Water Resources Systems

Modeling Techniques and Analysis

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Lecture No. # 10

Linear programming: simplex method (2)

Good morning and welcome to this the lecture number 10, of the course, Water Resource Systems Planning - Modeling Techniques and Analysis. In the previous lectures, we have been talking about the linear programming problem, and in the last lecture, I introduced the motivation for the simplex algorithm through a simple example. If you recall we talked about the basic variables, basic solution and non basic variables, basic feasible solution and so on. And then towards end of the last lecture, I introduced a problem, and then went on to write the first simplex tableau for that particular problem.

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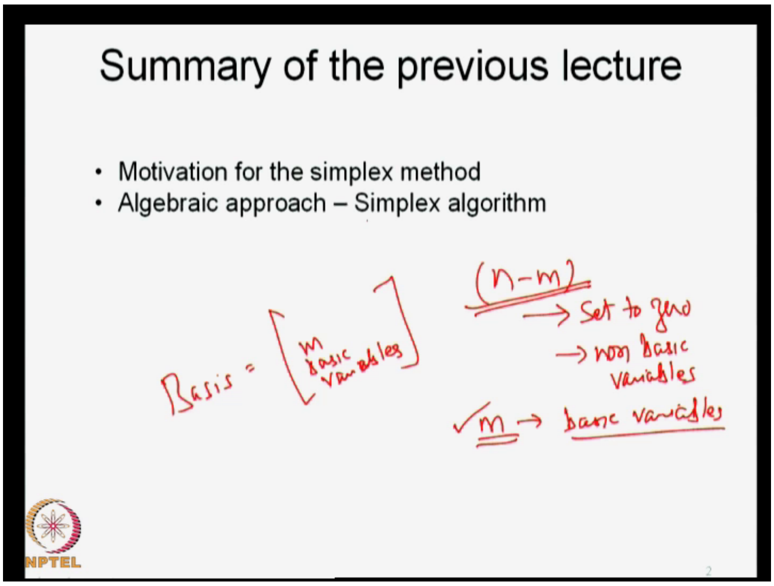

So, in the previous lecture, we have looked at the motivation for the simplex method through a simple example, and then we introduced here, what are the basic variables, what are the non basic variables, and so on. If you are call, we have n number of variables and m number of constraints. So, we have n minus m variables which are set to

0 and these are called as the non basic variables. So, those variables which set we set to 0 are called as the non basic variables. And then we solve for the m number of variables and these are called as the basic variables. So, there are m number of equality constraints, that is m number of equations by setting n minus m variables to 0, we solve for these m variables, and these variables for which we solve are called as a basic variables. And then we define the basis as consisting of the m basic variables. So, this is what we saw in the previous lecture. And then I started introducing the simplex algorithm, we will continue the discussion today from where we left in the previous lecture.

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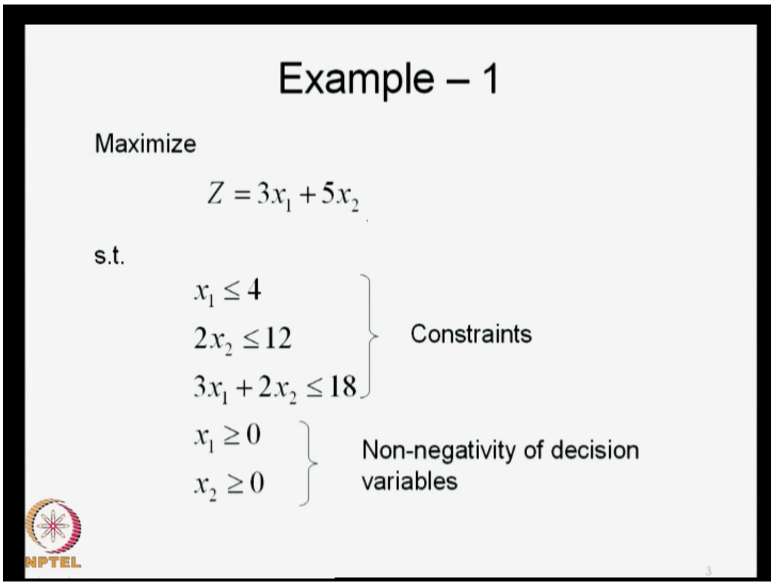

So, we were talking about this example, where we have the objective function to be of maximization type. So, maximize Z equal to 3x 1 plus 5x 2, re-call that this is same problem that we have solved in the graphical method. So, we will relate this problem by solving it through the simplex method and compare the solution with the with what we obtain in the graphical method. So, 3x 1 plus 5x 2, x 1 less than or equal to 4, 2x 2 less than or equal to 12, 3x 1 plus 2x 2 less than or equal to 18. These are the constraints, then we have the non-negativity of decision variables, x 1 greater than 0 greater than or equal to 0 and x 2 greater than or equal to 0.

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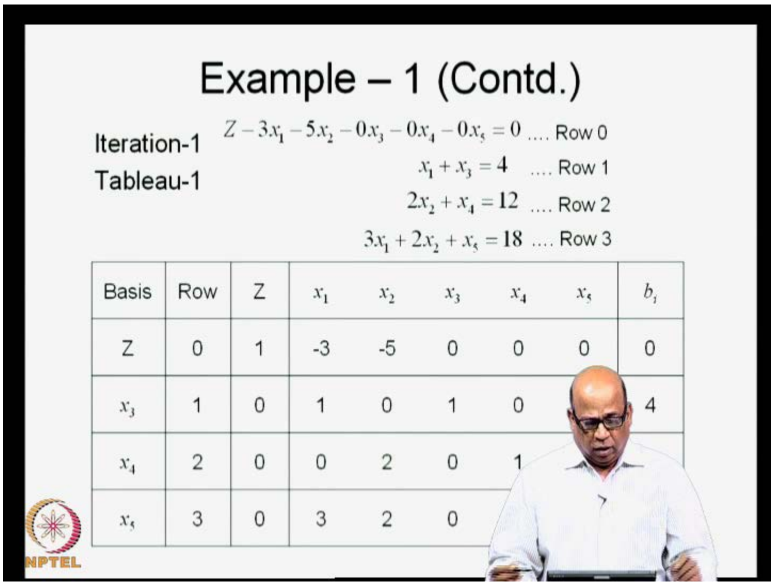

So, in the previous lecture, I just explain how we write the first simplex tableau, they are called as tableaus. So, we can Z use equation 1 or tableau 1 and so on. So, I will be sticking to the terminology iteration. So, from one iteration to the next iteration, how we proceed is what will see today. As I mention the objective function itself, we write it as another equation. So, your objective function was Z is equal to 3x 1 plus 5x 2, what we do is we write it as Z minus 3x 1 minus 5x 2 minus whatever slack variables we have use with 0 coefficients as first row. So, we write it as Row 0, and right hand side will be 0 for this. Then x 1 plus the x 3 is equal to 4, we have introduce the slack variable x 3 and that we called it as Row 1. 2x 2 plus x 4 is equal to 12, we have introduce the slack variable x 4, then 3x 1 plus 2x 2 plus x 5 is equal to 18, we have introduce the slack variable x 5. As I mention, we should start with an initial basic feasible solution. And typically in cases like this where you have 1 slack variable associated with each of the constraints. You start with setting the original variables, in this case x 1 and x 2, the original variables 2, 0 which means you choose the original variables to be non basic. Typically, what we are then doing is that we are starting the solution with the origin, x 1 equal to 0, x 2 equal to 0.

So, these are the non basic variables x 1 and x 2, and we choose the slack variables as the basic variables x 3, x 4 and x 5. So, the way we write is as I mention in the previous class, Z is always in the basis; then you have x 3, x 4, x 5 as the basic variables. So, the basis consisting of Z, x 3, x 4, x 5; then we write this column will identify the rows - Row 0, 1, 2 and 3. Then we pick up the coefficients of each of the variables including Z.

So, Z we write the coefficient of Z in each of the rows. As you can see, in Row 0 Z has a coefficient of 1, in all other rows Z has a coefficient of 0, and therefore, we write 1, 0, 0, 0. Similarly, corresponding to each of the variables x 1, x 2, x 3, x 4, x 5 etcetera, we write the coefficients in the corresponding rows. So, look at x 1, in Row 0 it has a coefficient of minus 3 and in Row 1 it has a coefficient of 1 plus 1, in Row 2 it has a coefficient of 2, I am sorry where writing from here, that is x 3 in a… I am sorry here you just look at the coefficients of x 1, let me repeat that. In the coefficient of x 1 in Row 0 is minus 3, in Row 1 it is plus 1, in Row 2 it is 0, in Row 3 it is 3. This is what you write here. Then x 2 similarly, in Row 0 it is minus 5, in Row 1 it is 0, in Row 2 it is 2 and in Row 3 it is 2 and so on, you will write all of these. And x 3, x 4, x 5 being the slack variables will have a coefficient of 1 exactly in one of the rows and in all other rows they will have coefficients of 0s. The b I here is the right hand side value of each of the constraints or each of the equations here. So, for Row 0 the b I is 0, for Row 1 b I is 4, this is 12 and this is 18. So, this is how you write a tableau, and in fact, this is how you start the iterations. This is the first iteration and the first tableau is like this.

At any iteration, the first question that you would ask is - is the solution optimal? Re-call that, the solution is read like this. This is the variables Z is equal to 0, x 3 is equal to 4, x 4 is equal to 12 and x 5 is equal to 18. With the non basic variables x 1 is equal to 0 and x 2 is equal to 0 as the solution. So, the solution in any iteration can be read from the variable here and the value here, variable value, variable value and so on. Why x 3 will be equal to 4? You look at this, we have set x 1 is equal to 0; therefore x 3 will be equal to 4. We have set x 1 equal to 0, x 2 equal to 0 therefore x 4 is equal to 12 and so on. So, this is how you read the solutions in any of the tableau. From the basis, you pick up the variable and from the right hand side you pick up the associated value. When once we construct this table, we asks the question, is the solution optimal? Whether the current solution is optimal or not is decided based on the coefficients in the Z row. If there is at least 1 coefficient here which has a negative value, it means that the Z value can be further improve by bringing that particular variable into the basis. The current basis consists of x 3, x 4 and x 5. There are two variables with a negative value.

So, if there is at least one variable with a negative coefficient in the Z row or the 0 eth row, the solution the current solution is not optimal. And therefore, you need to improve the solution. So, the answer to the first question in this particular case namely, whether the solution is optimal, the answer is this solution is not optimal, because there is at least one variable with a coefficient with a negative coefficient in Row 0. Then if the solution is not optimal, we need to bring exactly one of the currently non basic variables into the basis. So, x 1 and x 2 are the currently non basic variables. We bring either x 1 or x 2 in to the basis. So, the next question that we have to ask is, out of the currently non basic variables which among the currently non basic variables which one has to enter the basis. And which among the currently basic variables will make way for the entering variables. The basis the variable that enters the basis is called as the entering variable, the variable that exceeds the basis is called as the departing variable. So, we have to identify which is the entering variable and which is the departing variable.

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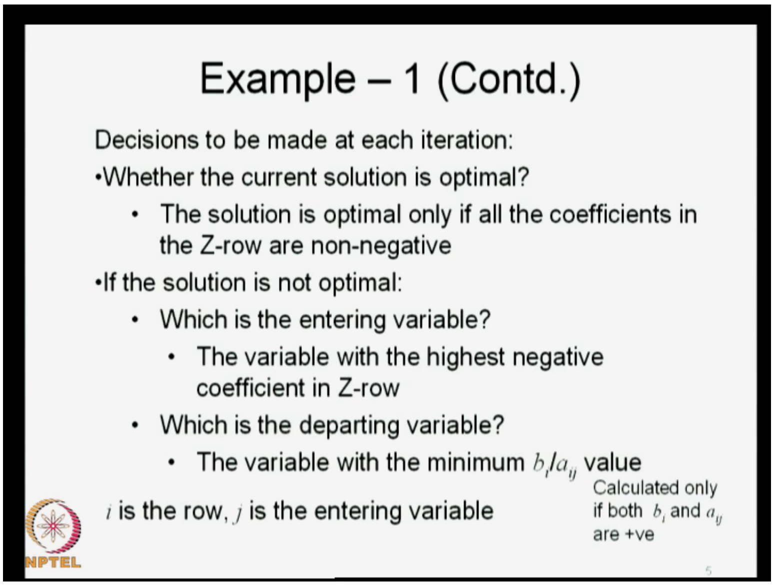

So, at each iteration then we ask these questions. Whether the current solution is optimal? The solution is optimal only if all the coefficients in the Z row are non-negative; as I set all these coefficients here must be non-negative. If there non if there at least one value which is negative, it means that the solution can be improve further, and therefore, that is on the current the current solution is not the optimal solution. Then, if the solution is not optimal, we have to identify which is the variable that has to enter the basis, which is not currently in the basis and therefore, it is the non basic variable, which is the variable that has to enter the basis. So, we have to identify which is the entering variable.

As I mention in the last class, you look at the coefficients here, minus 3 minus 5, you are interested in improving the Z value the fastest, and therefore, the coefficient the variable which has the highest negative coefficient will be the entering variable. So, we identify the variable with the highest negative coefficient in the Z row as the entering variable. Then, once we identify this, we have to see, which is the departing variable? For the departing variable, we compute what are called as the b I by a ij values. But first let us start with this entering variable; I will explain how we identify the departing variable, once we identify the entering variable.

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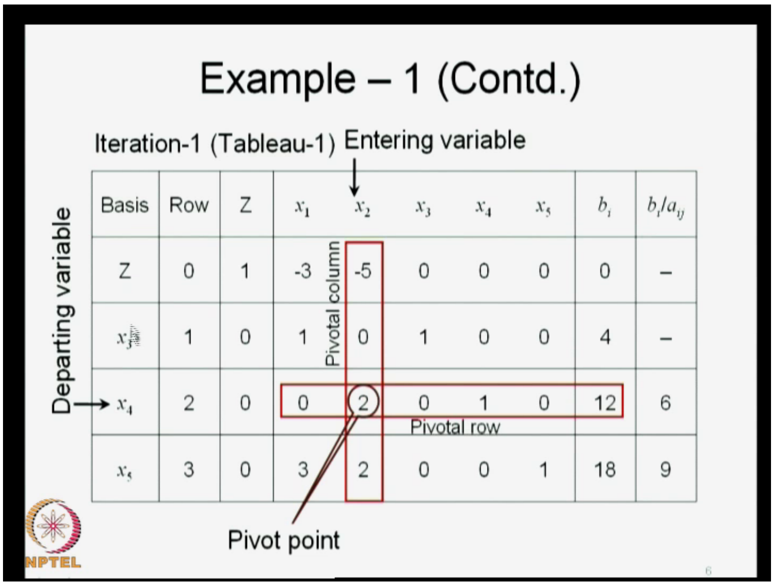

Basis	Row	Z	x_1	x_2	x_3	x_4	x_5	b_i	b_i/a_{ij}
Z	0	1	-3	-5	0	0	0	0	–
x_3	1	0	1	0	1	0	0	4	–
x_4	2	0	0	2	0	1	0	12	6
x_5	3	0	3	2	0	0	1	18	9

So, this is the iteration number 1, this was the table that we formulated. We now identify x 2 as having the highest negative coefficient in the Z row or the 0 th row. And therefore, this becomes the entering variable. So, we identify this as the entering variable, and then we put corresponding to that column, we mark it like this, and this is called as the pivotal column. So, we identify the entering variable, and the column associated with the entering variable is called as the pivotal column. You have the b I value is here which are the right hand side values. Now, what I going to do now is to identify which is the departing variable, then in the previous lecture, you re-call what we discussed about, which is the variable that departs? The variable which reaches 0 first as you start increasing the value of x 2, there is one of the currently basic variable which will reach 0 first, and that is what will that is the variable that will depart and we have to identify that. For that what we do is, you calculate b I by a ij values.

Now, these, this variable we are denoting the variable by j and I is the row. So, b I by a ij which means the right hand side value of the i th row by the coefficient of the j eth variable. Now, the j eth variable is the entering variable. So, we are saying entering variable is j. So, we are calculating 0 by minus 5, whenever it is the positive value. So, we calculate this as the variable with the minimum b I by a ij is the departing value, where i is the row and j is the entering variable. This is calculated only if both b I and a ij are positive. So, this is what we do b I and then entering variable, we have identified the column, and we start calculating b I by a ij. This part you understand correctly then everything is simple. So, we calculate b I by a ij, if both of these are non-negative. So, here it is negative, therefore you do not calculate, here it is 0, therefore you do not calculate. That is 4 by 0 is infinity do not calculate that. Then 12 divided by 2 is 6 and 18 divided by 2 is 9. So, this is have you calculate b I by a ij.

So, a ij s are the coefficients in the column of the entering variable. These are the a ij s. Now, you have two values of b I by a I j s. The variable corresponding to the minimum of the b I by a ij will depart, we have identify the entering variable, now we are going to identify the departing variable. So, the departing variable is that variable which has a minimum value of b I by a ij among all the values that you have calculated. Now, these are positive values. So, this becomes the departing variable. And then we identify this as the pivotal row. So, you identified pivotal column associated with the entering variable. You identify now a pivotal row associated with departing variable. And the departing variable is identified based on the minimum value of b I by a ij and b I by a ij is calculated only when it is positive.

So, we identified the entering variable, we identify the departing variable. Now, which means that, will formulate the next basis for the next iteration as Z x 3 and x 2 here in place of x 4. So, x 4 goes out and x 2 comes in. And then x 5 remains as it is. So, at every iteration exactly one currently non basic variable enters the basis and exactly one currently basic variable leaves the basis. So, we identify the entering variable and the departing variable.

Now, we write the next tableau by using the Gauss Jordan elimination method. So, we do the actually canonical form, we write the tableau in canonical form by using the Gauss Jordan elimination method. So, I will explain how we do this. This is the pivotal row, first we re-write the pivotal row, by simply dividing… Instantly this element here which is at the intersection of the pivotal column and the pivotal row is called as the pivot point or simply pivot it is called as the pivot. So, we go to the next iteration, first what we do is we r e-write the first basis, Z, x 3 and x 2 and x 5, 0, 1, 2, 3 this is just the row identification, Z this remains the same. First we write re-write the pivotal row by dividing the pivotal row by pivot value. So, we write it as 0 by 2, 2 by 2, 0 by 2, 1 by 2, 0 by 2, 12 by 2 up to b I. So, this is what we do here.

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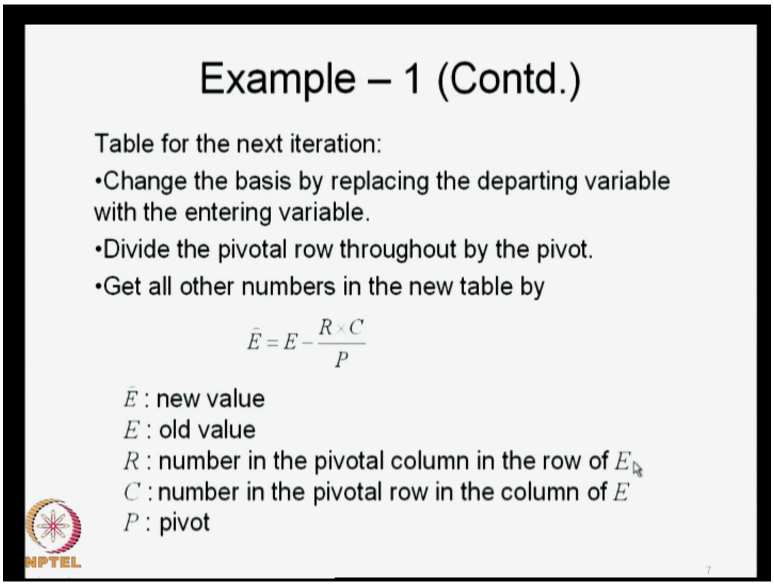

Let me explain it this way. We are now going to write the tableau for the next iteration. The first step that we do is that we change the basis by replacing the departing variable with the entering variable. So, in this case the departing variable was x 4 and the entering variable was x 2. So, we re-write the basis by replacing the departing variable with the entering variable. Then we divide the pivotal row throughout by the pivot. We will first do that and then explain. So, we wrote Z, x 3, x 2 x 5 and this remains the same, this remains the same. We first re-write this row which was corresponding to the pivotal row by dividing the pivot which was 2. So, from here, I am dividing this row by 2, 0 by 2, 2 by 2, 0 by 2, 1 by 2, 0 by 2, 12 by 2. That is what we write here; 0, 1, 0, 1 by 2 and 0, and 6. So, up to the b I column I re-wrote. So, this is the first column to be written.

Then you look at this, there are all other elements, all these other elements have to be transform to new elements some. Now, what I am doing is the Gauss Jordan elimination method, but I will tell you how to transform this straight away. So, for all other numbers, the new table we get it like, this is a older number and this is the number in the pivotal column in the row of E. So, any number you have that row there, and there is a pivotal column. So, you have one element; let us say, you want to transfer to transform this number. So, you have this number to be transformed that becomes old E that is E, and this is the column of the particular element, and you have an element in the pivotal column of that particular row, and that is what we denote it as R.

Similarly, number in the pivotal row in the column of E; so, let us say you are trying to convert this transform this, this is a there is a column and then you have a pivotal row and there is a element here. So, this is how we use is. This is the old number minus R into C divided by P and P is the pivot itself. Let us say, I want to convert this number 0. So, 0 minus minus 5 into12 divided by 2. So, this is how we convert. Let us say, you want to convert this number, it is 0 minus minus 0, 0 minus 0 into 0 divided by 2. If you want to convert this, 1 minus 0 into 2 divided by 2; so, you identify corresponding to that number, you identify the pivotal column element and pivotal row element and divided by the pivot. So, this is what it says, E minus R into C divided by P. So, this you can get used you can start doing mechanically.

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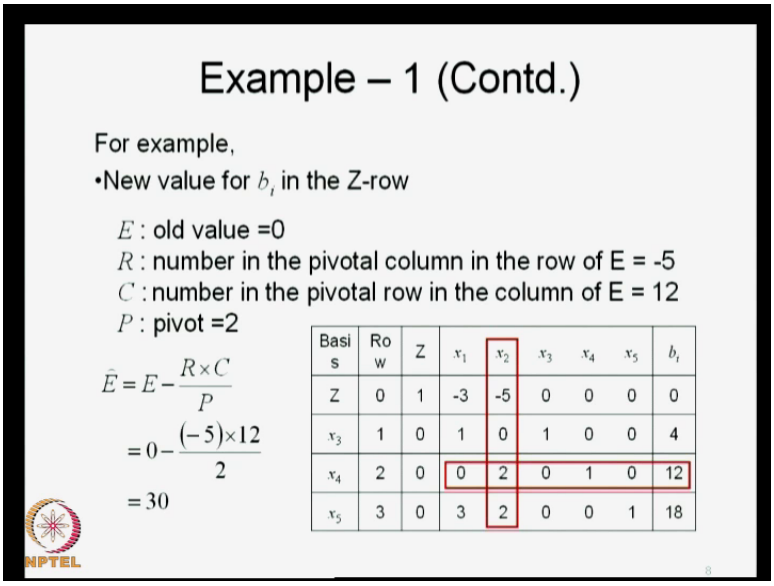

So, let us say that you you have this table, I will show you how to convert this. Let us say that you want to convert this 0 here. In 0 you identified the row. So, this is the row of this particular 0. And there is an element associated with the pivotal column in that row which is minus 5. And there is an element associated with the pivotal row in that column which is 12. So, this would be 0 which is the old number minus minus 5 into 12 divided by 2, this 2 is the pivot, that will become 30. So, this number will be replace by 30.

Similarly, let say that you want to convert this number 12 not except the pivotal row. Pivotal row you do not touch, because you already divided. Let us say, you want to convert 4 here; then what what it will be 4 minus 0 into 12 divided by 2 that is have you convert. The moment is there this element or this element is 0 that number remains the same. So, like this convert each of these numbers up to Z row you do not touch, up to the Z column you do not touch. Now, all of these coefficients including the b I coefficients, you convert using this simple expression. Let us say, you want to convert this number now, 18, let me write that, so that. So, 18 minus, this is 12 into 2, this is this 2, this 12 divided by this 2, that is the pivotal element. So, this would be 6. So, this 18 gets converted as 6, like this you write all the elements. So, this is what we do.

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Example – 1 (Contd.)

Iteration-2 (Tableau-2)

Basis	Row	Z	x_1	x_2	x_3	x_4	x_5	b_i
Z	0	1	-3	0	0	5/2	0	30
x_3	1	0	1	0	1	0	0	4
x_2	2	0	0	1	0	1/2	0	6
x_5	3	0	3	0	0	-1	1	6

So, this 0 become 30 by the way have shown, and this 4 becomes 4 only, and you look at minus 3 for example, minus 3 minus minus 5 into 0 divided by 12. So, as I said if either this is 0 or this is 0, this number remains the same. So, that number remains the same as minus 3. So, this is minus 3.

And then you convert this x 2 as 0. Let us say minus 5. So, how do I convert that? Minus 5 minus minus 5 itself, because that is the row into 2 divided by 2 into 2 divided by 2. So, that will be minus 5 plus 5 that will become 0. So, like this, you simply convert, let us again show of how do I get 5 by 2 here, a corresponding to x 4. x 4 was 0 here, 0 minus minus 5 into 1 divided by 2. So, that will be 5 by 2. That is what you get here. Like this, except the pivotal row you convert all the other elements. That is have you write the tableau 2. Once you complete all the elements, you should be able to read the solution. How do you read the solution? Z is equal to 30, x 3 is equal to 4, x 2 is equal to 6, x 5 is equal to 6, this is the solution. And of course, the current non basic variables which are x 1 and x 4 they will be at 0 level, the current non basic variable will be always at 0 level.

So, this is a tableau number 2. What did way achieving tableau 2? We started with Z is equal to 0 and tableau number 1, we have achieve the solution of Z is equal to 30. So, from 0 you have achieved an improvement, the the Z value which is the objective function value as to 30 now. We ask the question, is the solution optimal? This solution is again not optimal, because there is one variable here which has a negative coefficient. Because there is at least one variable which is the has the negative coefficient, the solution can be further improved and therefore, the current solution is not optimal. Then we have to identify the entering variable. Entering variable is that variable which has currently the highest negative coefficient in the Z row or the 0 eth row. And in this particular case, there is only one such variable and therefore, that becomes an entering variable.

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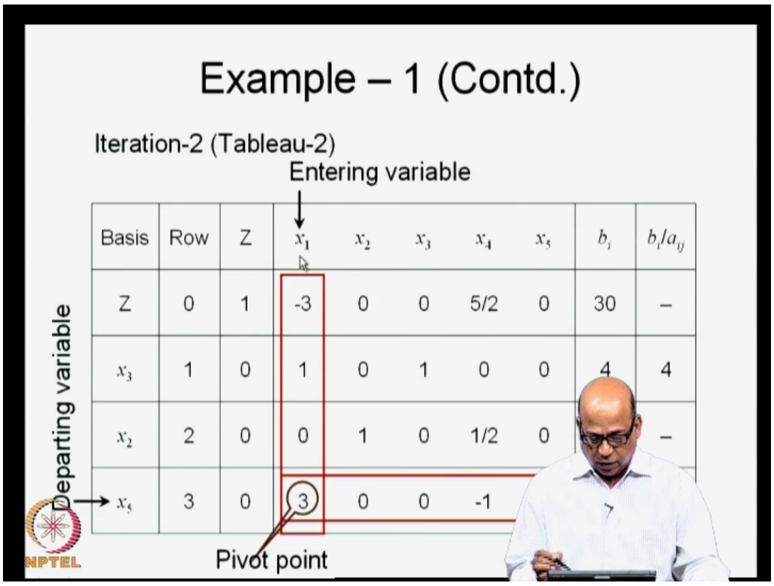

So, identity this has the entering variable. Once you identify the entering variable, you calculate b I by a ij if it is positive, this is not positive therefore do not calculate, 4 divided by 1 4, 6 divided by 0 do not calculate, 6 divided by 3 2. So, among the b I by a ij s whichever variable has among those variables for which the b I by a ij is positive. The variable that has a minimal b I by a ij value will become a departing variable. So, therefore, this becomes a departing variable, and this is the pivotal row, this is the pivotal column, and this is the pivot. Then what it is that we do? We first divide the pivotal row by the pivotal element and re-write that. So, your new basis will be Z, x 3, x 2 and x 1 will come here, x 5 will go out, x 1 will come here. So, we re-write the basis, this remains the same, this remains the same, and first we re-write the pivotal row here by dividing the pivotal row elements here by pivot point. So, let us look at this.

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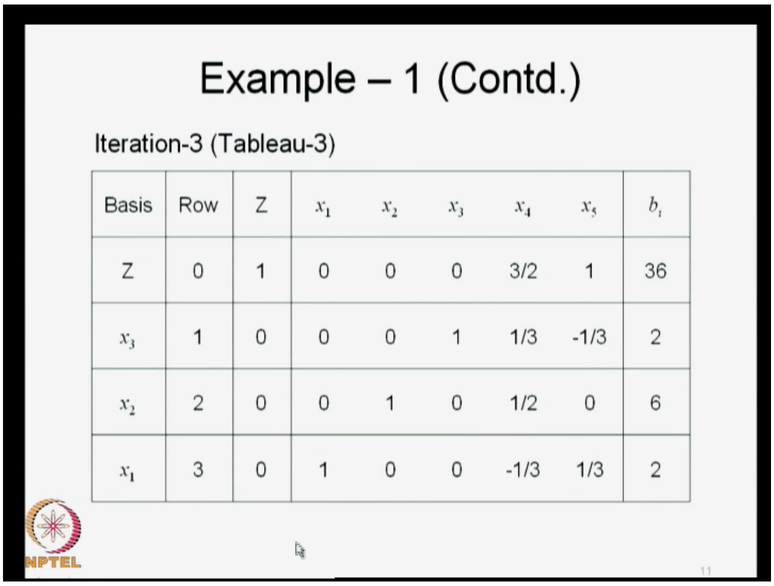

Example – 1 (Contd.)

Iteration-3 (Tableau-3)

Basis	Row	Z	x_1	x_2	x_3	x_4	x_5	b_i
Z	0	1	0	0	0	3/2	1	36
x_3	1	0	0	0	1	1/3	-1/3	2
x_2	2	0	0	1	0	1/2	0	6
x_1	3	0	1	0	0	-1/3	1/3	2

So, this would be 3 by 3, and 0 by 3, 0 by 3, and minus 1 by 3, and plus 1 by 3, and this will become 2. So, this is how the first transformation you do. After that each of this other numbers you start calculating by using our formula that we have use. Let us say you want to convert this number, 30 minus minus 3 into 6 divided by 3. So, that will become 36. 4 minus 1 into 6 divided by 3; so, 4 minus 1 into 6 divided by 3 which is 2. So, that is have you get 2 here. Like this, you convert each of these numbers and write the tableau number 3 much the same way as you did tableau number 1 to tableau number 2. The same formula you use and then get the tableau number 3.

Now, you look at the solution, Z is equal to 36, x 3 is equal to 2, x 2 is equal to 6 and x 1 is equal to 2. This becomes the solution. Is a solution optimal? The first question that we ask after preparing the tableau is, whether this solution is optimal? Look at the coefficients of the 0 eth row - coefficients in the 0 eth row of all these variables. There is no variable which has a negative coefficient here, which means that - the current basis x 3, x 2, x 1 are the current basic variables, and x 4 and x 5 are the non basic variables. If you bring in either x 4 or x 5 into the basic variables, the Z value decrease, because they have positive coefficients here. And therefore, because there is no variable which has a negative coefficient in the 0 eth row, this solution is in fact a optimal solution, because this solution cannot be improved further by bringing in the any of the currently non basic variables into the basis, and what is the optimal solution? Optimal solution is Z is equal to 36. That is an objective function value. x 3 is equal to 2, x 2 is equal to 6, x 1 is equal to 2, this is how you read the optimal solution. This is the same problem that we discussed earlier in the praxis two lectures ago, when I introduce the graphical solution.

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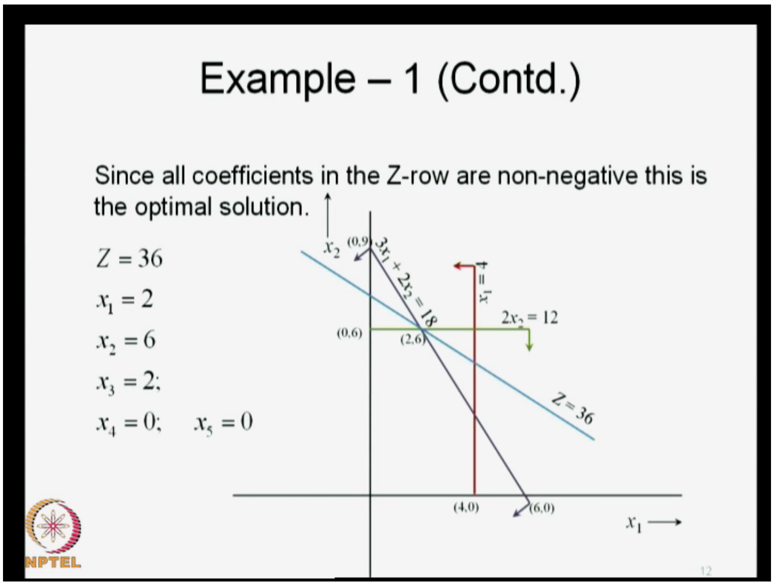

And what did we obtain their, this is a graphical solution for the same problem. Re-call that - this is your feasible space, and then this is a feasible space and we finally obtain the optimal solution as this particular point 2, 6 and Z value was 36. So, the x 1 value was 2 here, x 2 was 6 and of course, x 4, x 5 will be 0 at this point, and that is exactly what we obtain here. x 1 is 2, x 2 is 6 and Z value is 36. So, this is how you proceed from 1 iteration to another iteration; essentially, in the simplex method what we did is, we started with this point which was x 1 is equal to 0, x 2 is equal to 0, in the first tableau here iteration number 1 you look at this solution x 1 was 0, x 2 was 0, because this was this were you are non basic variables which were Z to 0. Then we went to iteration 2. In iteration 2, what are the solutions? You got x 2 is equal to 6, but x 1 was still 0, x 1 is 0, because is still non basic variable. So, x 2 is equal to 6 and x 1 is equal to 0 was a solution.

You look at the graph, varies that point, from this point we went to this point 0 and 6, then from this point to this point we went, 2 comma 6. Notice that from this point we did not go here, we went here. That is because this is the direction in which the Z value increase faster than, if you are taken this particular direction, and that is what we decided based on the departing variable. So, from which point to which point you are moving will be decided in the simplex tableau both by the entering variable as well as the departing variable. So, in the iteration 1 we started with this solution, iteration 2 we went to this solution, and iteration 3 we went this solution. Without computing the other 2 points, we decided that this is the optimal solution. The solution cannot be improve further.

Now, this is the 2 by 2 that is a two variable problem. So, this is very simple to explain. But in large engineering problem has have been repeatedly you will have thousands of constraints, thousands of variable etcetera, it is not possible for you to trays from which point to which point you are going, but the algorithm will make sure that you are proceeding in an optimal manner; that means, instead of going here, if you are gone here what would happen? You would have to go here and then here. Whereas, you when directly here and then came to this point.

So, in a large space of feasible solutions, you proceed from one corner to another corner in an optimal manner, and until you hit the optimal solution, and also re-call that in the simplex tableau, simplex iteration number 3, the moment you reached here, you know this is the optimal point, you need not have to enumerate the other solutions. So, you know which when you have it the optimal point and there you terminate the solution. So far, so go, we took a simple problem, and then we know how to formulate the simplex tableaus, and how to move from the one iteration to another iteration. Now, there are a few questions that will arise, even in the simple problem. First we ask, whether the solution is optimal and we could resolve that the solution is not optimal, because there are at least one variable which has a negative coefficient in the Z row, and therefore, that is not the optimal solution.

And then we identified the entering variable. How did we identify the entering variable? We said that the particular variable which has the highest negative coefficient in the Z row that will be the entering variable. What if more than one variable have the same highest negative coefficient? That means, there is a phi, phi for a entering variable. What happens if there is a phi for a entering variable? The phi for the entering variable is broken arbitrarily; you can bring in any of the variable. Let us say that in this particular example; we had a tie between x 2 and x 4, let us say both of which had the same negative coefficient to which also happens to the highest negative coefficient. In which case, you brake the arbitrarily, bring either x 2 or x 4. And you will see that immediately the next iteration, the other variable also comes into the basis, if that is the that is leading to the optimal solution.

So, the tie breaking for the entering variable is done arbitrarily. Alright then we went into the departing variable. Departing variable we calculated b I by a ij s, and then looked at that particular variable which has the minimum b j by a ij value. What if what if there is a tie here? That means in the departing variable there is a tie. In general, whenever there is a tie in the departing variable, this leads to what is called as degenerates solutions or what is called as the degeneracy. So, entering variable if there is a tie do not worry, simply take arbitrarily one of the variables, the other variable also subsequently enters the basis, if that is in fact leading to optimal solution. Existing variable or the departing variable if there is a tie that leads to degeneracy; I will explain what it is means here. Let us look at this.

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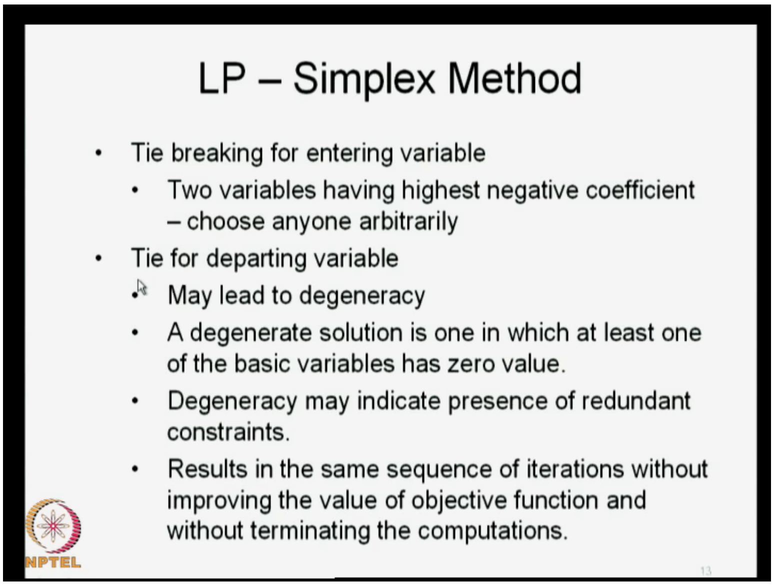

So, when there is a tie for departing variable, it may lead to what is called as degeneracy. A degenerate solution is one in which at least one of the basic variables has a 0 value in the Z row. Now, in general, the degeneracy will indicate the presence of redundant constraints. Let us say that you had some constraints even with n greater than or equal to m; that means number of constraints, number of equations is less than the number of variable. Even there you may have certain redundant constraints, and that may lead to degeneracy. What happens when you have degenerate solutions? Let us say that you had a tie between two variables for the departing variables, and you broke the tie arbitrarily and broad one of the took out one of the variables as the departing variable. In the next iteration, you may see that you get the other variable also as the departing variable and then you get struck in a loop. So, you the solution does not improve, but you keep on getting one entering variable and another departing variable.

So, the optimality condition is not satisfied. However, the iterations will keep on continuing in the same same manner, one variable goes out the same variable comes in again in the next iteration, again the same variable goes out in the next iteration, etcetera. Like this, it is keeps on happening and you are cot in a loop. So, the degeneracy in general causes cycling. So, unable to improve the value of the objective function, however, the solution is not terminated yet, because the optimality condition is not obtained. And in general, whenever such a thing happens, if we are able to catch the degeneracy, it indicates that there are redundant constraints. And therefore, the problem is not really well formulated; you will have to re-look at the problem formulation.

Then another feature is you look at any of this iterations, let us say, x 3, x 2 and x 1, you look at the coefficient of x 3 under x 3 that will be 1, and the coefficient of x 3 corresponding to other two basic variables should be 0. So, x 3 has a coefficient of 1, and x 1 and x 2 are other two basic variables. So, x 3 has a coefficient of 0 under x 1, and x 3 has a coefficient of 0 at under x 2. Similarly, x 2 will have coefficient of 1 under x 2, and 0 under x 1 and x 3. x 1 will have coefficient of 1 under x 1 and 0, and 0 here.

So, the basic variables among themselves will constitutes a unit matrix 0 0 1 0 1 0 1 0 0 like this. This is iteration 3, let us look at iteration 2now. Iteration 2 is this, which are the basic variables x 3, x 2 and x 5. So, x 3 will have under x 3 it will have 1, x 2 it will have 0 and x 4… I am sorry you are looking at x 3, and x 3, x 2 and x 5 are the basic variables. So, x 3 under x 3 will have 1, under x 2 it will have 0, and under x 5 it will have 0. Similarly x 2, it will have 1 coefficient under x 2, but under x 3 it will have under x 3 it will have 0, under x 5 it will have zero. So, at any iteration, the basic variables among themselves will constitute a unit matrix, like this. So, this is one of checks that you do. So, that you are show that the calculations that you are doing especially on the transformation of elements, those calculations are correct.

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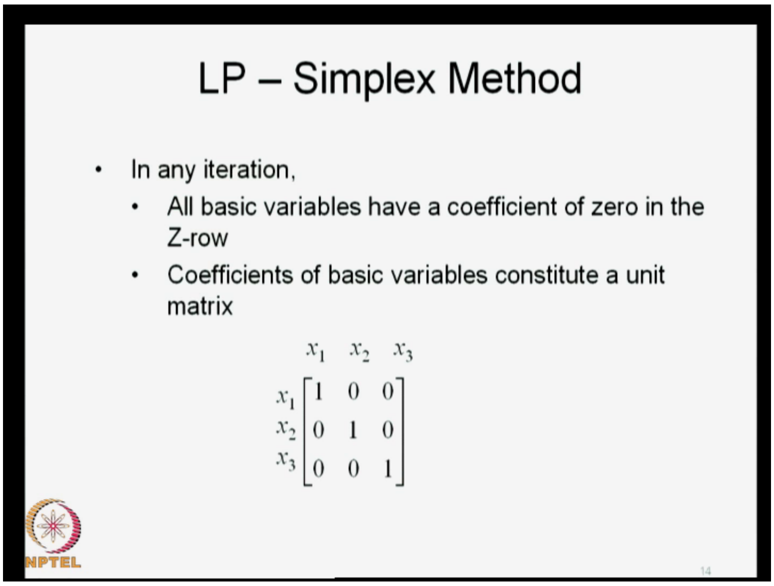

So, what the check is that, if you have x 1 x 2 x 3 as basic variables under x 1 x 2 x 3 the coefficient should look something like this, x 1 will have coefficient of 1, and in other under other basic variables it will have coefficients of 0. When we looked at the graphical solution, I had also introduce, what is called as a unbounded solution and an infeasible solution and also multiple solutions. So, if you re-call, when do you have multiple solutions? In the graphical solution, you just re-call that when the Z line is moving parallel to one of the edges of the basic feasible feasible space, one of the edges of the basic feasible space and it is increasing in that direction, and just at the optimal solution point the Z line coincides completely with one particular edge, which means I am sorry which means that any point along that line along - that particular edge is also an optimal solution, and therefore you have an multiple solution.

So, we must able to identify multiple solutions in the simplex method. Then we also know that if your feasible space is in fact unbounded in the direction in which Z increasing in the maximization problem. If we are Z is increasing the in the particular direction in which the feasible space is not bounded, then it leads to unbounded solutions, because Z can be increased up to infinity without violating any of the constraints, which means feasible there is no bounded feasible space, and that we should be able to identify in the simplex method. Then we also know in the graphical method, we also saw that there may not be a feasible space possible at all, which means that the region that intersecting may not be found or the region that is intersecting may not be in the first quadrant, because thus violating your non negativity conditions and so on. So, we must be able to identify infeasible solution, because as I mean repeating repeatedly saying that the sizes of the l p problems will be quite large in an engineering application, and therefore, you will not have control as you had in the graphical solutions. So, you may not be able to say with the problem is feasible or not.

So, in any iteration as you progress from iteration to iteration etcetera. You should be able to identify that it is either unbounded solution or it may be a infeasible solution or you may have multiple solution. Once you reach the optimal solution, it should be able to also identify multiple solutions. Let us see how we do all of this. So, you look at the final optimal solution here.

In the iteration 2, in the previous example, we decided that this solution is not optimal and made this as the entering variable, because this as the negative coefficient and there are no other negative coefficients. Therefore, this is also the variable which has the highest negative coefficient, and we made this as the entering variable, we made this as the departing variable, reconstructed the next tableau that is for iteration 3, and then we look at all this variable, all this coefficients here. Because there is no variables which has a negative coefficient in the Z row or the 0 eth row, we said that this is the optimal solution. So, the optimality criteria here is that there is should be no variable which has a negative coefficient. Because if there is a negative coefficient that negative that variable with the negative coefficient can be broad back into the basis, and we can improve the value of Z. In this iteration, in the final iteration x 4 and x 5 are the non basic variables. x 3 x 2 x 1 are the basic variables x 4 and x 5 are the non basic variables. Look at the coefficients of x 4 and x 5, this is positive 3 by 2 and this is positive 1, and therefore, they cannot be broad into basis without improving to improve the value of Z. I repeat that if you bring in either x 4 or x 5, the Z value will only decrease, and therefore, this cannot be broad back into basis.

However, if in a problem the currently non basic variable in this particular case x 5 or x 5 x 4 or x 5, has a coefficient of 0 here in the final tableau, when your optimality criteria is might, it has a coefficient of 0, one of the non basic variables, one or more of the non basic variables has a value of 0. What is that meant? That means that this basic variable which has a which has the coefficient of 0 can be broad into the basis, but the objective function value will not change. Z will still remains 36, because these are the coefficient of 0 and you bringing a coefficient of 0 by using all your criteria of departing variable etcetera, one of the variable will go out this variable will come in and then you see that the objective function value will not improve, which means that you could capture two solutions.

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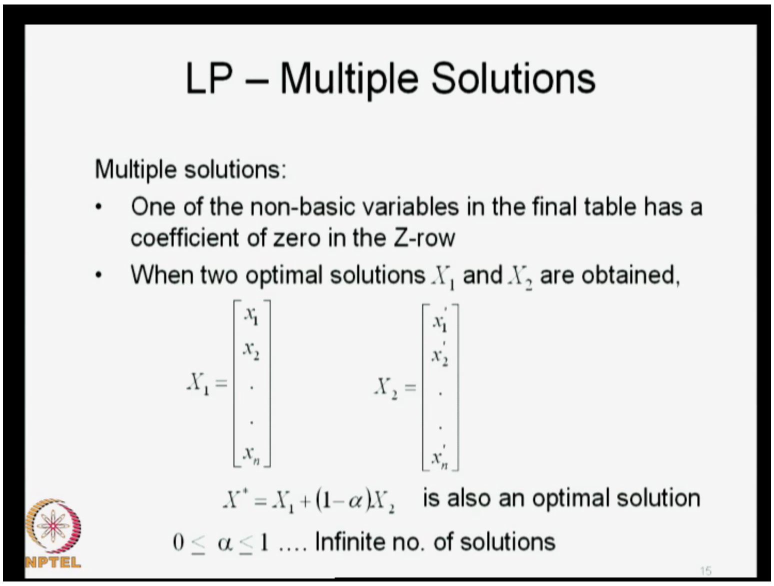

So, the multiple solutions is identified, if one of the non basic variables in the final table has a coefficient of 0 in the Z row. This indicates that that particular non basic variable which has a coefficient of 0 can be broad into the basis, it was the non basic variable, you can make it a basic variable without changing the value of Z, which means than you got two solutions. You look at this, let us say that this a solution in which you can bring in x 4 here into the basis back, because it is as the coefficient of 0. So, this was an original optimal solution by bringing x 4 into the basis, again you get another optimal solution still with Z is equal to 36. So, you can capture two solutions. Once you have two solutions this is 1 of the optimal solutions x 1 is equal to x 1 etcetera, x n and x 2 is equal to x 1 dash x 2 dash etcetera, x n dash, you got two solutions. How did we obtain two solutions? From the optimal tableau here the last tableau, you identified that particular non-basic variable and then broad that non-basic variable which has the coefficient of 0 in the Z row, and generated another solution, this was 1 solution and you generated another solution which still has Z is equal to 36, but basis will be different and therefore, identified two solutions and these are the two solutions. If you once you get two solutions, you can generate infinite number of solutions.

So, you capture two solutions by going to the next iteration by bringing that particular non-basic variable into the basis that particular non-basic variable which has the coefficient of 0 in the 0 Z row. You bring that into the basis generate another solution that another solution still has the same optimal solution, optimal objective function value, and therefore, this forms an alternate solution and once you capture this two alternate solutions, you can generate infinitely many number of solutions. So, you can generate another set of solutions as x 1 plus 1 minus alpha x 2. So, x 1 is this solution, x 2 is this solution, you take 1 minus alpha x 2 which is alpha with alpha varying between 0 and 1. Essentially, what we are doing is that we are saying you got this solution, you got this solution, you are generating all these solution, and this is along the line, and therefore, you can write this as x 1 plus 1 minus alpha x 2 of course, x 1 will be x 1 comma x 2 here and that is small x 1 comma x 2 and small x 1 dash x 2 dash. So, let me write this is x 1 x 2 and this would be x 1 dash x 2 dash.

So, what we are saying is that if this is an optimal solution, and this is an optimal solution, with a given Z value the same that value will remain here the line joining this two will also be all the points along this line, will also be optimal solutions. This is for a two variable problem, the same logic extends to n number of variables, and for the n number of variables, you are writing it as x 1 plus 1 minus alpha x 2 where x 1 and x 2 will be vectors. And by choosing an appropriate value of alpha, you can generate different solutions. So, keep on choosing different alpha, you will get different sets of solution. Remember all of these solutions are optimal solutions. At all of them will in fact lead to the same Z value. And this provides enormous flexibility in decision making, because you are sticking to given optimal function optimal value of the objective function, but you have an enormous flexibility in the decision variables themselves. So, you you have getting different sets of solutions all of which lead to the same objective function value. Therefore, capturing the multiple solutions is an important step when we are doing, when we are dealing with a problem.

And so, when we reach the optimal solution that is the last tableau, you made the decision that this is the optimal solution, because there is no variable which has the negative coefficient in the Z row. At that time, you also look at whether there are any variables which are currently non basic variables in the final tableau in the final iteration, there are some non basic variables, whether out these non basic variables any one of any one or more of them have a coefficient of 0 in the Z row. So, you have reach the optimal solution, but you are you are allotted to the fact that there is one of at least one of non-basic variables which has the coefficient of 0. The moment you identify that it means that there is a multiple solution possible, there are multiple solution possible. And therefore, you proceed to the next iteration. So, you do not stop the computations there although you you know that Z is equal to that particular value that you reach in the optimal in the final tableau is in fact the optimal solution, but do not terminate there. You go to the next iteration. How do you go the next iteration? You make this particular non-basic variable which has the coefficient of 0 in the 0 eth row or the Z row, you make that as the entering variable, apply your criteria for the departing variable, make one of the variable is to be departing. So, proceed essentially to the next iteration. And you do all your computation, transformation etcetera, you will realize that the Z value remains the same, but you got to different solution, which means that you capture, starting with this solution x 1, you went to the next iteration, you captured another solution x 2, both of which lead to same optimal objective values. Once you get these two solutions you can generate an infinite number of solutions by using x 1 plus 1 minus alpha into x 2 read by appropriately choosing alpha. So, you can get infinitely large number of solutions.

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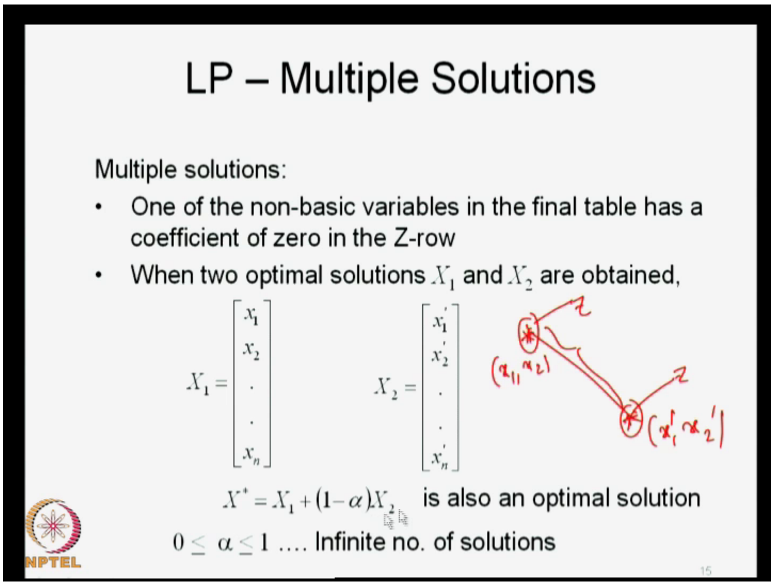

And this has huge of physical significance, because as I have been mentioning, let us say that you talking about physical dimensions, and one optimal solutions who got a certain physical dimensions. And another optimal solutions solution got another set of physical dimensions. Both of these have the same objective function value. And therefore, you can choose any of these or an infinitely large number of other solutions. All of which leading to the same optimal solution, and therefore, these will have a great flexibility in decision making.

So, we have dealt with today, the simplex method and we have chosen a simple problem to demonstrate how the simplex method works. We start with an initial basic feasible solution. And typically we start with putting the initial original decision variables to be 0 which means that we choose those original decision variables x 1 and x 2 in the case that I explained with as non-basic variables. So, the original variables you start set them as non-basic variables which means essentially you are starting with the origin, x 1 is equal to 0, x 2 is equal to 0 in that example that we discussed. And then start improving the solutions. So, you move from one iteration to another iteration and keep improving the solution by every time asking the question, whether the solution is optimal? If the solution is not optimal, you identified an entering variable. The entering variable is identified by the highest negative coefficient in the Z row by that variable. I repeat the entering variable is that variable which has the highest negative coefficient in the Z row. Once you identified in the entering variable, you compute the b I by a ij s, that is the last column which is computed only for positive values. And then look at the minimum positive value and variable corresponding to the minimum positive value of the b I by a ij will be the departing variables. So, you re-write the basis.

And then you identified the pivotal column, pivotal row, identified the pivot - first transform the pivotal row by dividing the each of the elements of the pivotal row by the pivot element. And then transform each of other elements by the procedure that I just mention, and formulate the next iteration tableau. Like this from one iteration to another iteration, you keep an progressing, until you hit the optimal solution. The optimal solution is achieved when all coefficients in the Z row are non negative. That means, there is no coefficient which has a negative value, and therefore you cannot further improve the Z value, and that is where you terminate the solution.

When you hit the optimality criteria, you also look at the currently non-basic variables and what is happening to the coefficients of the currently non-basic variables. If the currently non-basic if at least one of the currently non-basic variable variables has a coefficient of 0 in the final tableau, it means that this variable which is the currently non-basic variable can be brought into the basis without changing the objective function value Z, which means that you can generate alternate solutions. And once you generate two solutions, you can generate infinitely many solutions. And this multiple solutions have a enormous physical significance, because you can play around with the solution. That means instead of having one optimal solution, you have large number of optimal solution. So, you can pick up, pick up any of these solutions for physical implementation and so on. So, we will continue this discussion perhaps for in the next lecture I will start with an example where you can capture the multiple solutions in an l p problem. Thank you for your attention.