Stochastic Hydrology

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Lecture No. # 03

Independence; Functions of Random Variables

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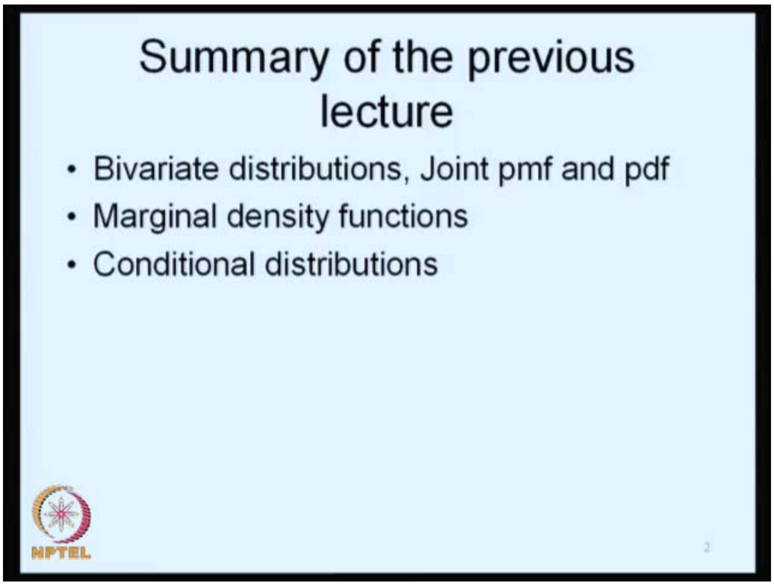

Good morning and welcome to this the third lecture in the course of stochastic hydrology. In the previous class we have introduced the concept of bivariate distributions, where we are interested in the simultaneous behavior of two random variables x and y, and we introduced the concept of joint probability mass function, and the probability joint probability density function. And then we went on to see the distribution of one variable irrespective of the other variable, and we called it as marginal density function or marginal probability mass function. And then we introduce the concept of conditional distributions where we are talking about the distribution of one variable x conditioned on the other variable y.

So, the distinguishing features among these are that in bivariate joint mass probability mass function and probability density functions, we are talking about the simultaneous behavior of two random variables x and y, and in the case of marginal distribution we are talking about the distribution of one variable irrespective of the other variable. And in the case of conditional distributions, we are talking about the distribution of one variable x conditioned on the other variable taking on specific values. And we also introduce the concept where the conditions placed on the variable y may be such that y belongs to a certain region, in which case we integrate over that region both the joint probability density function as well as the marginal density function. Now, today we will introduce the important concept of independence of two random variables, and then we will go on to see how we derive the probability distributions for functions of random variables.

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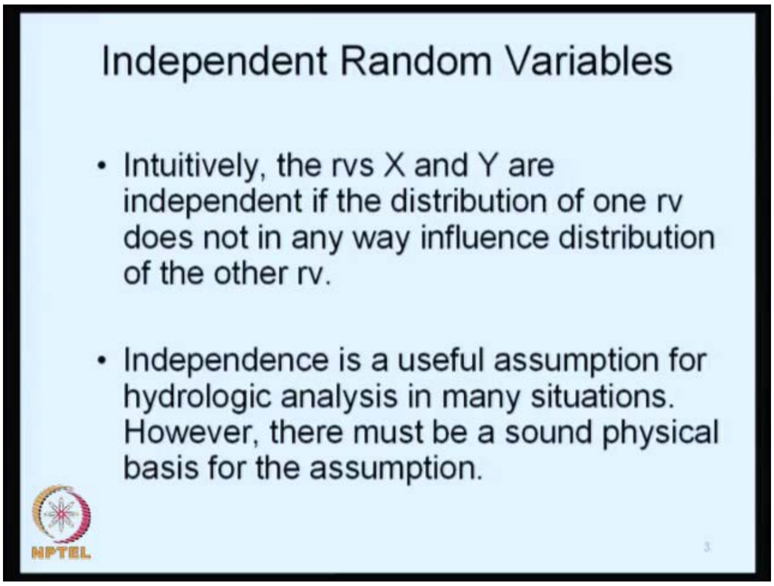

So, independent random variables intuitively you know X and Y are independent when the outcome of one of the random variables does not at all influence the outcome of the other random variables. So, we say the two random variables are independent if the distribution of one random variable does not in any way influence the distribution of other random variable. Now in many situations independence is a very useful assumption and specially in hydrologic analysis, but you must not forget that there must be a sound physical basis for this assumption for example, if physically the two are in two are dependent on each other the two variables are dependent on each and although stochastically you may say you may show that the two variables turn out to be independent based on the data that you have collected and based on the distribution that you fit and so on.

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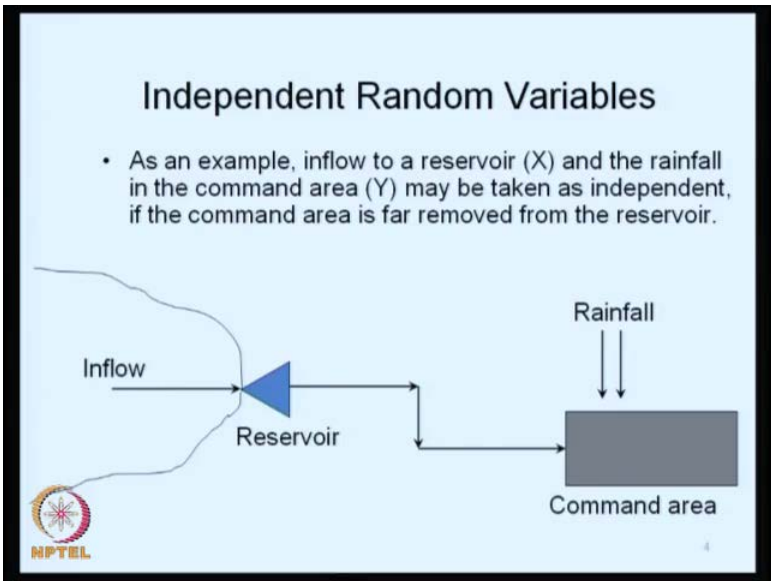

But there must be a sound physical basis for the assumption and only then you should proceed with the with taking of independence of the two random variables for example, we will take the example where we are talking about inflow to the reservoir and that is a command area where there is a rainfall that is occurring and you are interested in the two random variables inflow and the rainfall in the command area. Note that the inflow here in the catchment area of the reservoir is generated by the rainfall in the catchment area the precipitation in the catchment area and the rainfall in the command area, if the command area is far removed from the catchment area may be assumed independent of the inflow that comes to reservoir here now if; however, the rainfall here and the rainfall here are governed by the same processes and they are in the same hydrologic region then we cannot assume the rainfall and the inflow here as independent.

So, there must be a case for you to consider the inflow here and the rainfall here as two independent random variables based on whether or not they belong to different hydrologic regions and only then you will be able to take them as independent variables. You take another example, where in the class in the lecture number one we introduce this problem where we are interested in the water quality of a stream which is primarily governed by the stream flows and the affluent load that is the pollutant load that comes at various locations.

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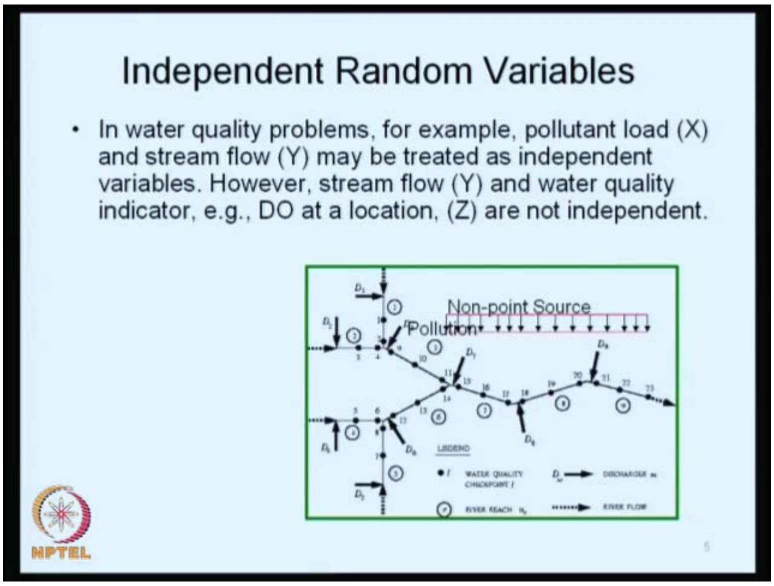

So, the pollutant load and stream flow may be treated as independent variables, because pollutant load is essentially from anthropogenic activities although it may be a random variable and the stream flow is essentially a natural process. So, the pollutant load which is one of the random variables x and the stream flow which is other random variable y may be treated as independent variables however stream flow and water quality indicator let us say we are interested in the desired oxygen at a particular location here. Now the water quality indicator at this point cannot be treated as independent of the stream flow, because it is directly influenced by the stream flow.

So, this is how first we need to make a case for whether the two random variables can be assumed to be independent of each other once you have a physical base physical basis on which of you may make the assumption that yes the two random variables can be assumed to be independent. Then we go on to examine whether stochastically we can treat them as independent or not. Now when we say two random variables are independent what does this mean? this means that the distribution of one of the random variables, let say x is in no way influenced by the distribution of the other random variables. So, if you look at the conditional distribution conditional density let say g of x given y conditional density of x conditioned on y taking on a certain value y will be the same as the marginal density of x because that the density of x is not influenced at all by the conditions placed on y if the two random variables are independent.

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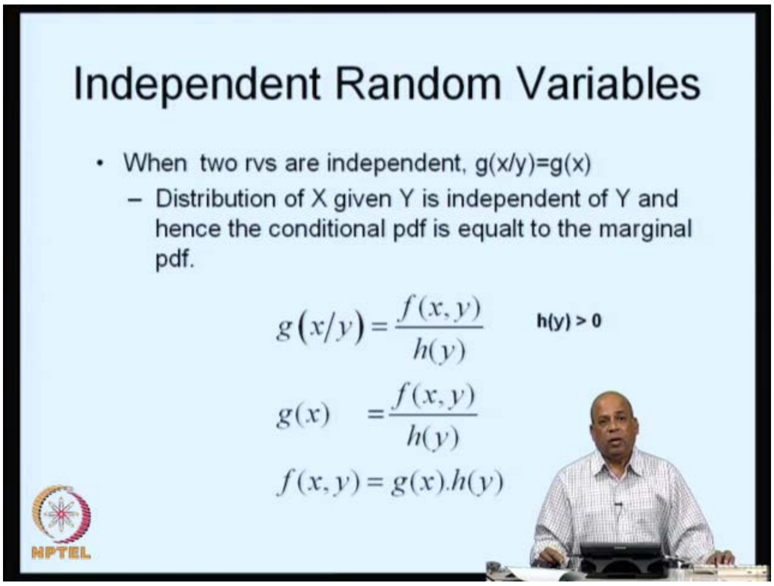

And therefore, we can write the conditional density as equal to the marginal density itself and from the definition of the conditional density recall that g of x given y is given by the joint density f of x given y by the marginal density h of y for h of y strictly positive values now because the two random variables are independent g of x given y will be the same as g of x and therefore, we write g of x is equal to f of x comma y which is a joint density function divided by the marginal density of y. And form this we write f of x comma y will be equal to g of x into h of y now this is both the necessary as well as sufficient condition for the two random variables to be independent. So, we state more formally that the two random variables x and y are stochastically independent, if and only if the joint density f of x comma y is given by the product of the marginal densities g of x and h of y.

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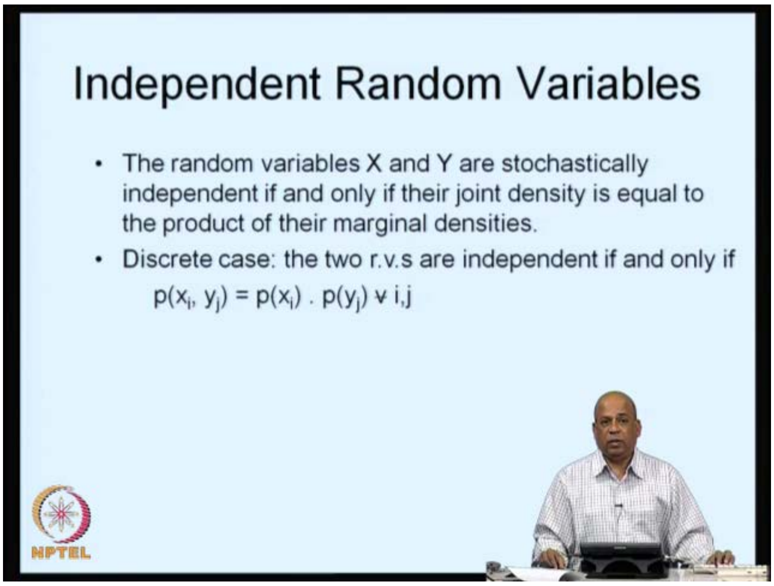

Now, in the case of discrete random variables the two random variables are independent. If and only, if the probability x i comma y j which is a joint probability mass function is equal to probability of x i into probability of y j for all i and j, which is stating in words that joint probability mass function must be equal to the product of the marginal probability mass function e of x i into p of y j the product of the two marginal probability mass functions.

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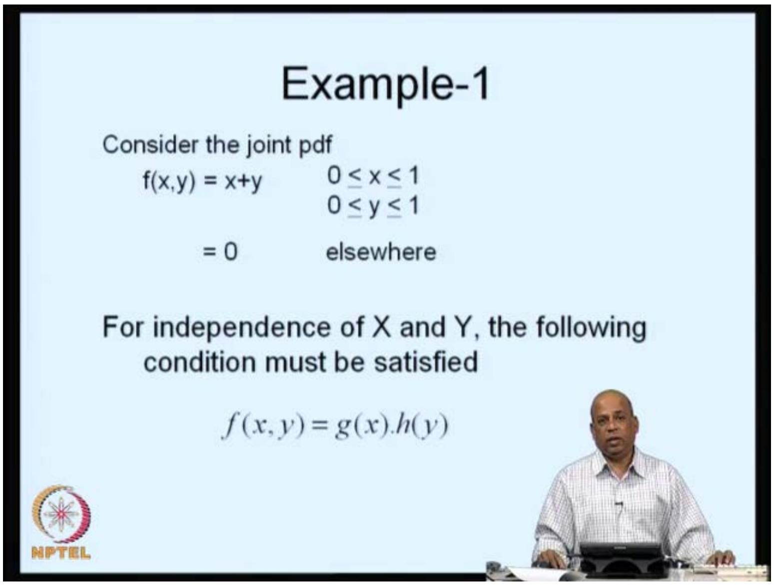

Let us see an example let us say f of x y that is a joint density function is given by x plus y for these limits x varying between 0 and 1 and y varying between 0 and 1 now for independence of x and y we have seen that the condition that needs to be satisfied is f of x y must be equal to g of x into h of y. So, let us first determine g of x recall that the marginal density g of x is integral of the joint density over y with respect to y.

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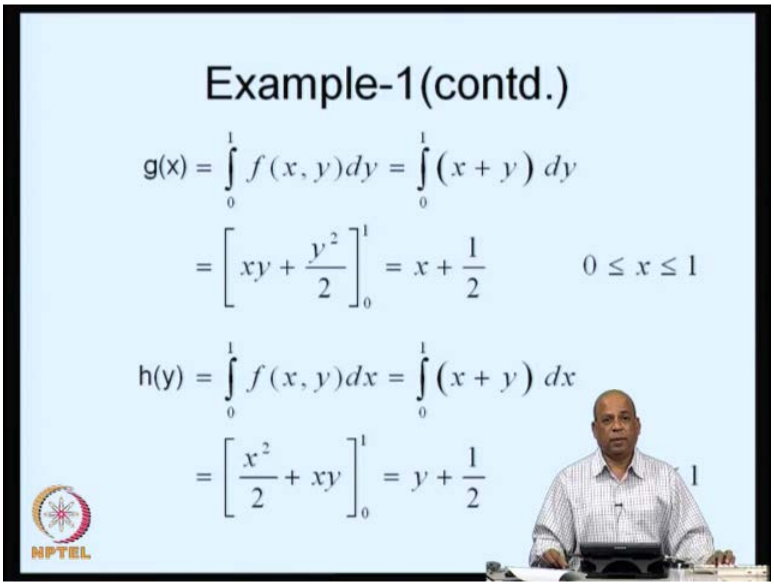

So, when we integrate this these are the limits for y and then when we integrate this you get x plus 1 by 2 as the marginal density of x, similarly marginal density of y is h of y you integrate over x the joint density function and you will get y plus 1 by 2. This is valid over the region 0 to 1 x varying between 0 to 1, 0 and 1 and y varying between 0 and 1 for h of y. Now, your f of x y was x plus y and you got g of x is equal to x plus 1 by 2 and h of y is equal to y plus 1 by 2.

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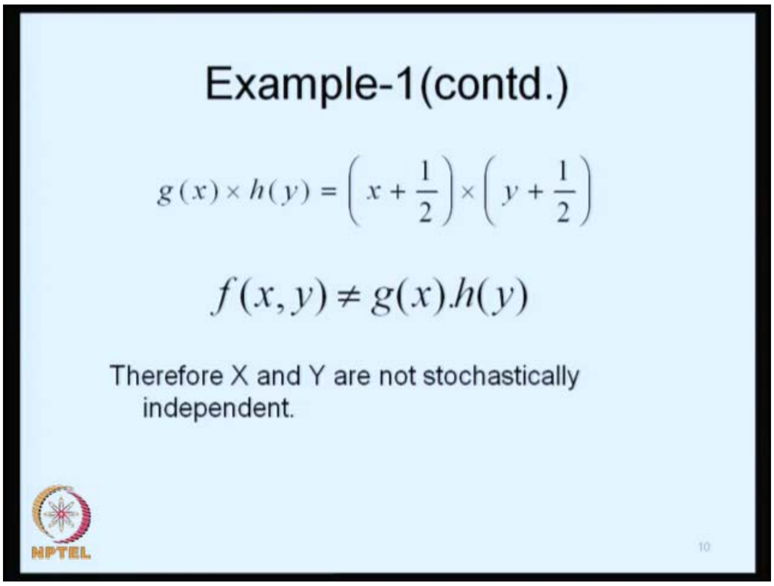

Obviously, this means that f of x y is not equal to g of x into h of y, and therefore x and y are not stochastically independent.So, to summarize them first you must have a physical basis to at least suspect that the two random variables can be taken as stochastically independent can be taken as independent, and then you examine for the stochastic independence by checking, if the joint density function f of x comma y is in fact equal to the marginal density product of the marginal densities g of x and h of y.

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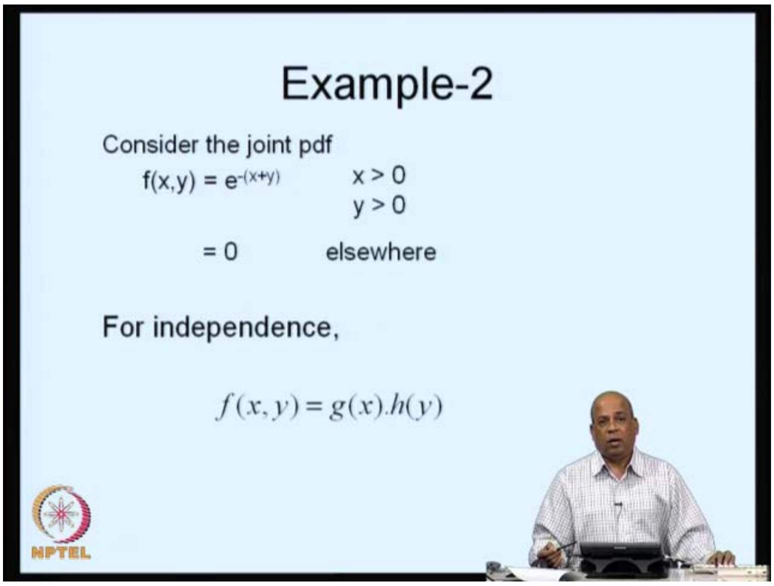

Look at this example, f of x y is equal to e to the power minus x plus y for x and y both positive values. Now we will check again the condition f of x y is equal to g of x into h of y. So, first you obtain g of x again integrate the joint density over y 0 to infinity. So, from here you get g of x is equal to e to the power minus x, and x greater than 0.

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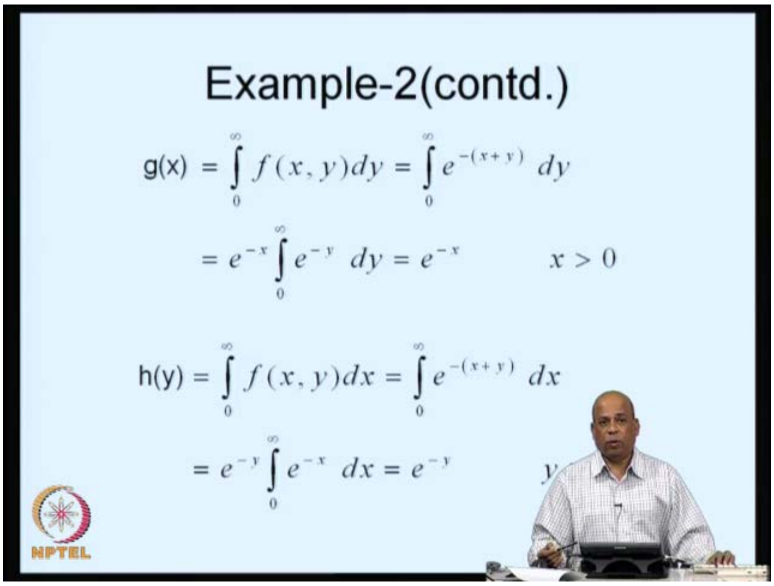

Similarly, h of y you for getting h of y you integrate the joint density over x with respect to x. So, you will get h of y is equal to e to the power minus y for positive values of y. So, you get g of x into h of y is equal to e to the power minus x into e to the power minus y which is e to power minus x plus y and therefore, f of x comma y is equal to g of x into h of y and therefore, x and y are stochastically independent.

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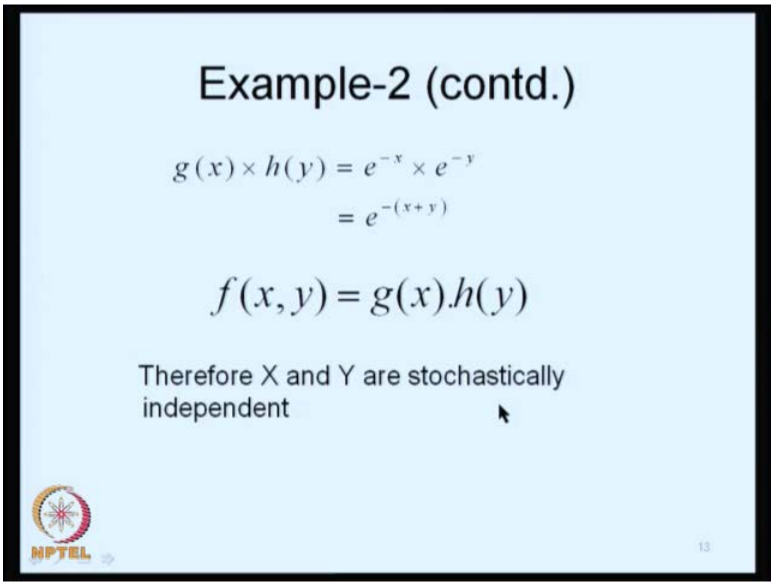

So, the independence assumptions always implies that the joint density of the two random variables f of x comma y is a is equal to the product of a marginal densities f of x g of x. So, f of x comma y must be equal to g of x into h of y, now we will go on to determining the distributions of functions of random variables. Now these situations arise more quite often in hydrology where we are interested in getting the distributions of functions of random variable, let say there is a stream here and there is another stream here the stream flow here is denoted by the random variable x and the stream flow here is denoted by the random variable y.

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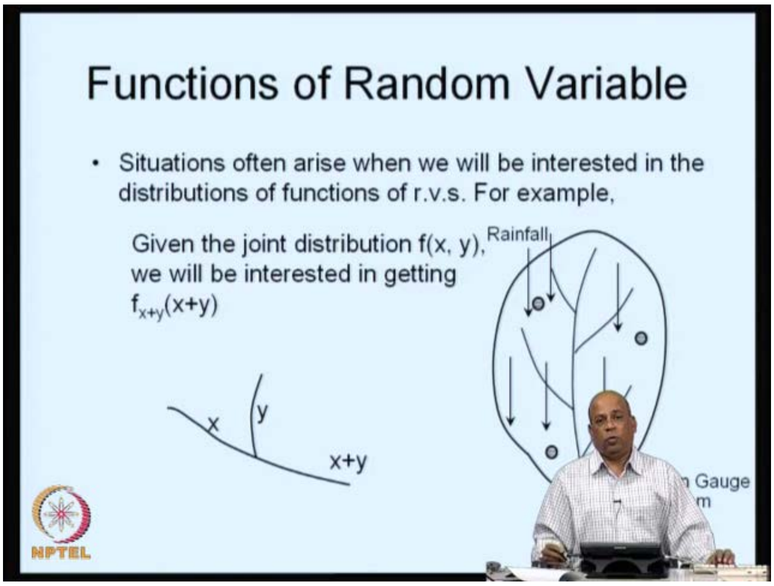

And we may be interested in the distribution of x plus y which is the combined stream flow that is coming down stream of this location. So, if you know the density of x and the density of y how do we determine the density of x plus y that is a question here. Similarly you look at another example where there is a rainfall occurring in the catchment area and then there is a associated stream flow here now we may have a functional relationship between the stream flow at this point and the rainfall in the catchment area let say we express this stream flow y as a function of x let say y is equal to x square minus four x plus one or some function and we may have the density function or we may have estimated the probability densities of the rainfall in the catchment area and we would be interested in getting the probability densities of the stream flow at this location. So, such situations lead to the derived densities for the functions of random variables. We will see how this can be obtained. So, if you have x as a discrete random variable which means that x can take on only finite number of values or accountably infinite number of values and you define a function y is equal to h of x. So, because x takes only discrete values y also takes on only discrete values.

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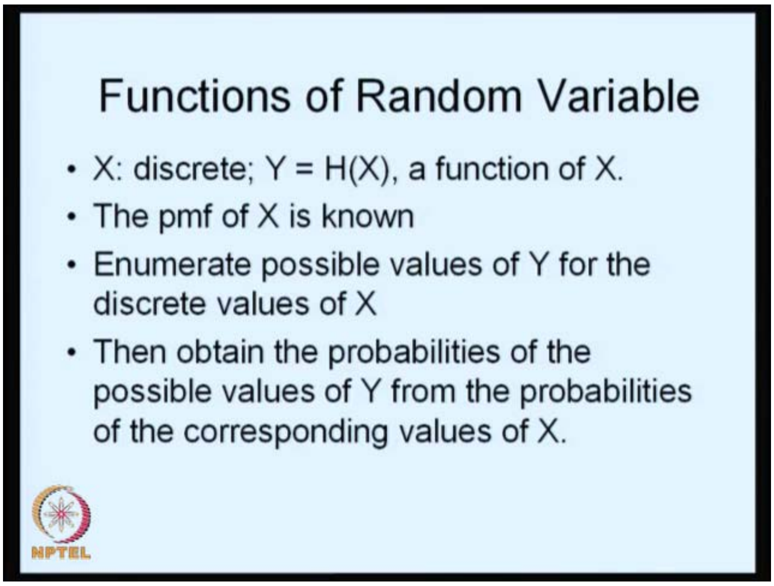

And we know the probability mass function of x. So, in such a situation what we do is that associated with each of the discrete value of x you obtain the discrete value of y and then knowing the probability with which that particular discrete value of x occurs the corresponding probabilities of y can be directly obtained now it is possible that the same value of y occurs for different discrete value of x in which case that particular value of y can occur for x is equal to x 1 x is equal to x 2 and. So, on. And the associated probabilities need to be added up. So, in the case of discrete random variables when you have a function of the discrete random variable the procedure is fairly straight forward you simply innumerate all the possible values of y for the discrete values of x and then we obtain the probabilities of all this possible values of y from the probabilities of the corresponding values of x.

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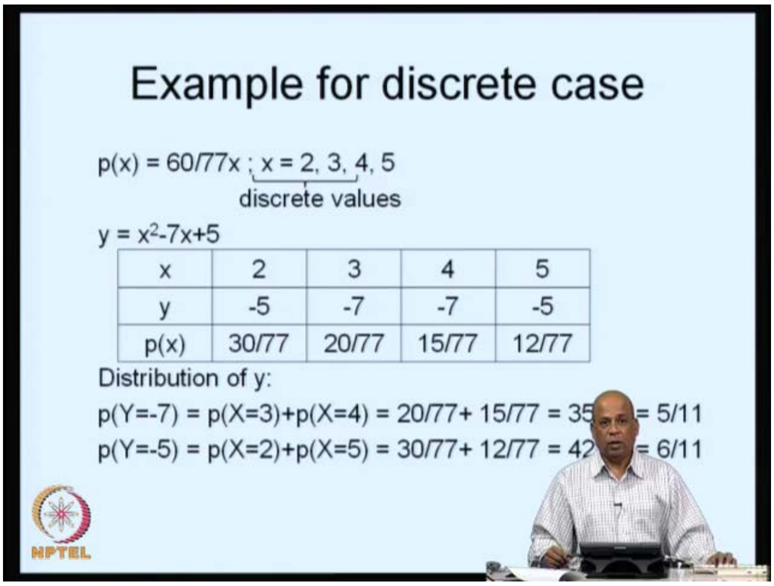

See for example, we have the probability mass function p of x is equal to 60 divided by 77 x takes on value 2, 3, 4 and 5. So, x takes on only the discrete values and we define a function y is equal to x square minus 7 x plus 5. So, knowing the probability mass function p of x we are interested in getting the probability mass function of the random variable y. So, what? we do is that associated with each of the discrete when values that x can take namely 2, 3, 4 and 5 you enumerate the values that can take for example, when x is equal to 2 y will be equal to minus 5 substituting in this expression here. Similarly x is equal to 3 leads to y is equal to minus 7 x is equal to 4 leads to minus 7 and x is equal to 5 leads to minus 5. So, y can take on only two values namely minus 5 and minus 7. So, once we define the probability of y is equal to minus 5 and probability of y is equal to minus 7 we would have defined the probability mass function for y. So, let us see what for what values of x does y take on value of minus 5 let us say now y is equal to minus 5 occurs for value of x is equal to 2 and x is equal to 5. So, probability of y is equal to minus 5 will be equal to probability of x is equal to 2 plus probability of x is equal to 5 which turns out to be 6 by 11 as you can see here.

Similarly, the value of minus 7 y takes on a value of minus 7 when x takes on a value of 3 or x takes on a value of 4 and therefore, the probability of y is equal to minus 7 turns out to be probability of x is equal to 3 plus probability of x is equal to 4 which will be 5 divided by 11 note that, because it is a probability mass function the sum of all the probability that you thus obtain should be equal to one which in this particular case it is. So, in the case of discrete random variables I repeat that you simple enumerate the possible values of y which is a function of x using the functional relationship knowing the discrete values that the x. The random variable x can take associated with each of the random variable discrete value of the random variable x, you enumerate the value that the random variable y can take and then associate probabilities for each of the values that the random variable y can take.

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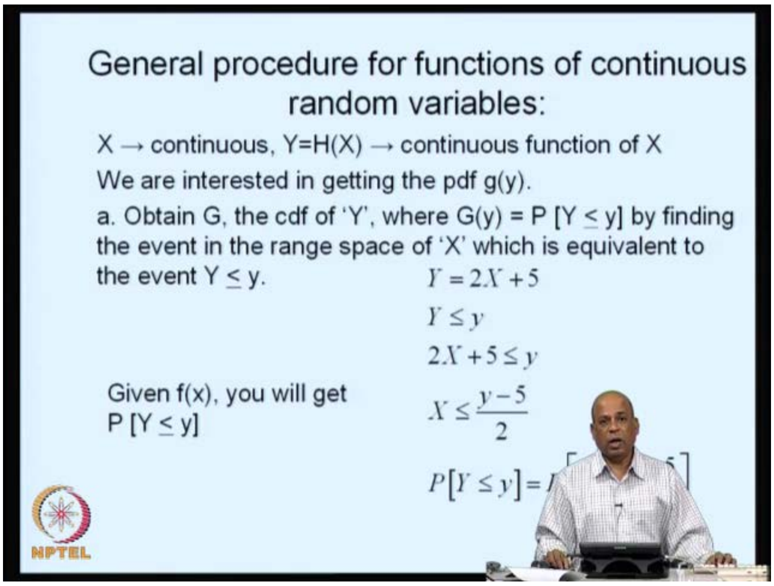

Now, we will go to the case of continuous random variable let say you have the random variable x as the continuous random variable and you define a function y is equal to h of x which is a continuous function of x and we are interested in getting the pdf of y which is the marginal pdf g of y what? we will do here is first we obtain the cdf the cumulative distribution function of y where from definition g of y is equal to probability y being less than equal to y. How do we define this you know the functional nature y is equal to h of x and on this we identify in the range space of x we identify the event corresponding to y less than equal to y for example, let us say y is equal to 2 x plus 5 some simple linear function will take and we are interested in the event y being less than equal to y remember the small y indicates the value of that the y takes and therefore, we can write this as 2 x plus 5 is less than equal to y, y is a values and from this we write x is less than equal to y minus 5 divided by 2 . So, the event y being less than or equal to y is now written as x being less than equal to y minus 5 by 2 and then we can get the probability probabilityof y being less than equal to y as probability of x being less than equal to y minus phi by 2, which we can obtain because you know the probability density function of x. So, form the probability density function f of x we will get probability of y being less than equal to y. Now this defines the cdf of y which is capital g of y once we know the cdf of y you can obtain the pdf of y how do we do that we simply differentiate the cdf with respect to y to get the associated pdf.

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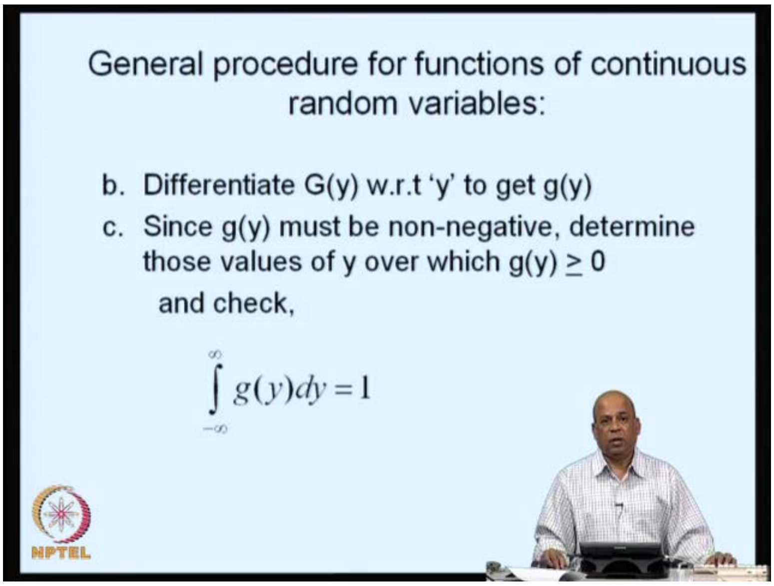

So, we obtain the g of y which is the marginal density of y from the cdf g of y by differentiating with respect to y then we have to examine whether this g of y in facts satisfies all the conditions of a pdf and this must be non negative and also the integral over the entire region must be equal to 1. So, we choose the limits of y form the functional relationships and the associated limits of x we choose those limits of y over which g of y thus obtained is non negative and then we examine over that particular range whether the integral of g of y is equal to 1. So, the procedure is simple that from the range space x we obtain an equivalent event of probability of y being less than the equivalent event of y being less than or equal to y and then get the cdf of y which is G of Y differentiate the G of Y which is a cdf to obtain the marginal density g of y once we obtain the marginal density is g of y we will fix the limits of y or which this is g of y is non negative. And then over those limits we will examine whether the integral minus infinity to plus infinity g of y d y is. In fact, equal to 1. Let us take an example let us say f of x is equal to x by 2 over this region x varying between 0 and 2 and we define a simple function h x h of x is equal to 4 x plus 1 or we define y is equal to 4 x plus 1. So, first we get and we are interested in getting the pdf of y, y is equal to h of x.

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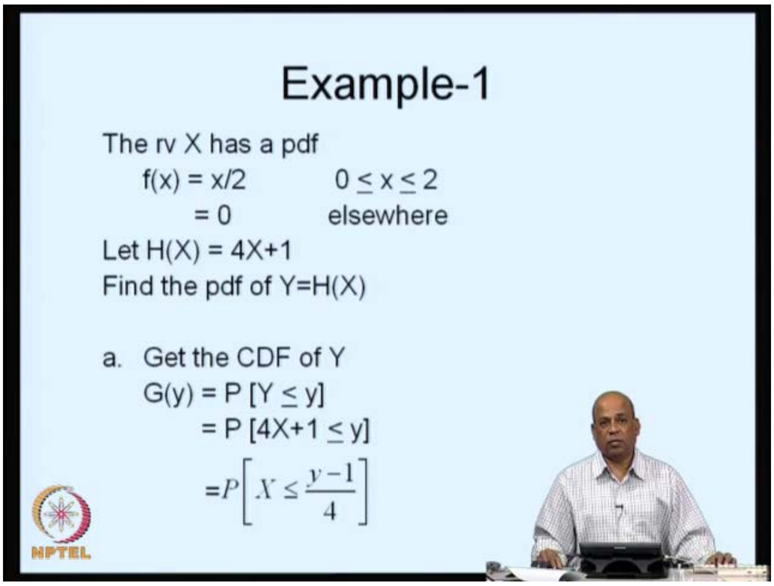

So, first we will obtain the cdf of y form the definition g of y is equal to probability of y being less than equal to y and what is y? y is 4 x plus 1. So, 4 x plus 1 is less than equal to y from which we write probability of this is equal to probability of x being less than or equal to y minus 1 divided by 4, now you know the f of x which is the density function of x and you are interested in getting the probability of x being less than or equal to y minus 1 by 4 and x varies from 0 to 2.

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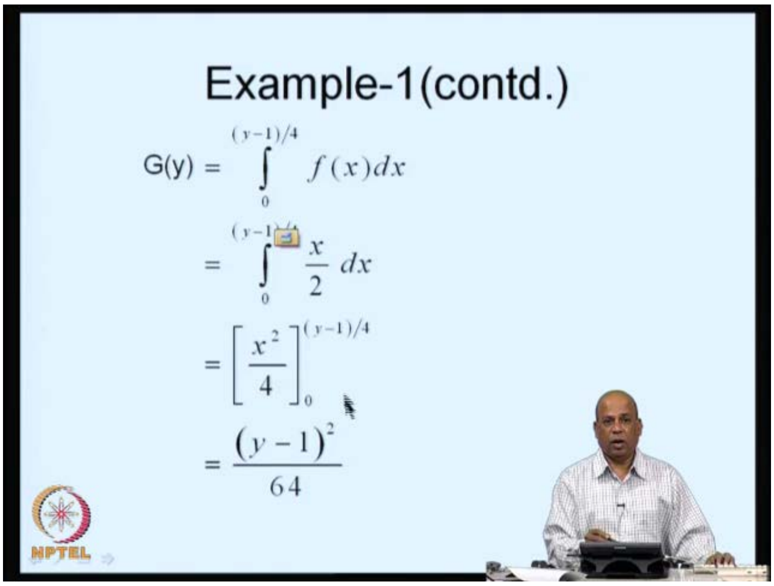

So, we integrate the pdf f of x between 0 and y minus 1 by 4 to obtain the G of Y. So, we obtain the G of Y by this procedure as y minus 1 the whole square divided by 64, now we differentiate this to obtain the g of y which is the marginal density of y. So, marginal density of y is equal to the differential of the cdf with respect to y and that we obtain as y minus 1 divided by 32.

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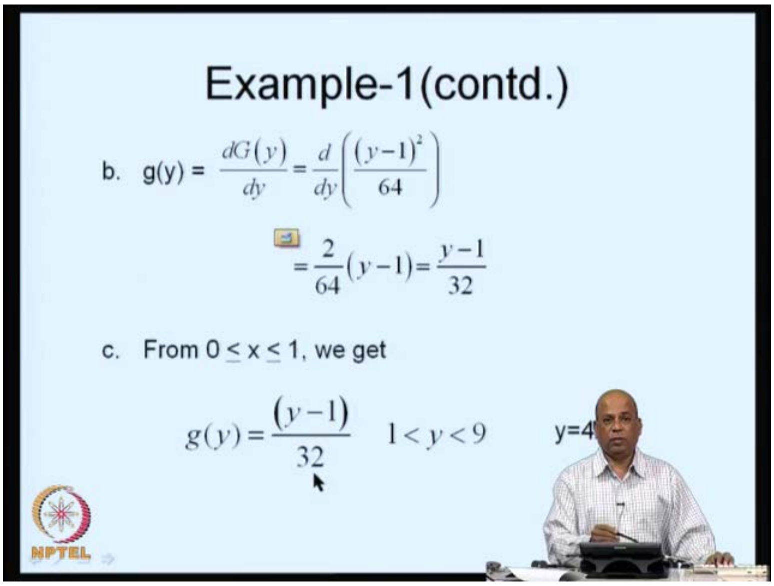

So, our original variable x takes on values between 0 and 1 and y is equal to 4 x plus 1 is a function from this we get the limits of y as y varying between 1 and 9 verify that within this region where y takes on values between 1 and 9 the function g of y is non negative you should put y is equal to 1 you get a 0 value, if you get if you put y is equal to 9 you get a non negative value here and therefore, this g of y is non negative over the entire region as obtained here.

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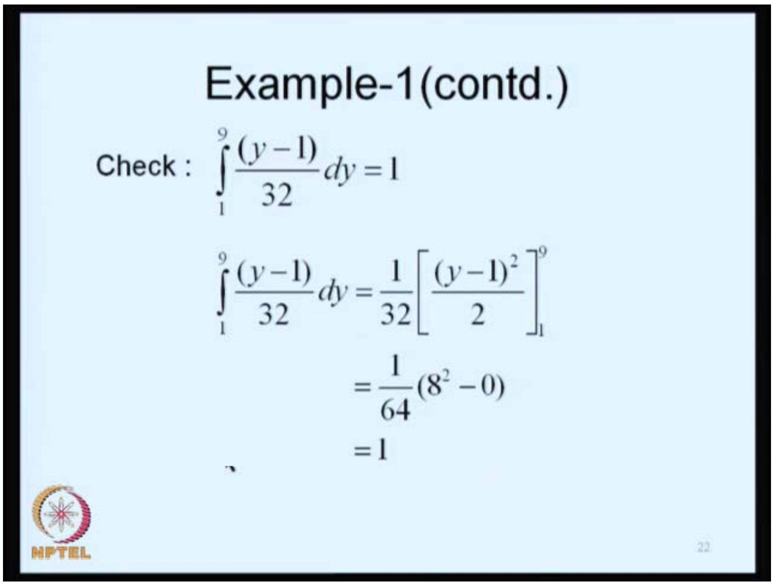

Then we will also verify that the pdf. So, obtained g of y the integral of that with respect to y over this entire region must be equal to 1. So, this we verify and that is integral of g of y with respect to y over the region 1 to 9 must be equal to 1, which in this case turns out to be true. So, in the case of continuous random variables we first obtain the G of Y differentiate that to get the g of y which is a marginal density of y and then obtain the limits of y over which the g of y thus obtained is non negative and then we verify that the integral of g of y over this region that we have identified is in fact, equal to 1.

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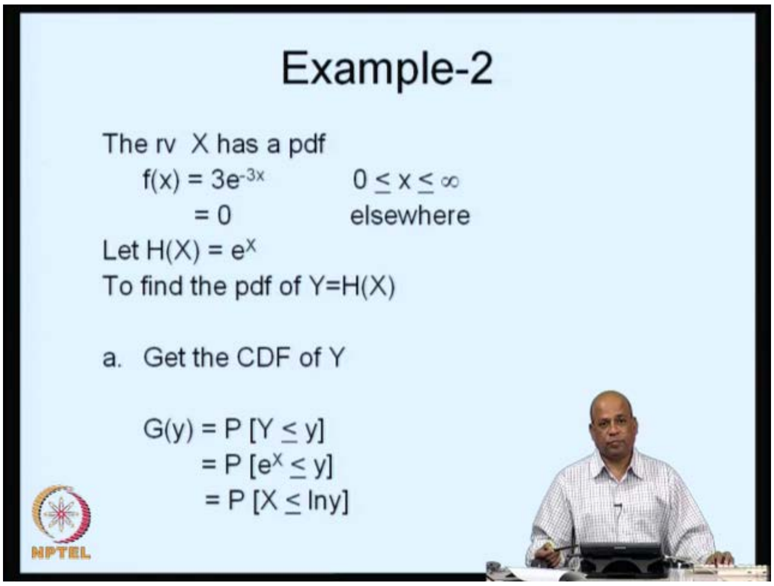

Let’s take another small simple example this is in fact, the exponential distribution lambda e to the power of lambda x which will introduce later on. So, we take f of x is equal to 3 e to the power minus 3 x where x is greater than 0 and we take h of x is equal to e to the power x this is x is positive here. So, h of x is equal to e to the power x and we are interested in getting y is equal to h of x getting the pdf getting the pdf of y is equal to h of x. So, first as did in the previous example first we obtain the cdf of y by identifying the event y is less than or equal to y the equivalent event of y is equal less than or equal to y in the range space of x. So, this we write it as e to the power x e to the power of x less than or equal to y and by taking the logs we write this as probability of x being less than equal to log y.

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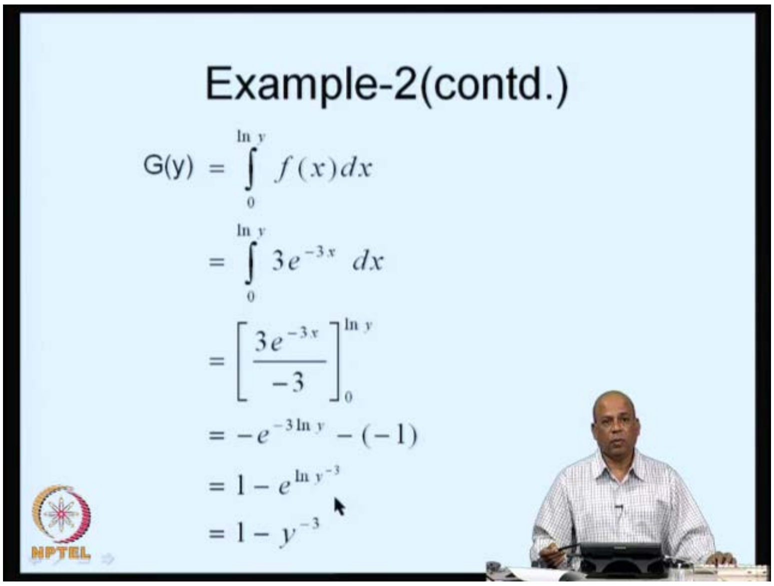

So, g of y then is probability of x being less than or equal to log y and therefore, you integrate between 0 and log y the f of x that you have here. So, we integrate between 0 and log y the marginal pdf of x to obtain the G of Y this is straightforward integration putting the limits. So, we obtain G of Y is equal to 1 minus y to the power of minus 3.

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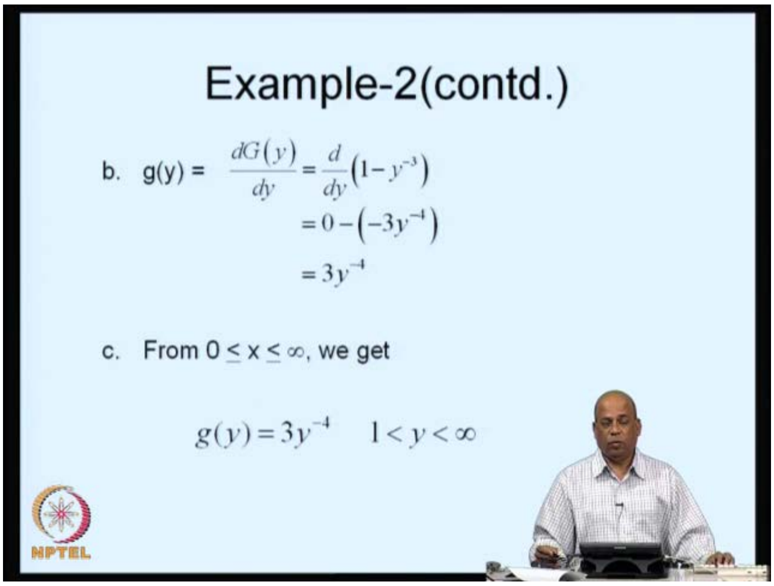

Now, we differentiate this to obtain the marginal pdf of y. So, when we differentiate this you get the pdf as 3 y to the power minus 4 then from the limit that x is positive you get g of y is equal to3 y to the power minus 4 in the range between y is equal to 1 and y is equal to infinity. I introduce this example with a purpose which we will see presently. So, to repeat again for the continuous variable we obtain first the cdf of y which is a function of x and the pdf of x is known once you obtain the cdf of y you get the pdf of y by differentiating the cdf identify the limits over which this pdf is non negative and then verify that the integral of the pdf over this region that you have just indentified is. In fact, equal to 1.

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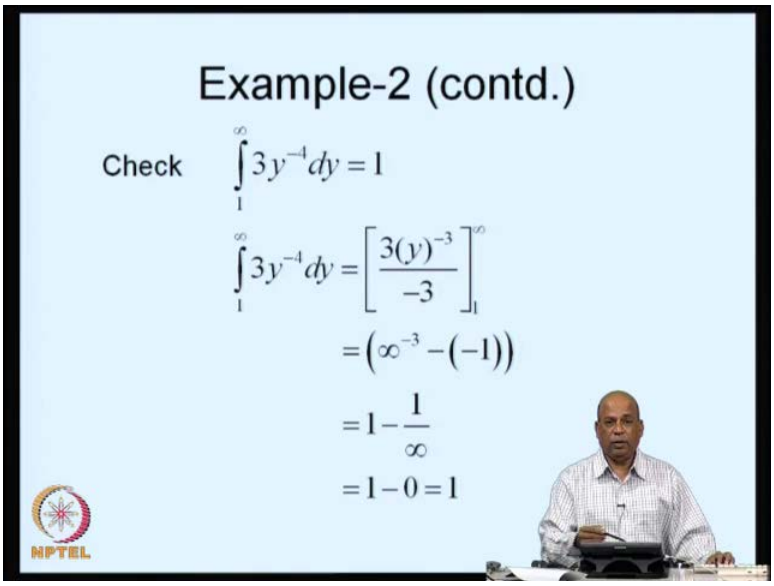

Now, this can be more generalized as we will presently see, but this slide shows that the pdf that we obtain three by 2 to the power minus 4 is satisfies the condition that over the region that you have indentified 1 to infinity the integral must be equal to 1, now we will go to some generalization in the case of monotonic functions the examples that we just did and the procedure that we just introduced can be generalized and we can write this in a quite an elegant fashion.

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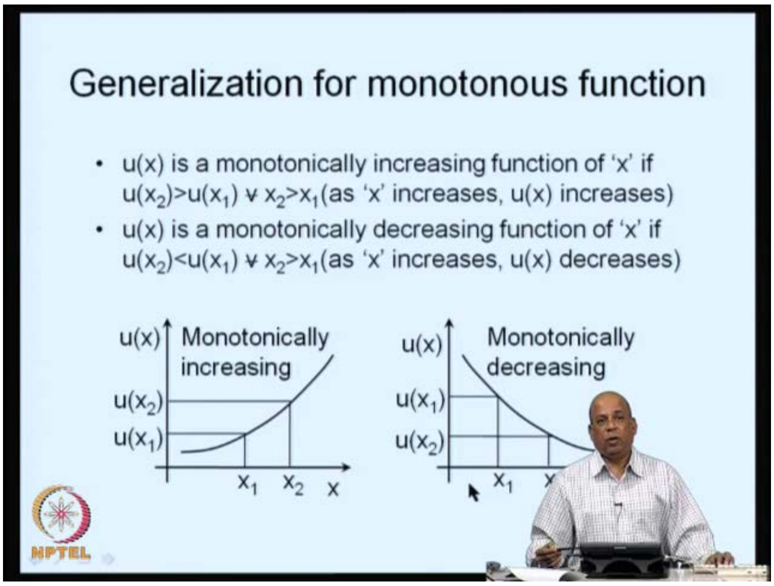

First we will see what are the monotonic functions we say u x is a monotonically increasing function of x if as x increases u of x increases for example, you take a function like this now x 1 and x 2 are here. So, x 2 is greater than x 1 your function corresponding to x 1 is u x 1 function value corresponding to x 2 is u x 2. So, u x 2 u of x 2 is greater than u of x 1 as it is in this particular case for all values of x 1 greater than or equal to x 2 in such a case we call this function as a monotonically increasing function, similarly for a monotonically decreasing function we have the condition that as x increases the function value decreases. So, we write this as u x 2 less than u x 1 for all x 2 greater than x 1. So, as x increases here the function value decreases.

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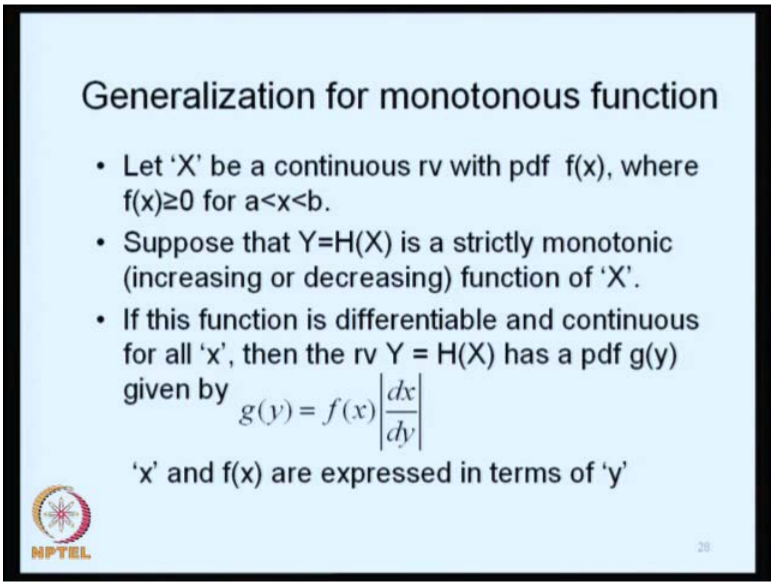

Now, when we have monotonically increasing or decreasing functions the procedure for obtaining the pdf of y which is a monotonically increasing or decreasing function of x can be simplified and we write it as follows suppose that y is equal to h of x is a strictly monotonic function it may be either increasing or decreasing function of x and we know the function that the pdf of x then the pdf g of y can be straight away written as equal to f of x into the absolute value of dx by d y which is the differential of x with respect to y. We are talking about a single random variable and a function of a single random variable. Now in this case both x and f of x we express in terms of y we have the functional relationship here y is equal to h of x from which we can express x in terms of y and then that we put in the pdf f of x and therefore, we express pdf f of x also in terms of y. So, we express both x and f x f of x in terms of y and then obtain g of y is equal to f of x which is a function of y and then absolute value of dx by d y by differentiating x which was expressed in terms of y with respect to y. We will not go into the proof of this, but we will take this result and then start applying to numerical examples let say that you take again the same example that we just explained f of x is equal to 3 e to the power minus three x as you can see and then we define a function h of x is equal to e to the power x.

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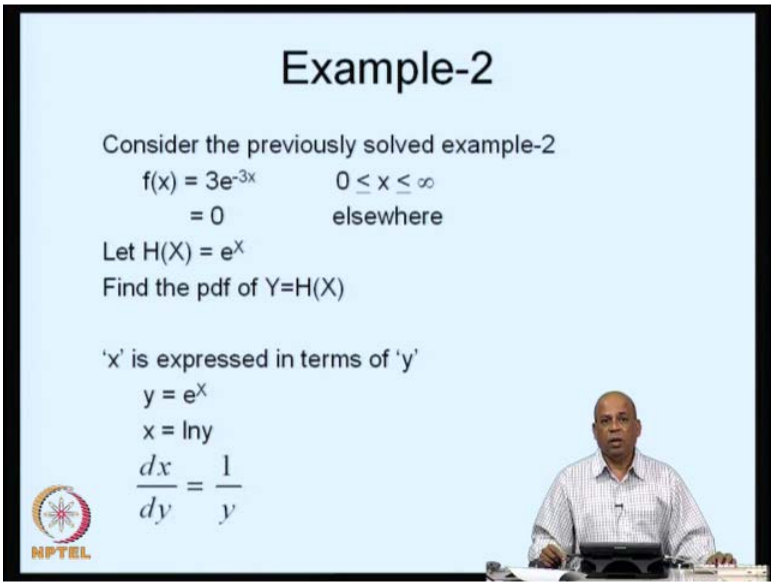

Remember here when we said monotonically increasing or decreasing function we were talking about the transformation the pdf of which you are interested in. So, we must examine for the monotonicity of the function h of x and not f of x. So, y is equal to h of x is a function in which you are interested in. So, y is equal to h of x which is e to the power of x for x greater than 0 and this is a monotonically increasing function as you can verify. So, first we will express x in terms of y, y is equal to e to the power of x and therefore, x is equal to log y and then we get dx by d y is equal to 1 over y. Similarly we have to express f of x also in terms of y, y is equal to e to the power x. So, we express f of x as y is equal to e to the power minus 3 x and then x is log y from which you get f of x is equal to 3 y to the power minus 3.

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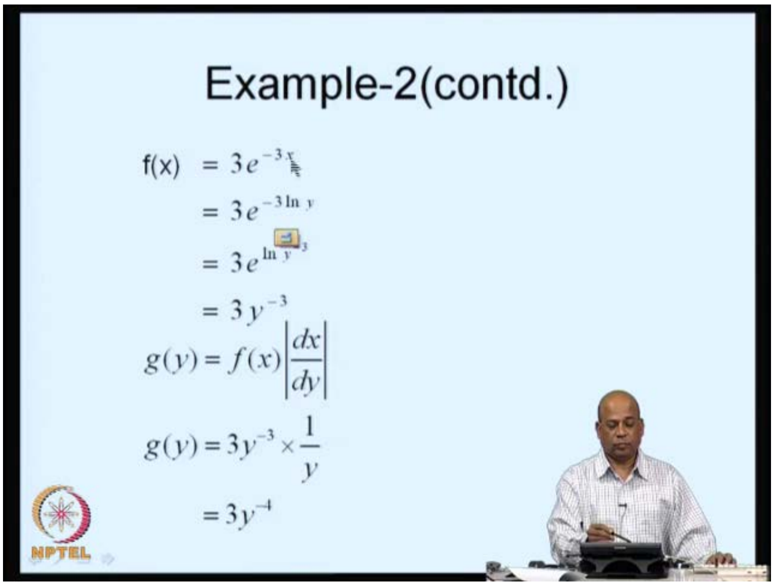

Because the transformation y is equal to h of x e is a monotonically increasing function we use this expression, because it is a monotonic function whether it is increasing or decreasing, but in this particular case it is increasing function g of y is equal to f of x absolute value of dx by d y and therefore, you get g of y is equal to 3 y is equal to power minus 4.

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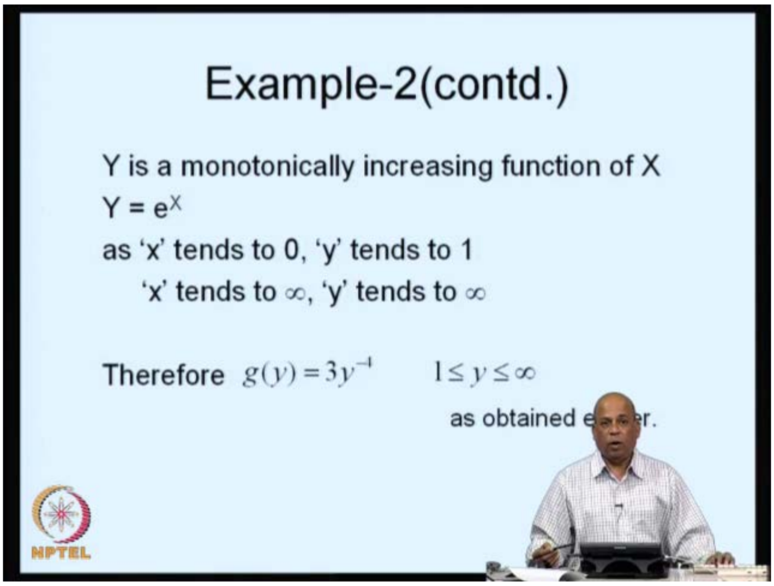

We obtain the relationship or the range for y as here the range for y will be h of b less than equal to y less than equal to h of a if it is a monotonically decreasing function and h of a less than equal to y less than equal to h of b if it is a monotonically increasing function. Where h of a and h of b are obtain from these values of a and b this can be verified here. So, we obtain the limits as y going between one and infinity as was obtain in the example discussed earlier.

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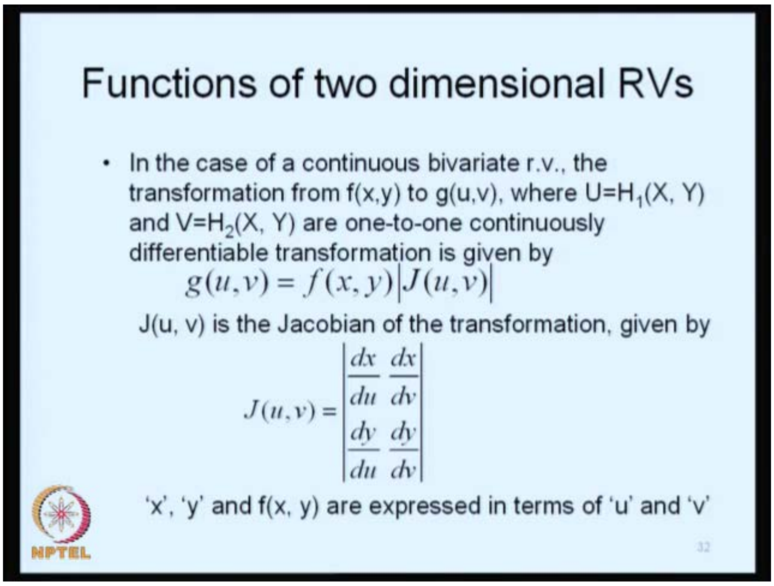

The same concept we extent to functions of two dimensional random variablesso in the case of a continuous bivariate random variable we have the joint density f of x y and we are interested in getting the joint density g of u comma v where u is a continuous 1 to 1 function of x comma y and v is another continuous 1 to 1 function h 2 of x comma y. So, u and v are both functions of x and y in such a case the joint density g of u comma v is given by f of x y into the determinant j of u comma v where j u v is the jacobian of the transformation this is the absolute value of jacobian the jacobian u comma v is given by dx by d u dx by d v and d y by du and d y by d v the determinant of that. So, this is the jacobian of the transformation we take the absolute value of the jacobian and then write g of u comma v as equal to f of x comma y into the absolute value of jacobian u comma v. Again as we did earlier x y and f of x y are all expressed in terms of u and v .So, we use the functional transformations here u is equal to h 1 of x comma y and v is equal to h 2 x comma y and express x y and f of x y as they are expressed in terms of u comma v u and v. So, we will take this example to drive home this point f of x y which is a joint density function is given by 3 by 2 x square plus y square for the range x varying between 0 and 1 and y varying between 0 and 1.

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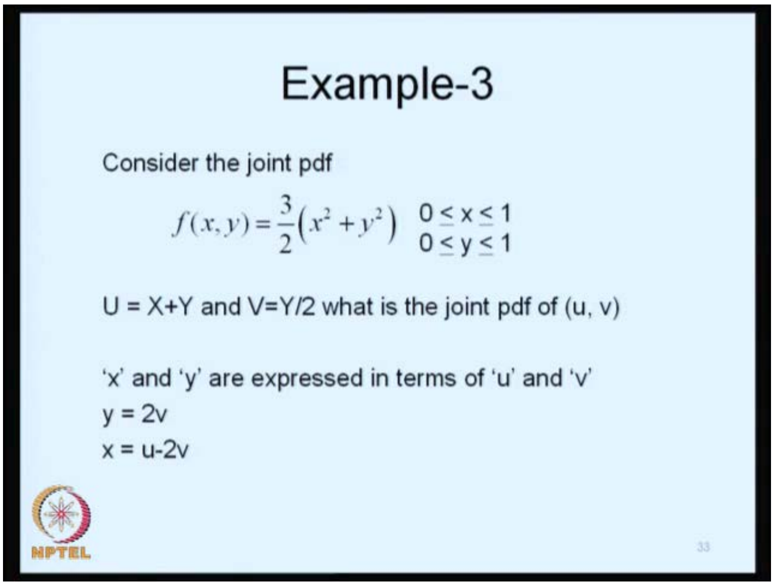

We will define the functions u is equal to x plus y and v is equal to y by 2 and we are interested in obtaining the joint density u comma v remember here both u and v are functions of x and y although v does not contain the x term, but in general u and v can be functions of x and y. So, first we will express x and y in terms of u and v. So, from here v is equal to y by 2 we write y is equal to 2 v and u is equal to x plus y we write x is equal to u minus 2 v because this is y is equal to 2 v. So, we write x is equal to u minus 2 v. So, once we express x and y in terms of u and v we substitute these in the expression f of x y and express f of x y again in terms of u and v. So, we write f of x y in terms of u and v by looking at f of x y is equal to 3 by 2 x square plus y square and therefore, you express f of x y as 3 by 2 u square minus 4 u v plus 8 v square then we get the jacobian.

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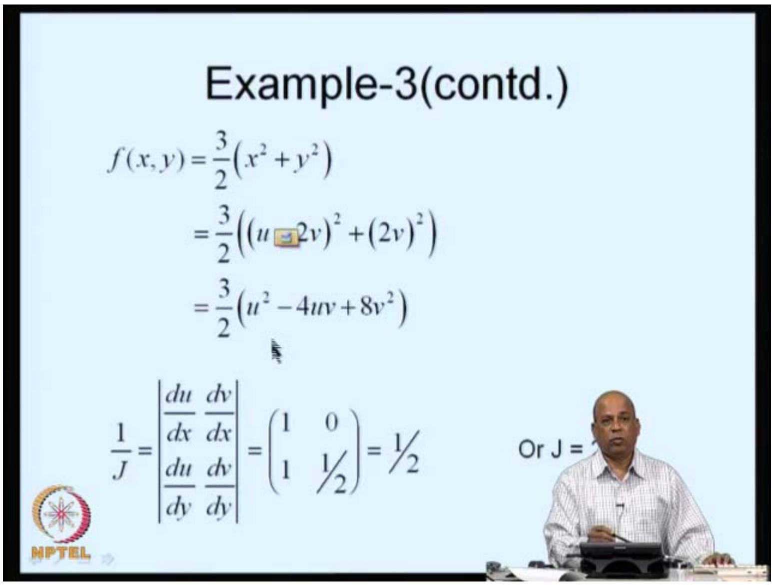

Now, it is always convenient in most situation it is convenient to get 1 by j rather than j as it is in this particular case. So, 1 by j we express it as d u by dx d v dx d u by d y d v by d y and from your expression for u and v you can obtain this differentials. So, 1 by j d u by dx transfer to be 1 d v by dx transfer to be 0, because v is equal to y by 2 and d u by d y is equal to 1 and d v by d y is equal to half. So, from this the determinant this is a determinant here will be equal to 1 by 2 and therefore, j transfer to be 2. So, we have expressed f of x y in terms of u and v and you have obtained the jacobian which is 2 and therefore, you get the joint density of u and v as f of x y into absolute value of the jacobian which will be equal to 3 by 2 this is f of x y here multiplied by 2 which is a jacobian.

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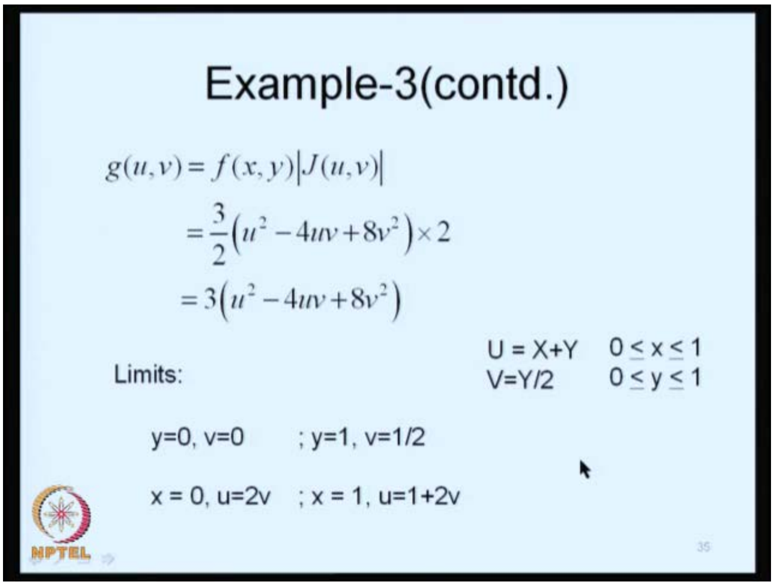

Now, we need to fix the limits these limits are fixed from the fact that u is equal to x plus y and v is equal to y by 2 and x takes on values between 0 and 1 and y takes on values between 0 and 1 .So, we fix the limits as y when y is equal to 0 v will be equal to 0 and when y is equal to 1 v is equal to 1 by 2 and x when x is equal to 0 u is equal to 2 v look at this u is equal to 2 v x is equal to 0. So, therefore, u is equal to y which is equal to 2 v and when x is equal to 1 u will be equal to 1 plus 2 v.

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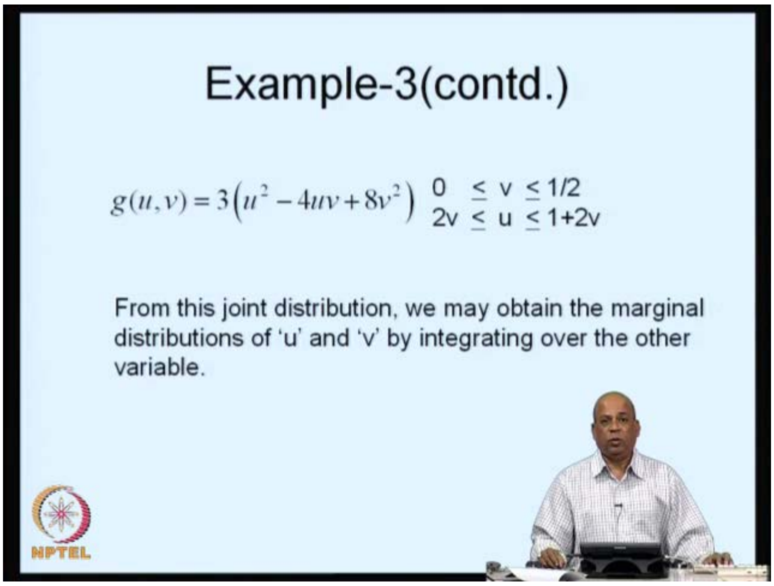

So, the limits for this are expressed as u going v going between 0 and ½ and u going between 2 v and 1 plus 2 v. Now, once we obtain the joint distribution we can now obtain the marginal distributions of u and v by integrating over the other variable as we did in the previous class.

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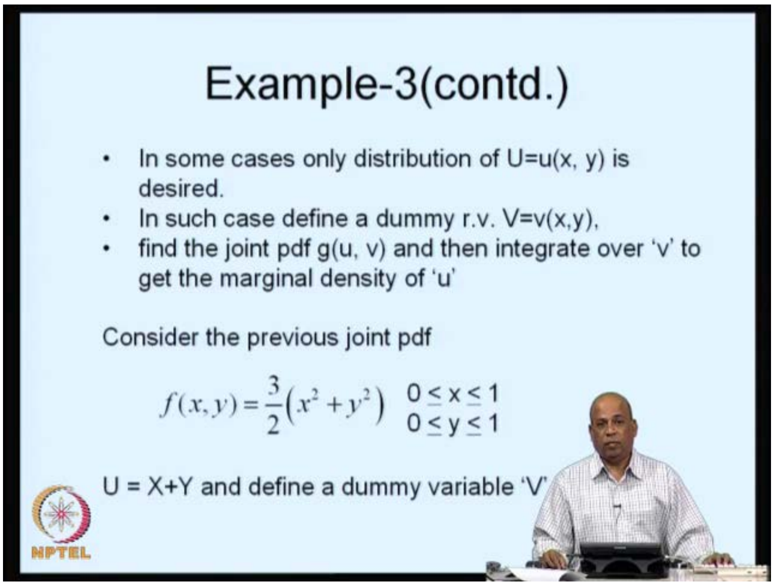

Now, in some cases we are given the joint density function, but we are given only one function, let say that you have f of x y which is a joint density function of x and y and you are interested in the distribution of u is equal to u of x comma y that is no other v in such case we introduce a dummy variable v is equal to v of x comma y some convenient variable we introduce convenient function we introduce and then obtain follow the same procedure as we did just now for the two functions u and v and obtain the density function the first the joint density function g of u comma v and then we integrate over v to obtain the density of u. So, when we are given only one function you can still follow the same procedure as we followed for two functions by artificially introducing another function which is a convenient function. So, that it does not introduce any complexities in your computations and then obtain the joint density function by following the same procedure once you obtain the joint density function you integrate over v and then the marginal density function for example, you take the same pdf as in the previous example, but instead of having two functions we are interested in only u is equal to x plus y, let say we are interested in the density function of u where u is equal to x plus y then we define a dummy variable as v is equal to y and obtain the joint density function of g of e comma v and by integrating over the density function y.

So,. So, far what? we studied in this course is given a random variable we know how to define the density function and then we have gone on to bivariate random variables and introduce the concept of concepts of joint joint density functions the marginal density function and the conditional density functions then we introduce the concept of independent random variable which we said the independence is it always ensures that the joint density function f of x comma y is a product of the marginal density functions g of x and in and h of a then we looked at functions of random variables and saw how we define the density functions of functions of random variables. Let say y is equal to a h of x is a function and we know the pdf of x we know now how to determine the density function of y this we extended also to monotonically increasing and decreasing functions where we generalize the procedure then we introduce functions of two random variables x comma y are the two random variables x and y are the two random variables and we are talking about the functions h 1 of x comma y and h 2 of x comma y and then we saw how we determine the joint densities of these functions of random variables. Now with this background now we will see we will slightly go to slightly different topic which in which we will start talking about moments of the descriptions to understand the moments first we will understand basic definitions we will just go through some basic definitions. You have a random variable x and it can take on assume certain values now the collection of all possible values that this random variable can assume is called as a population. So, we will introduce these concepts first.

Let's say we are talking about a page being randomly picked up from a thick book consisting of let us say 1000 pages. So, a page that is randomly picked off is a page in the book is the random variable what are all the possible values that the random variable can assume? it is the entire set of the pages let us say 1 to 1000. So, the random variable can take on any value between 1 to 1000, now the set comprising of all possible values that the random variable can take is called as the population then any subset of that population is called as a sample, let us say we are talking about pages in a particular chapter the pages in a particular chapter is a subset of the pages in the entire book. So, the sample indicates is subset of the population then we have what is called as a realization? realization is the actual outcome of the experiments that you have carried.

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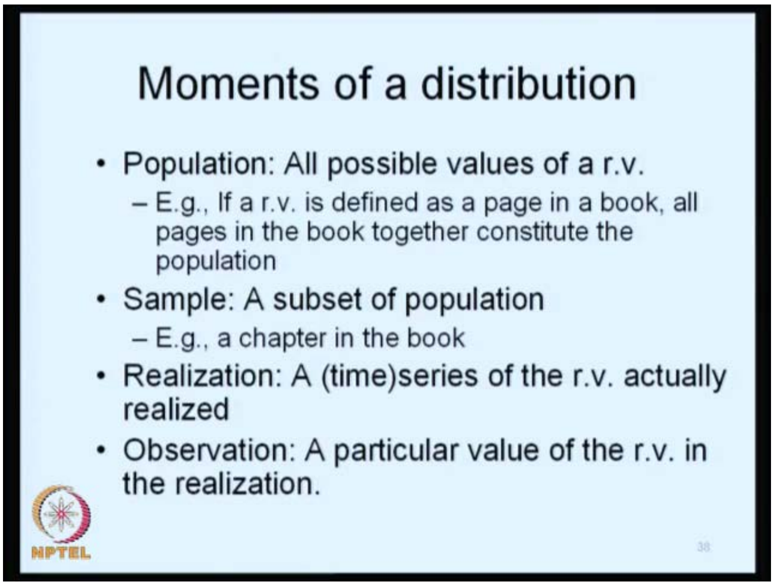

For example you have actually randomly drawn certain pages and then you get the series of such values that the random variable has actually taken and this is called as a realization and observation is a particular value of the random variable in the realization. So, you have from population you have a subset of the population which is called as a sample and we have a realization which is generally used in the time series sense where the value that the random has actually taken is called as a realization and a particular value in the realization let say you are talking about flow in June month of 1977 which has been observed already a particular value in the realization is called as the observation.

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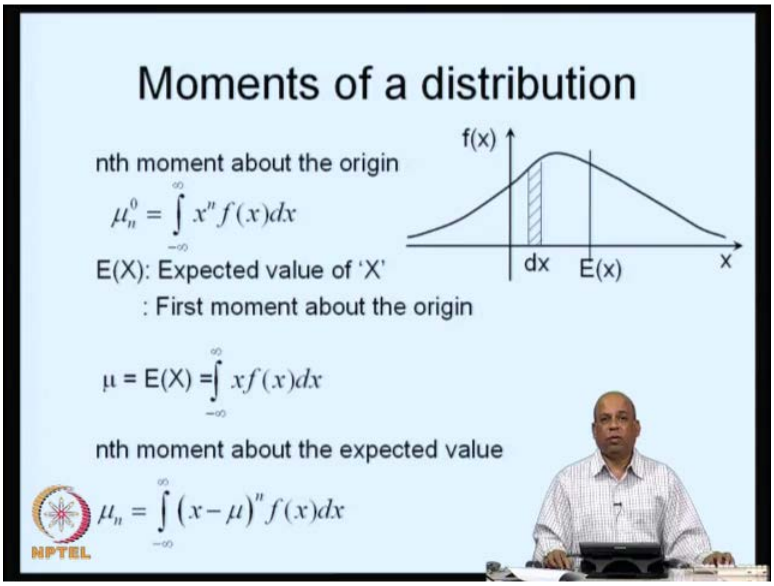

Now, with this now we introduce the concepts of moments of a distribution this is much similar to the moments that we talk about in mechanics, let us say you have the distribution here f of x and this is the origin 0 0. So, we define the n-eth moment about the origin denoted as mu n o as minus infinity to plus infinity this entire region x to the power n into f of x dx where f of x is the pdf. So, we are defining the moments of probability density functions. So, the moment the n-eth moment about the origin is simply minus infinity to plus infinity x to the power n f of x dx lets take the first moment, before that what happens when n is equal to 0; that means, we are talking about the 0-eth moment what happens when x is n is equal to 0 it is minus infinity to plus infinity x to the power 0 which is 1 f of x dx that will turn out to be 1,because that is a area under the f of x. So, the first moment about the origin when n is equal to 1 is called as the expected value of x. So, we are putting n is equal to 1 here. So, mu 1 0 which is denoted as mu simply is equal to expected value of x minus infinity to plus infinity x of x into f of x dx we are putting n is equal to 1. So, this is x into f of x dx we will come to the implications of this later on in subsequent classes, but this is just the expected value of x.

Once we know the expected value let say we fixed a expected value here. Then we start taking moments about the expected value which come in very handy for our analysis .So, first we define the moments of the distribution about the origin and obtain the expected value the other moments about the origin are also important and significant, but once we define the expected value of x we also start taking moments about expected value itself. So, we define the n-eth moment about the expected value which is denoted as mu we write it as mu and is equal to x minus mu to the power n f of x dx. So, this defines the n-eth moment about the expected value or about the mean this expected value is also called as the mean. Now the expected value has interesting properties first by definition expected value of x is equal to x f of x dx between minus infinity to plus infinity this is how we obtain the expected value.

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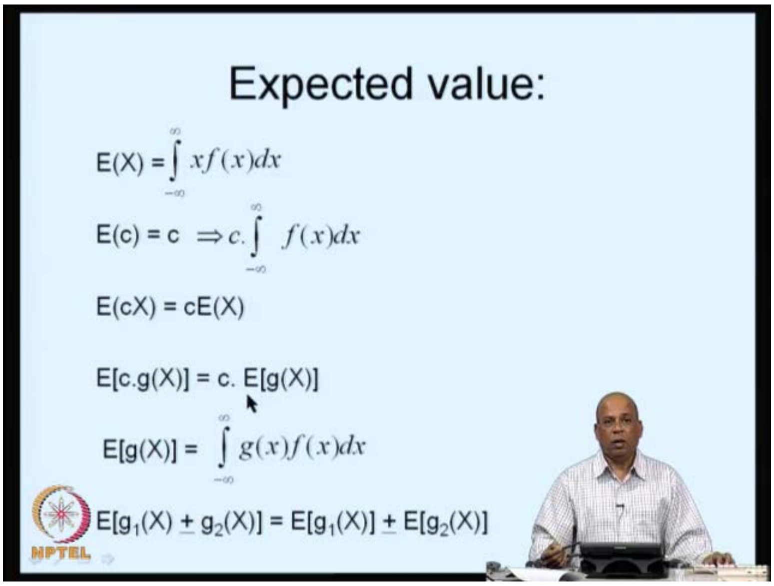

Now, expected value of c is equal to c into integral of minus infinity to plus infinity f of x dx that is instead of expected value of x we are talking about the expected value of a constant minus infinity to plus infinity f of x dx is equal to 1 and therefore, expected value of c turns out to be c itself. So, expected value of a constant is the constant itself which is quite obvious in the sense that we are given a random variable whose probability density function is known and you are asking for expected value of 5. So, expected value of 5 b 5 itself then from this it also follows that expected value of c into x is equal to c into expected value of x. We also have to see this particular result which is interesting where we are taking about the expected value of a function of x g of x is equal to minus infinity to plus infinity g of x, f of x dx and from this we again get expected value of a constant into g of x is equal to c into expected value of x.

Then we also have expected value of g 1 of x plus or minus g 2 of x must be is equal to expected value of g 1 of x plus or minus expected value of g 2 of x, now these are some interesting results that we will be using in many of the applications subsequently. So, this is the first moment that we are talking about. So, x f of x dx similar to the first moment we can start talking about higher moments and in general when we go to higher moments we start talking about moments about the expected value itself. So, we start talking about the second moment about the expected value third moment about the expected value and then we we will see what exactly they indicate. So, in the next class we will introduce higher moments and solve some examples and we also introduce the population moments and their associated sample estimates because we will have always a sample which is a subset of a population with us and based on these samples we need to make the decisions on the on the population.

So, in the next class we will introduce higher moments about the expected values and then we go on to provide the sample estimates for these moments, then we will also take examples of continuous probability density functions that are commonly used in hydrology. So, thank you for your attention we will meet again. Thank you.