints are called as the binding constraints or tight constraints, there is no slack available at all, it is a tight constraint in the optimal solution it has been completely stretched to its limit such constraints are called as tight constraints.

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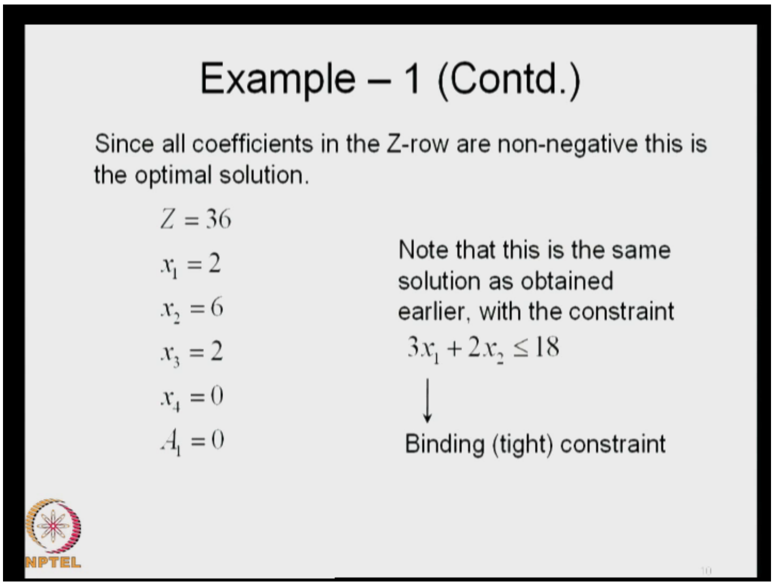

So in fact, these are important, in it is important to identify which are the tight constraints, which are the slack constraints etcetera when you will do the sensitivity analysis, which we will see later on to some extent, look at the other constraint which you had x 1 is less than or equal to 4, in the optimal solution you got x 1 is equal to 2 which means there is still some slack available there.

So, x 1 less than or equal to 4 was your original constraint, in the optimal solution you got x 1 is equal to 2, so there is still a slack of 2, that is available, x 3 is equal to 2. Because x 1 plus x 3, you may write as equal to 4 in that particular case and therefore, the slack variable which is non zero in this particular case has taken a value of 2. So, that slackness is still available to be satisfied in that particular constraint, that is how you should look at and in fact, when we do some applications etcetera, hopefully we will cover this aspect in slightly, but right now you just keep in mind that the constraints, which are stretched to the full extent in the final optimal solution are called as a tight constraints or binding constraints.

And there will be some binding constraints, there will be some flat constraints, the constraints, which we are not completely stretched in the optimal solution are called as a slack or loose constraints. So, there will be in typically there will be some binding constraints, there will be some loose constraints and so on, alright. Now, what we saw was that, multiple solutions exist and we know how to capture the multiple solutions. And we introduce the artificial variables to account for greater than or equal to and equal to constraints, just to make sure that you have an initial basic feasible solution to start of the computations.

In the graphical method, I introduced several other special cases for example, you may have a unbounded solution, where the feasible region is unbounded in the direction in which the Z is increasing for a maximization problem and therefore, Z can be increased up to infinity, without highlighting any of the constraints and such problems lead to unbounded solution.

And there may be also in feasible solutions, where the set of constraints is such that, there is no intersecting region that these constraints define in the feasible region. And therefore, you cannot find an intersecting feasible region for the type of set for the set of constraints that you have in the original problem. Identification of unbounded solution as well as in feasible solution is as important as identifying multiple solutions.

In fact, both unbounded solutions and the in feasible solutions always indicate that the problem is not correctly formulated that may be certain constraints that have not been correctly put and so on. So, the moment you come across unbounded solution or the infeasible solution you should look at your problem formulation, whether the physical features of a problems have been correctly translated into mathematical expressions and mathematical expressions in terms of the constraints and so on.

So, typically if you have an unbounded solution or the in feasible solution, it always indicates that the problem is yield formulated. How do we find an unbounded solution? If you recall your entering variable you identified based on the highest negative coefficient in the Z-row. This indicates that, the rate of increasing the objective function is much higher, if you bring in this particular variable into the basis compared to bringing in any other currently non basic variable that is how you identified the entering variable? Then, you go to the departing variable. How did you identify the departing variable? It is that particular value that particular variable, which reaches the 0 first as you start increasing the value of this freshly entering variable.

So, as you start increasing this variable the one of the non basic variables will attain one of the currently basic variables will attain 0 first and that becomes your departing variable. Let us say that, a situation exist where you keep on increasing the currently entering variable without making any of these currently currently basic variables to 0; that means, you keep on increasing the basic currently entering basic variable without making any of the basic variables to reach 0; that means, what you can increase this particular variable, which has been identified as a entering variable up to infinity and in a maximization problem, that leads to the objective function going to infinity.

So, if at any iteration you came across a situation where you have been able to identify an entering variable, but you cannot identify any exiting variable any departing variable that indicates that the objective function value can be increased up to (()) infinity without violating any of a conditions; that means, that the solution is unbounded. So, the unbounded solution is obtained or identified the moment you come across a situation, where you are able to identify an entering variable and therefore, the solution is not optimal, it is saying that the solution can be further improved however, you are unable to identify any departing variable example.

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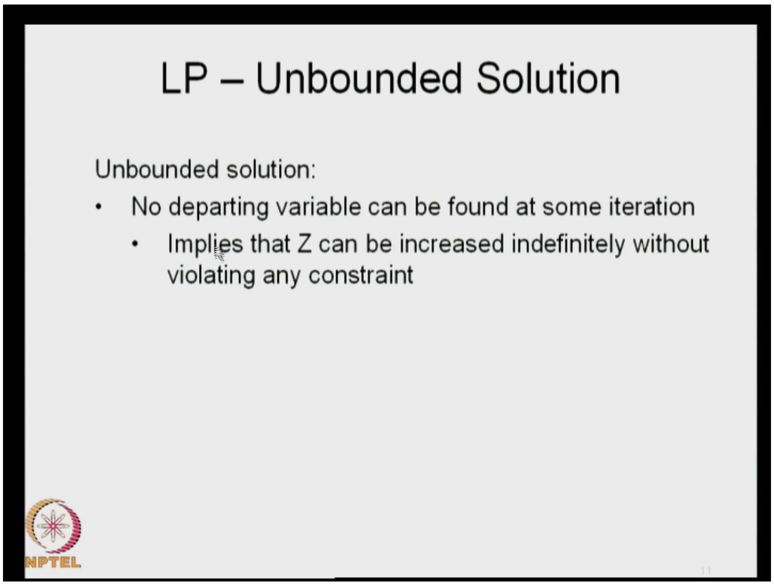

So, this unbounded solution is a situation, where no departing variable can be found at some iteration you have not optimal solution yet. So, yet any intermediate iteration you come across a situation, where departing variable cannot be found such a situation leads to an unbounded solution. Let us look at this example Z is equal to x 1 plus x 2 will write x 1 greater than or equal to 5 and x 2 less than or equal to 10. Just looking at the problem you know that it is a unbounded solution, because this constraint says that x 1 is greater than or equal to 5 and there is no other constraint, which will restrict the variable x 1.

You can keep on increasing x x 1 and you would like to do that, because the objective function is such that as we increase x 1, your Z will keep on increasing and you are looking at a maximization problem therefore, you would like to increase the x 1 as much as possible and you start increasing from 5, there is no restriction there is no upper bound on x 1.

So, you keep on increasing x 1 up to infinity, without any without violation of any constraint and therefore, this is the, obviously, and unbounded problem. Let us see, what happens if you solve such a problem in using simplex algorithms, so this we know it is a unbounded problem, let us solve this problem with the simplex algorithm and see what happens for such problems. So, this is if you solve it by graphical method not solve just try depict it by graphical method, you you have this x 1 is less than or equal to 10, so this is bounded here. But, this says x 2, I am sorry x 1 is greater than or equal to 5. So, this region to the right of this line x 1 is equal to 5 is all feasible and x 2 less than or equal to 10, so region lower than this particular line is all feasible. So, the feasible region here is as shown.

So, all of this let us say you extend this, so all of this region is feasible and Z is increasing in this direction. So, as we increase Z it is moving like this. So, it moves like this in that particular direction, so it is moving in this direction. So, there is no restriction at all you keep on moving still there would be at least one point and it keeps increasing in this particular direction as x 1 increases.

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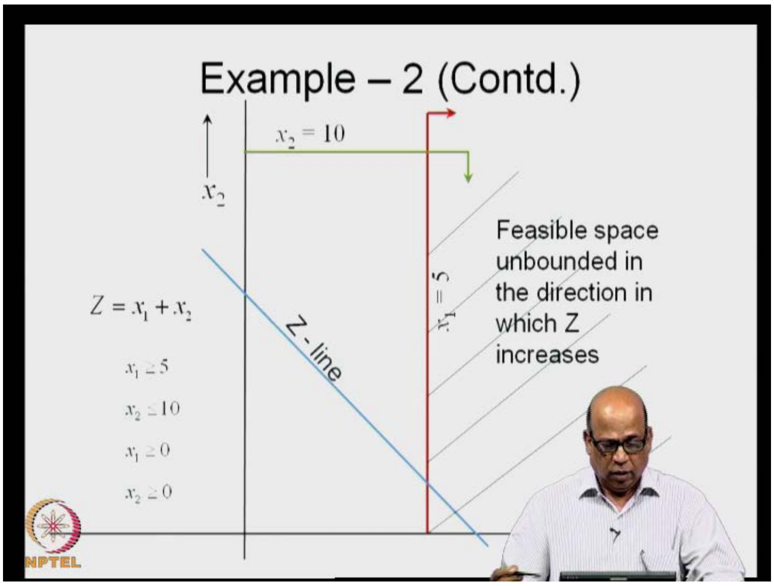

So, you can keep on increasing x 1 without violating any conditions and therefore, x 1 can go up to infinity and Z will reach infinity at that particular point and therefore, this is a unbounded solution. Let us solve this using simplex method, x 1 is greater than or equal to 5 I will write it as x 1 minus x 3 is equal to 5, this is the surplus variable because it is a greater than or equal to constraint and then x 2 is less than or equal to 10, I write it as x 2 plus x 4 is equal to 10 this is slack variable.

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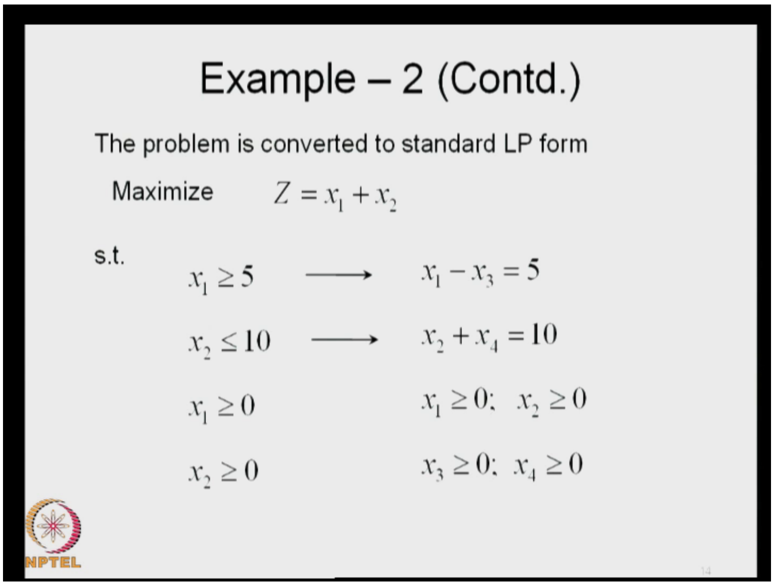

And your objective function is Z is equal to x 1 plus x 2, all these are non negative variable, because you have a minus x 3 is here that means, you have a greater than or equal to constraint, you add an artificial variable here.

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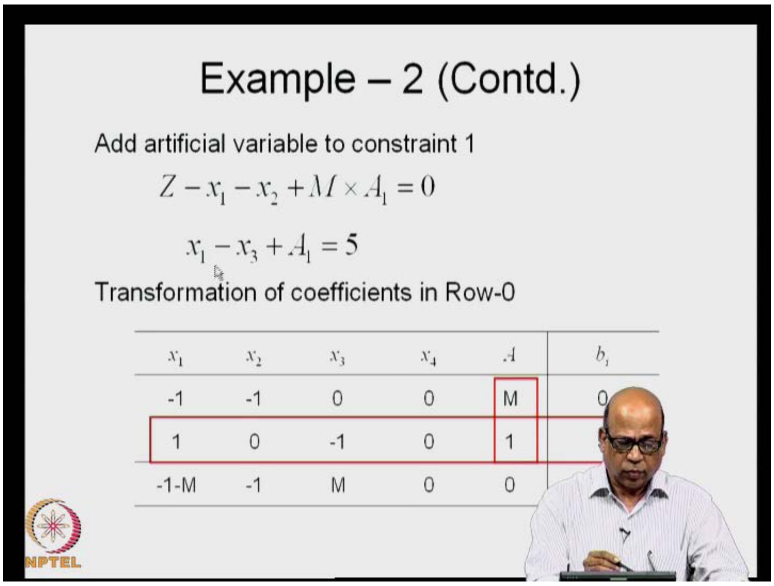

So, will add an artificial variable and write this constraint as x 1 minus x 3 plus A 1 is equal to 5. So, this is the artificial variable that we have added, because you added artificial variable you penalize the objective function, so you would have written Z is equal to x 1 plus x 2 plus 0 x 3 plus 0 x 4 minus M into A 1, because you are penalizing the objective function and therefore, when you write the Z-row it becomes plus M A 1. So, Z-row is written as Z minus x 1 minus x 2 plus M A 1 is equal to 0.

Now, you take all these variables, so you are first transforming the coefficients of Z-row, so this is the Z-row this is the row containing the artificial variable, you do all these transformation like we did in the previous example and rewrite the transformed row, so this is the transformed row minus 1 minus M minus 1 M 0 0 minus M minus 5 M. So, this becomes the transform Z-row, this is what we used in the first simplex tableau. So, we write the simplex tableau row 0 1 2, now this is row 0 minus 1 minus M minus 1 M 0 0 minus 5 M. So, this is minus 1 minus M minus 1 M 0 0 minus 5 M and row 1 is your x 1 minus x 3 plus A 1, this is the row 1.

So, x 1 plus 0 x 2 minus x 3 0 x 4 plus A 1 is equal to 5, like this you write row 1 and row 2. So, you identify the entering variable, this is the entering variable minus 1 minus M, because this is the highest negative value and this becomes entering and this becomes departing, because b i by a i j is minimal among all the values that you put calculate, in fact, this is only value and therefore, this becomes departing variable, you rewrite the basis Z x 1 x 4 x 1 came in the place of A 1.

So, this becomes the new basis, we identify the entering variable identify the departing variable go to the next iteration, you see that this is how you the new new table, you get the entering variable there is one more variable which has 1 variable here, which has a negative value namely x 3.

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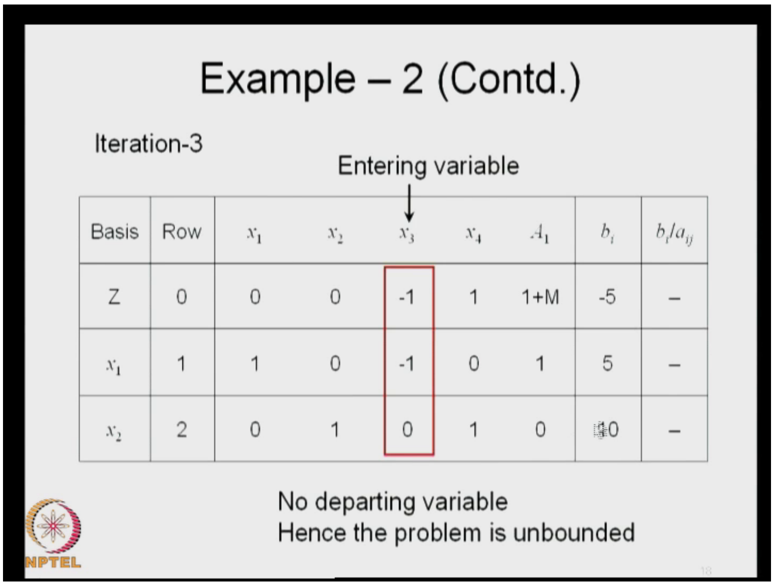

Basis	Row	x_1	x_2	x_3	x_4	A_1	b_i	b_i/a_{ij}
Z	0	0	0	-1	1	1+M	-5	–
x_1	1	1	0	-1	0	1	5	–
x_2	2	0	1	0	1	0	10	–

Therefore, you could get an entering variable. However, you want to now identify which is the departing variable x 3 wants to come in here into the basis and therefore, one of these x 1 or x 2 has to go out. But, when you want to see which is the departing variable you want to calculate b i by A i j for non negative values non zero and non negative values, it is not possible. So, none of these currently basic variables would like to go out now.

So, there is no departing variable, which means what the solution is not optimal here, because there is a negative coefficients sitting in the Z-row and therefore, the objective function value can be further improved by bringing in this as the basic variable. However, the currently basic variables x 1 and x 2 neither of them can be identified as departing variables.

So, this means that neither x 1 nor x 2 reaches 0 as you start increasing x 3. And therefore, there is no departing variable, that is identified and therefore, the problem becomes unbounded. So, this is how you identified the unbounded solution, at any iteration. This is iteration 3 in the particular case, you have not yet reach the optimal solution you have been able to identify the entering variable, but there is no departing variable, that you you can identify at such situation you just terminate the computation say that the problem is unbounded.

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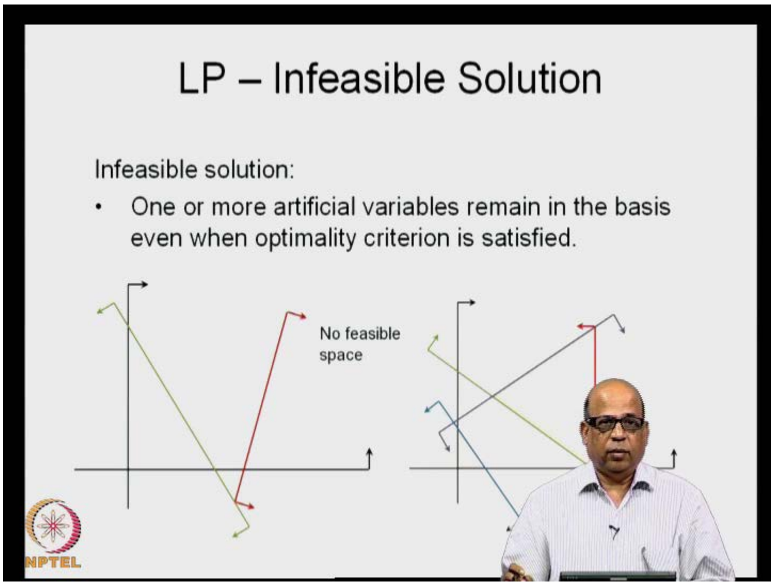

So, this is how you identify the unbounded solution let us look at the other case where we talked about in feasible solutions. Recall that, in the in feasible solution or in feasible problems you are not able to get any feasible space, so one constraint may be looking at this direction the other constraint may be looking at this direction, so there is no intersection intersecting region.

And or you may have a set of constraints, which define some intersecting region here and another set which define another intersecting region here etcetera, but there is no common intersecting region in the feasible region, feasible quadrant here. And therefore, anytime you are unable to find a feasible solution, feasible space that problem is an in feasible problem. This you could do in the graphical case because, there are only two variables and you could see the intersection visually. However, in large problems you must be able to identify the in feasible solution through the simplex algorithm.

In the case, where we introduced artificial variables in both the problems, you just see what happen to the artificial variables; let us say that in this particular case, you had introduced the artificial A 1. In the next iteration A 1 are still in the basis in the next iteration a 1 has gone out. So, A 1 leaves the basis sooner than later if there is a feasible solution, the artificial variables will not sit in the basis for a long time in fact, even if they sit in the basis for a long time, in the optimal solution they will never out, they will never.

So, in the optimal solution, you saw that A 1 is 0, similarly in this particular case you have put a artificial variable and then you look at the final solution, the artificial variable not not exactly the final solution, because unbounded, but the A 1 has left the basis already, so A 1 has left the basis. So, whenever you introduce the artificial variables, if the problem is feasible, the artificial variables leave the basis sooner than later. In fact, in most of the cases if you have 1 or 2 artificial variables within first little iteration, the artificial variables will all leave the basis.

And therefore, they become non basic variables therefore, the solution in the solution they will have artificial variables equal to 0, if the problem is in feasible then, when you reach the optimality criteria, that is that the coefficients in the Z-row associated with each of the variables, are all non negative, there is no negative coefficient there. If you have attained this optimality criteria, but in the basis still one or more of the artificial variables are sitting. That means, what the artificial variables have a non zero solution. So, you have met the optimality criteria, but the optimal the artificial variables are still in the basis. Such solution such situations we lead you to identify this as an infeasible solution, why it is because the artificial variables have taken a non zero value.

And the moment an artificial variable takes a non zero value your objective function gets penalized and therefore, the objective function will be pulled down much further than, if you had only x 1 and x 2, which are the initial basis initial decisions variables, if you taken 0 value. So, Z will become negative in a maximization problem. And therefore, this is the in feasible solution, so you identify an in feasible solution, by a situation in the simplex algorithm, where you have hit the optimal optimality criteria, but the one or more artificial variables are still sitting in the basis that is how you identified the in feasible solution, let us look at this particular example.

You are looking at minimization problem minimize Z is equal to 3 x 1 plus 5x 2, subject to x 1 plus x 2 less than or equal to 4 and 2 x 1 plus 2 x 2 greater than or equal to 32. Just look at these two constraints, one is saying x 1 plus x 2 less than or equal to 4, another is saying x 1 and x 2 greater than or equal to 16. If you just divided by 2 all throughout, so this requires at x 1 plus x 2 to be greater than or equal to 16, this requires at x 1 plus x 2 to be less than or equal to 4.

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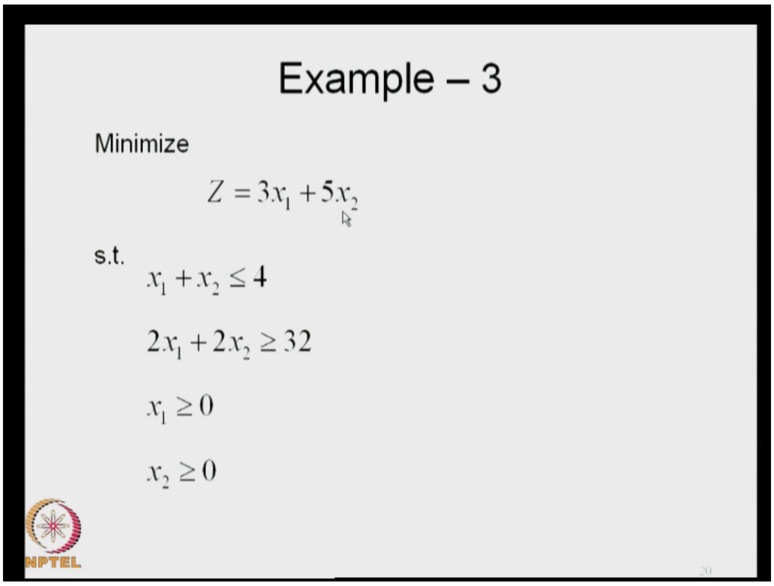

And therefore, they are conflicting with each other they are looking at different sides of the quadrant, different sides one is looking towards the right, another is looking towards the left and there is no intersecting region and obviously, this is the in feasible problem. If you plot the graph this is how it looks, so 2 x 1 plus 2 x 2 greater than or equal to 32, it is this region. And x 1 plus x 2 less than or equal to 4 is this region and therefore, there is no intersecting region possible at all, and therefore, this is the infeasible problem. Let us solve this using our simplex algorithm to see what happens so, you have maximize Z is equal to 3 x 1 plus 5x 2 subject to x 1 plus x 2 less than or equal to 4, so I will add a slack variable making it x 1 plus x 2 plus x 3 is equal to 4, 2 x 1 plus 2 x 2 greater than equal to 32, I will write it as 2 x 1 plus 2 x 2 minus x 4 is equal to 32, I am detecting a surplus variable and all these are non negative.

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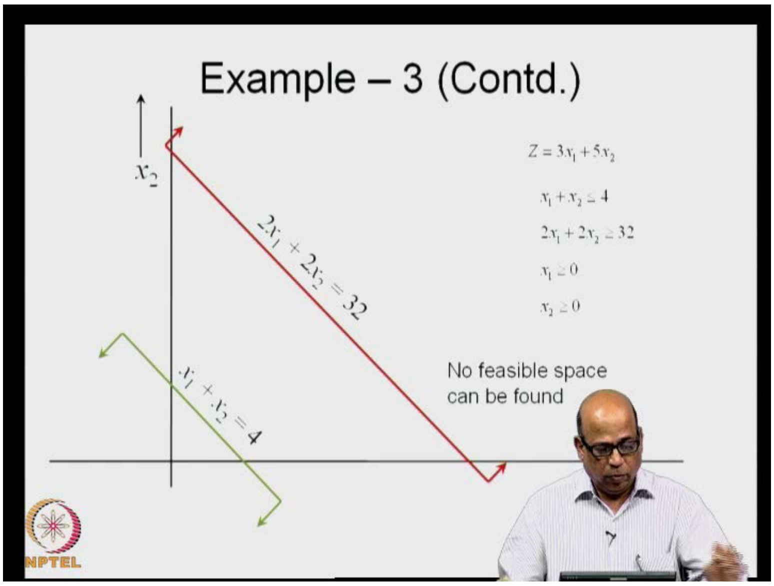

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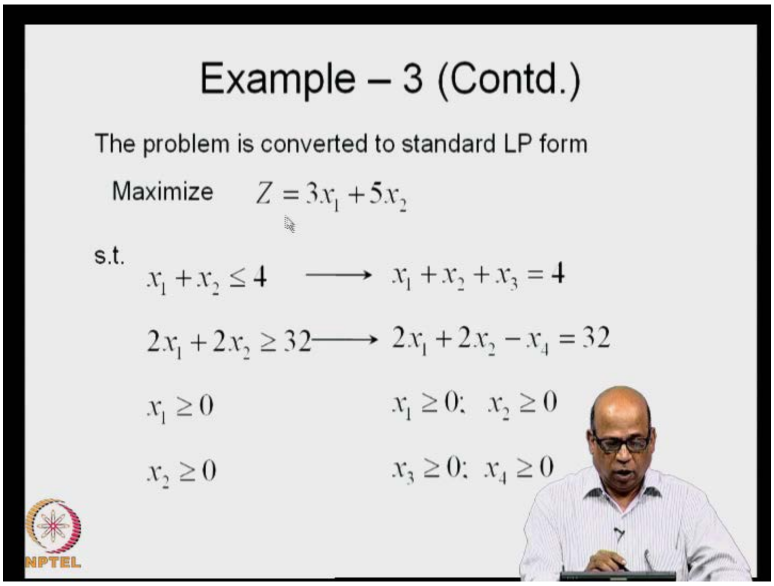

Now, corresponding to this now, I will add a artificial variable penalize the objective function and transform the Z-row using all our earlier methods your Z-row is Z minus 3 x 1 minus 5 x 2 minus that is plus M into A 1 is equal to 0, then the row containing the artificial variable will be 2 x 1 pus 2 x 2 minus x 4 plus A 1 is equal to 32, so there is a row containing the artificial variable. So, you transform like, we did earlier and this will be transform row minus 3 minus 2 M minus 5 minus 2 M 0 M 0 minus 32 M. Use this transform Z-row now in the first simplex tableau, so this is the transform Z-row and this is your row 1 and this is row 2 in their original form.

And then, you you solve using your using your optimality criteria you see that minus 5 minus 2 M is higher negative value and therefore, this becomes a entering variable. And then, this becomes a departing variable x 3 is a departing variable, so your basis gets defined as Z x 2 and A 1, so the Z x 2 and A 1 and when you do the when you do the transformation, you see that all the coefficients in the Z-row in iteration 2 are all non negative, remember M is a positive quantity and a very large number at that. So, this this is also positive and these are all non negative and therefore, you have reach the, you have satisfied the optimality criteria; that means, it is says that the solution cannot be further improved.

However in the basis there is A 1 sitting here; that means, a artificial variable is taking a non zero value a 1 is 24 here and the Z value here is 20 minus 24 M, where M is a very large quantity, which means Z is a very large negative value. So, although you have satisfied the optimality criteria here, because there is an artificial variable that is sitting in the basis this is an in feasible solution. So, you identify the in feasible solution, the moment you hit the optimality criteria and still there is a at least one artificial variable sitting in the final tableau in the basis, that leads to an in feasible solution, so this is how you identified in feasible solution.

So, you know now, how to identify multiple solutions you identify the multiple solutions, where the coefficient of one of the non basic variables in the final tableau by final tableau, I mean when you have met the optimality criteria, when the coefficient in the Z-row corresponding to one of the non basic variables is 0, that is when you when you say that there are multiple solutions.

And you know how to generate multiple solutions. You identify the unbounded solution, where in any of a intermediate iterations, remember you are not hit the optimal solution yet, in the case of unbounded solution.

In any of a intermediate tableaus, you are able to identify an entering variable, but there is no corresponding departing variable, which means that the solution can be increased to infinity for a maximization problem. And therefore, that becomes unbounded solution then you know how to identify an in feasible solution, you have hit the optimality criteria, but one of a at least one of a artificial variables is still in the basis and therefore, it becomes in feasible solution.

So, all these special cases you know, how to identify through simplex algorithm. Now, we will go to a interesting feature of a linear programming problems corresponding to any problem that, we have dealt with let us say that, you dealt with the last problem which was in feasible corresponding to any problem. You can formulate a corresponding problem called as a dual problem.

So, the problem that you are originally having can be termed as a primal problem and associated with every primal problem, you can formulate a dual problem. Now the duality concept of duality is an important concept in linear programming problems. In fact, it appears to be too theoretical, because you know you have been able to solve the primal problem, you will able to also solve a dual problem, there is the corresponding there is a relationship between primal problem and a dual problem, so a new entrant to this area will always wonder, why consider duality at all it is a it is a very elegant theory of a linear programming problems, but a new entrant or an inexperienced user of linear programming problem will not be able to appreciate it so much, because it appears to be slightly determinant concept.

But, you know this is a very sound theory and the (()) that, there is nothing more practical than a sound theory, so as long as a theory is sound and it is good and you know it is elegant in mathematical form, it is really practical. So, let us go through what do you mean by a dual problem associated with a primal problem and later on we will see how we use the dual problems in the L P situation in the linear programming problems. There is a concept called as sensitivity analysis, there is the topic called as sensitivity analysis, in the linear programming problems, what do we mean by that? Let us say that, you solved a problem, you take that Z is equal to 36 problem, where you have been able to achieve a optimal solution.

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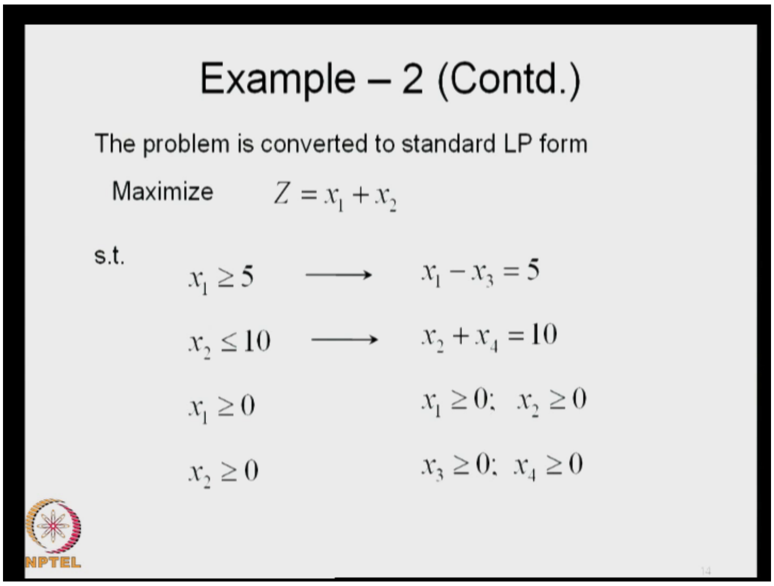

Let us say we are looking at this particular problem Z is equal to 36 and you got the solution x 1 is equal to 2, x 2 is equal to 6 x 3 is equal to 2 and so on. This is the optimal objective function value that you got corresponding to the set of constraints, so these were the sets of constraints.

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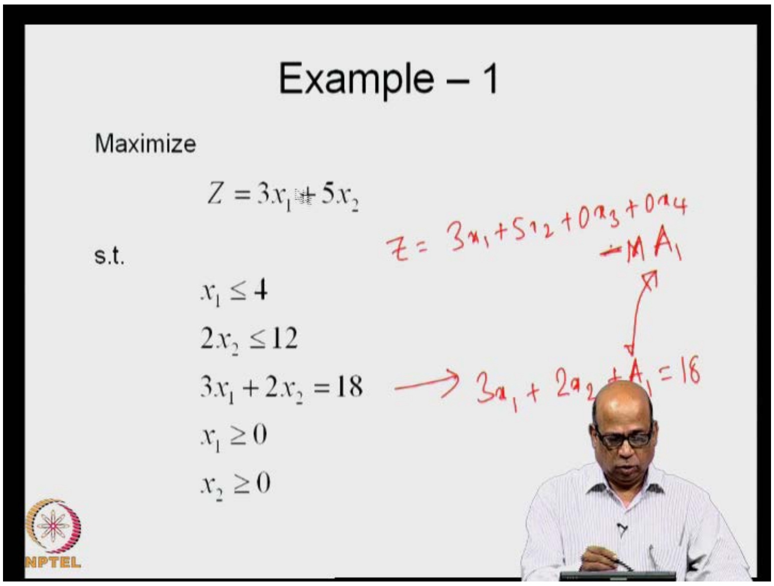

So, I will write here optimal Z is equal to 36 and let us say this was a less than or equal to constraint, less than or equal to 18 and you saw subsequently, when we solve this with the equality constraint that this solution also lead lead to Z is equal to 36. We know that this is the binding constraint, in resource allocation problems particularly this binding constraints are important because, they represent mostly some constraints on the available resources. Let us say that, we want to ask the question this was binding, which means that 18 have become restrictive now. What happens if I increase this right hand side value from 18 to let us say 20 what happens to my objective function?

Which means we are seeking the sensitivity of the objective function to changes in the right hand values of this constraints of the constraints, that is one type of sensitivity problem, so in the linear programming problems, there is a important topic associated with the sensitivity of the solution, one is obtaining the optimal solution itself, but you may start asking questions on the optimal solutions, you may say instead of right hand side being 18, if it was 16, if my resource available was much more restrictive than 18, let us say it was 16, then how much do I sacrifice on the objective function and so on.

There are several questions that you ask this is maximization, therefore let us say that instead of 3 it was 2, what happens to my objective function? All such questions, you can answer using the final optimal solution itself, you got the optimal solution. Let us say that I change it to 16, you do not have to resolve the problem by reintroducing this constraint, from the optimal solution itself, you should be able to answer questions on sensitivity of the solution various small changes that you make, either on the right hand side of the objective constraints or in the coefficients in the objective functions or in the coefficients of the constraints that itself and so on.

So, you will be able to answer questions, such as how sensitive is my objective function optimal objective function value or the optimal solution to small changes that can happen on the right hand side of the constraints, on the coefficients of the constraints or on the coefficients on the objective function and so on.

The concept of duality that we are going to introduce now is closely associated with the sensitivity analysis. All though you can still do the sensitivity analysis without understanding the concept of duality, so duality is a strong theoretical aspect of a linear programming and it can also be used for in many practical situations, we will introduce the concept of duality in the in the how we formulate dual problems associated with a primal problem. This will do in the next lecture. So, essentially in today's lecture, we started of with the multiple solutions, identification of multiple solutions, which was covered in the last lecture.

And then we went on to identify unbounded solutions and then we also examine the problem of unbounded, problem related to unbounded solution. Recall that unbounded solution, you can identify when there is no departing variable at any iteration. Then you identified an in feasible solution in feasible solution, you identify when you meet the optimality criteria in the simplex algorithm, but there is a artificial variable setting. And we have also seen how you have introduced the artificial variables in the case of equality constraints and greater than or equal to constraints. And remember always that, you introduce the artificial variables just to make sure that you get an initial basic feasible solution to start of the computations.

And the artificial variables should sooner than later leave the basis. If they do not leave the basis even in the optimal, even when the optimality criteria are met, it indicates that the problem is infeasible. So, in the next lecture, we will introduce the concept of duality and see how we formulate a dual problem associated with any given linear programming problem.

Thank you for your attention we will continue the discussion.