Stochastic Structural Dynamics

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Module No. # 04

Lecture No. # 14

Random Vibrations of MDOF Systems-2

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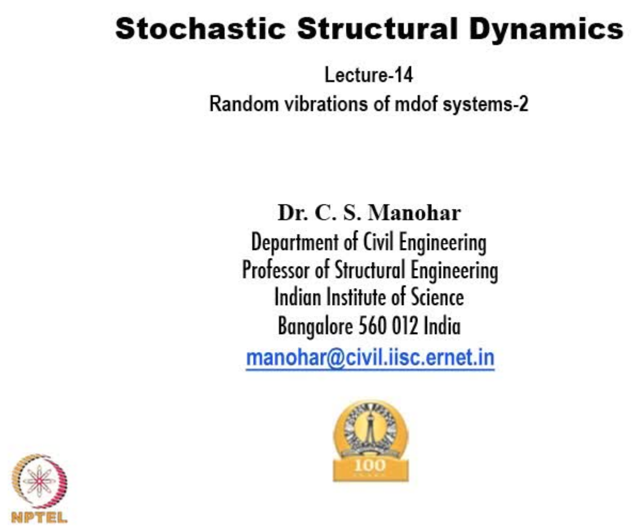

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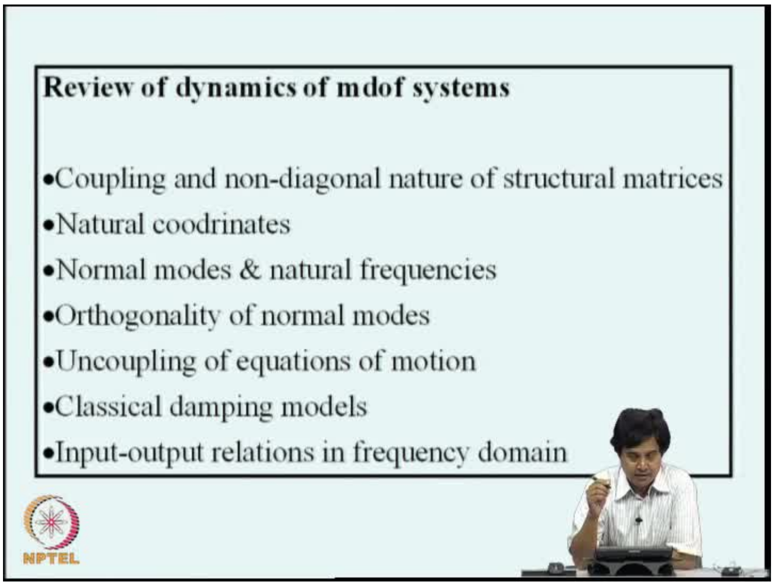

We are discussing random vibration of discrete multi-degree freedom systems, that is, multi degree freedom system with finite degrees of freedom. So, in the last class, we looked at the nature of equations of motion for multi degree freedom systems, and we noticed, that many cases or in almost all cases, the coordinates would be coupled and the coupling between coordinates is manifest as one of the one or more of the structural matrices being non-diagonal in nature.

We showed that through a transformation, we can remove this coupling, that means, we can diagonalize the structural matrices using the matrix of Eigen vectors associated with K and M matrices, that helped us to define to notion of normal modes and natural frequencies. And these normal modes, had an interesting property, almost useful property, namely the orthogonaility property, that helped us to uncoupled the equation of motion. And we had a problem in dealing with damping matrices, because the model matrix derived from undamped free vibration analysis, would not uncouple the damping matrix unless damping matrix is of a certain kind.

So, we assume that, the damping matrix is that we are going to consider are such that the undamped normal modes uncouple the equations of motion in presence of damping. Towards the end, we also derived the input output relations in the frequency domain. So, will continue from here; we will spends some more time on input output relations in time and frequency domains for deterministic excitations and then, analyze the response of the system for random excitations.

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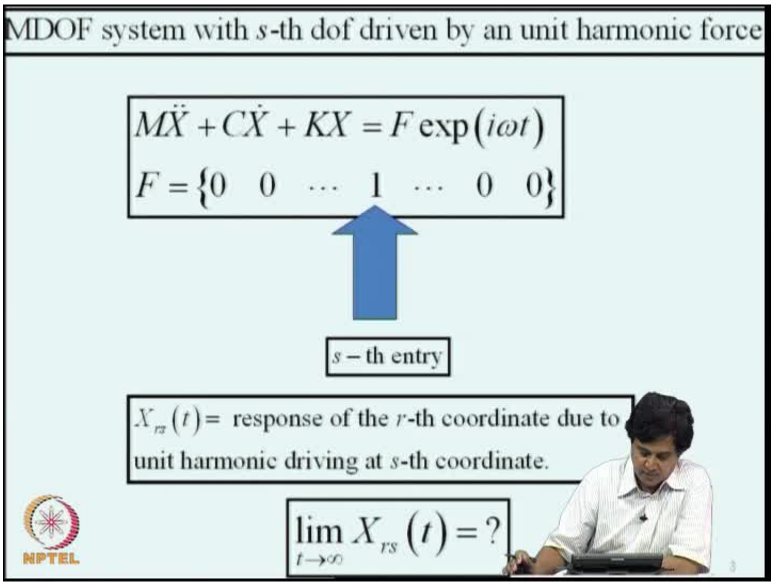

Now, let us consider a multi-degree freedom system with s-th degree of freedom driven by an unit harmonic force, that means, I consider the equation of motion to be in the form M X double dot plus C X dot plus K X equal to F into e raise to i omega t; this F is a vector, whose entries are 0 except for the s-th entry, which is 1. So, we call X r s of t as response of the r-th coordinate due to unit harmonic driving at s-th coordinate. So, we are interested in steady state response. So, the question that we are trying to answer now is what would be the response at say r-th degree of freedom as t tends to infinity when we are driving the system at s-th degree using an harmonic excitation.

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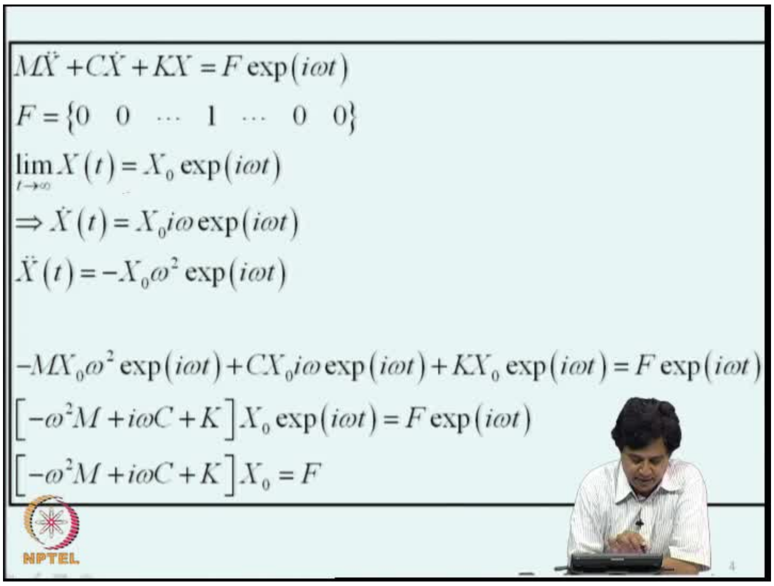

So, we can start by assuming that the response of the system remains harmonicated driving frequency; so we can assume that as t tends to infinity, x of t is some x naught into e raise to i omega t; this would make X dot as X naught i omega e raise to i omega t and similarly, acceleration is derived as shown here. So, if you now substitute this into this equation- the governing equation - so I get this is the acceleration term; this is the damping term; this is the stiffness term; this is the driving term. So, if we rearrange, we get this equation, where we get a matrix now minus omega square M plus i omega C plus K into X naught into e raise to i omega t equal to this; e raise to i omega t cannot be 0. So, from this, we conclude that this matrix into X naught is equal to F.

So, this reassembles the equilibrium equation that we get in static problems like, X equal to p except that, the stiffness matrix here is now function of damping matrix and the mass matrix and also it has a frequency parameter appearing here and it is complex valued. So, this matrix, we call it as dynamic stiffness matrix; it is analogous to the static stiffness matrix; the prefix dynamic converts the fact that, the matrix is function of driving frequency, damping matrix and mass matrix.

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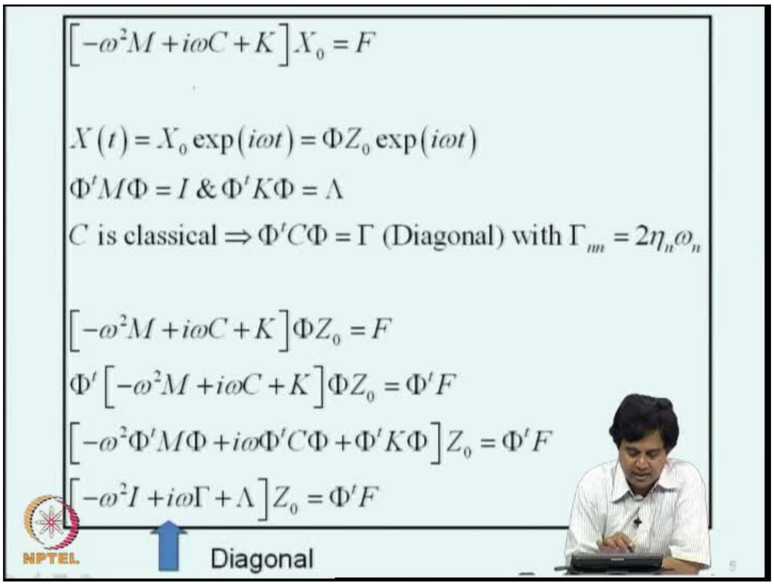

So, we can get X naught by inverting this dynamic stiffness matrix that is the direct approach or we can try to see if we can diagonalize this matrix. So, to do that we assume X of t is X naught e raise to i omega t and for X of t, I use the transformation phi Z of t, where phi is the matrix of Eigen vectors. So, X of t is phi of Z naught e raise to i omega t; so this phi matrix has this orthogonaility property, namely of phi transpose M phi i and phi transpose K phi is a diagonal matrix with the diagonal entry being the square of the natural frequency. We also assume that C is classical, that would mean phi transpose C phi is also diagonal we and we denote the diagonal entry in the form 2 eta n omega n.

So, we substitute this into this equation now and we used orthogonaility property; we pre-multiply by phi transpose; so we get this. Now, by taking phi inside here and phi transpose inside here, I get this as minus omega square phi transpose M phi plus i omega phi transpose C phi plus phi transpose K phi into Z naught is now phi transpose F. Now, phi transpose M phi is diagonal; it is an identity matrix, because that is how we have normalized the model vectors and phi transpose C phi I have written as capital gamma; it is a diagonal matrix with diagonal entry being 2 theta n omega n capital lambda is again diagonal, the diagonal entry being omega n square.

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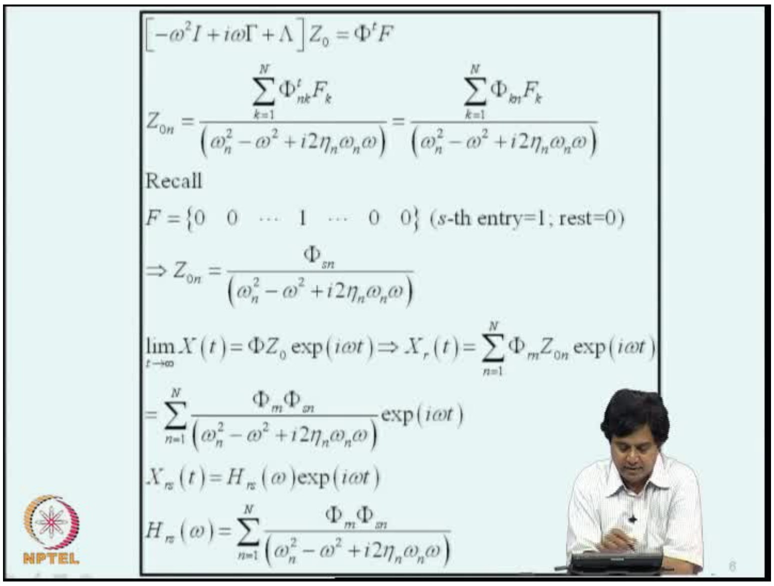

So, this matrix is now diagonal; so we can easily invert it; this is straight forward exercise. So, if we now write in long hand, so if I now consider the n-th element in this vector Z naught, it will be given by phi transpose N of k divided by the diagonal entry in this dynamic stiffness matrix in the transform coordinates system; so I get this expression. Now, the numerator here can further be simplified, because we have taken F to be such that, all the entry except the s-th entry is 1. So, therefore, this summation really contributes to only one term and that is when K equal to s, F K is 1 so I get here in the numerator simplify s n. So, Z 0 n is given by this; Z this prefix 0 subscripts 0 indicates that we are talking out amplitude Z of t and n is the contribution from the n- th mode.

So, now, we can write the expression for response in the original coordinate system. So, as t tends to infinity, I have response as phi Z naught e raise to i omega t. Therefore, if you take now the r-th element, it is given by the summation n equal to 1 phi r n Z 0 n exponential i omega t; for Z 0 n, I have just now derived this expression; so if I plug it here, I get this expression and if you now look at the amplitude of the response here, I can write it has H r s of omega and that H r s of omega is now displayed in terms of a model summation.

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$$X_{rs}(t) = \sum_{n=1}^{N} \frac{\Phi_{rn}\Phi_{sn}}{\left(\omega_n^2 - \omega^2 + i2\eta_n\omega_n\omega\right)} \exp(i\omega t)$$

$$H_{rs}(\omega) = \sum_{n=1}^{N} \frac{\Phi_{rn}\Phi_{sn}}{\left(\omega_n^2 - \omega^2 + i2\eta_n\omega_n\omega\right)}$$

Remarks

- $X_{rs}(t) = X_{sr}(t)$
- $H_{rs}(\omega) = H_{sr}(\omega)$
- $\left[H(\omega)\right] = \left[H_{rs}(\omega)\right]$
- $\left[H(\omega)\right]$ is symmetric but not Hermitian
- $\left[H(\omega)\right] = \left[-\omega^2 M + i\omega C + k\right]^{-1} = \left[\sum_{n=1}^{N} \frac{\Phi_{rn}\Phi_{sn}}{\left(\omega_n^2 - \omega^2 + i2\eta_n\omega_n\omega\right)}\right]$

This is quite different from considering the inverse of this matrix and looking at the r s-th element. This is a Brute forced inversion, whereas here it is done using the normal modes. So, I have X r s of t is given by this term multiplied by e raise to i omega t; so this is our frequency response function. So, H r s of omega means response at r-th degree of freedom, when s-th degree of freedom is driven harmonically at omega unit harmonic driving at s-th coordinate. So, you can see here that X r s of t is same as X r of t; this known as reciprocity relation is a kind of symmetric that we expect in these problems; this frequency response function is also symmetric. If I now assemble all these H r s of omega for r equal to 1 to n and s equal to 1 to n in a square matrix, we can call this matrix as matrix of frequency response functions.

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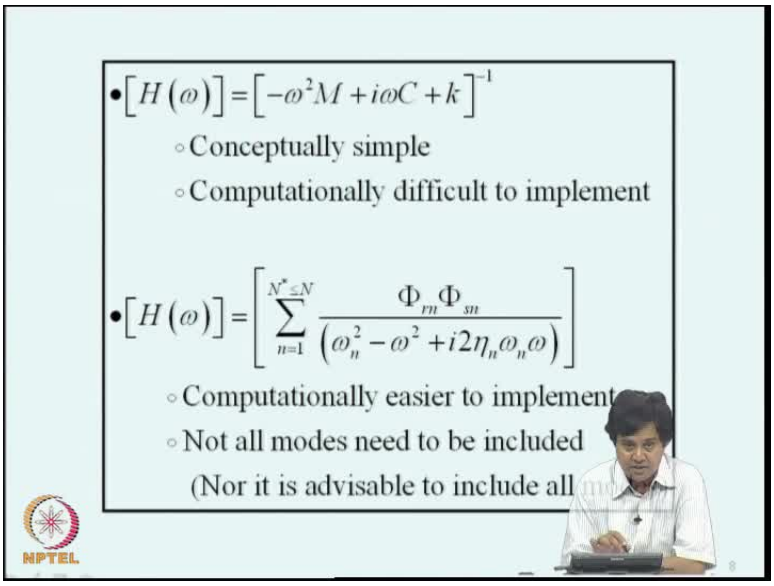

It is symmetric, but it is not Hermitian, that is, H r s of omega is H r Hs r of omega; it is not a conjugate of that. Now, we have two representations for H of omega: one is the direct inversion of the dynamic stiffness matrix or in terms of the model summations. The direct inversion of stiffness matrix appears conceptually simple; there is no coordinate transformation you have to simply invert a matrix, but it is computationally difficult to implement; in computation will seldom like to invert matrices. So, this is not a preferred way of computing frequency response function. The calculation frequency response function as a summation or model contribution is computationally easier to implement and an important feature that we have to notice here is that this model summation, although it is written as n equal to 1 to capital M, it is not necessary that we need to include all the modes in our calculation.

In many applications, the first few modes contribute significantly to the response and we can actual ignore the contribution from higher modes or apply correction for that for ignoring and I will take up that issue later in the course. But right now what is important to notice that in this summation, we need not include all the modes, in fact it is not advisable to include all the modes, because when you make discrete models for continuum problems using say finite element method or any of the method of residuals the accuracy of higher modes are much less in comparison with accuracy of the lower modes. So, there is no reason why we should include all the modes. In a model with say n degrees of freedom, we can trust only one-tenth of the modes, typically is a thumb rule so in that sense, summing our all modes is a formal representation in it is seldom down in actual practice.

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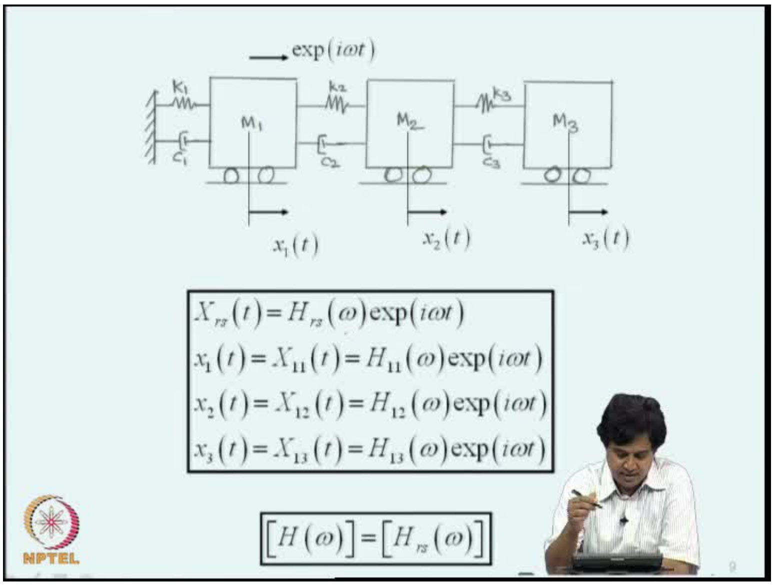

Now, to see simple example illustration of what we discussed, suppose we consider a three degree freedom system, suppose if I drive the first degree of freedom harmonically as e raise to i omega t, now I have X r X r s of t is H r s omega exponential i omega t; suppose I consider now response, these are all steady state response; so x 1 of t, I call it as capital X 11 of t, which is written as H 11 of omega e raise to i omega t this means that, I am driving at the first node and measuring the response at the same node; x 2 of t in this case will be X 2 1 of t, because I am measuring response at 2 and driving at 1.

Similarly, this is X 3 1 of t; so this is a matrix of the frequency response functions and to construct these elements, you need to do this exercise of driving each of the degree of freedom harmonically and finding out the response at all other DOFs and assembling the solution that you get. For each one of such calculations, you would be able to fill up one row and one column of the frequency response function.

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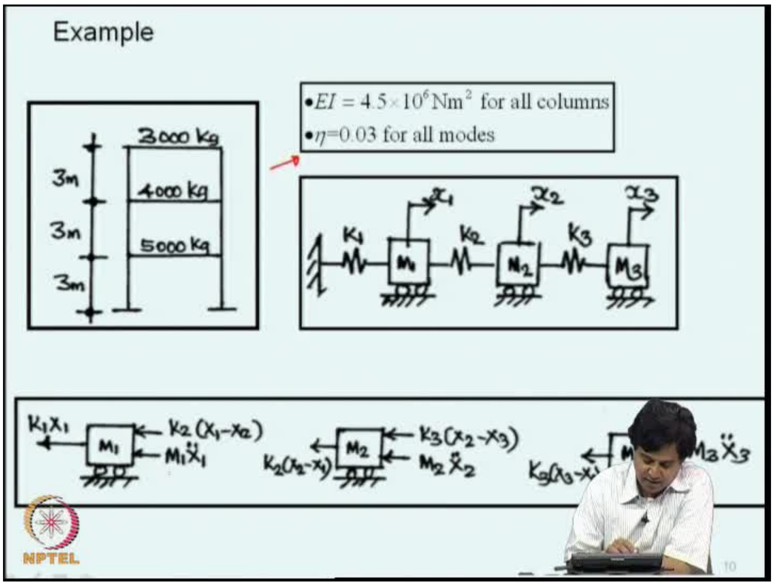

A simple numerical example, we consider a three storey building frame; we use shear frame model, that means, we assume that slabs move parallel to each other, resembling shared information, this move like slices of bread and the harmonic displacement. So, we are interested in dynamic of this frame basically in horizontal direction in it is one plane. So, given that we can simplify we can make a simplify model for this structure in a three degree freedom system, where this M 1, M 2, M 3 are respectively the masses of these three slabs; we assume the slabs are plot more heavier than the columns and the contribution to stiffness is made essentially by columns, because we assume that the slabs are infinitely rigid in their own planes.

So, K 1 would be for example, 2 into 12 e i by l q; K 2 will be similarly 2 into 12 e i by l q and so on and so forth. So, I have suggested some numbers, e i is a flexural rigidity of each of the columns, it can be taken as this number and another data that we have is that damping is 3 percent for all the modes means the three modes that are possible in this model. So, we can draw the free body diagram for M 1, M 2, M 3 and represent all the forces; so we can setup the equation of motion by summing the forces in horizontal direction and we get three equations.

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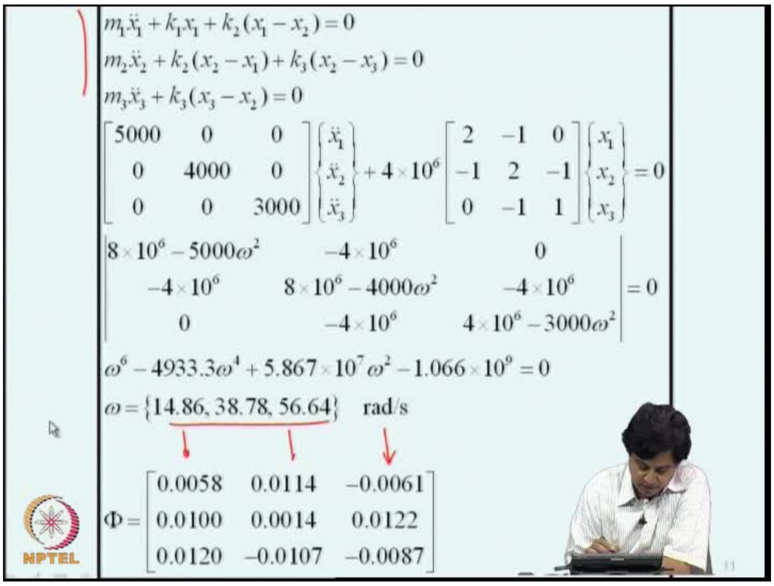

These are the three equations and we can arrange this in matrix form; this is a mass matrix; this is a stiffness matrix; we can see that mass matrix is diagonal, stiffness matrix is fully populated and it is non-diagonal it is symmetric; mass is also symmetric.

So, we can now consider the Eigen value problem associated with this K and M matrices and the characteristic equation in this case is given by this, omega square as square of the natural frequencies and if you expand this determinant, you get a sixth order polynomial in omega or a cubic polynomial in omega square and if you solve this equation, we get three roots displayed here and associated with each of this Eigen values, I can compute the Eigen vectors; this is the first Eigen vector; this is the second Eigen vector and this is the third Eigen vector.

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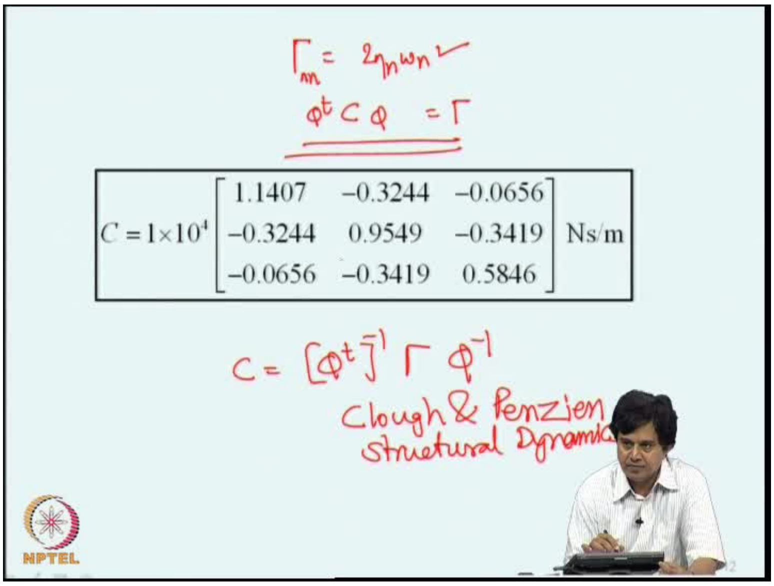

This matrix 5 matrix is now normalized with respect to mass matrix and based on the value of damping given as three percent for all the modes, we can construct the C matrix. So, we are given gamma as gamma n n is 2 eta n omega n and I have phi transpose C phi as gamma; so gamma is given. So, from this equation, we can compute gamma. Brute force method would be C will be phi transpose inverse gamma phi inverse. Actually we can avoid inverting the model matrix by using orthogonality; I would not get into those detail it is possible, you can see the discussion in the book by clough and penzien structural dynamics.

Many case we can get the C matrix; we need C matrix if you are going to compute F r f by direct inversion of the dynamic stiffness matrix; otherwise, this matrix need not be computed we can directly use the etas and compute the F r f purely by manipulating the expressions in the transformed domain. So, this is the c matrix is provided for sake of completion; the c matrix again is symmetric and fully populated.

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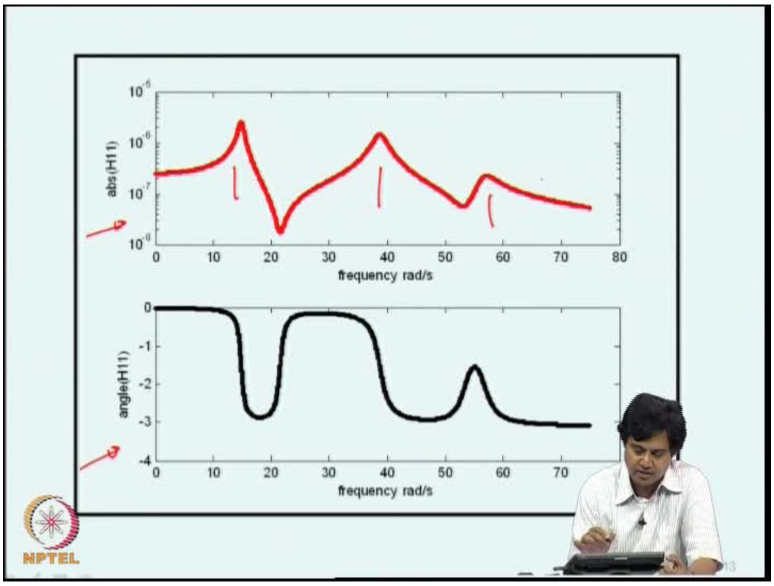

If you carry out the numeric, this is plot of H 1 1 of omega; the elements of frequency response function functions are elements are F r f matrix of complex value; so I would displayed here the absolute value and the phase. So, x-axis shows their frequency and this is the amplitude of the frequency response function and we see that it peaks at the three natural frequencies, which we have computed and this is the phase angle associated with this F r f.

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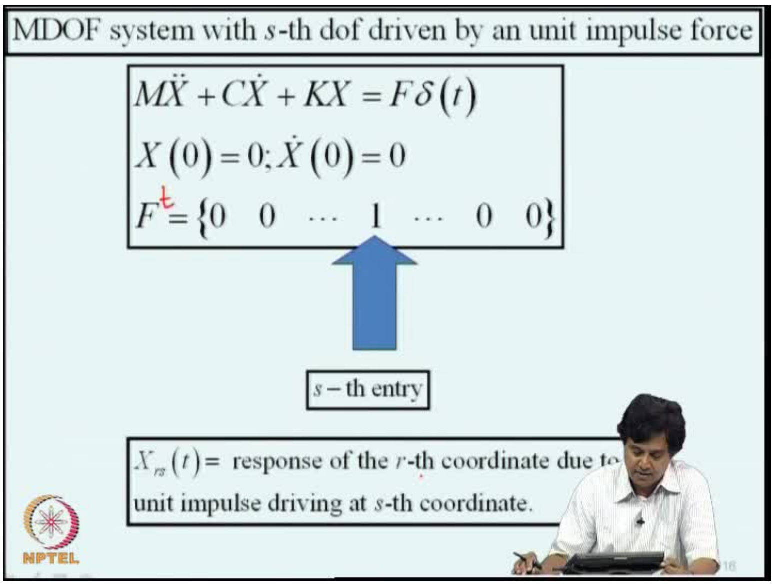

So, in multi-degree freedom system say with three degrees of freedom, there are three regions, where resonance can occur; this is in contrast with single degree freedom system, where there can occur one resonance. This is plot of H 1 3; this is amplitude; this is the phase; this is for H 1 2 of omega this kind of gives an idea how this functions look like. We can now consider the problem in time domain; instead of, driving harmonically, we can now apply impulsive forces.

Suppose, I now consider say multi-degree freedom system with s-th degree of freedom driven by an unit impulse force, so the equation would M X double dot plus C X dot plus K X into F of delta of t, where delta of t is a direct delta function and this F is a vector; this is a vector, whose elements are all 0 except the s-th element, which is unit. So, again we denote X r s of t as the response of r-th coordinate due to unit impulse driving at the s-th coordinate. So, this analysis is quite similar to the analysis that we just completed in frequency domain, except that now instead of applying harmonic excitation, we are applying and unit impulse.

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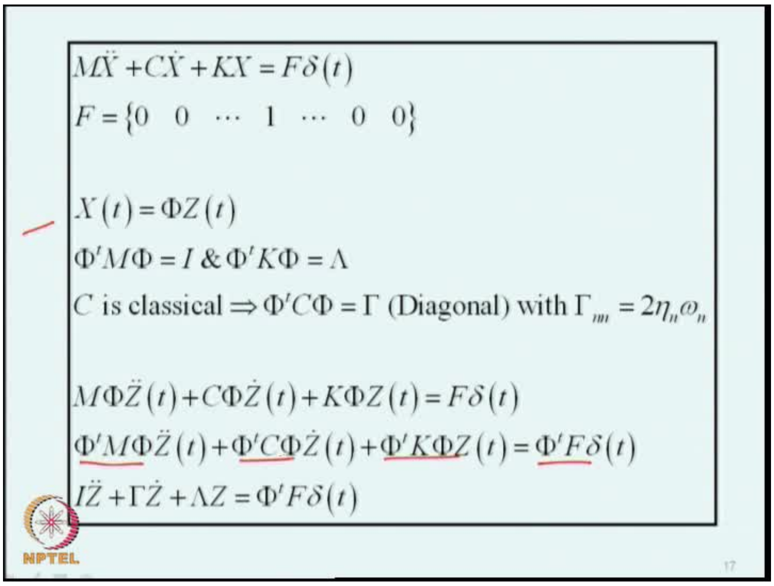

So, we have to here we have to conduct transient analysis and steady state system goes to 0. So, what is what is of interest here is the transient response. So, here again we make the transformation X of t is phi Z of t, where phi is the model matrix having this orthogonaility properties phi transpose M phi is i and phi transpose K phi is lambda and C is again taken to be classical with phi transpose C phi being a diagonal matrix as before. Now, I substitute this into this, I get this equation M phi Z double dot plus C phi Z dot plus K phi Z is equal to F delta of t; I pre-multiply now by phi transpose I get this; these matrices, these are the mass stiffness matrix in the transform coordinate system; this a generalize force in the transform coordinate system; this phi transpose M phi is an identity matrix, because that is how we have would normalize the model matrix; this is phi transpose C phi is gamma Z naught Z dot and phi transpose K phi is capital I.

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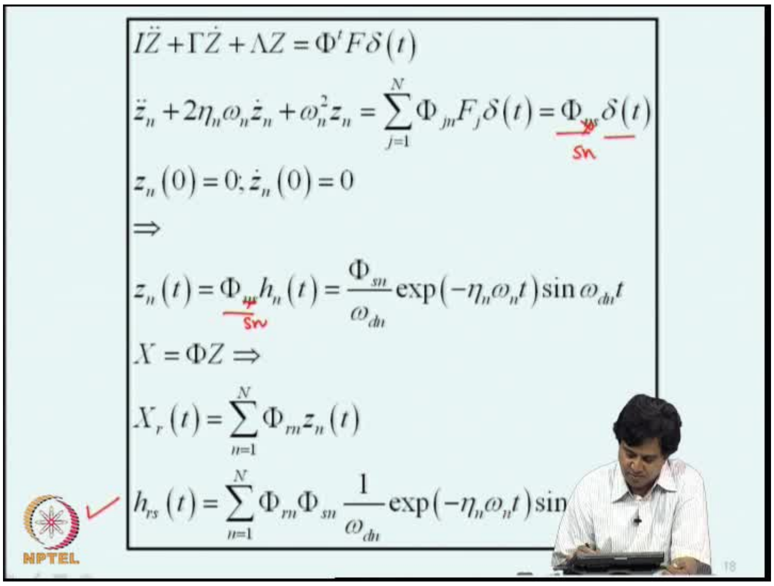

So, these equations are now uncoupled, so I can take up individual equations and try to solve them. So, if I now consider the n-th equation, I get Z n double dot plus 2 eta n omega n Z n dot plus omega n square Z n is summation j equal to one to n phi j n f j direct delta of t. Now, this f j is 0, except for when j equal to s when it is unity; therefore, this will contribute only when j is equal to s and I get this expression.

So, this is system, where assuming the system is starting from rest. So, consequently now Z n of t is obtained has phi s n by omega d n exponential minus eta n omega n t sin omega D n t, because the response of this system to unit impulse response is H n of t; so it is scale version of that multiplied by this and we get this. Now, if you return to the physical coordinates, X is equal to phi Z and look at the r-th coordinate, this says summation N equal to 1 to n phi r n Z n of t; for Z n of t, I will substitute this and I get this expression.

So, this is the response of r-th coordinate due to a unit impulse applied at s-th coordinate; so it is generalization of notion of impulse response function for a multi degree freedom system; this is now expressed in term in terms of a model summation. So, all the parameter that appear on the right hand side are essentially obtained in our free vibration analysis; these are the elements of model matrix; omega d n is a damp natural frequency of the n-th degree of freedom; eta n is the damping in the n th coordinate; omega n is the natural frequency in the n th model.

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$$X_r(t) = h_{rs}(t) = \sum_{n=1}^{N} \Phi_{rn}\Phi_{sn}\frac{1}{\omega_{dn}}\exp(-\eta_n\omega_n t)\sin\omega_{dn}t$$

Remarks

- $h_{rs}(t) = h_{sr}(t)$
- $[h(t)] = [h_{rs}(t)]$ = Matrix of impulse response functions
- $[h(t)] = [h(t)]^t$
- Not all modes need to be included in the summation
- If an arbitrary load $f_s(\tau)$ is applied at the s-th dof (instead of unit impulse excitation)

$$X_{rs}(t) = \int_0^t h_{rs}(t-\tau)f_s(\tau)d\tau$$

$$= \int_0^t f_s(\tau)\left\{\sum_{n=1}^{N}\Phi_{rn}\Phi_{sn}\frac{1}{\omega_{dn}}\exp\left[-\eta_n\omega_n(t-\tau)\right]\sin\omega_{dn}(t-\tau)\right\}d\tau$$

So, I can now construct this solution at r-th degree of freedom, this is what I get. Here again we can notice that this function is symmetric and if you assemble all elements of this impulse response functions into a square matrix, we call this matrix H of t as matrix of impulse response functions. So, the notion of an impulse response function for a single degree freedom system now gets generalized and we get instead of getting as scalar function, we get a matrix and this function this matrix is symmetric. Again as I pointed out already, we need not include all the modes in these model summations; we can include the first few modes on which we have greater trust.

Now, once I find the impulse response function, I can use the Duhamel's integral concept and if I now assume that instead of unit impulse, there is a load F s of tau, I applied at the s-th degree of freedom, instead of unit impulse excitation, then the response, the remaining all other things remaining the same, that means, all other degrees of freedom are not driven and systems start from rest, I get the response in this form. So, knowing the impulse response function, I can construct the response to an arbitrary load. So, here again I get a convolution integral, but this is now this gets buried inside; now there is a summation here that needs to be factored in while interpreting this integral as a convolution integral.

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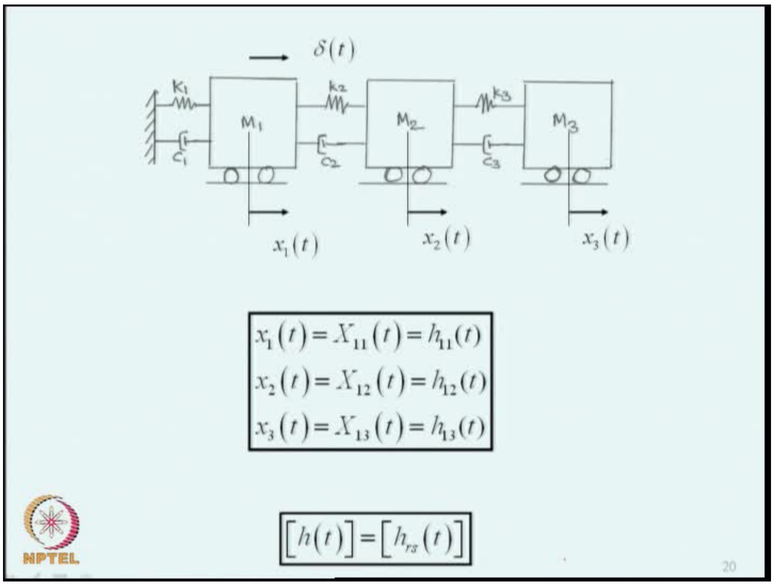

So, we can consider the same example, instead of applying unit harmonic excitation, now I applied unit impulse and I solve this problem and I get X 1, X 2, X 3 that leads to h 1 1, h 1 2 and h 1 3 of t 1 column and one row in our impulse response function matrix. So, if I repeat this exercise, by that I mean after completing this, you now apply unit impulse here and measure the response, then you apply unit impulse here and measure the response, you will get a 3 by 3 matrix of impulse response functions and this constitutes one of the way of studying dynamical system in laboratory and these impulses are applied through instrumented hammers and measurement of this impulse response function is an important activity in a dynamics laboratory.

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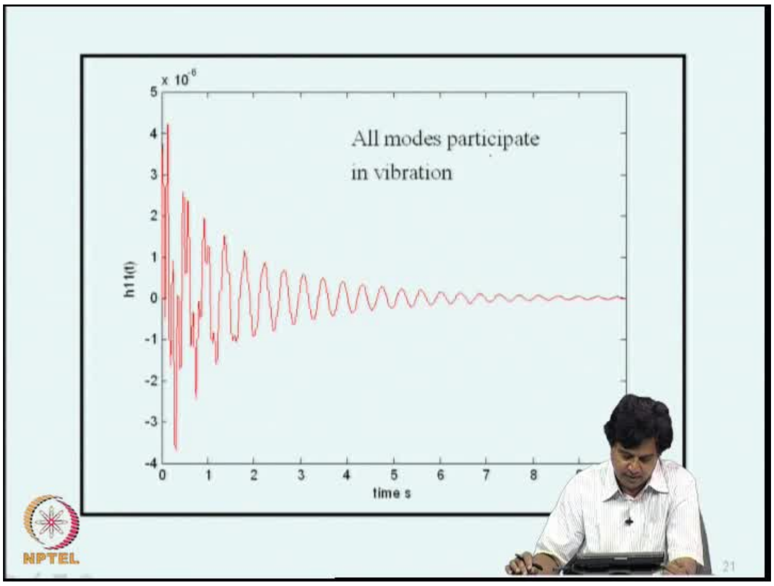

So, I have numerically solved the equation using this summation. We have already seen what are the eigen vectors and damping and all the numbers we have computed; so if I plug that in, I get this is the element h 1 1 of t and as expected as time tends to infinity the system comes to rest. But the transient here you can see that they consist of more than one modes, there are more than one harmonics, which are getting super post here and actually all the modes are participating in the free vibration decay; whereas in a single degree freedom system, you apply in unit impulse, the decay of the oscillations there will contain only one frequency that is the system natural frequency in that case, but whereas here, the response would consist of contributions from all the three modes.

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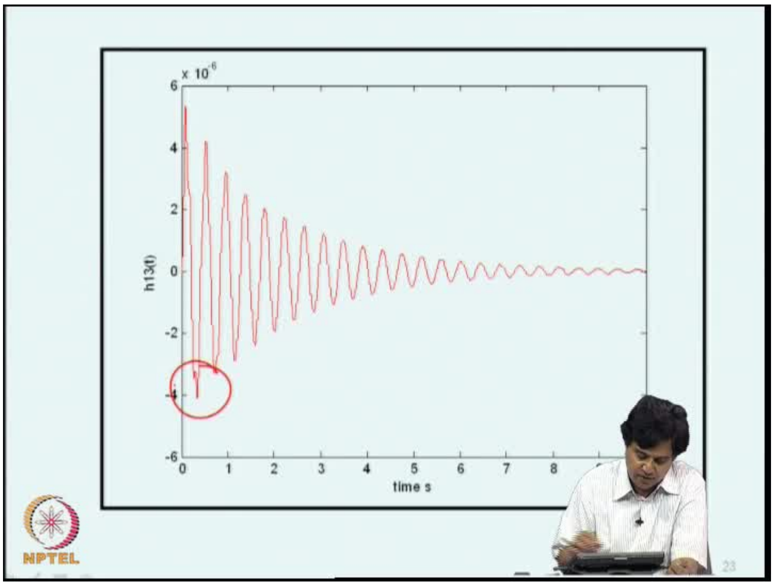

This is plot of h 1 2 of t, here again there are participation of different modes, for instants here you can see that. But this decay is dominated by a single mode as you can see here and this is for the third degree of freedom system. Initially, there is some influence of more than one mode, but eventually the decays at basically one frequency.

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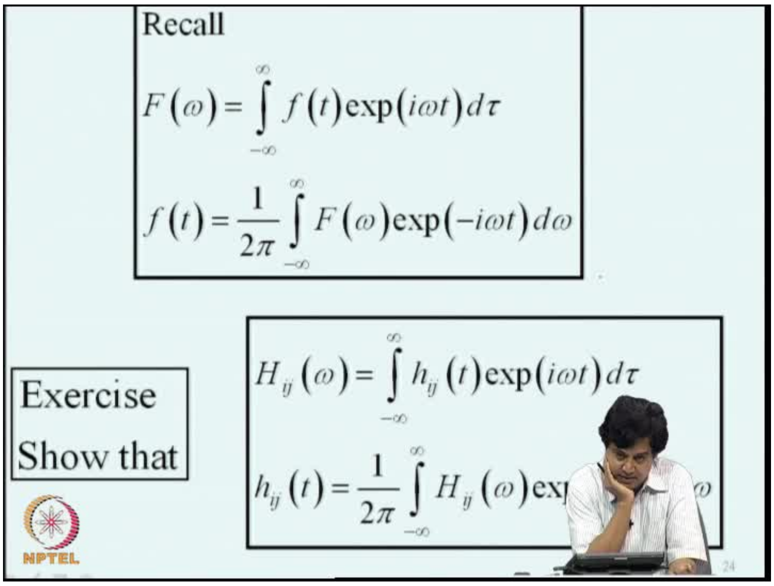

We have already seen in the discussion of single degree freedom system that the impulse response function and complex frequency response function form a Fourier transform pair and that property continues even for discrete multi-degree freedom systems. And if you recall, the definition of a Fourier transform pair for function F of t and F of omega we can show that the impulse response function H i j of t and complex frequency response function H i j of omega form a Fourier transform pair. So, I leave this is an exercise for you to do this.

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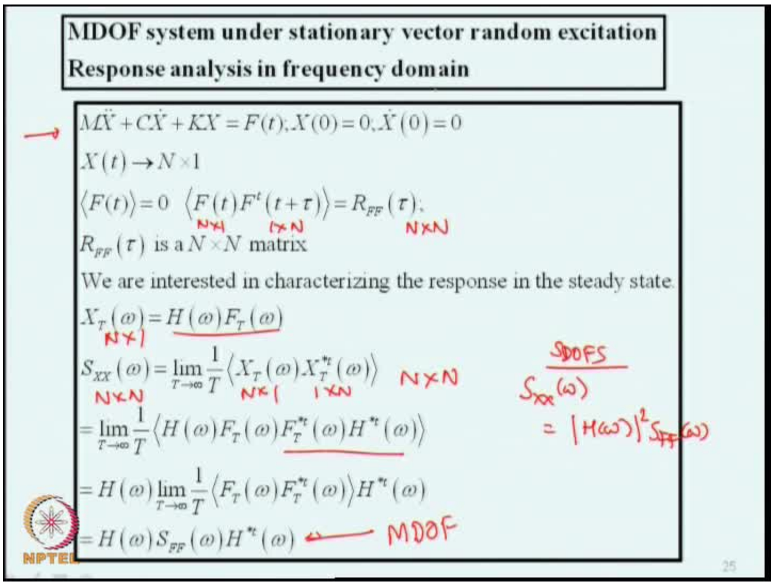

Now, equipped with this background on deterministic analysis of multi-degree freedom systems in time and frequency domain, we are now ready to you consider the problem of random vibration of multi-degree freedom system. So, we will start by considering MDOF systems under stationary vector random excitation. So, we will consider first the response analysis in the frequency domain. So, we consider the equation of motion M X double dot plus C X dot plus K equal to F of t and we assume that the system start from rest, X of t is the n cross on vector that means in the n degree freedom system. We assume that the mean of this process vector random process is 0 and its covariance is given by R f f of tau. So, this is a n cross n matrix, because this is n cross1; this is 1 cross n and we get this square matrix; this superscript t is the transposition operation.

So, R f f of tau is actually n cross n matrix, where assuming that all degrees of freedom are driven by one component of F of t. So, we are basically interested in characterizing the response of this system in the steady state. So, in the steady state, we can write X t of omega is h of omega into F t of omega, where H of omega is a matrix, X t of omega is the Fourier transform of a truncated time history of X of t, where between 0 to capital T it is equal to X of t and beyond that, it is 0 and this we have already discussed when we defined Fourier transforms of a periodic signals. So, the same ideas being used here and this is a definition of the matrix of power spectral density functions, since X of t is now n cross 1 vector, this power spectral density function will be a n by n matrix and this is defined as expectation of 1 by t x t of omega X t star transpose omega as t tends to infinity; so this is conjugation and transposition.

So, this is n cross 1; this is 1 cross n; so the product is n cross n. Now, for X t of omega i will use this; I write H of omega F t of omega and for X t star t of omega, I write this I get this. Now, the in this expression, the randomness is involved with F t of omega and its conjugate transpose. So, H of omega and H star t of omega are deterministic; so the expectation operator can be taken inside; if I do that, I get the output power spectral density function is now given by H s into H dot t. So, this is a generalization of what we got for single degree freedom system as H of omega whole square S f f of omega; this what we saw for single degree freedom systems; now this is for M D of system. The elements of H of omega, I have already describe how to compute them, either directly in terms of M c k or in terms of the natural frequency is a mode shape.

Essentially if you are looking response in the steady state, the power spectral density function is a complete description. So, this this analysis completes the analysis that we intended to do the outside, that means, we are find out the output power spectral density function matrix of output power spectral density function.

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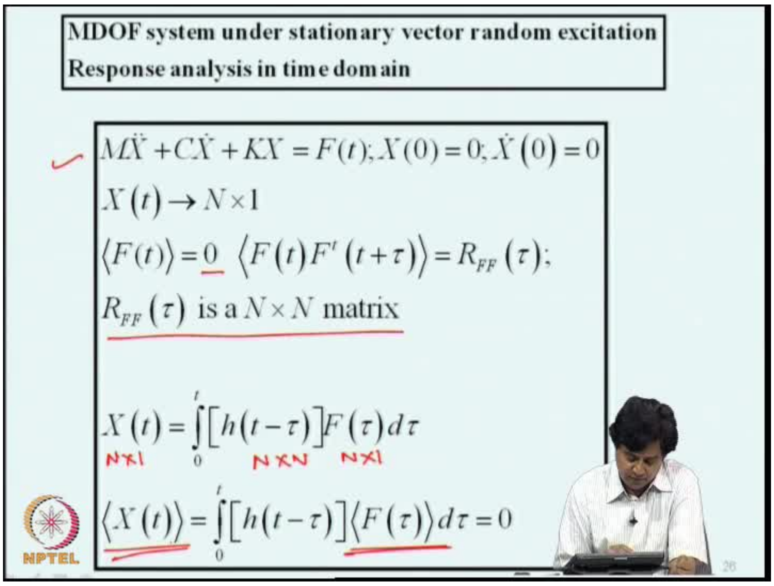

Now, we can repeat this exercise by considering the response analysis in time domain. Here, of course, there will be transient state and steady state; the steady state, of course should correspond to what we are seeing just now, but time domain analysis also includes transient; so that we will see how it emerges. So, we consider again the same system m x double dot plus C X dot plus K X is equal to F of t and F of t has 0 mean. And covariance matrix R f f of tau, now this vector X of t is now given in terms of matrix of impulse response functions. So, this is n cross 1; this is n cross n; this is n cross 1. F of tau is a random process vector random process. So, now, if I take the expectation on this, I get expected value of X of t is this matrix H of t into expected value of F tau and this is given to be 0, therefore 0 mean X of t becomes 0.

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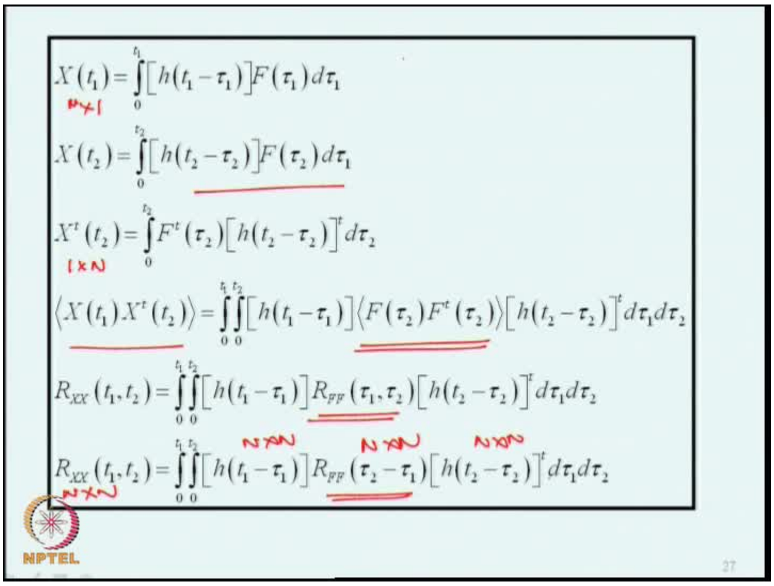

Now, we can consider the response at t equal to t 1, I get x of t1 is 0 to t 1 into this; I can also write the response at t equal to t 2, I get this expression; if I now take transposition of this, I get X t of t 2 as F t of tau 2 into h of t 2 minus tau 2 transpose; this is actually symmetric matrix so we can omit this t, but although I have written it. Now, if I multiply this is n cross 1 so this is one cross n; so I can multiply that; so if I do that, I get double integral and if I take the expectation and take the expectation operator inside; I get this expression shown here and this expected value is nothing but R f f of (tau 1, tau 2) and since F of tau is given to be stationary, I get this.

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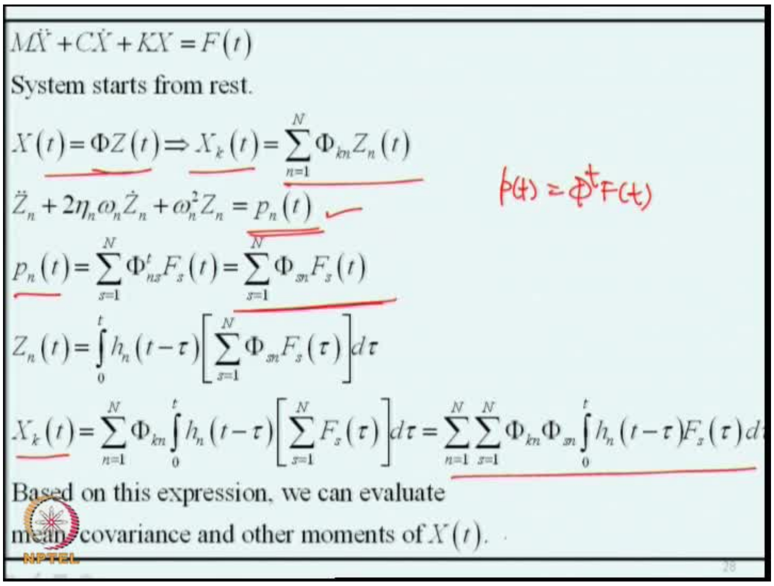

So, this is n cross n; this is n cross n; this is n cross n; this n cross n; so this is the response characterization using matrix notations. So, this again completes the response in analysis in time domain. We can redo this analysis by considering the solution in terms of normal modes, suppose before I do the random vibration analysis, if I were to make the transformation X of t is phi Z of t and for K-th degree of freedom, I get this summation and this Z n of t is governed by this single degree freedom system and p n of t is the generalized force and we have seen that p of t is phi transpose F of t; so using that, I get for the n-th element here, I get this summation.

So, Z n of t now in scalar notations in scalar form is now given by H n of t minus tau into this forcing function. X k of t therefore is obtained by using this transformation; all the modes need to be summed; if I do this, I get X k of t in terms of a double summation. Now, based on this expression, we can of course evaluate the mean and the covariance and other moments of X of t. Here, I have got the response characterization in terms of the natural frequencies and normal modes of the system.

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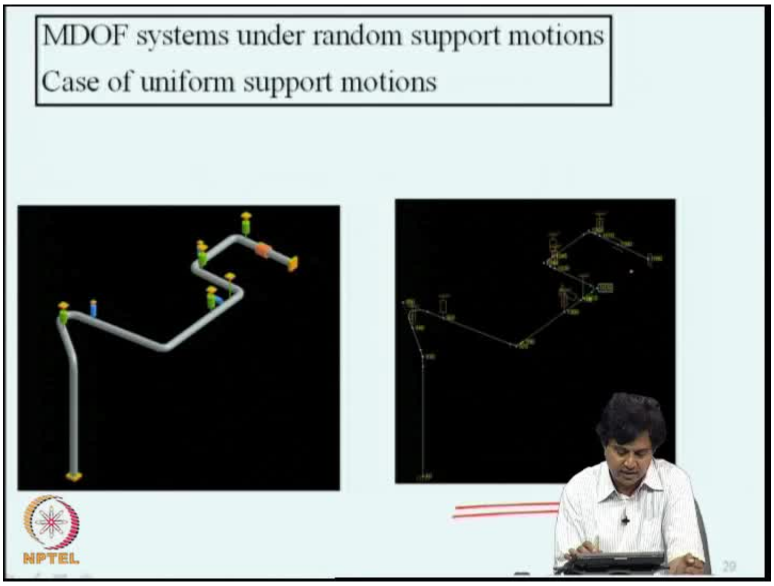

As an illustration of these formulations we will now consider how do how do multi-degree freedom systems response to earthquake like support motions. Here I have shown a typical pipeline in industrial structure and these orange elements are the supports and these supports receive differential support motion in the event of an earthquake, so this piping is housed inside a civil structure. So, in the event of an earthquake, the civil structure oscillates and these points are of the piping are connected to the slabs and beams and walls of the civil structure, and due to oscillation of this building components this pipe also oscillates and since this connections are different, elevation are different points on the structure; this supports suffer different support motions.

So, if you make a finite element model for this, this is this second view graph shows the how a finite element model is made using line elements, but this aspect is not important for our discussion, this is shown for completeness.

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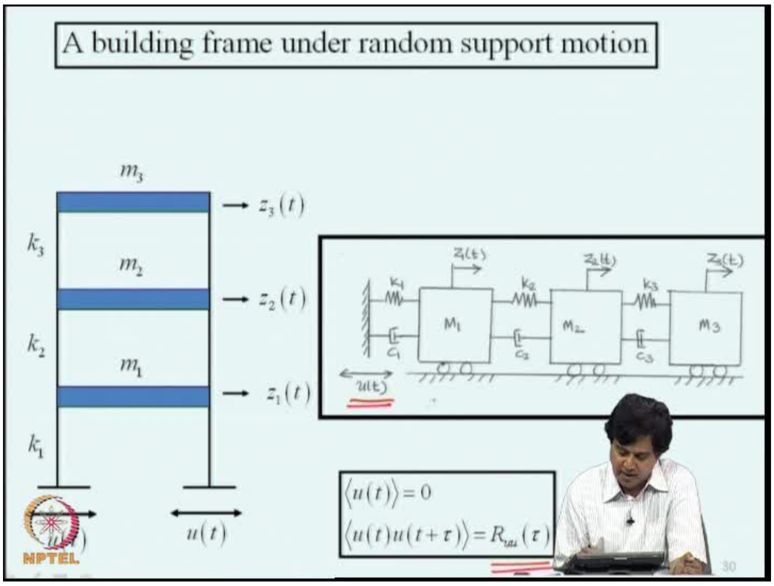

A simpler class of problem would be a building frame, which is subjected to support displacement u of t and again, if you assume slabs are rigid and heavier compared to column and columns are flexible, etcetera the usual share building model assumptions we can approximate this by a three degree freedom system, where the support motions now appear here has support displacements. So, this u of t now is scalar random processes, its mean is 0 and suppose if it is stationary random process, its auto covariance is given by R u u of tau.

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$$m_1\ddot{z}_1 + c_1(\dot{z}_1 - \dot{u}) + c_2(\dot{z}_1 - \dot{z}_2) + k_1(z_1 - u) + k_2(z_1 - z_2) = 0$$
$$m_2\ddot{z}_2 + c_2(\dot{z}_2 - \dot{z}_1) + c_3(\dot{z}_2 - \dot{z}_3) + k_2(z_2 - z_1) + k_3(z_2 - z_3) = 0$$
$$m_3\ddot{z}_3 + c_3(\dot{z}_3 - \dot{z}_2) + k_3(z_3 - z_2) = 0$$

$$x_1 = z_1 - u$$
$$x_2 = z_2 - u$$
$$x_3 = z_3 - u$$

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + c_2(\dot{x}_1 - \dot{x}_2) + k_1x_1 + k_2(x_1 - x_2) = -m_1\ddot{u}$$
$$m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + c_3(\dot{x}_2 - \dot{x}_3) + k_2(x_2 - x_1) + k_3(x_2 - x_3) = -m_2\ddot{u}$$
$$m_3\ddot{x}_3 + c_3(\dot{x}_3 - \dot{x}_2) + k_3(x_3 - x_2) = -m_3\ddot{u}$$

So, we can write the equation of motion. Now, u of t is appearing here; this we have seen in the previous lecture. So, if I now introduce the x 1, x 2, x 3 coordinate, which defines the relative displacement with respect to the ground, they related to the displacement of the masses with respect to the ground, I get these as an equation of motion and the effect of support motion appear as external forcing m 1 u double dot m 2 u double dot and m 3 u double dot.

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$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} =$$
$$-\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

So, I can cause this equation in the matrix form, this m x double dot plus c x dot plus k x is equal to minus m this column of unit elements 1 1 1 into u double dot t. There is only one time history here, but difference multiples of this acceleration m 1 into u double dot acts on first floor, m 2 into u double dot acts on second floor and so on. Now, we need to analyze this equation, when u double dot of t is a random process.

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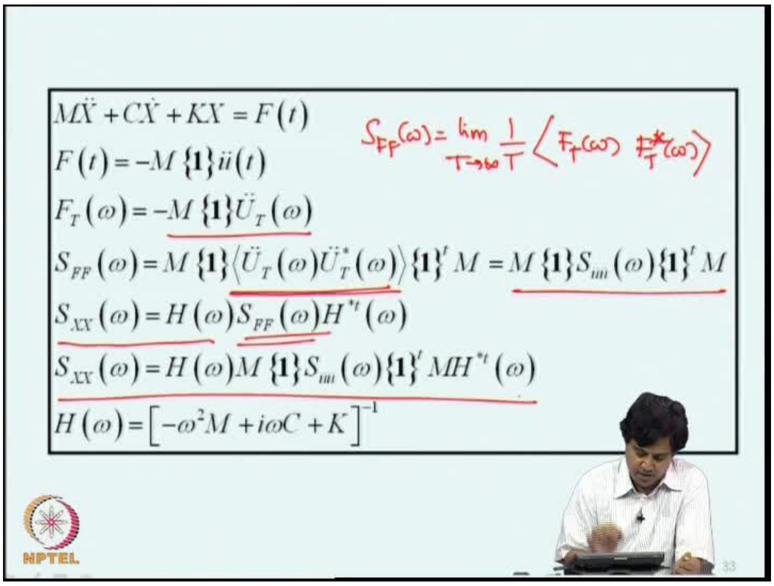

So, we can follow the matrix notations to start with, suppose if I write the equation of M X double dot plus C X dot plus K X is equal to F of t, where F of t minus m unit vector column vector u double dot of t. Now, if you are interested in analysis in frequency domain, I can consider the Fourier transform of the input F t omega as this and if I want the power spectral density of the input effective forces, I get m unity; I have to multiply F t of omega by F t star of omega and take the expectation. So, what I am doing is limit t tending to infinity 1 by t expected value of F t of omega F t star of omega.

So, this is contribution this is F t of omega; F t star of omega will come as U T star transpose of omega is unit vector transpose, M transpose is unity. So, the expectation operator goes applies only on the random components. So, I get S $_{X\,X}$ of S $_{F\,F}$ of omega as this M into unity S u double dot of omega unity transpose M. So, this is the power spectral density function of the excitation. Now, output power spectral density is H S f f of omega into H star t of omega. So, I have now this expression; I plug it in here and this is the p s d that I am looking for.

So, if you are doing a coding on computer, this is very easy to implement. H of omega here is the inverse of the dynamic stiffness matrix and we already seen that this can be computed in terms of the normal modes of the natural frequency and normal modes of the system.

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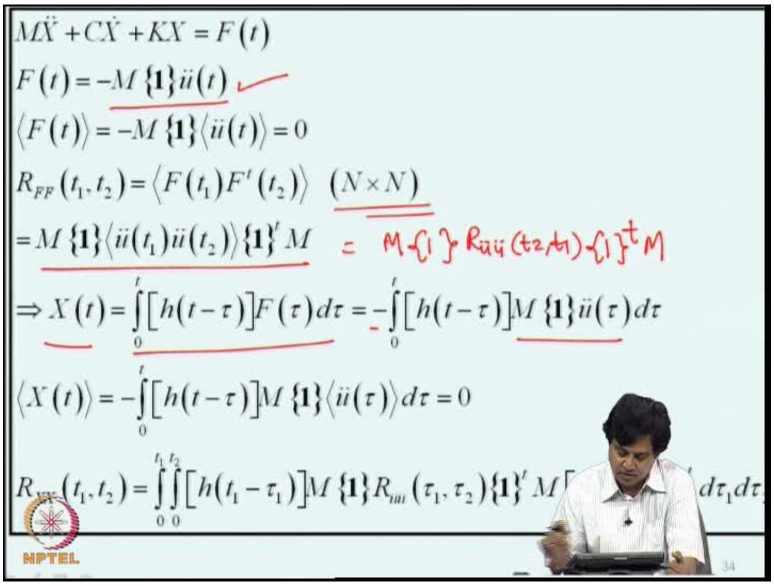

Now, you want to do a time domain analysis, again the analysis is straight forward. F of t is now this; mean is 0, because mean of u of t 0. If you want now the matrix of auto covariance function of F of t, I have to find expectation of f of t 1 into f transpose of t 2 this is the n by n matrix. So, for F of t I have got this expression; now if I substitute this here, I get this expression; this is F of t into F t of t 2. So, this is what I am getting and expectation runs only on U double dot of t. So, this is R f f of (t, t 2) so this nothing but M unity R u double dot of (t 2, t 1) into unity transpose into M.

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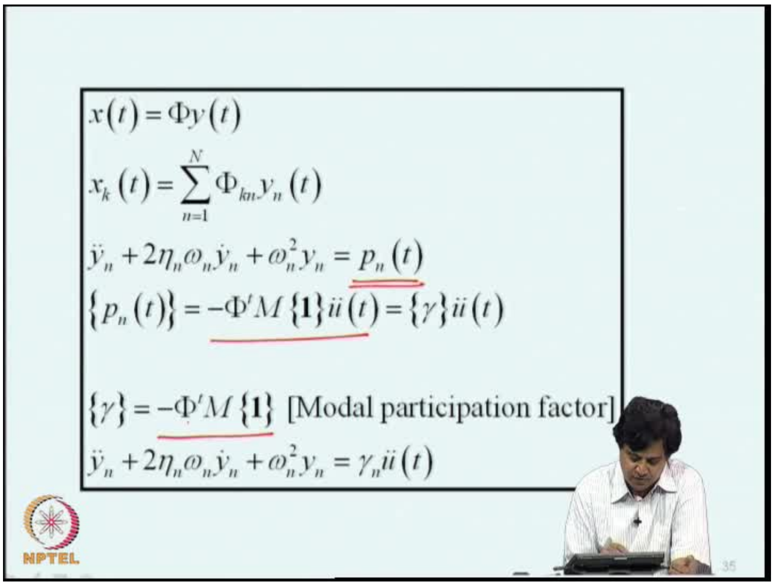

Now, I know X of t is given by this; this is a matrix of impulse response function; this is the applied expectation. Now, f of tau itself if given in terms of M into unity u double dot of tau d dot of t minus sin here and expected value of X of t is 0. And if you want now the auto covariance of X of t, you have to take expected value of X of t into X of t 2 transpose and if you do that, you get this expression. These are all in now matrix notation and we can evaluate this to derive the covariance. In terms of the generalize coordinates, I can make the transformation X is equal to phi y; I am using y, because Z I have used for total displacement.

So, if the X K of t is given by this now; so we can find out the expression for the effective force for n-th coordinate n-th degree of freedom by using this relation and this quantity phi transpose M into unity, this one is given a name it is known as modal participation factor. So, I get the equation for n-th generalize coordinate as Y n double dot plus 2 eta n omega n Y n dot plus omega n square Y n into gamma n u double dot of t. For different values of this n, the right hand side is implies scale by this participation factor. The time history remains the same, but it is factored by multiplied by gamma 1 gamma 2 or whatever, the n-th mode it is gamma n. This gamma itself is function of the model matrix and the mass matrix of the system.

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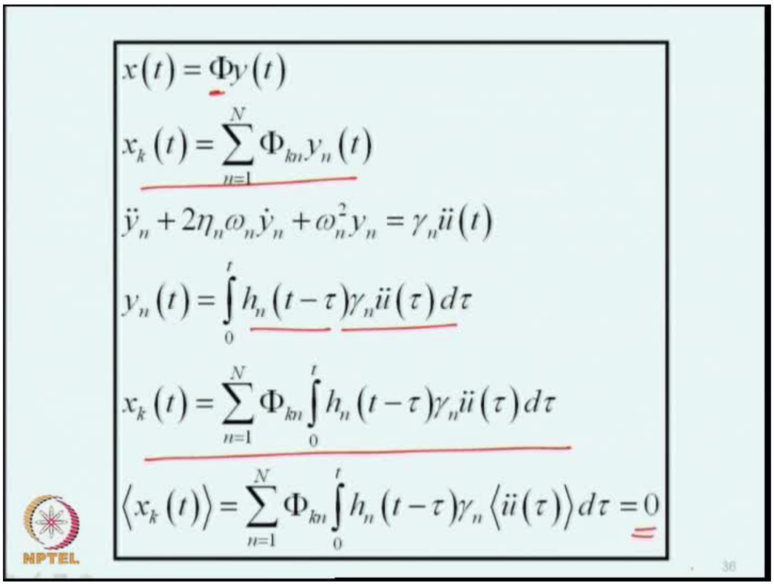

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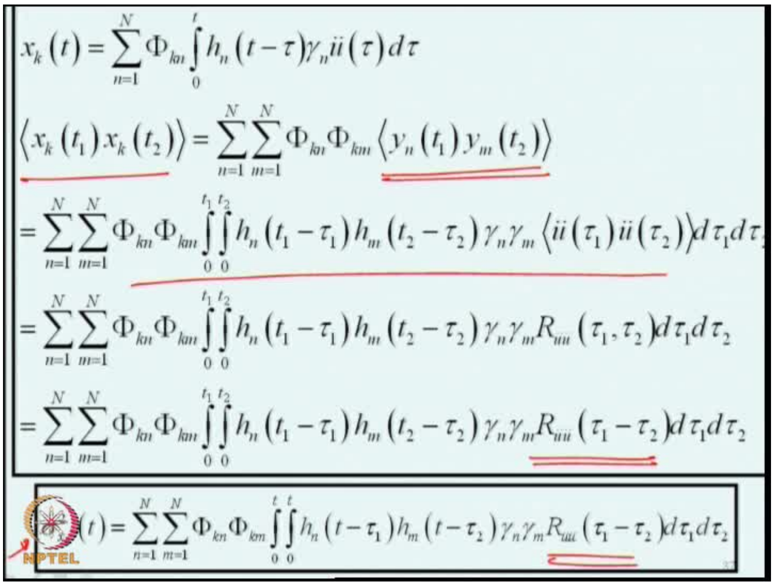

So, X of t is this; this is the transformation modal matrix and X K of t is this. So, if I now write Y n of t in terms of impulse response of n-th degree of freedom system, I get this expression; so substituting this into expression for X K of t, I get X K of t to be given by this. Now, I can launch the calculation of response moments; so you want now the mean, you take the expectation of this expected value of u double dot of t is given to be 0; therefore, expected value of X K of t is 0. And if you multiply now X K of t 1 into X K of t 2, you get the you need to compute the expected value of Y n of t 1 and Y m of t 2 and if we do that, I get this double summation and this expected value of u double dot of tau 1 u double of tau 2 is this, because u of t is given to be stationary. So, this is actually the auto covariance of response that I am looking for. In this effective t 1 equal to t 2, I get the variance, which is given by this expression.

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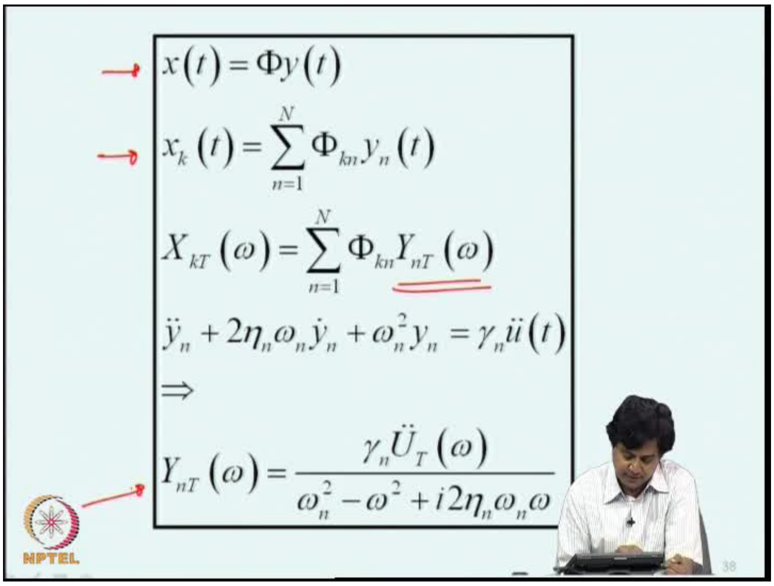

So, if you know now the auto covariance, we have to now the task is to reduce the evaluation of these integrals. This analysis can also done in frequency domain; to see that, we again begin with the transformation X K of t, X of t and X K of t is given by this. Now, I take the Fourier transform of this; I get this expression in terms of the Fourier transform of the generalize coordinates the truncated Fourier transform of the truncated time histories and the expression for Y n t of omega itself can be derived by using the governing equation and one gets in terms of participation factor and Fourier transform of the u double dot of t and this is the expression that we get.

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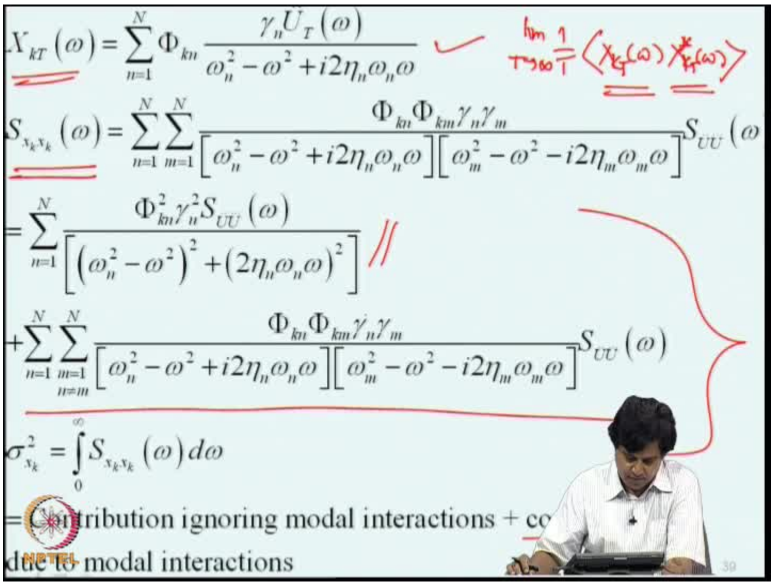

So, if I now substitute the expression for the generalize coordinate into the expression for the physical coordinate, I get this expression. Based on this, now I can compute the power spectral density of X K of t, that is, actually limit t tending to infinity of 1 by t of expected value of X K T of omega X K T of omega conjugated. So, I have find out this expression; I can find this conjugation and multiply; if I do this, this one single summation become double summation and I get this as the output power spectral density function. This is the quantity that we are looking for.

In this expression, we can do the following, we can sum up all the diagonal terms, that means, I will take the summation when n equal to m so i get n equal to n i get this this term, where n equal to m therefore, all this quantity now become real and this is the expression. These are the half diagonal terms contribution from half diagonal terms; so I can now find the variance of the process X K of t by finding area under this power spectral density function area. Under this first term can be viewed as contributions that ignore the modal interaction; this second term, where n naught equal to M is contributions due to covariance or cross covariance between different normal modes, then the different general generalize coordinates.

Therefore, this can be viewed as some kind of correction due to modal interaction. Now, this description becomes important later when we discuss relationship between response spectrum base analysis and power spectral density function power spectral density function based analysis for earthquake response structures. There is a question on rules of combining modal contribution arises and those rules are essentially formulated by investigating these expressions and we will written to this later, when we will discuss the modal combination rules in response spectrum base analysis.

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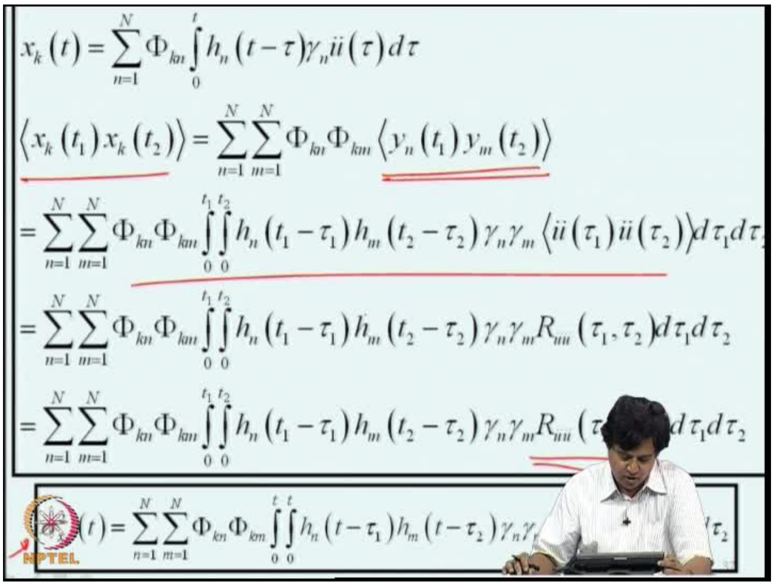

This kind of division into one that ignores modal contribution and one that correct it can also be done in time domain. Here again in this summation also, I can first summed the terms, where n equal to m and get the second set of term which represents cross correlation between different modes. So, if all follow our calculation right as t tends to infinity, whatever we get here in time domain analysis should match with this analysis; this is valid only for steady state. So, such a match we have already seen for single degree freedom system under random excitations.

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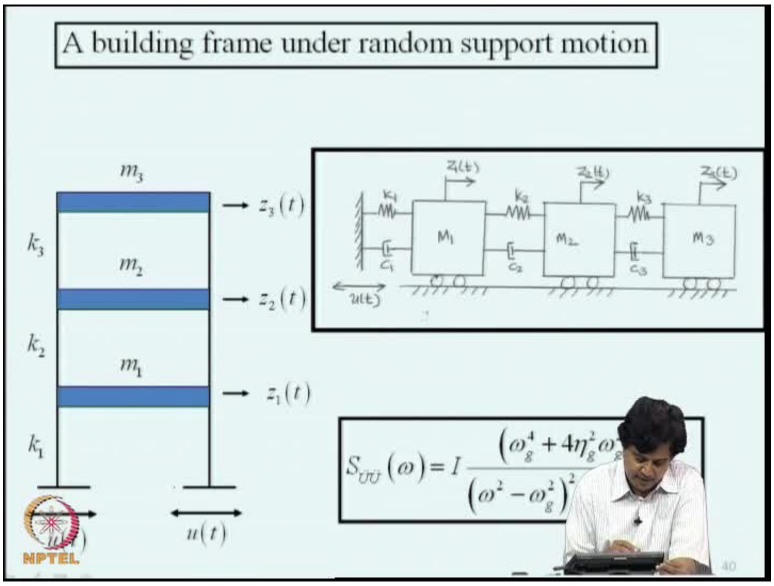

So, to illustrate this idea, I have considered a building frame under random support motion and this support acceleration is modeled as a Kanai Tajimi power spectral density function. So, this this itself if you recall was obtained by passing a white noise through a soiled layer and the absolute acceleration at the top of the soil layer was taken as the input to this building.

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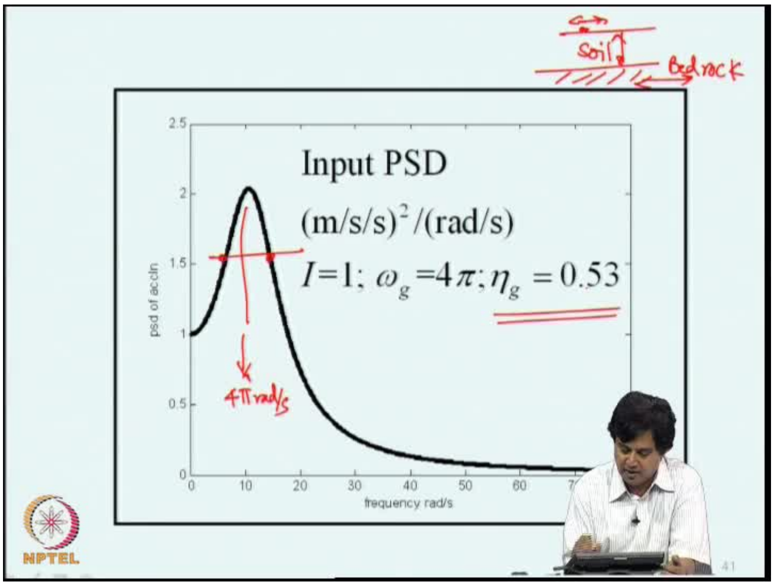

Now, if you do the random vibration analysis in this case, this is actually the input power spectral density that is actually use in numerical work the Kanai i th power spectral kanai

Tajimi power spectral density function as I mentioned, applies a unit applies a white noise at the bedrock level; this is bedrock; this is soil and we are interested in how this top of the soil layer oscillate and we have modeled this soil layer as a single degree freedom system. Consequently when you look at the power spectral density of the absolute acceleration at the top of the soil layer, it will have one resonance peak; this is this peak corresponds to I think I have taken as four pi radian per second and this bandwidth corresponds to damping in the soil layer and this is kept at fairly high value 0.53. So, this is the power spectral density function, for that we are assuming for the ground acceleration.

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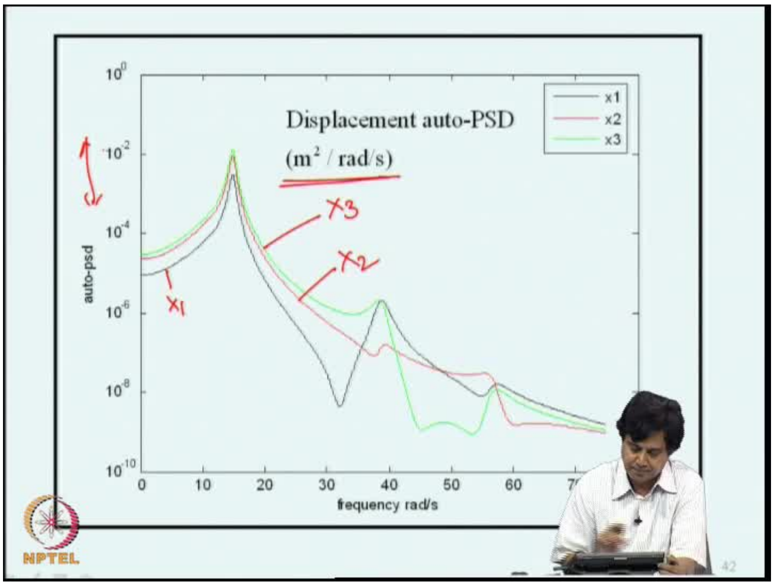

If you carry out the response analysis using the method that I just now outlined, I get the power spectral density function; the auto power spectral density function for the three degrees of freedom x 1, x 2, x 3. So, the black one is for x 1, the red 1 for x 2, green one for x 3. So, this is the displacement auto power spectral density function for the three coordinates and the power spectral density function itself is expressed in these unit.

So, you can see here, the power spectral density function has you could expect is dominated by first mode, because the first natural frequencies around 14 radian per second and soil frequency is around 4 pi, which again close some thirteen radian per second. So the first mode dominates the behavior and second and third modes make marginal contribution to the total response; mind you mind you that this y-axis is on log scale.

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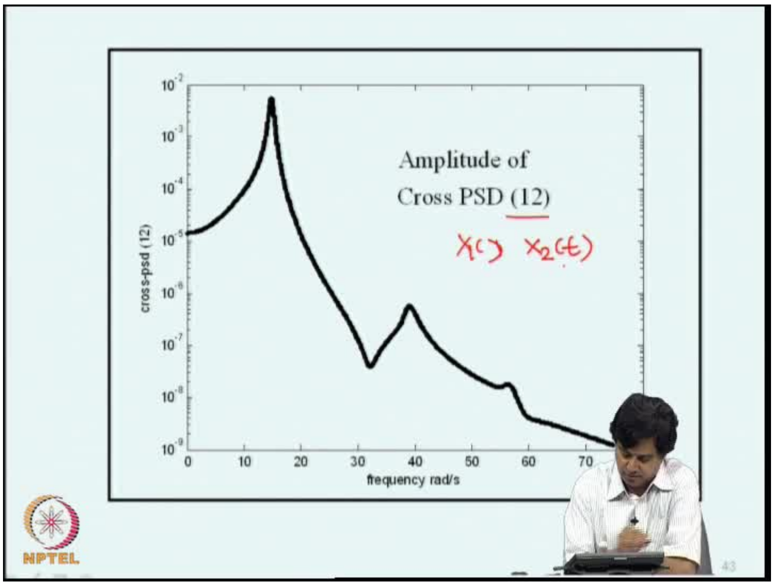

We can also look at the cross power spectral density function between response at different floor levels; that cross power spectral density function would be a complex value of quantity; therefore, we can talk of an amplitude function and a phase function. So, what is shown here is the amplitude of cross power spectral density function between x 1 of t and x 2 of t, the phase spectrum is shown here.

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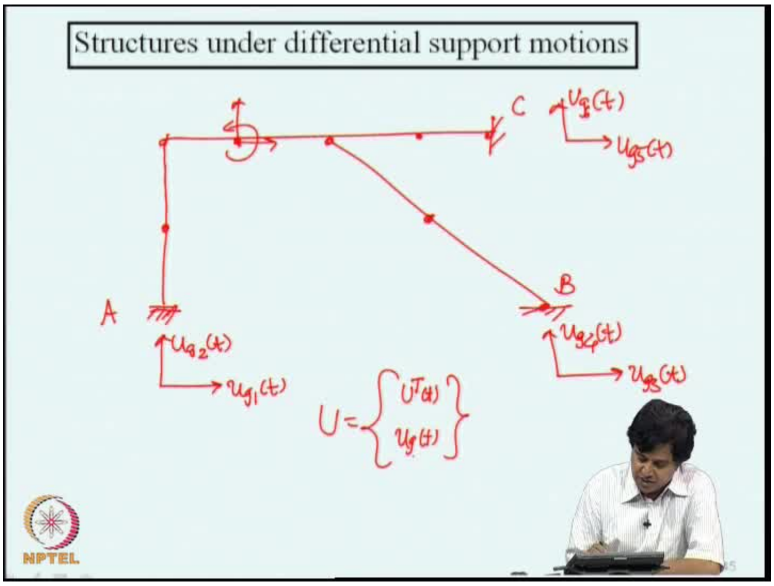

So, this this complete analysis of you know structures such as this, where all the supports of the structure essentially receive identical support motion. A generalization of this problem would be when different supports receive different support motion. So, we will briefly indicate what exactly is the problem there and consider the detail discussion in the next lecture. So, here what we are considering is a structure something like this; suppose we have made a computational modal say using finite element method, this A B C are supports; suppose we are assume that these supports are subjected to displacements, there are six displacement components, which are different from each other. So, this I will call it as U g 1 of t, U g 2 of t, U g 3 of t u and so on and so forth. This is a frame structure made up of beam elements; so the degrees of freedom that a node here will have are two translations and one rotation. So, if you look at this entire structural modal, there are now degrees of freedom associated with the ground; these points as I call it as ground and degrees of freedom associated with the super structure.

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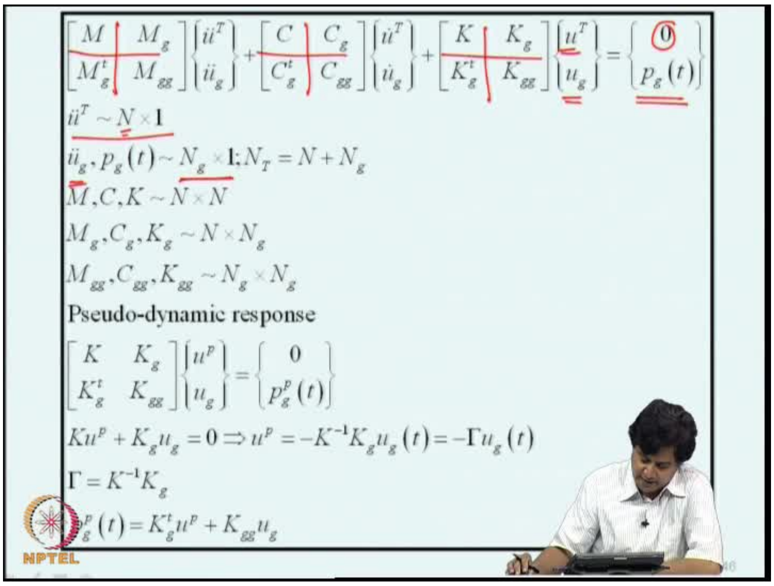

So, the total degree of freedom can be classified has some u t of t and u g of t; this superscript capital T indicates total response and u g of t are the applied support displacements. So, if you write now the equilibrium equation for the entire structure, I get the equation standard form M u double dot plus u dot plus K u equal to some F. But I have now partitioned the displacement degrees of freedom into u t and u g of t; these are unknown degrees of freedom of the super structure and u g of t is the applied support displacements. This u g of this u t there, let us assume that there are capital N number of unknowns, whereas this u g there are n g number of degrees of freedom; this P g of t , here on the right hand side P g of t are the reactions.

There is no external force other than the support motion therefore, this element is 0. So, based on this, partitioning of the displacement vector I can also partition now the mass, the damping and stiffness matrices and we can consider this equations separately. This m c and k that appear here are n cross n, and this M g C g K g are n cross N g and this M g g C g g K g g are n g cross n g matrices.

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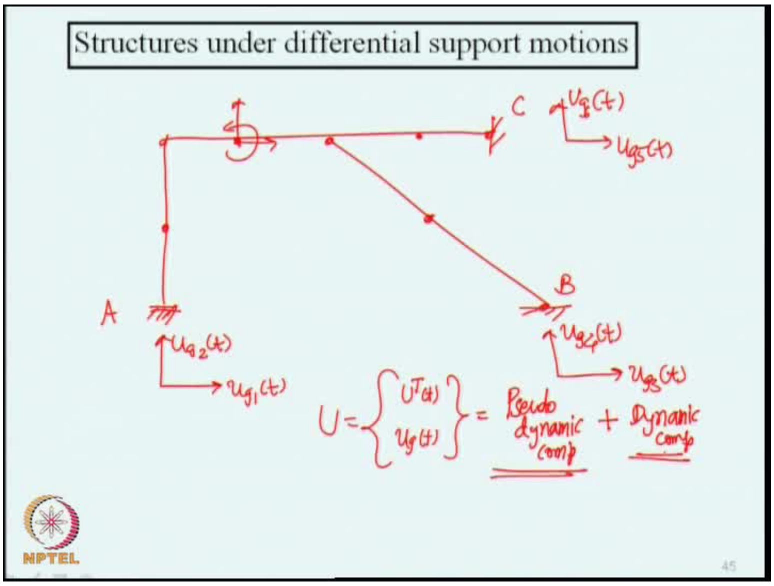

Now, before we look into the detail analysis, we can consider certain response features without actually looking into the solution of the equilibrium equations. Imagine that these support displacements are applied statically then there will be stresses in the structure because of differential support displacements; even purely under static conditions. Now, if these support displacements are now dynamic, there will be a pseudo dynamic response, which is purely calls purely due to the differential displacements and which is not affected by the inertia of this structure.

So, the total response can be decomposed into two vectors: a pseudo dynamic component and a dynamic component. In analysis of structures of this kind, it is important to delineate the contributions to the response from the pseudo dynamic component and dynamic component. So, we when we analyze this problem therefore, we need to formulate the problem in such a way that you will be able to output the properties of pseudo dynamic component of the response and the dynamic component of the response. So, this problem we will formulate in the next lecture and we will conclude this lecture at this stage.