Stochastic Hydrology

Prof. P.P. Mujumdar
Department of Civil Engineering
Indian Institute of Science, Bangalore

Module No. # 05

Lecture No. # 23

Markov Chains-II

Good morning and welcome to this the lecture number twenty three of the course stochastic hydrology. In the last lecture we introduced the concept of Markov chains. If you recall we said Markov chains are the stochastic processes where we can write probability of X t given the entire history of process X t minus 1 X t minus 2 etc up to X 0 as equal to probability of X t given X t minus 1 which means the memory of the process is limited to only what has happened in the last time step. These kinds of Markov chains we use when we are analyzing let us say the drought runs and the flow sequences and specially the weather conditions. What is the probability that tomorrow will be a rainy day given that today is a rainy day? And what is the probability that next year the drought year when this year is not a drought year? And questions like that. So, Markov chains are very useful and powerful techniques. Markov chains is a good technique to analyze several of hydrologic problems.

We also introduced the concept of transition probabilities where we are talking about the probability of the process transiting from a given state in time period t to time period t plus 1 to another state in time period t plus 1. For example, we may be saying that what is the probability that starting with a dry day today the process goes to a dry day tomorrow? Or starting with a dry day today, state being in dry day today the process goes to a wet day tomorrow?

So, these kinds of probabilities which relate the transition from a given state in a particular time period to a given state in the next time period; these probabilities are called as the transition probabilities and starting with transition probabilities we introduce the concept of the probability vector.

What do you mean by probability vector? At a particular time step n it is it consists of the probabilities of going into certain states in that particular time step. We wrote P to the power n or P of n as the probability vector at time step n and the vector consists of limits P j n. That is the probability of going into state j in time step n. From that we develop the concept of steady state probabilities or they are also called as limiting probabilities.

As the process goes far into the further into time then the probabilities vectors converge to the steady state probabilities in which case we wrote if you recall small P is equal to small P into capital P. Capital P is the transition probabilities, small P is the probability vector. The determination of steady state probabilities is important when we are looking at the Markov chains.

The steady state probabilities will indicate for a given process what is the probability that it goes into state j when the process has reached steady state For example; you are looking at the reservoir levels. So, you may be interested in the steady state probabilities that the reservoir will be between 50 percent and 75 percent. Or if you are looking at the drought drought lengths so, you may be interested in what is the probability that a given year will be in drought situation. So, these are the steady state probabilities. So, given thus a transition probabilities we must be in a position to determine the steady state probabilities.

So, what we will do in today’s class is staring with those definitions that we introduced in the last lecture; we will solve some examples to make sure that we understand the concepts that we we have introduced in the last lecture.

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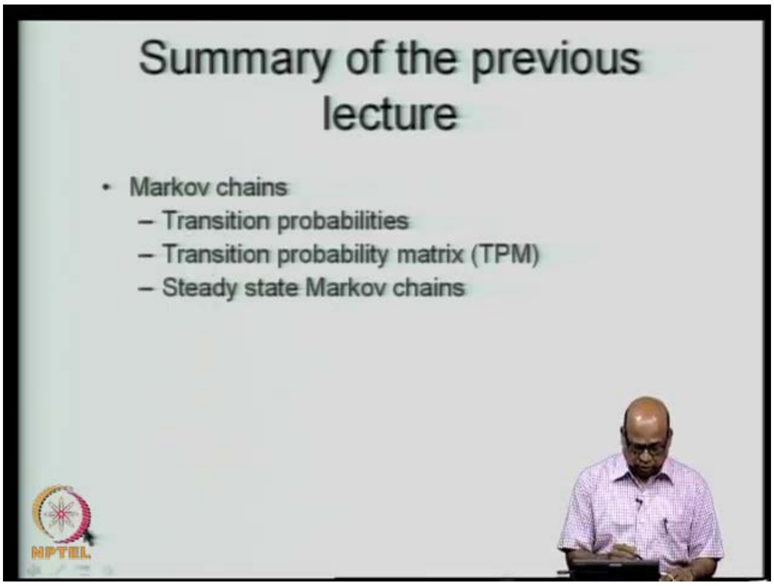

So, let me begin with summary of the last lecture again. We have introduced the concept of the transition probability matrix and the steady state Markov chains or the steady state probabilities.

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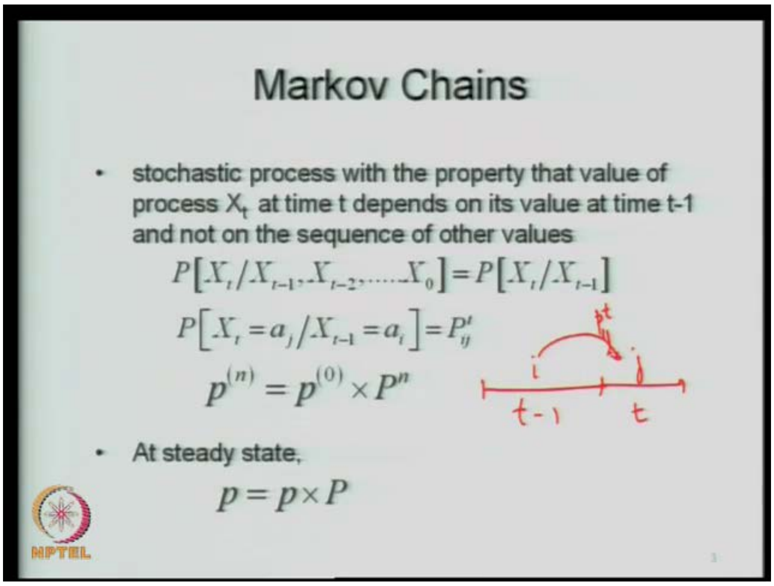

So, if you recall we wrote the Markov chain as probability of X t given X t minus 1 X t minus 2 etc X 0 as equal to probability of X t given X t minus 1. We also called this as the 1 step Markov chain or single step Markov chain and typically we write the states as probability that X t is equal to a j. That is, it indicates that in time period t the process has gone to state a j given that in time period t minus 1, it was in state a i and this probability we indicate it as P i j t.

This indicates the probability of transiting from the state i in period t minus 1 to state j in period t. So, if you are taking 2 time periods, 2 adjacent time period it was in state i in period t minus 1 and it goes to a state j in period t minus period t with a transition probability P i j t or relate this with let us say t minus 1 was in the state, dry state which means there was no rain there are 2 states for example, no rain or rain or dry and wet. So, it was in no rain condition.

What is the probability that it goes to a no rain condition? Or what is the condition starting with the no rain condition in time period t minus 1 it goes to a rain condition or wet condition in period t?

So, these transitions is governed by the transition probabilities denoted as P i j t. So, this is the transition probabilities. Then we defined the n step or probability vector as equal to P 0 into P to the power n where capital P is a transition probability. So, P to the power n, P 0 is the probability vector at times 0. So, given P 0 we must be able to get the probability vector at any given a time step n.

Now, as n increases; the dependence of P to the power n on P 0 decreases and in fact, P to the P n that is the probability vector at time step n converges to a certain steady state values and at that time we call, we say that the Markov chain has reached the steady state and then we recon the probabilities that state as steady state probabilities. Steady state probabilities will be given by P is equal to P into capital p. So, the solution of this will give you p, but, typically as I said in the last lecture you keep on getting P of n with increasing n let us say you get P 2, then P 4, P 16 like this you keep on getting P 32 and so on. P 16 into 6 P 16 to the power 2 as n keeps on increasing you will see that the P to the P of n will converge to a certain steady state values.

So, instead of using this you can use this and keep on increasing n until you obtain a probability vector which converges and that is when you can call those probabilities as steady state probability. You can also use this to verify that. We will do that through a numerical example. So, let us now get used to the concept of transition probabilities. So, essentially what we do in the transition probabilities is we assign probability for the transition from one step to another step.

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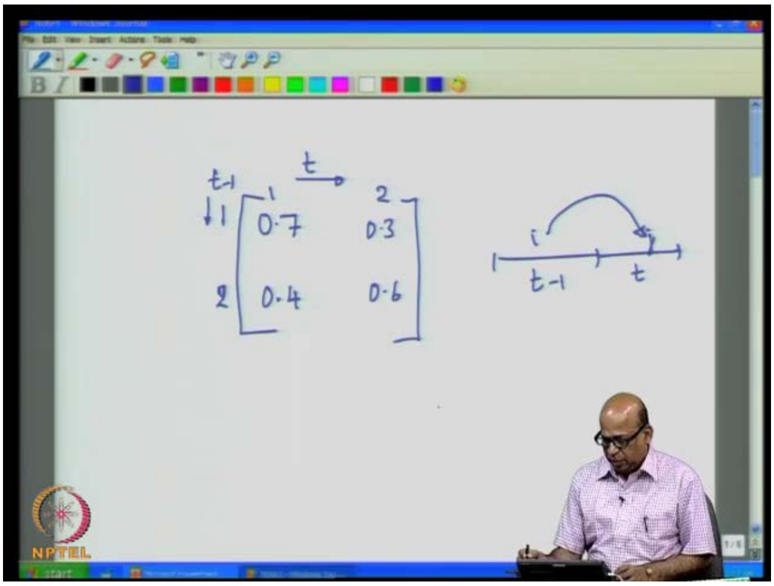

Say for example, we are talking about let's get use to this concept first that we are saying that there are two stages or two steps I am sorry two states one and two. Let us say this state one and state two. This refers to the time period t or time period t minus 1. Let us say we are talking about the transition from period t minus 1 to period t. So, this states are for period t minus 1 and these states are for period t. You will have again states one and two. Let us say you are talking about the day being a dry day or a wet day. I may say the process is the weather state, weather process is state one if it is a dry day. It is in state two if it is a wet day.

Then the probability that from a dry day it goes to dry day. That is given that today is a dry day what is the probability that tomorrow is a dry day? We may say this is .7. That means, it is the probability that it goes from a dry day today to a dry day tomorrow is .7.

Then this has to be 0.3 remember the transition probability matrixes are stochastic matrixes where all the rows add up to one. So, the probability that it goes from state 1 to state 2 will be 1 minus of this if you have because you have only 2 states. So, this will be .3.

Let us say that you are already in state two. That is it is a wet day what is the probability that from wet day you go to a dry day. So, two one that transition let us say it was point four. So, this was .4 then this will be 0.6 that is from state two to state two. You must always keep in mind that we are talking about two adjacent time period t minus 1 to t and this is the transition probability that we are talking about, from 1 to 2 or 1 2 to 2 2 to 1 etc.

So, you will have four such probabilities in this case. Now, another important aspect that we must remember is when we formulated the TPM that is the transition probability matrix; we had m by m probabilities.

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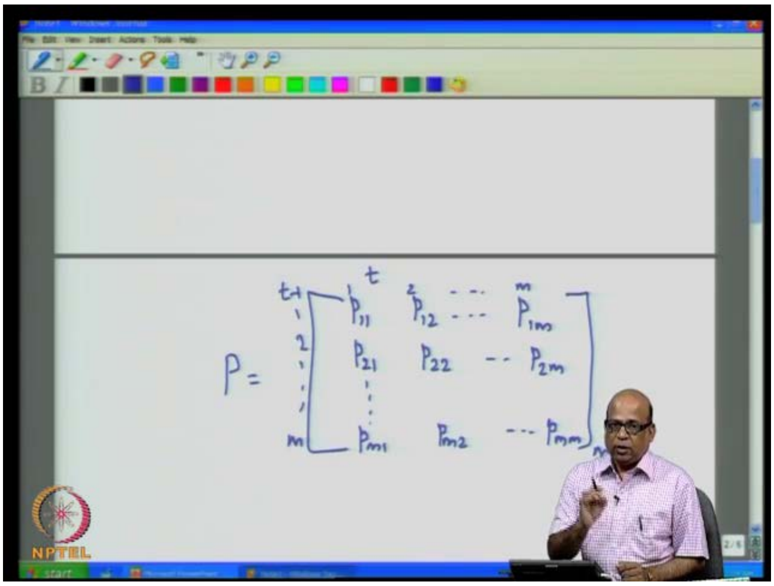

Let us say I write P is equal to if you recall P 1 1 P 1 2 etcetera P 1 m you have m steps m straights and P 2 1 P 2 2 etc P 2 m and so on. So, you have P m 1 P m 2 etc P m m. So, the transition probability matrix is m by m matrix where m is the number of states and this you can write this as t minus 1 one 2 3 etc m. So, in the time period t minus 1 it occupies the state 1 2 3 etc up to m. Then these probabilities give the probability of the transition from a given state in t minus 1 to a given state in period t. This is the general notation. So, it is an m by m matrix where m is the number of states. From the data we know how to determine this transition probability.

Yesterday in the last lecture I explain that from the data you should be able to arrive at each of these transition probabilities. How many such values you need to estimate? There are m square values, but, the last value in any of the rows can be obtained by using the remaining m minus 1 values.

So, you will have m into m minus 1 number of values that you need to estimate from the data. So, given the historical data you discretize that. Let us say you are talking about the stream flows at a particular location, you identify the states that you are interested in depending on let us say I call it as a state number one when the flow is between 0 to 100. I call it as state number 2 when it is between hundred and 200 etc like this you identify the state’s first, states of the Markov chain. And then you look at the historical data assign states corresponding to the actual flow value that you have obtain and then look at the transition from one time period to another time period how the transition has taken place for starting with the state number one in let us say month of June. Where did it go? Did it go to class number one, class number two? How many times did it go to class number one? etc like this one. The relative frequency approach you will obtain the transitions the transition probabilities P 1 1 P 1 2 etc. So, from the historical data you would have estimated the transitional probabilities.

The transition probabilities are the most single most important requirements for analyzing the Markov chains, single step Markov chains. So, let us do a simple example now. We will look at the weather conditions. The dry and wet weather conditions and see how we analyze that.

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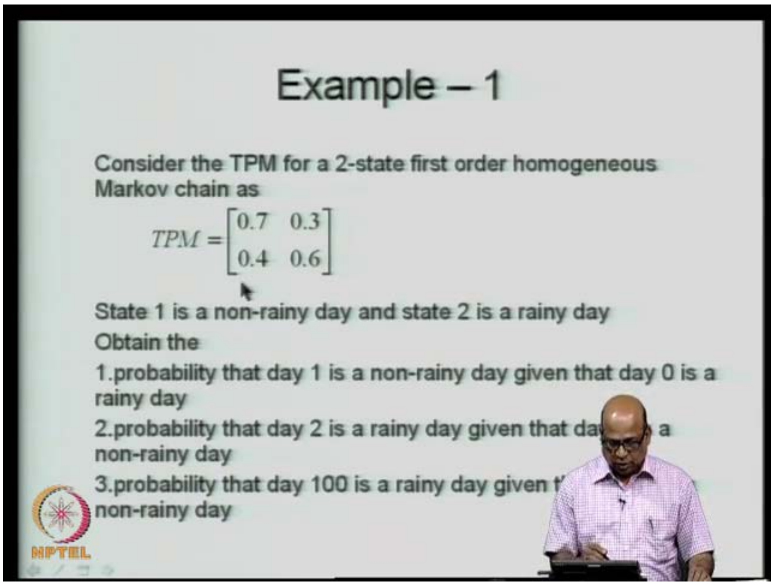

Let us say we have 2 states and the state one is the non rainy day and state two is a rainy day. We also can call it as a dry day and a wet day. Now, we will obtain first the probability that day one is a non rainy day given that day 0 is a rainy day and the transition probability matrix is given here. So, this is state 1 state 2 and this is state 1 state 2. So, starting with state 1 in any day, the probabilities that it goes to state 1 the next day is .7. Starting with state 1 in any day the probability that it goes to state 2 in the next day is 0.3 like that. So, we are given that day 0 is a rainy day.

So, this is non rainy this is rainy. So, we know that day 0 is in non rainy is a rainy day. So, you are looking at this state now. So, you are in stage 2 in day 0. Then what is the probability that day one is a non rainy day? So, going from a rainy day to a non rainy day this is 0.4 itself. So, directly from the transition probability you know that the probabilities of going to a non rainy day is 0.4 here.

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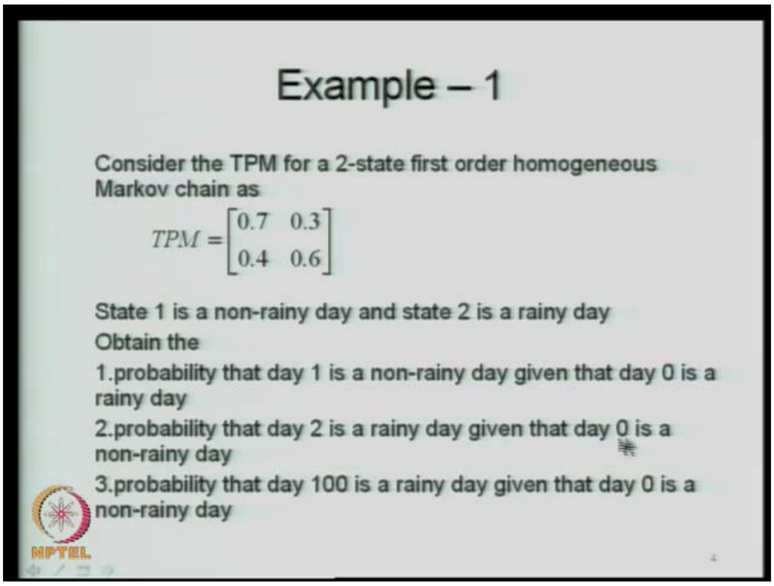

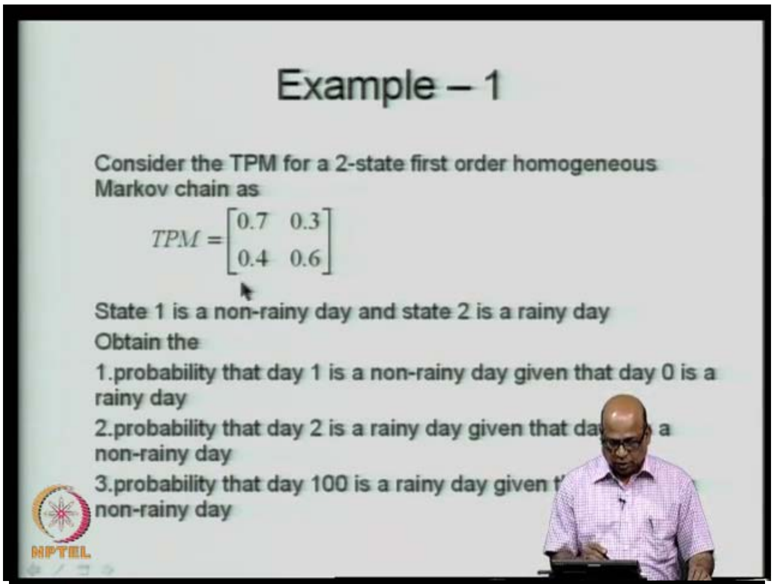

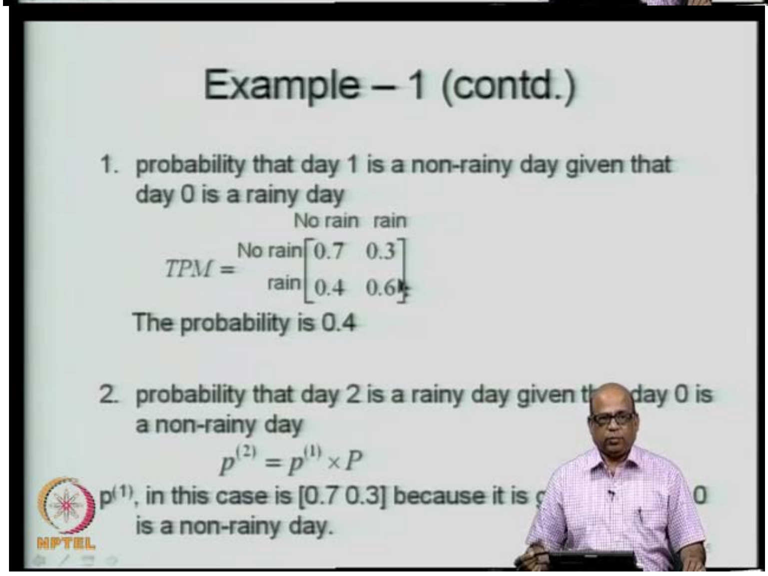

So, that is what is written here probability that day one is a non rainy day given that day 0 is a rainy day. So, you have no rain, rain, no rain, rain these are the 2 states. So, from a rainy day you want to go to a non rainy day; so, the probabilities are .4. Similarly, from a rainy day if you want to go to a rainy day probability is .6. So, this is the interpretation of the transition probabilities. Now, we will look at the second problem which is second sub problem. So, to say probability that day two is a rainy day given that day 0 is a non rainy day.

So, you are starting with a non rainy day, you are given that you are in a non rainy day what is the probability that day 2 not the next two, but, day 2 is a non rainy day? I am sorry day 2 is a rainy day? For this, what what is that we need to do? We must get the probability vector of day number 2.

So, day 0 is given and you need to get probability vector of day 2. What does the probability vector give? The probability vector at a particular time step gives the probability of being in a given state in that time step. So, P of n I repeat P of n is equal to is it gives P j of n. It is a row vector consisting of P j of n which indicated that in the time step n the probability of being in state j is given by P j of n. So, for any time step we need to determine the probability vector for that particular time step. To determine the probability that it the process occupies a particular state in time.

So, in this particular problem what do we do? We obtain the probability vector for day number 2 given that day number 0 is a non rainy day because you are given that day 0 is a non rainy day. You are looking at this vector .7 and .3. So, P 1 in this case is directly 0.7 and 0.3 because you are given that you are in non rainy day.

So, what is the probability that in the day one you go to a non rainy day? It is .7. What is the probability that you go to rainy day in day number 1? It is .3. So, P 1 in this case is .7 .3 itself because it is given that day 0 is a non rainy day and P 2 we write it as P 1 into P. Recall that we wrote P to the power n is equal to P of n is equal to P of n minus 1 into p.

So, P 1 is .7 .3 because we are looking at, we are given that day 0 was not rainy day then we know the probabilities of going into a non rainy or rainy day on day number 1 and therefore, P 1 is given. So, we will go to P 2. P 2 is calculated as P 1 into P where P is the transition probabilities

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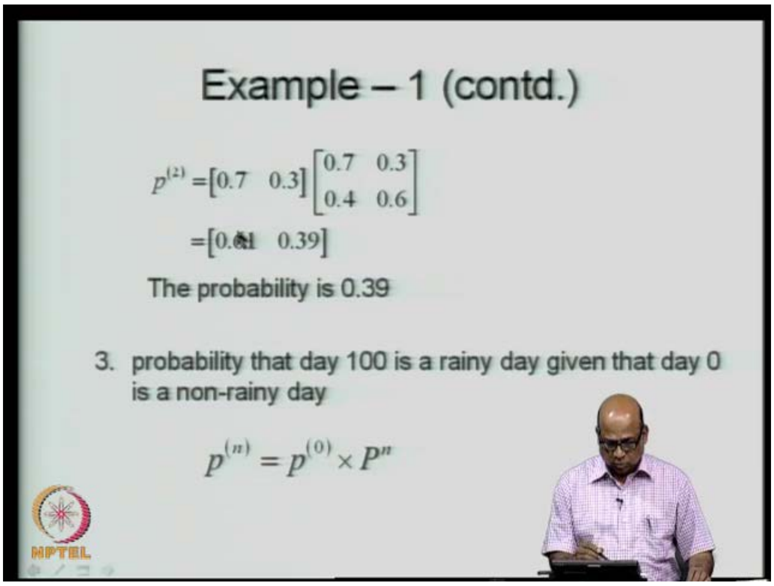

So, we calculate P 2 as .7 .3 and the transition probability which is 0.61 and 0.39 what does this mean? This is no rain and rain. So, this is no rain and rain and we are interested in getting the probability that day two will be a rainy day. So, day two is a rainy day is what we are interested in.

So, day 2 is a rainy day is .39, day 2 is a non rainy day is 0.61. So, this is probability vector and therefore, the required probability is 0.39. Then we will look at the third sub problem which is probability that day 100 is a rainy day. That means, far into the future where given that day 0 is a non rainy day, we want to get the probability that day hundred is a non rainy day.

I am sorry Day 100 is a rainy day either way you know once you get the probability vector for any particular day. You will be able to tell what is a probability. That it is a non rainy day or also what is the probability that it is a rainy day. So, recall that we wrote P of n is equal to P of 0 into P to the power n. That is starting with probability of probability vector of a time step 0. We write this to obtain the vector probability vector for time step n. So, we need P to the power n.

Actually what we are doing now is, you can write P 1 first the from P 1 you obtain P 2 from P 2 you obtain P 3 etc like this P of n is equal to P to the power 0 into P to the power n. So, as you progress well into the as Markov chain process; progresses well into the time then the P to the power n P of n converges to steady state probability and therefore, P to the power n is what will be governing the steady state probability.

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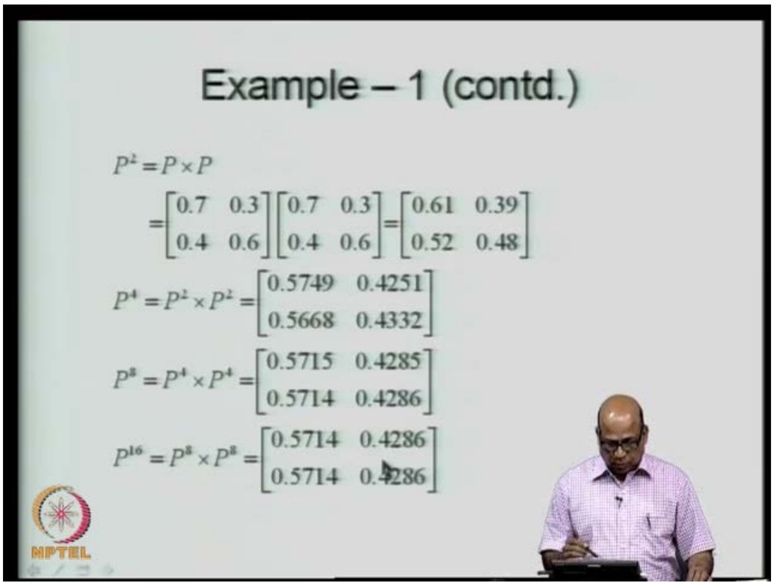

So, let us look at P to the power n. So, P square, this is the transition probability P square is equal to P into P which is this P into P. You get 0.61 0.39 0.52 0.48 when you multiply P into P then P to the power 4 which means we are obtaining the probability vector at time step 4 at day number 4 when P 4 will be .7. That is P square whatever we obtain you multiply it again by P square. So, you get this vector this matrix I am sorry then P to the power 8 which is P to the power 4 into P to the power 4.

So, 0.571 5.4 285 etc like this then we will take P to the power 16 which is P to the power 8 into P to the power eight that is 0.5714 0.4286 0.5714 0.4286 what did what what is happening here? These values are slowly converging. Not only that the individual values were converging, but, the column each of the column converges to a particular value 0.5714 in this case and here 0.4286 in this case. What does this mean? This means that the probability of non rainy day here. Look at P 16 probability of non rainy day is 0.5714 0.5714 irrespective of what was the starting point here.

Similarly, the probability of rainy day is 0.4 to .86 irrespective of what are the starting point and therefore, this reaches a steady state and in fact, the probability vector at any given time step will then be 0.5714 0.4 286.

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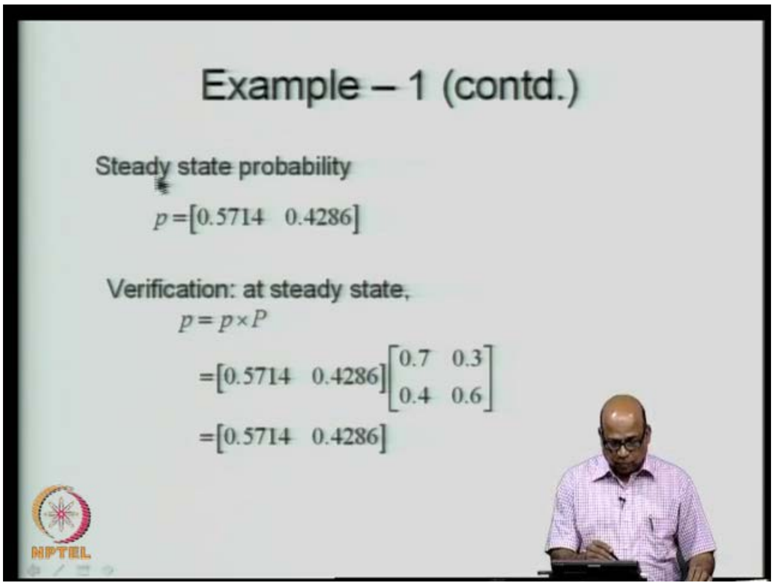

So, the probability has converged to these values and therefore, we write the steady state probability as P is equal to 0.5714 and 0.44286.

Let me just check what to we said this is a non rainy day and this is a rainy day. So, this is probability of a non rainy day and this is the probability of rainy day. This means that for this particular type particular example we can say that the probability of the 100th day or day number 100 being a non rainy day 0.5714 and the probability of day number 100 being a rainy day 0.4286.

We had the conditions for the steady state as P is equal to P into capital P. Now, this is what we obtain by a simply raising the powers of capital P which is a transition probability and that is how we obtain this. Let us try to verify whether the P that we obtain. In fact, satisfies these conditions. So, P is equal to P into capital p. So, this 0.714 into 0.4286 into the transition probability 0.7 0.3 0.4 0.6. So, 0.5714 into 0.7plus 0.4286 into 0.4 that gives you 0.5714 itself then 0.574 into 0.33 plus 0.42 0.6 into 0.6 that gives you 0.4 to 86 etc.

So, this is the steady state probability vector. So, this is how we obtain the steady state probabilities. As I mentioned, you could have straight away used this and then obtained the steady state probability. But, in certain higher dimensions this becomes slightly cumbersome. So, the easier way of doing it is.

Simply keep on obtaining the powers of the transition probabilities and see that the transition probability matrix when it is raised to higher powers, it converges it to certain value. The both, the column converges it to the certain values and then you say that steady state probabilities are achieved. But, then you should verify that this condition is. In fact, satisfied by the steady state probabilities that you are you also obtained.

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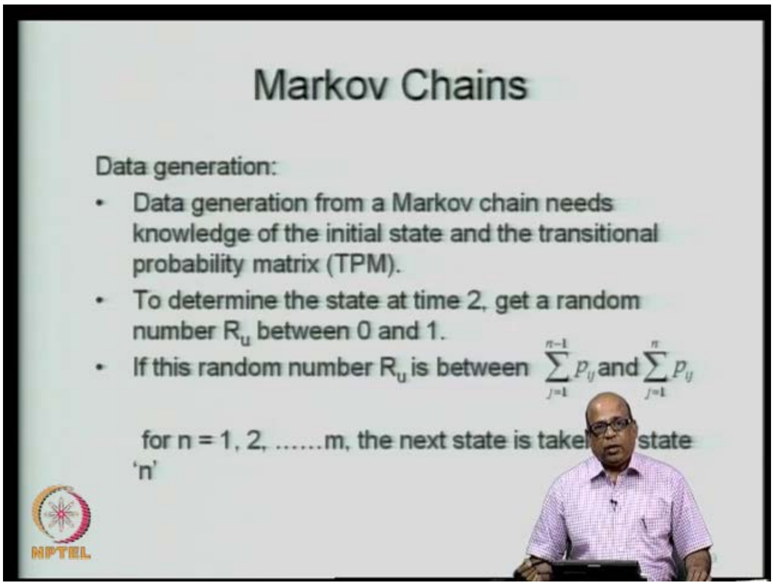

Let us look at another example. Now where we will use the transition probabilities to generate the data? What did we do earlier cases? We used Arma models to generate the data. We used the Markov process the Thomas Fiering model to generate data. We also used in our earlier examples; we use the probabilities distributions when the process is uncorrelated. We use probabilities distributions to generate data for example; we have studied how to generate data belonging into gamma distribution.

How to generate data belonging to exponential distribution and so on? Now let us say that we have a Fit Markov chain. What do I mean by a fitting a Markov chain? That we assume that the process follows Markov chain and then we have obtained the transition probabilities. So, you have discredited or you have identified the states to which the process goes into in every time period.

Time period t time period t plus 1 time period t plus 2 etc. So, you have for every time period you have identified the states to which the process goes in and the you have determine the transition using the historical data or otherwise now we we need to determine or we need to generate the states to which the process is likely to go in into the future. Let us say that you had two states; one being a dry state and two being a wet state. Then can we generate a sequence much the same way as we generate a sequence of flow or much the same way as we generate sequence of data belonging to certain distributions and so on. Given the Markov chain can we generate a sequence of states that the process is likely to go in for example?

Next ten periods; I want to say that it went it goes into state 1. It goes into state 2 etc not with the probabilities, but, now we will say that it is going into state 1 going into state and so on. So, starting with the transition probability matrix for a given Markov chain, we now simulate or we now generate the states that the process is going into in future. So, that is the exercise we will do.

So, for the data generation from a Markov chain what we need? We need the initial state; that means, the starting state whether it was a dry state or a wet state in that example and then the transitional probability matrix as I said. The TPM is both referred to as transition matrix transition probability matrix as well as plus transition probability matrix.

So, TPM then the initial state is known. Let us say that the time period t is equal to 1 state is known. Then we want to determine the state at time two. That means, we want to generate the state at time 2. What do we do? We get a random number r u between 0 and 1 you recall that any CDF that we talk about the cumulative probability of distribution functions. Cumulative distribution functions if you take number’s randomly from a CDF. They follow uniformly distributed random number, the uniform distribution with 0 between 0 and one. So, we generate r u. How do we generate r u? As I mentioned in the last class you have a calculator and most of the calculators we have, we will have random number’s. This we have done in the case where we studied the Thomas Fiering model.

Let us say I pick up a random number. This is it comes to be 0.767. The next number will come to 0.650 and so on. So, you can use the calculators. Most of the calculator will have random number generators and these random numbers are uniformly distributed random numbers between 0 and 1. So, we generate random numbers and then look at the random numbers, compare it with the cumulative transition probability matrix. What do I mean by cumulative transition probability matrix? The number that you generate which is between 0 and 1 if this are r u is between the summation P i j up to n minus 1 where n is a state. So, n minus 1 and if this lies between the cumulative transition probabilities up to n minus 1 and n cumulative probabilities up to n. Then, we assign the class interval n for that.

So, this is how we generate. Now, let us say that the first class interval you generate as 2. Then this becomes the initial class for the next value again you generate a random number, then look at for the next state for the, for the associated state where does it belong in which interval of the accumulative transition probability. It belongs then assign that particular class interval and so on.

So, like this you are now generating the states of the Markov chain and not the exact values as we did in our transition, as we did in our earlier data generation problems. So, we are now generating the states of the Markov chain that will occur in future.

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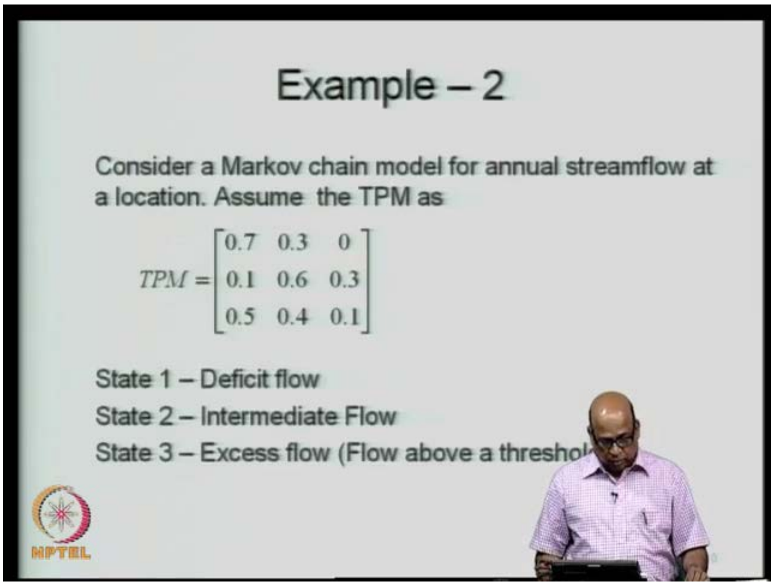

So, let us do an example for that. You consider the Markov chain of annual stream flow at a location and let us say we have 3 states for the annual stream flows. We say it is a state. It is in state 1 when there is a deficit flow we may assign that as soon as the flow falls below a particular threshold. We call it as a deficit flow. So, that is state 1.

Then state 3, we say it is a excess flow flow above a particular threshold. We call it as excess flow and then we also define state 2 as intermediate flow. Then we have the transition probability. So, state 1 state 2 state 3 here then state 1 state 2 state 3 here. These are annual flows. So, given that this year is a deficit year in terms of flow what is the probability that the next year will also be a deficit year given that this year is a deficit year?

What is the probability that next year the flows will be intermediate given that this is the deficit year? What is the probability that next year will be an excess flow? So, these are the probability, this is how we formulate the transition probability matrix. So, there are 3 states here time period t minus 1 to time period t. So, this gives you the transition probability of going from deficit to intermediate deficit to excess flow and so on. So, this last row here starts with excess flow, excess to deficit excess to intermediate and excess to excess. So, that is how you get the transition probability. Now, given that these are the transition probability we want to generate a sequence of states for example, next ten year we want to say that it is a deficit years it is a intermediate flow year it is a excess flow here etc.

Like this, we want to generate for the next ten years. Let us say how do we do this first we consider the cumulative transition probability matrix. What do I mean by cumulative? You keep on adding across the rows. So, this will be 0.71 0.0 and 1.0 then 0.7 this is 0.1 0.71 0.0 0.5 0.9. That is I am adding these two numbers 0.5 0.9 and 0.101 0.0 will be the maximum number.

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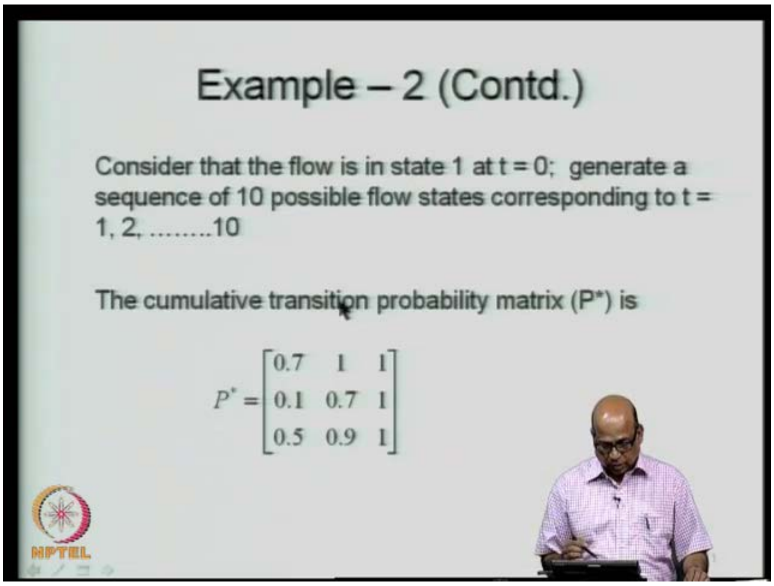

So, you get the cumulative transition probabilities as 0.7 sorry 0.71 and and 1 once you reach one you written it as 1 then 0.1 0.71 0.5 0.9 and one. So, these are the cumulative transition probability. So, this is state 1 state 2 state 3 any uniformly distributed random number that you pick. If it is less than 0.7 when you were in state 1 let us say you start with t is equal to 1 and you are in state 1 and you generate a uniformly distributed random number, if by uniformly distributed random number that you get is less than 0.7, less than equal to .7, you assign class interval 1 to that if it is between .7 and 1 assign class 2 to that and so on. So, class 3 does not appear in this particular case.

So, if you were in class 2 and you obtain a uniformly distributed random number which is less than equal to .1, assigned class interval 1 the uniformly distributed random number is between .1 and .7 assigned class interval two if it is between 0.7 and one assigned class interval three.

This is how you keep on assigning the class intervals and the class interval that you. So, assign for the next time period becomes an initial class interval of the next time period and this is how you use the cumulative transition probability matrix along with the uniformly distributed random number to generate a sequence of state into which the Markov chain is going. So, this is what we will do.

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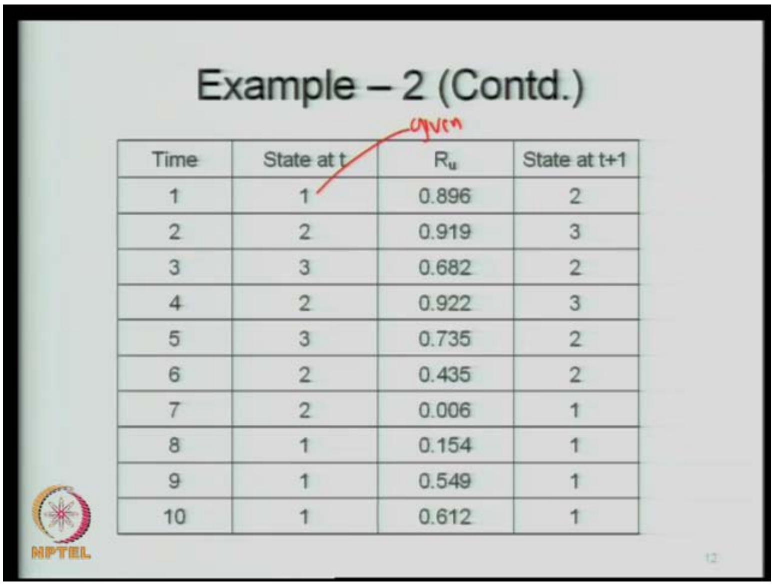

Time	State at t	R_u	State at t+1
1	1	0.896	2
2	2	0.919	3
3	3	0.682	2
4	2	0.922	3
5	3	0.735	2
6	2	0.435	2
7	2	0.006	1
8	1	0.154	1
9	1	0.549	1
10	1	0.612	1

Let us say at time one we assume this is a starting state that is assumed or given let us say this is given state. Then you generate a uniformly distributed random number from the calculator. So, this is from the calculator or otherwise if you are using mat lab program you can generate if you are using excel you can generate the uniformly distributed random random numbers.

Most of the scientific programs, all most of them have random number generators which provide you uniformly distributed random numbers in the interval 0 and one. So, you get 0.896. So, look at this, you have the cumulative probability transition probability matrix you are in state 1 and you are generating for state 2 that is for period 2 you were here. So, you got a number of 0.896 and .896 falls in this interval that is between class 1 and class 2 you are assigned class 2 to that.

So, that is how you get class 2 in the next time period this class 2 becomes your starting point for time period 2. So, starting with time period 2 you want to generate for time period three. So, in the time period 2 you got the class interval 2 and you generate another random number this comes to be 0.919.

So, you were here you are in class interval 2 and 0.919. So, you are here between 0.7 and 1 therefore, you assign class interval three. So, that is how you go to class interval 3 state three and that state 3 becomes your initial point for a period 3 using which you want to generate for period 4. Again you generate another random number, it becomes it is .82. So, you are in class interval three.

So, you are here 0.68 2 falls in this region and therefore, you assign class interval 2 and that is how you get the state 2 here and like that you continue. So, you generate ten values like this and you can in facet generate a large numbers of values much the same as we did in for your Arma models or Markov model Thomas Fiering model or data generation using using the probability distributions and so on. So, this is how we generate states of the Markov chain.

The Markov chain that we have been discussing are are all 1 time step Markov chain which means that we are writing probability of X t given X t minus 1 etc is equal to probability X t given X t minus 1 which means the memory of the process is limited to what has happened during the last time period.

There may be situation where the state at a particular time depends not only on the previous one, but, also on previous 2 or previous 3 time periods. It is often advantages to convert such a multi step Markov chain into single step Markov chain. For example, if you are given that the weather today depends not only on weather yesterday, but, also on weather day before yesterday. That means, on the past 2 days whether the weather today is dependent in which case we must be able to define the states properly in appropriate manner.

So, that we can read this also as a single step Markov chain. Let us look at an example where the weather today depends during the last 2 days and then how we convert that in to a single step Markov chain.

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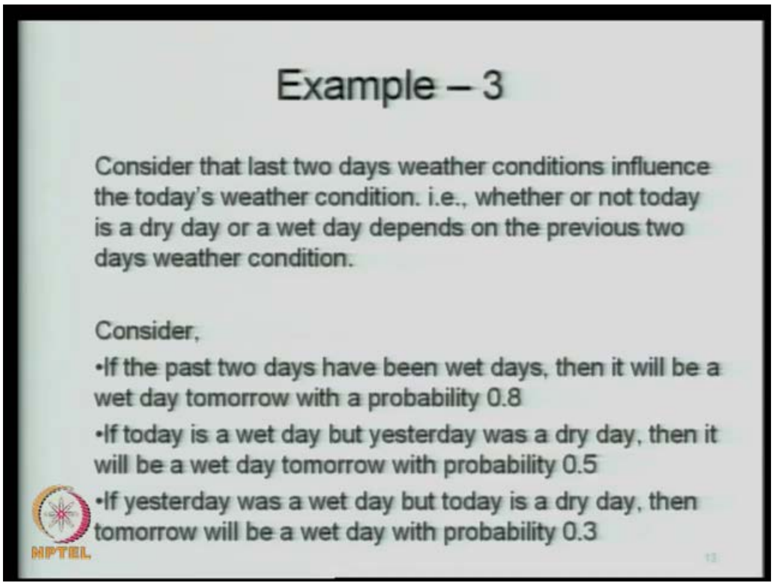

So, consider that last two days weather conditions influence the two days weather conditions, influence the two days weather condition. That is whether or not today is a dry day or a wet day depends on the previous two days weather conditions not just the previous days. In addition if the past two day has been wet days then it will be wet day tomorrow with the probability of 0.8. So, yesterday and day before both were wet day by wet day. I mean it is a rainy day and the probability that today will also be a rainy day or today will also be a wet day is 0.8.

If today is a wet day, but, yesterday was a, we are using today and yesterday. So, let e put it this way if today and yesterday were both rainy days or both wet days then it will be a wet day tomorrow with the probability of 0.8. the If today is a wet day, but, yesterday was a dry day then it will be a wet day tomorrow with the probability of 0.5

If yesterday was a wet day, but, today is a dry day then tomorrow will be a wet day with the probability of 0.3. Then if the last two days both have been dry days then tomorrow will be a wet day with the probability of 0.1. Remember we are talking about tomorrow being a wet day. All these probability are for tomorrow being a wet day.

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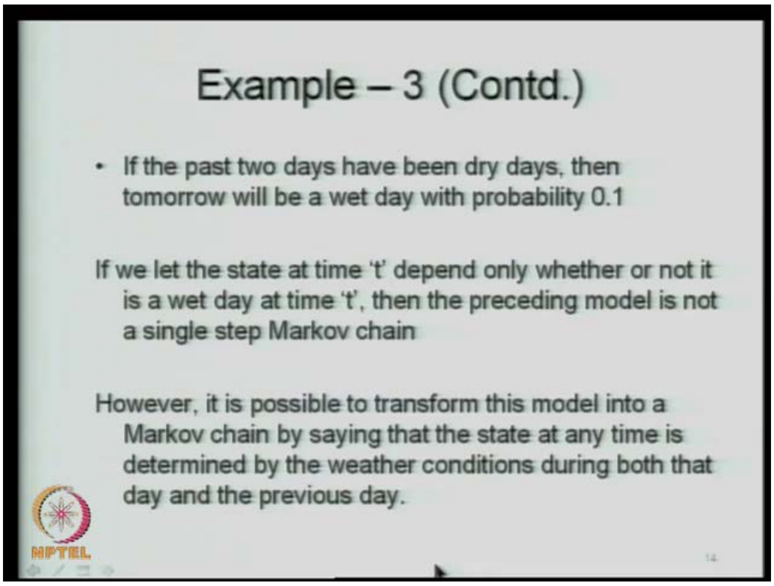

Now, how do we convert this into a single step Markov chain? Let us say if we let the same state at time P depend only on whether or not it is a wet day at time t that is time t and time e minus 1, let us say only we are looking at just one time step ahead that is this is t minus 1.

Then the preceding model is not a single step Markov chain because we are given that it depends on 2 day weather. However, for this problem it is possible to transform this model into a Markov chain. There is into a single step Markov chain. By Markov chain I mean a single step Markov chain by saying that the state at any time is by whether condition by both the, that day and the previous day.

So, we are considering two days weather to determine the state. For example, I may say that state one is when both yesterday and today were rainy days state 2 is yesterday was a rainy day. But, today is not a rainy day state 3 is yesterday was a non rainy day. Today is a rainy day and so on. Like this we can define the states and then look at how the transitions take place. So, this is how we define articulately the states and then at the transition probability using the condition that we are given earlier.

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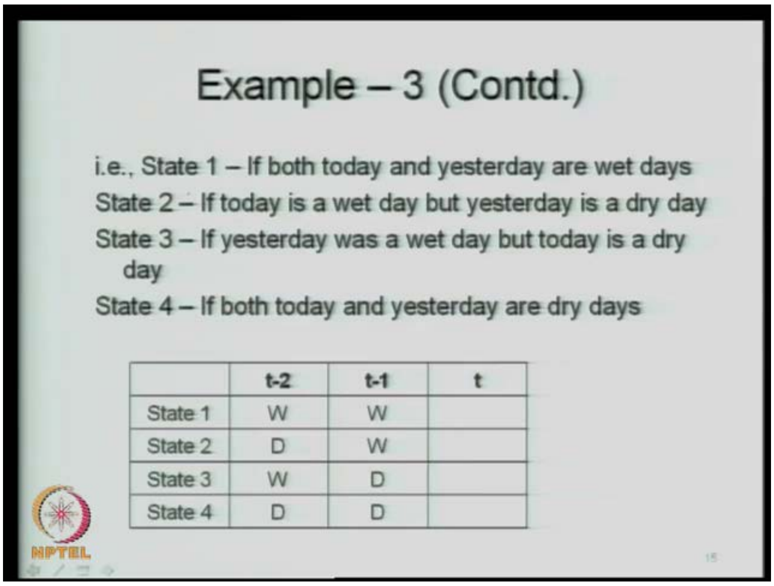

So, let us see what we do here. State one we define if both today and yesterday are wet days then I call it state 1 and if today is a wet day, but, yesterday is a dry day I call it as state 2. If yesterday was a wet day, but, today is a dry day then I call it as state three then if both today and yesterday are dry days then I call it as state four.

So, how do I write here? State 1 in general I write t minus 1 and t minus 2 and t is some particular day that we are talking about. So, the previous 2 days t minus 1 t minus 2. So, we are in state 1 if both t minus 2 and t minus are w w by w i mean it is a wet condition then state 2 we are saying today is wet. But, yesterday is a dry which means t minus 1 is wet t minus 2 is dry.

Similarly, state 3 is t minus 1 is dry and t minus 2 is w and then state 4 is both of them are d and d. So, you are looking at what is likely to happen in t given t minus 1 and t minus two. So, state states are defined based on the previous 2 days t minus 1 and t minus 2 the how do we formulate the transition probability matrix now.

So, what what we were given remember is that if both the previous have rained the probability that it will rain today is 0.8 or both today and yesterday were raining is the probability that it will rain tomorrow is 0.8. Then if today is a wet day, but, yesterday was a dry day then it will be a wet day tomorrow with the probability of 0.5. So, is what we have been given with this? Now we should be able to formulate the transition probability matrix.

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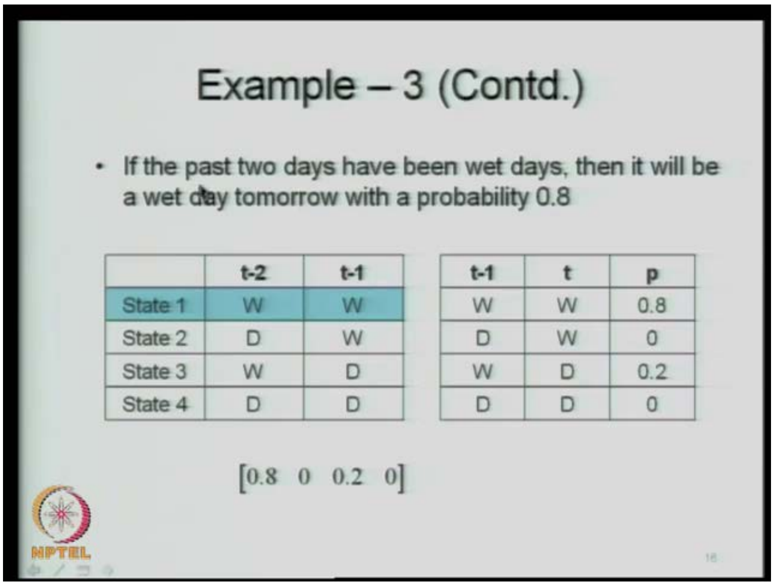

Let us see how we do this. So, if the past 2 days having wet days; I will demonstrate for state number one. So, when we say that we are in state number 1; what we are saying we are interested in getting the probabilities for t. So, you are considering t minus 1 and t minus two. So, when we are in state 1 both t minus 1 and t minus 2 have been wet days. So, we are in w.

Suppose you want to go to state 1 in t. What does that mean? State 1 in t means that t should be w and t minus 1 should be w and that is all. So, the 2 days must be w. So, what is the probability then that you have to go from w in the previous days to w in the next day? So, you have to look for the probability of the last two days being wet and going into wet days and that probability is given as 0.8 is the past 2 days have been wet days the it will be a wet day tomorrow with a probability of 0.8. So, this is what we are looking for. These two have been wet and you want to go into wet. This is state 1 transition. So, you want to go into wet and therefore, the probabilities 0.8.

You are in state 1 which means you are t minus 1 and t minus 2 are both w. What is the probability that you go to state 2? Now, this is state 1 you obtain 0.8. What is the probability that you go state 2 what is state 2. State 2 is d and w that is t minus 1 should be d and t should be w, but, here t minus 1 is w and therefore, you cannot go to t minus 1 being d here and therefore, the probability is 0.

You are in state 1 which is w. W you want to go to state 3 which means for t minus 1 should be w and t should be d this is possible and that will be simply from w. W you have 0.8 therefore, w d should be 1 minus 0.8 which is 0.2 you are in state 1 which is w w you want to go t state 3 state 4. I am sorry which is d it is not possible because t minus 1 is d here and t minus 1 is w here and therefore, this is not possible therefore, this is 0.

So, the state transition from state 1 to state 1, state transition from 1 to state 1 2 3 4 are given as 0.8 0.20. These are the probabilities. So, this is how we formulate the transitions corresponding to each of the 4 state. So, state 1 state 2 state 3 state 4. State 1 state 2 state 3 state 4. So, at every time you have to look at two time period. So, state 1 is defined by 2 time period state 2 is defined by 2 time period for example, state 2 to state 1 I want to go then what we will do this is t minus 1 is w and we need.

What do we need for state 2 to state 1 you are in state 2. So, you need another w here. So, d w and t minus 1 is w. So, you want another w here. So, d w to w. That is our transition that you need to look at. So, let us look at what is the transition d w to w? That is yesterday was a dry day d w, you want that is yesterday was a wet day and day before was a dry day. So, the probability will be .5 here like we have mentioned here this will be 0.5.

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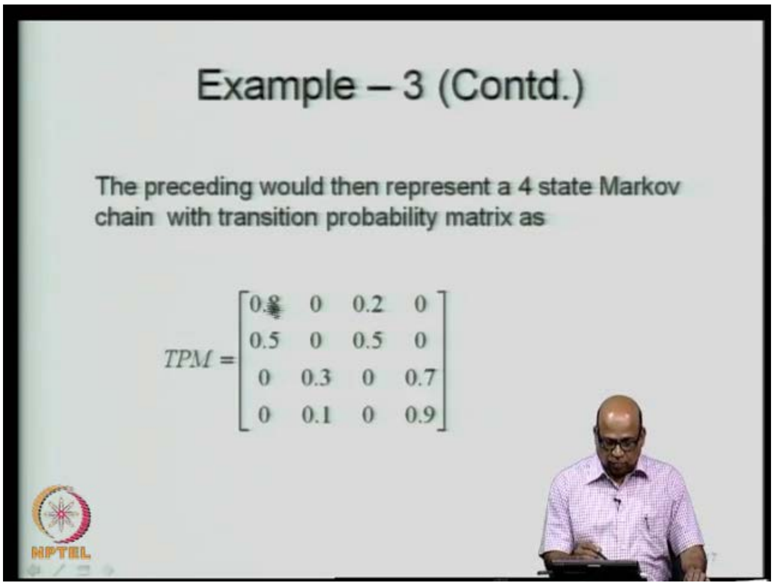

So, this is how you formulate the transition probabilities corresponding to each of the state. Assume take this particular state and look at what are the probabilities associated with going to state 1 state 2 state 3 state 4 and once you formulate the transition probabilities then you can answer questions such as What will be day number 4?, What is the probability that day number 4 will be in a dry state starting with a particular condition? So, this is what we will do now. So, the formulation of the transition probability matrix for this particular problem is depending on the two time steps t minus 1 and t minus 2 and look at the every time the two time steps and formulate the probabilities associated with .8. Remember the state one is w therefore, because you are looking at time period 3 this has to be w w and similarly, state 2 is d w. So, you must get d w here.

Similarly, state 3 is w d w d and so on. So, starting with any particular state you just move it and the look at whether you can get this kind of combination. If you can get the combination that is associated probabilities that is given and that is the probability that will get here.

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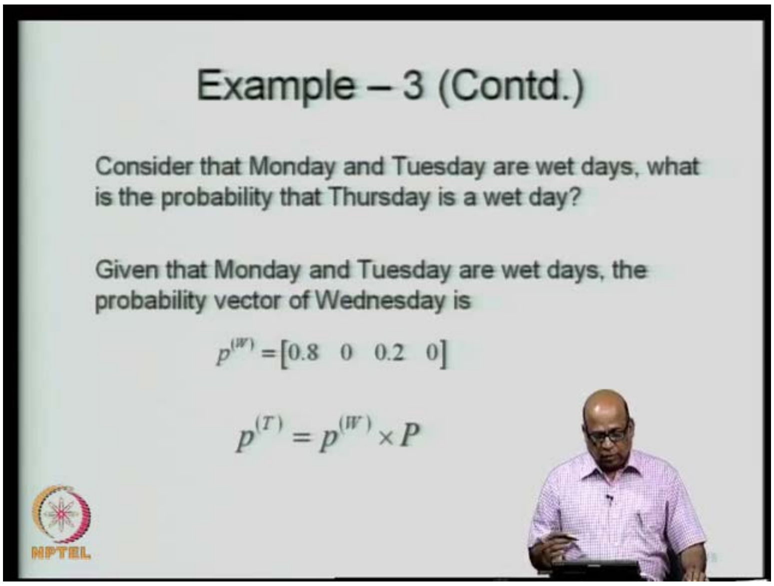

Now, let us say that Monday and Tuesday are wet days. Both the days it has rained. The, we want to say let us say we are planning for a cricket match or such thing on Thursday and Monday and Tuesday have being rainy days and we want to examine what is the probability that Thursday is also a wet day. So, given that Monday and Tuesday are wet days, the probabilities of vector of Wednesday you see what we need to do now it is a Markov chain Monday and Tuesday both have rained which means you are in state number one because both the days have been wet days. You are in state number one. You want to get the probability vector for Thursday. That means, what is the probability that Thursday is a dry day or Thursday is a wet day instead of jumping directly to Thursday which is also possible. But, what we will do is first we will see what happens on Wednesday first.

So, first we will get the Wednesday probability vector. So, because Monday and Tuesday both are wet days look at the transition probabilities Monday and Tuesday are both wet days which means that you are in state number one. So, the probability vector for the next day will be simply this because you have you have given to be in state number one.

So, I will write the probability vector as 0.8 0.20. So, this is how you obtain the probability vector for Wednesday.

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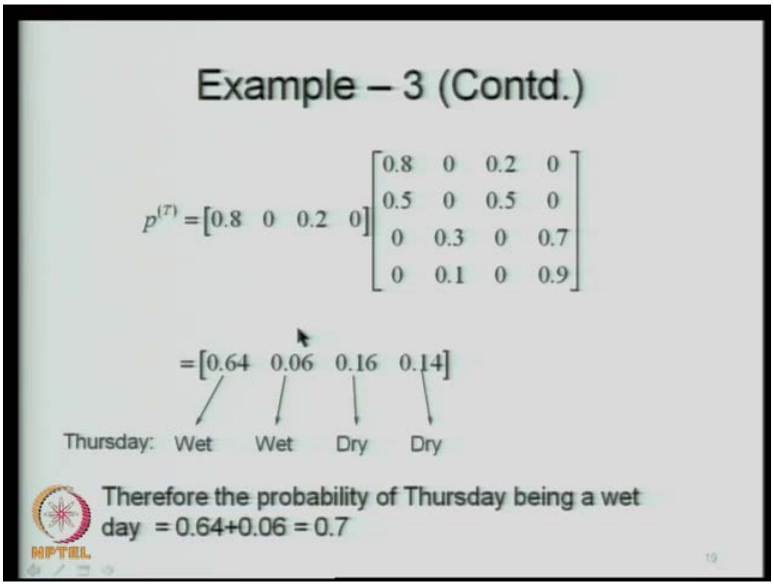

Then probability vector for Thursday is probability vector for Wednesday into the transition probability and so this you get as .8 etc. So, this is probability vector for Wednesday this is the transition probability. So, you get the probability vector for Thursday as 0.64 .06 .16 .14. So, this is wet, this is wet, this is dry, this is dry.

Therefore, the probability that on Thursday it will rain or on Thursday will be a wet day is probability of this going into wet and this being wet this is state number 1 state number 2. Both are resorting in wet conditions and this is 0.64 plus point naught 6 is equal to 0.7.

So, this is how you get the probability of a particular day being wet or dry starting with this.

So, in today’s class we have summarized through a number of examples. I specifically consider 3 examples; the concepts of transition probability, the steady state probabilities, the probability vector and how we convert a 2 step Markov chain into a single step Markov chain by appropriately defining the states.

So, this concludes the discussion on Markov chain. Of course, there are large numbers of advanced topics related with Markov chains. For example, we may talk about hidden Markov chains and then large number of applications of Markov chain which you know. Some of the application I may be discussing towards the end of this course, but, as far as theoretical coverage is concerned this concludes the discussion on Markov chain. In the next lecture we will start a new topic. Thank you for your attention.