Stochastic Hydrology

Prof. P. P. Mujumdar
Department of Civil Engineering
Indian Institute of Science, Bangalore

Lecture No. # 02

Bivariate Distributions

Good morning, and welcome to this the lecture number two of the course Stochastic Hydrology. In the last class which was the first class, we have seen some examples on which we can use, so the methods that will be discussed here. Some applications in hydrology and water resources where uncertainties are prominent, and we need methods to address the uncertainties. So, essentially was what we have covered in the last class was an introduction to the concepts of probability.

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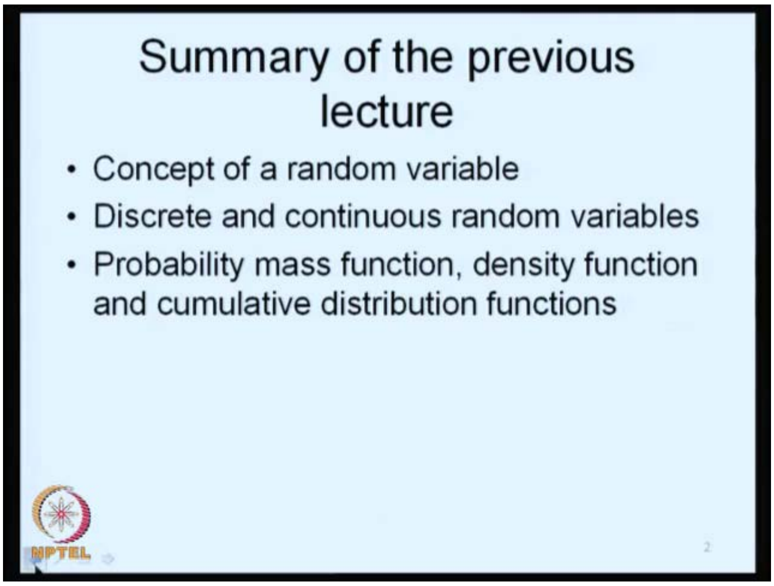

So, we went through the concept of random variable, and introduce your discrete and continuous random variables; recall that discrete to random variables can take on only discrete values, that is finite number of values or accountably infinite number of values, whereas the continuous random variables can take on values, along let say line or something infinite number of values they cannot same. Then we introduce the concept of the probability mass function for the discrete random variables. Typically we say probability of x taking on a specific value x i, that defines the probability mass function and then for the continuous random variables, we introduce the probability density functions, and for both continuous as well as random - discrete random variables, we introduced the accumulative distribution functions.

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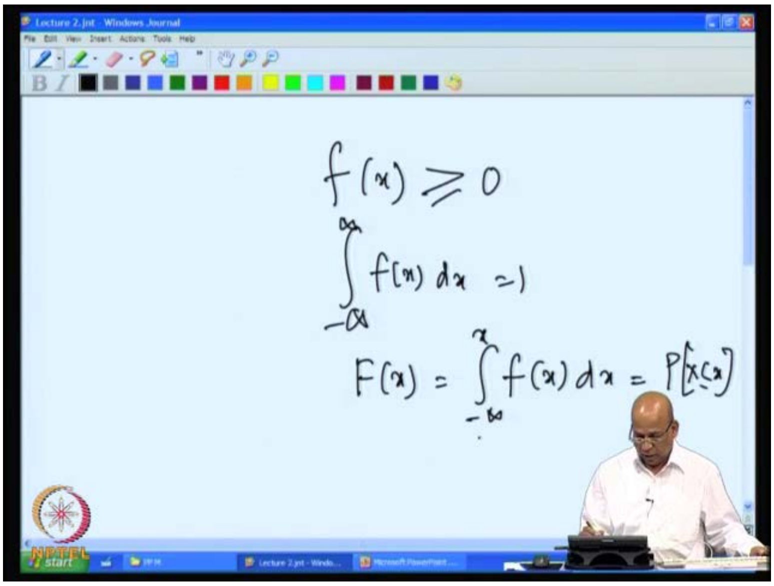

You also recall that we said for the continuous variable. We say f of x is the density function, where f of x must be non negative, and then the area under the curve minus infinity to plus infinity of f of x, with respect to x must be equal to 1 and then for the continuous random variables we also said the cdf, which gives the probability of x being less than or equal to x is given by the integral under the curve up to value of x of pdf, from this.

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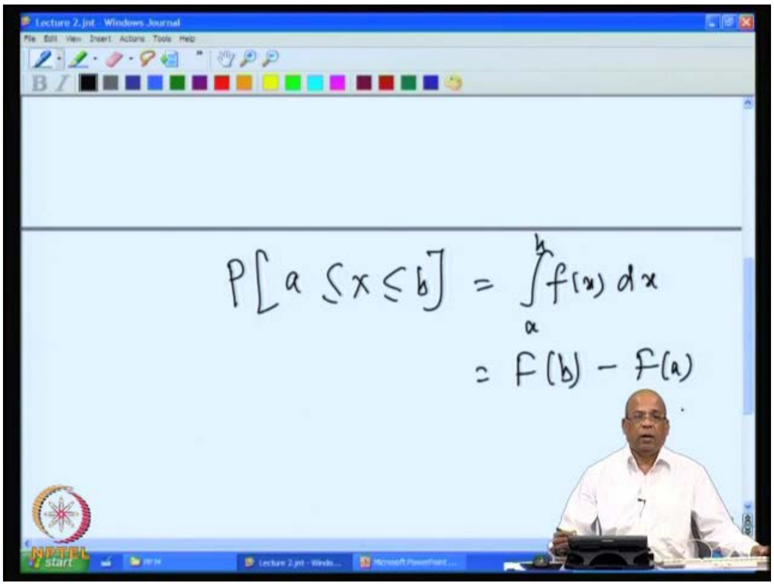

We introduce the concept that the probability that x takes on any value between the given values of a and b is simply given by a to b f of x dx, which from the definition of cdf turns out to be f of b minus f of a, which is the cdf value at the point b minus the cdf value at the point a. Then we examined a few applications related to these in terms of the numerical examples; simple numerical examples to drive home the point that we can estimate the probabilities from the given c d s. We also indicated that the pdf is not. In fact, the probability it is a probability density function and therefore, the area under the pdf for a given range provides a probability of the random variable taking on values in that range.

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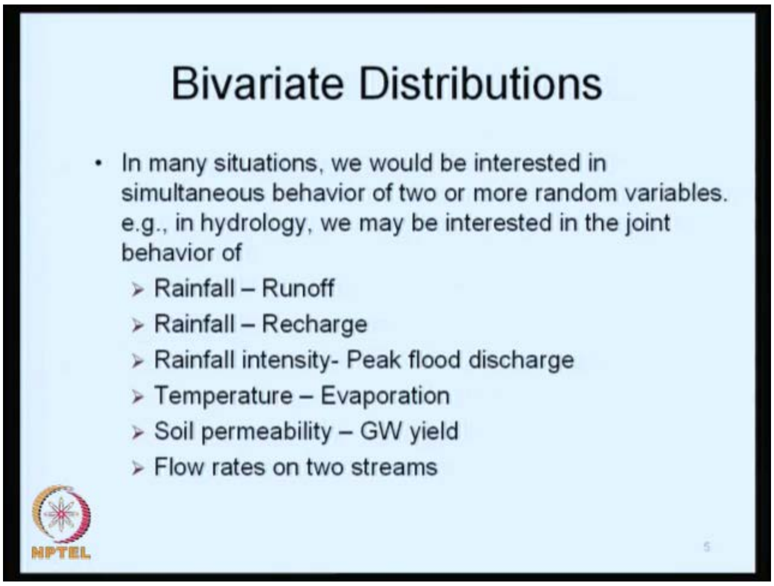

Now, we go on to bivariate distributions in the last class we covered single random variable the distributions of a single random variable in hydrology in many situations we come across problems where we would be interested in simultaneous behavior of two or more random variables. So, what we will do now is we will introduce the concept of a two-dimensional random variables first and then generalize it two-dimensional random variable typically the examples are that you know run off in a water shade maybe related to rainfall is in fact, related to rainfall and rainfall is a random variable runoff is a random variable. So, be interested in the simultaneous behavior of rainfall and runoff or rainfall and the ground water recharge.

Ground water recharge is a random variable, which is also governed by rainfall in some sense and rainfall is a random variable. So, we would be interested in the joint variations of rainfall and ground water recharge similarly, in the case of flood discharges. Let say you are talking about urban flooding, where the peak flood discharge is of interest and this is related to rainfall intensity and we would be interested in getting the joint distributions of rainfall intensity and the peek flood discharge or the joint variations of rainfall intensity and peak flood discharge.

Similarly, in the hydrologic models we would be interested in temperature and evaporation both of which are random variables, then soil permeability, and ground water yield and classic case is the flow rates on two adjacent streams where we may define q 1 as the flow way rate in one of the streams and q 2 as flow rate in another stream. The second stream and both of these are random variables, and we would be interested in the joint variations or the joint distributions of the 2 random variables q 1 and q 2. So, this brings us to the point that from the single dimension random variable we now, start talking about the joint distributions of 2 random variables.

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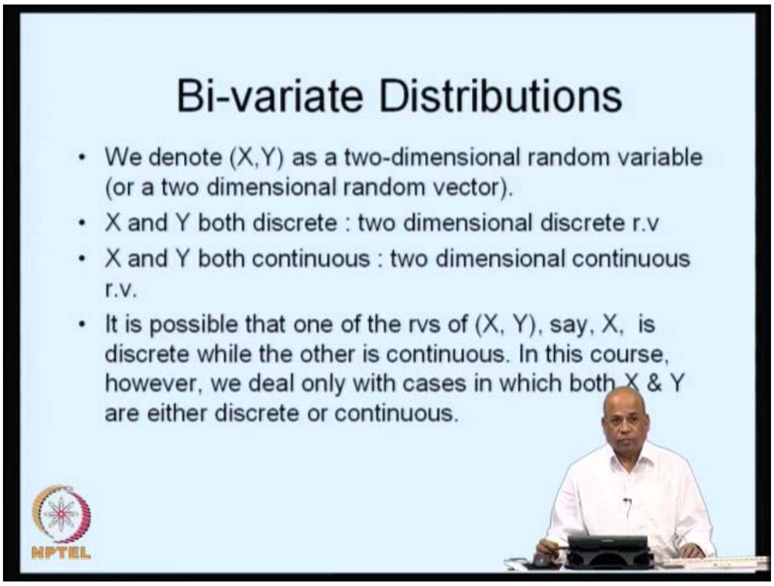

And we define the bivariate distributions, we start defining the bivariate distributions so, first from the single dimension random variable X, we move on to two-dimensional random variable denoted as X, Y this also called as a two-dimensional random vector, now we may have a case where both X and Y are discrete. Discrete random variables, and this will define a two-dimensional discrete random variable. Similarly when both X and Y are continuous we may get we define this as a two-dimensional continuous random variable. Now, in situations it is in some situations it is possible that one of the r v, let say X is discrete, while the other is continuous, Y is continuous such situations do exist, but in this course, we will not go into such random variables. We will deal with only cases where both X and Y are discrete or both X and Y are continuous.

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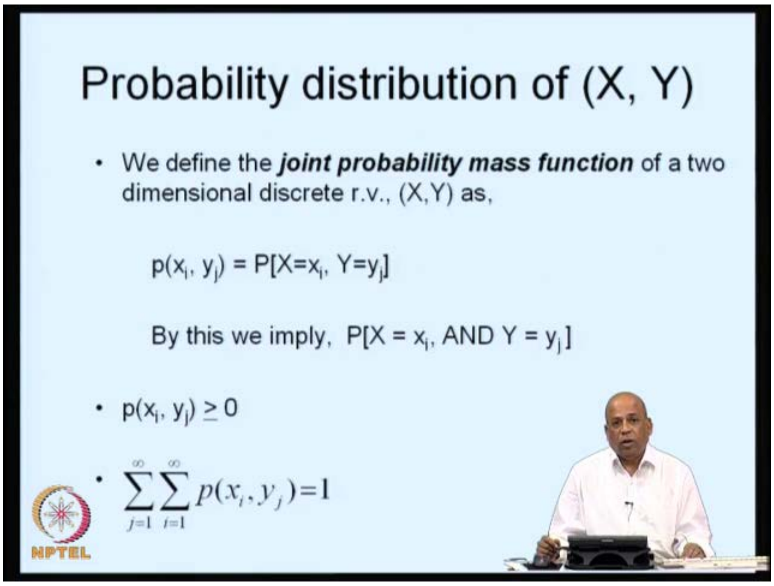

Then analogous to what we did in the single dimension random variables? we first take the discrete random variable and then define the probability mass function. So, in the case of a two dimensional discrete random variable, we call it as joint probability mass function and we define the probability x i, y j as probability of X is equal to x i, Y is equal to y j.

When we are talking about the joint distributions in this form we mean probability of X taking on a particular value of x i, and simultaneously Y taking on a particular value y j. So, this is what is meant by probability of x i, y j now, by the definition of probability; obviously, this probability of x i, y j is non negative and sum of all the probabilities over the entire region of x and y must be equal to 1. So, the probability mass function, which is for the two-dimensional random variable. We call it as joint probability mass function satisfies. These two conditions much the same way as the probability mass function of single random variable, satisfy probability of x i being greater equal than or equal to 0, and the sum over all possible values of x i must be equal to probability of x i must be equal to 1.

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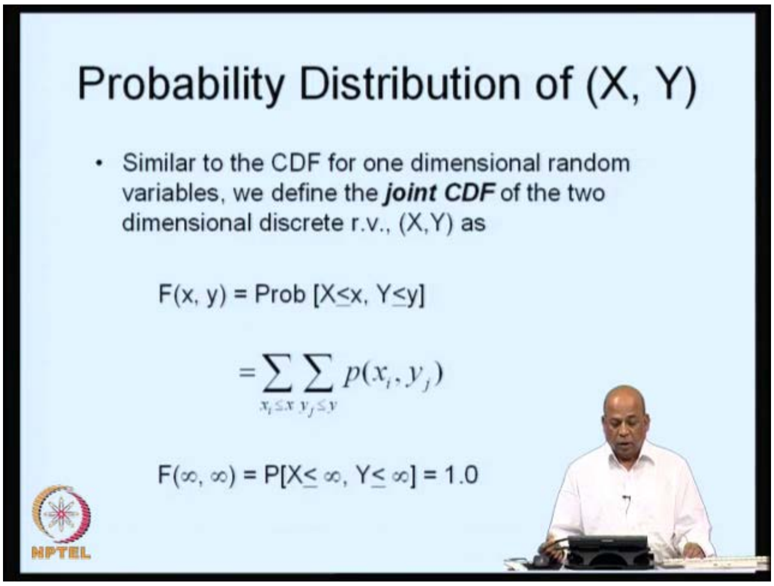

Then we define the cumulative distribution function as F of x, y is the probability that x is less than or equal to this given value of X, and Y is less than or equal to this given value of y, which means with the summation notation we sum over all those possible values of x i which are less than or equal to this given value of x, and all possible values of y j, which are less than or equal to this given y, we sum the probabilities of x i, y j. So, this gives the probability that X is less than or equal to x, and Y is less than or equal to y from this it is clear that F infinity that means, probability that X is less than or equal to infinity and Y is less than or equal to infinity must be; obviously, equal to 1, because you are summing up all the available probability all the probabilities over the entire region.

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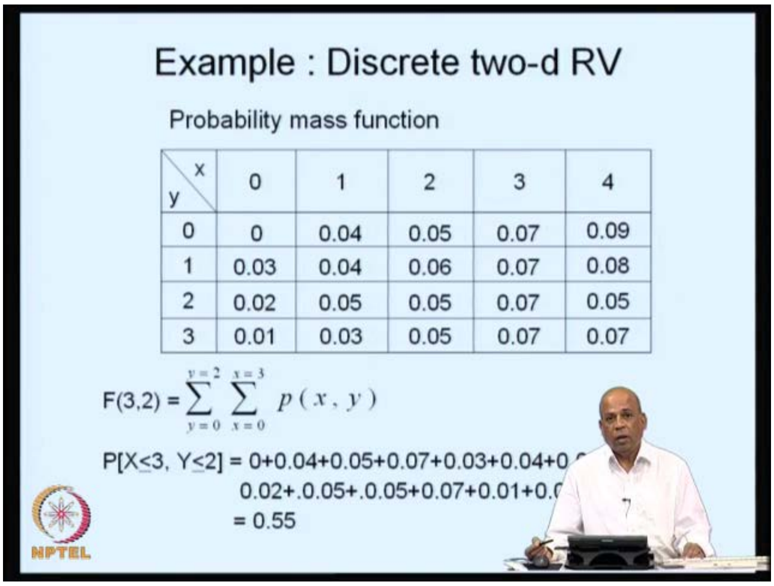

So, we will take; for example, x can take on 0, 1, 2, 3, 4 these are the discrete values. So, i is equal to 1, i is equal to 2, i is equal to 3, i is equal to 4, and i is equal to 5, similarly y can take on values 0, 1, 2, 3. j is equal to 1, j is equal to 2, j is equal to 3, and j is equal to 4. The numbers here in the body of the text for example, 0.04, 0.06, etc, these indicate the probabilities for example 0.04 indicates the probability that x is equal to 1 and y is equal to 1 that is 0.04, similarly probability that x is equal to 2 and y is equal to 2 is 0.05. So, in general these numbers indicate probability that x is equal to x i and y is equal to y j. So, this is the joint probability mass function of the two dimension random variable x, y when both x and y are discrete.

Now, from this we should be able to get the probabilities, let say we are interested in probability that x is less than or equal to 3 and y is less than or equal to 2. So, we identify the region in this range space of x y in which we are interested in for example, we identify the region, where x is less than or equal to 3 and y is less than or equal to 2. This region is that x will be less than or equal to 3 and y can be either 0 or 1 or 2 so, this entire region denotes x is less than or equal to 3 and y is less than or equal to 2. So, we sum over this region all the probabilities so, probability of x is less than or equal to 3 and y is less than or equal to 2 will be 0 plus 0.04 etc. So, this entire range we take the probabilities and sum it over and that is also equal to F of 3, 2, y goes from 0 to 2, and x goes from 0 to 3 of probability of x, y. As I said any of these numbers indicate probability the random variable. x taking on this particular value of x and y taking on this particular value of y. So, we get the associated probability as sum of all these probabilities, which will be equal to 0.55.

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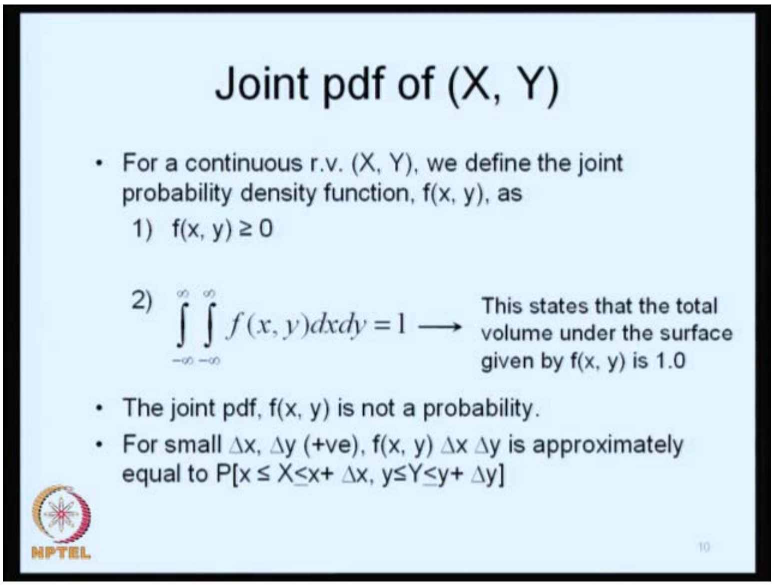

Then we come to the joint density function, when x y is a continuous random variable, two dimensional continuous random variable. We define the joint probability density function f of x, y now, similar to the single dimension density function any function, which satisfies x f of x, y being non negative and the total volume under the curve f of x, y. Under the surface f, f of x, y must be equal to 1. So, this implies the integral minus infinity to plus infinity minus infinity to plus infinity of the joint pdf. f of x, y with respect to x, and y must be equal to 1. So, the total volume under the surface given by f of x, y is 1.

Remember that this is a density function and therefore, it does not give probability; however, for small delta x and delta y or if you consider f of x y delta x delta y, which is actually the volume under f of x y over this range delta x delta y. This is approximately equal to the probability of x taking on a value between x and x plus delta x and y taking on a value between y and y plus delta y. So, similar to what we did in the single dimension random variable, if you take the volume under the joint pdf, f of x y over a particular region that volume gives the probability of x y the two dimensional random variable x y assuming values in that particular region. So, to get the probabilities as we did in the single dimension random variable. We identify the region in, which we are interested in, and get the volume under the surface f of x y in that region, and that gives you the probability that the two dimension random variable x y assumes values in that particular region.

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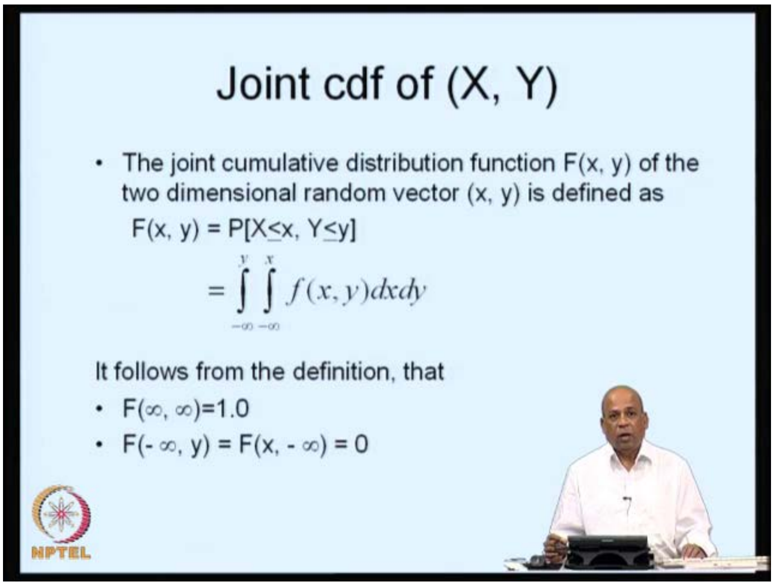

So, we defined now the joint cumulative density distribution function F of x, y. As F of x, y is equal to probability of X being less than or equal to x, Y being less than or equal to y, recall that the small x and small y are the values that the random variables can assume. So, for specified values of x and y the cdf F of x, y is defined as minus infinity to y minus infinity to x integral of that f of x y, dx dy. So, from this definition it is obvious that probability that x taking on value less than or equal to infinity and y taking on values less than or equal to infinity must be equal to 1. So, minus infinity to 1 minus infinity to 1, which is which follows from the definition of pdf. Similarly, F of minus infinity to y, which means x takes on value of x less than or equal to minus infinity and y must be equal to F of x, minus infinity both of these must be equal to 0 as it follows from this definition.

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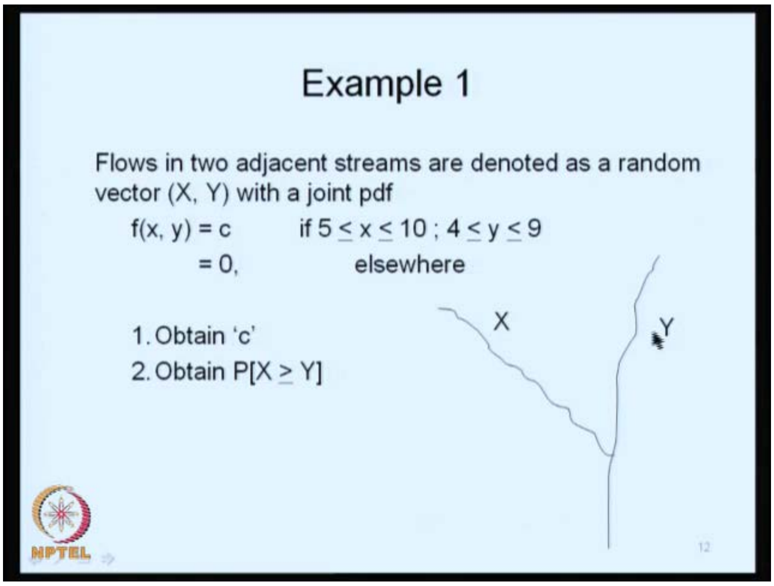

Now, with this we will just have a few examples; to see how the joint probability density function is applied. Let say we are considering flows in to streams two adjacent streams, one of the flows we indicate as the random variable x the other one as the random variable y. The joint density function f of x, y is given by a constant c, for the range x taking on values between 5 and 10, and y taking on values between 4 and 9 and this is 0 elsewhere so, first let us obtain c and then also, we will see how we get the probability that X is greater than or equal to Y. So, we may be interested in getting the probability that the flow in this particular stream is greater than the flow in this particular stream. These kinds of problems are important, because we may want to make decisions on let say you want to make builder, there are here or a there are here, then we would be interested in getting what is the probability that the flow in this stream is greater than the flow in this particular stream.

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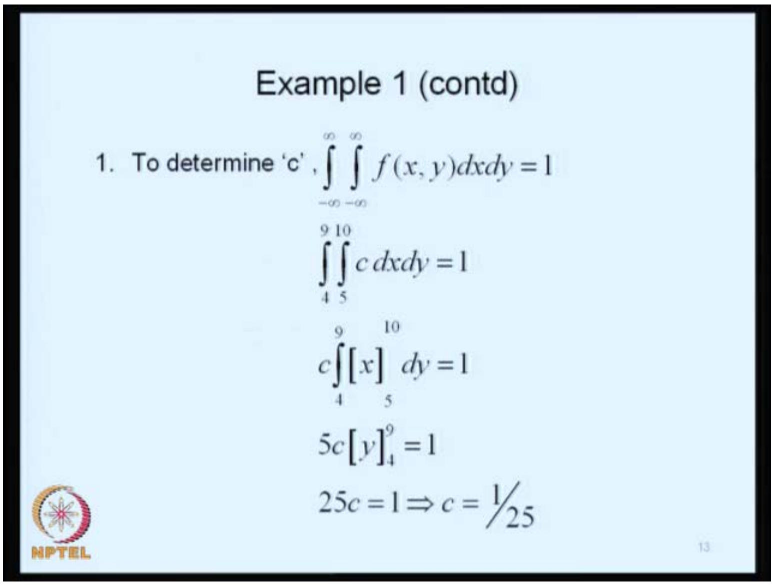

So, first to determine c we use the definition of the joint pdf the volume under the joint pdf for the region must be equal to 1. So, your x varies from 5 to 10 and y varies from 4 to 9 and your cdf is c constant dx dy this should be equal to 1. So, from this you get first two integrate with respect to x you get x varying between 5 and 10, and then integrate with respect to y that should be equal to 1. So, this is 5 c from this and then y taking on value between1 to 9. So, from this we get c is equal to 1 by 25. So, we have completely defined the joint pdf now as f of x, y is equal to 1 by 25 for the range x taking on values between 5 and 10, and y taking on values between 4 and 9.

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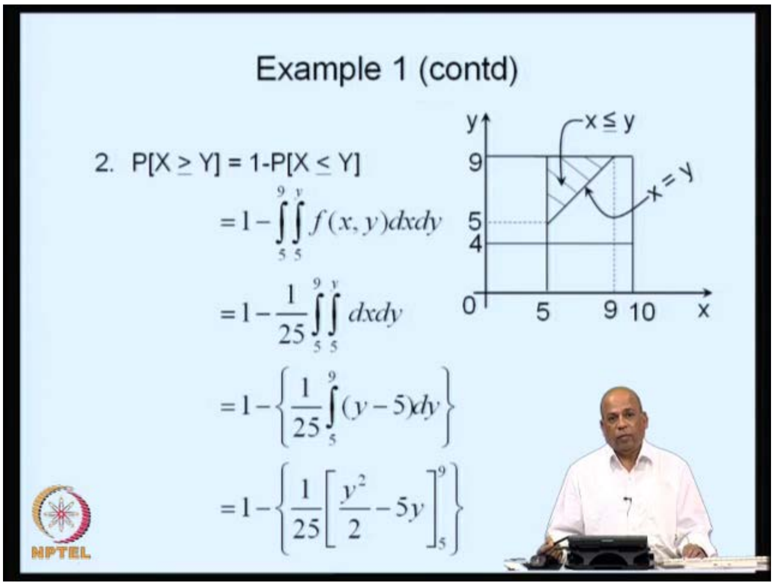

Then we looked at the problem. What is the probability that x is greater than or equal to y? Now, for this we first identify the range space over with f of x y is defined to be non negative, look at this figure now this is your y and this is x. So, y takes on values between 4 and 9, x takes on values between 5 and 10. So, this is the range over, which f of x y is non 0 it is of nonnegative, and we are interested in getting the probability that x is greater than or equal to y. So, first we will identify the region, where x is greater than or equal to y for that we draw a line x is equal to y in this region so, x is equal to y, because x the lower value of x is 5. So, we start with 5 and then draw a line x is equal to 5. In the region above this y will be greater than or equal to x, and in the region below this x will be greater than or equal to y. So, we are interested in probability that x is greater than or equal to y. So, we are actually interested in this region x is greater than or equal to y.

We use the fact that probability that x is greater than or equal to y can be written as 1 minus probability of x been less than or equal to above y. So, we can focus on this region, where x is less than or equal to y, and then get the probabilities. So, this we write it as 1 minus you look at this region we are allowing the y to very first. So, y varies from that is, where allowing x to vary first. So, x varies from 5 to y, let say I draw a line from here to here. So, at this line you have x is equal to y so, x goes from 5 to y. So, that is what to near it is x goes from 5 to y and y goes from 5 to 9. So, y goes from 5 to 9. So, we are focusing on this area now over this area we are integrating the function f of x, y. So, we do that f of x, y is 1 by 25, and then we get this is from 5 to 9 with respect to y.

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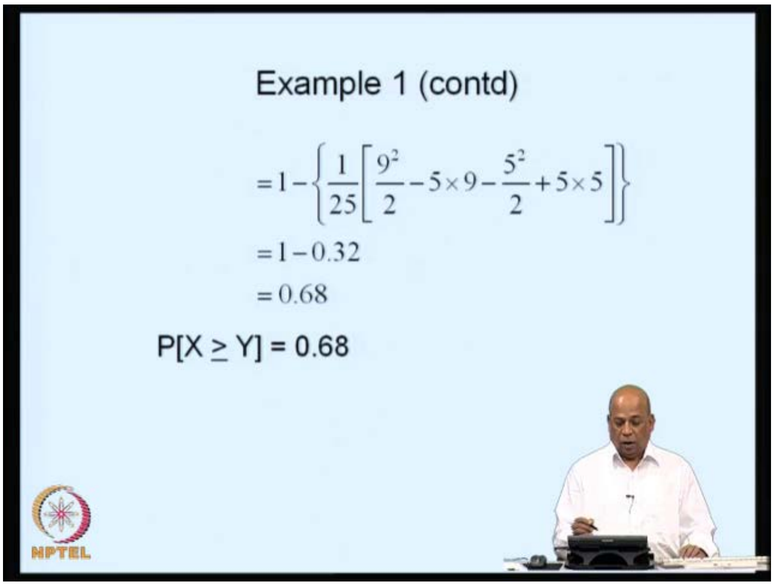

We get values like this, when we simplify this you get 1 minus 0.32. So, this 0.32 and you get 0.68. So, probability of x being greater than or equal to y you get it as 0.68.

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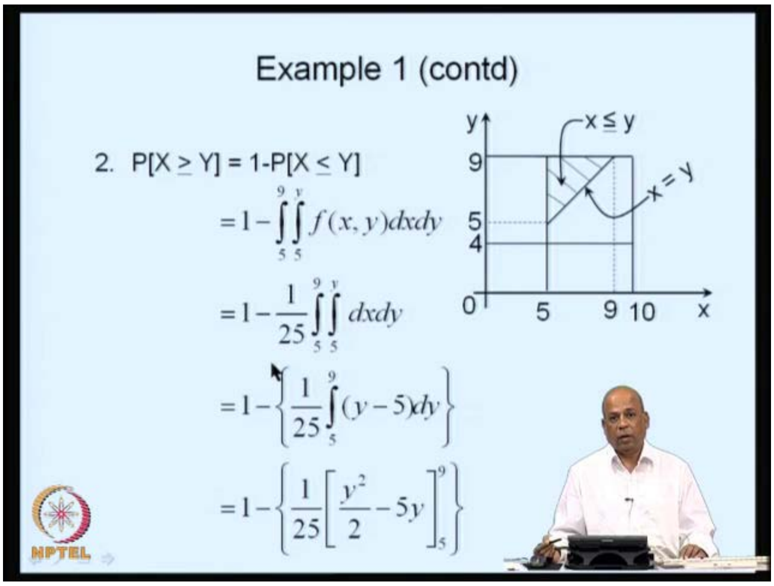

Just revisit the problem you can use it do it as a homework assignment, what we did is we integrated over this area. Now this is the region where x is greater than or equal to y. We would have got directly probability of x greater than or equal to y by integrating f of x y over this region. So, do this as a assignment for which what you need to do is that you take these two areas, up to this point and then the rectangle consisting of this. So, we need to define the region where x is greater than or equal to y, and then integrate f of x y over that particular region you must get the same answer 0.68.

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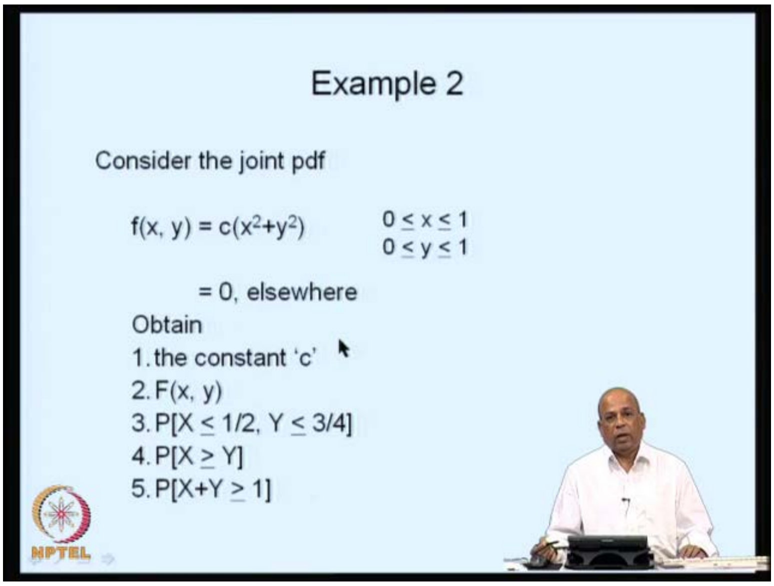

We will take another similar example, except that we are talking about different probabilities, there different types of probabilities. Let say f of x, y is equal to c x square plus y square for the region x taking on value between 0 and 1, and y taking on value between 0 and 1, it is 0 elsewhere. So, let us first get the constant c as we did in the previous example, then we get the joint cdf from, which we will get probability that x is less than or equal to half, and y is less than or equal to 3 by 4 we will also get probability that x is greater than or equal to y as we did just now and probability that x plus y is greater than or equal to 1.You must remember that, when we were talking about probabilities of the joint random variable taking on some specified values, you first identify the region in, which you are interested in and, then integrate the joint pdf over that particular region. So, identification of the region of interest in the two dimensions is what is important in this cases.

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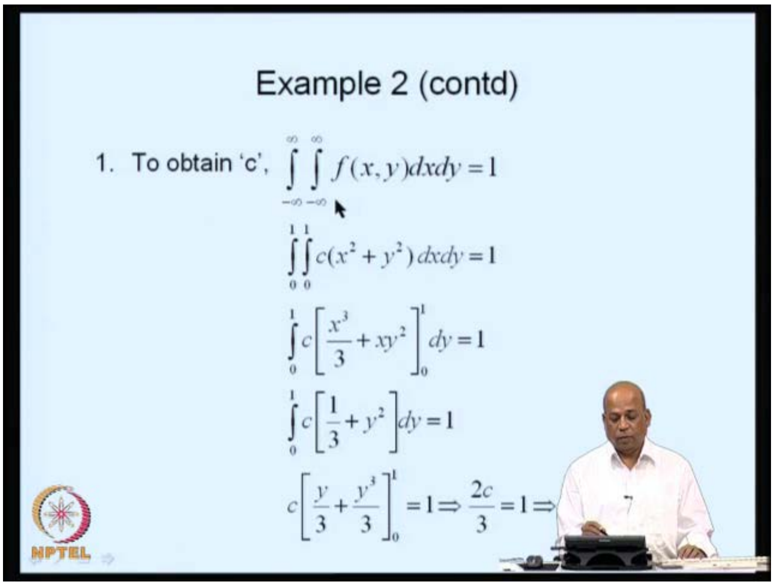

So, first will obtain c, we will simply use the definition of the joint pdf. So, the double integral of f of x, y dx dy over minus infinity to infinity, and minus infinity to infinity dx dy must be equal to 1, which means the total volume under the surface must be equal to 1. So, when we do this it is a fairly straight forward integration. So, I will not go into the details of this. So, you get c is equal to 3 by 2.

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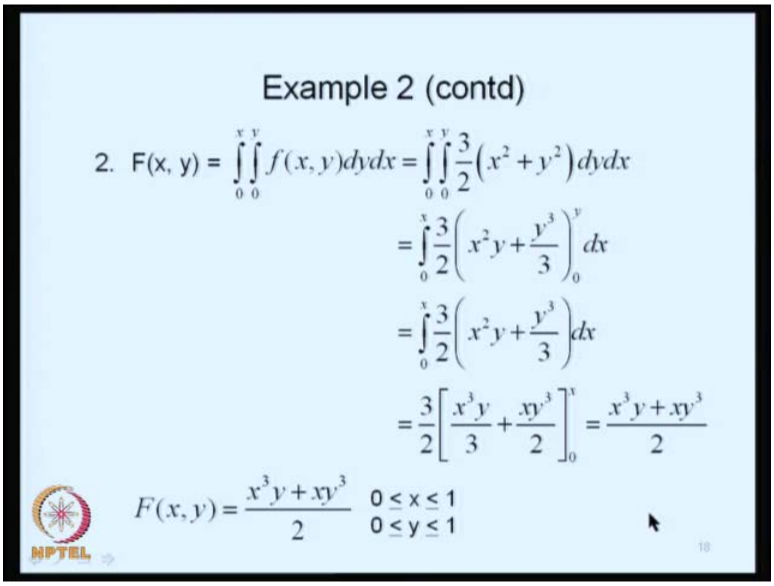

Once you get the constant c you have completely defined the joint pdf, then let see we how we get the joint cdf from this, because your x goes from 0 to 1, and y goes from 0 to

1 your lower limits for the integration will be 0 in both the cases x as well as y. So, first we will vary y and then vary x. So, f of x, y is integral of f of x y, that is the pdf with respect to y, and x the limits of integration are 0 to y for dy for the variable y and 0 to x for the variable x again this is a very straight forward integration, where you substitute this values and get the joint cdf, F of x y as x cube y plus x y cube divided by 2, remember whenever we define either the joint pdf or the joint cdf we must indicate the range over, which the expression is valid and it is understood that outside of this range the value is 0 or the value is the value is 0 in this particular case.

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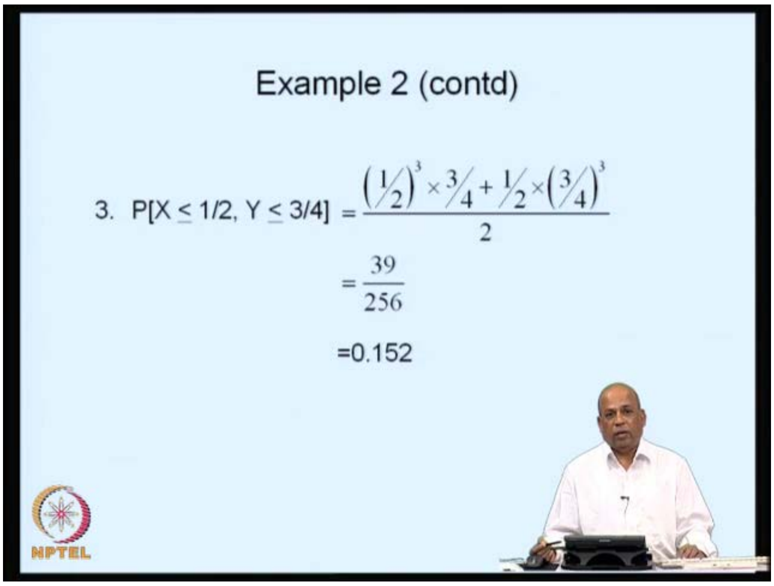

Then we go on to f of x being less than or equal to half and y being less than or equal to 3 by 4. This follows from definition of the cdf recall that the cdf provides us probability of x being less than or equal to a specified value of x and y being less than or equal to a specified value of y. So, in the expression of cdf we just obtained we provide we substitute x is equal to 1 by 2 and y is equal to 3 by 4, and that is what we do here this is just a expression of cdf as we obtained here and in this we substitute the values of x and y as provided here and we obtained the probability as 0.152.

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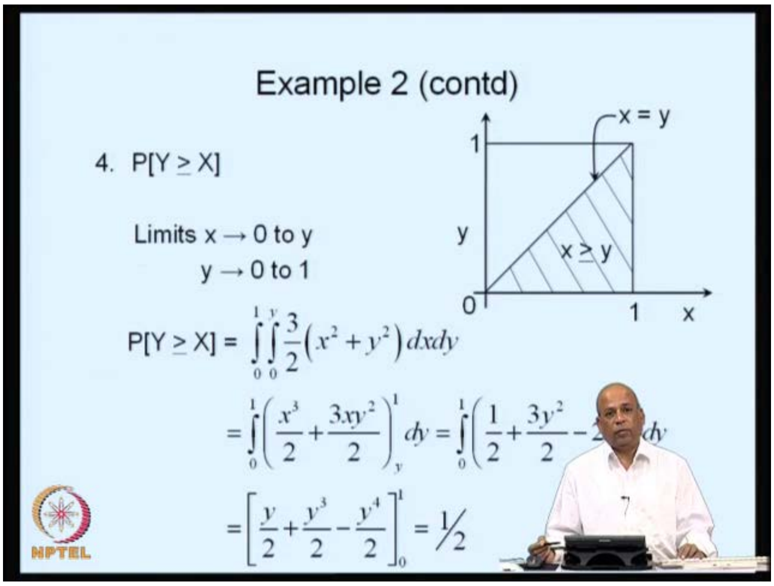

Then we look at the case probability of y being greater than or equal to x. Again as we did in the previous example we identify the region over, which we are interested in we are interested in the region y being greater than or equal to x. So, we draw a line x is equal to y and notice that in this region. x will be greater than or equal to y and in this region, y will be greater than or equal to x. So, we are actually interested in this particular region. So, once you identify the region you integrate the joint pdf over this region to obtain the probability of y being greater than or equal to x. So, in this region let say you take a line horizontal line. So, x horizontal strip actually you, where x varies from 0 to y on this line x is equal to y. So, x is equal to 0 here and x is equal to y here, and y varies in this region from 0 to 1. So, we fix a limits as x varying from 0 to y and y varying from 0 to 1.

And we obtain probability of y being greater than or equal to x by integrating the f of x y as we just obtained f of x y this is a joint pdf, which is defined over this region and integrate in this specified region to obtain the associated probabilities. So, here as you can see x varies from 0 to y from here to here and y varies from 0 to 1and by integrating you get probability of y being greater than equal to x as 1 by 2.

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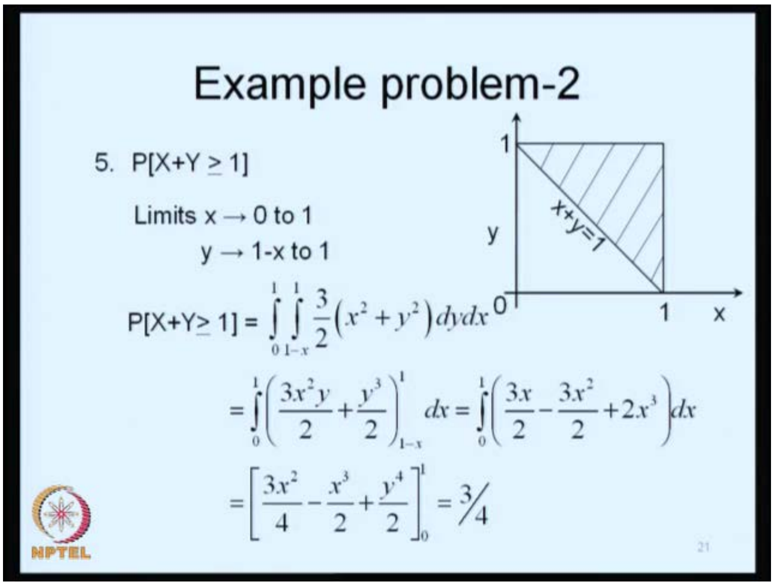

Next we see probability of x plus y being greater than or equal to 1 again we draw a line x plus y is equal to 1, and we looked at the region where x plus y is greater than or equal to 1; say for example, we are looking at y varying from 1 minus x to 1. In this region 1 minus x to 1 and x goes from 0 to 1. So, x is going from 0 to 1 and y goes from 1 minus x to 1. So, we integrate the joint pdf over this region 1 minus x to 1 and 0 to 1 and obtained the probability that x plus y is greater than or equal to 1 as 3 by 4.

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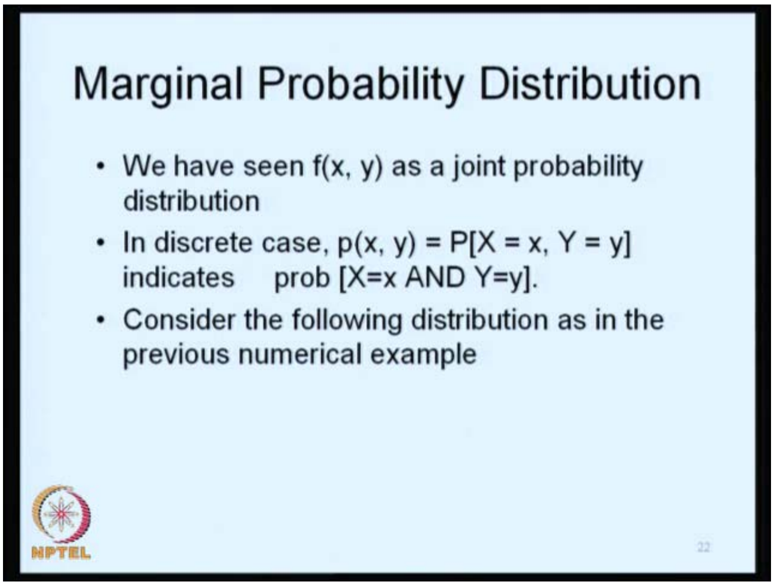

So what we just now did is that we defined the joint probability mass function in the case of a discrete random variable - discrete two dimensional random variable, and the joint probability density function in the case of the two dimensional continuous random variable. Now, let us see that if we are given the two dimensional marginal probability function or probability mass function, can we get back to the original distribution of the single dimension random variable.

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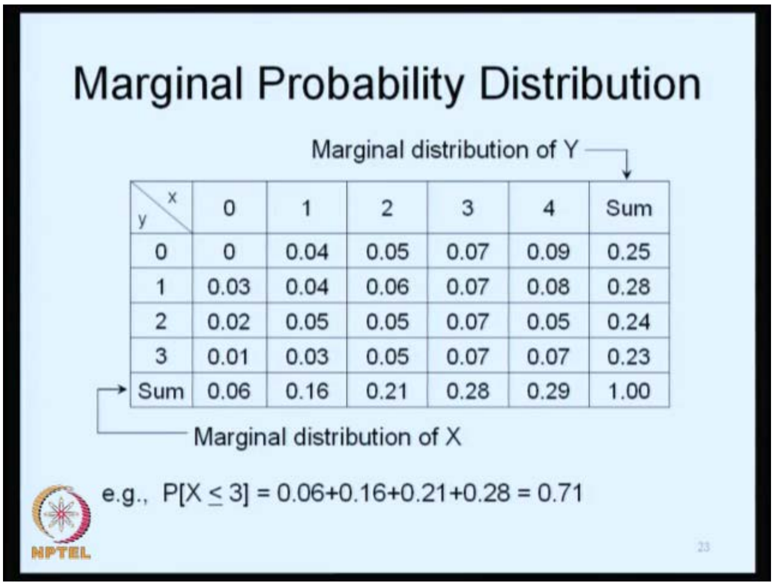

y \ x	0	1	2	3	4	Sum
0	0	0.04	0.05	0.07	0.09	0.25
1	0.03	0.04	0.06	0.07	0.08	0.28
2	0.02	0.05	0.05	0.07	0.05	0.24
3	0.01	0.03	0.05	0.07	0.07	0.23
Sum	0.06	0.16	0.21	0.28	0.29	1.00

Let us look at this example, let say you have the distribution from the same example that we discussed just now, for the discrete two dimensional random variable x takes on value 0, 1, 2, 3, 4 and y takes on value 0, 1, 2, 3 and these are the probabilities for example, this gives a probability that x is equal to 0 y is equal to 0 and so on. Now, let us look at the sum of these probabilities, that is we take the first row where y is taking on a value of 0, and add all these probabilities 0 plus 0.04 etc. We get 0.25. What does this indicate now? This indicates probability that y is equal to 0 and x is equal to 0 plus probability that y is equal to 0 and x is equal to 1 plus probability that y is equal to 0 and x is equal to 2 and so, on. So, this number 0.25, which is the marginal sum of all these probabilities here. In fact, indicates probability of y being equal to 0. Irrespective of the value of x similarly probability of y is equal to 1 is 0.28 probability of y is equal to 2 is 0.24 and so on. So, the marginal sums here. In fact, indicate probability of y is equal to 0, probability of y is equal to 1, probability of y is equal to 2, probability of y is equal to 3 and this is in fact, the marginal distribution of y.

Similarly, you look at the marginal sums over the columns here. So, this sum here indicates the probability of x is equal to 0 and y is equal to 0 plus x is equal to 0 and y is equal to 1, x is equal to 0 and y is equal to 2 etc. So, this is the probability of x is equal to 0 irrespective of the value of y, because we are summing over all the possible values of y and therefore, we obtain the probability distribution of x here. So, x is equal to 0 probability of x is equal to 1 probability of x is equal to 2 etc. So, we have defined the probability mass function of x here, and we have defined the probability mass function of y here. Once you get this you can talk about probabilities associated with one of the random variables, let say you are interested in probability of x being less than or equal to 3, which means you will pick up those probabilities along the probability mass function of x and look at 0.06 plus 0.16 plus 0.21 plus 0.28.

So, from the joint probability mass function of x, y you have now arrived at marginal distribution of x here and marginal distribution of y and therefore, you will be able to talk about probability associated with one of the random variables not both the random variables together. So, the difference here is that in the joint probability mass function, we talked about a simultaneous variation of the two variables for example, we were talking about probability of x is equal to x and y is equal to y, where as in the marginal probability distribution we are talking about probability of a particular variable irrespective of the value that the other random variable takes.

Similarly, we do this for now these are the important points here in the marginal probability distribution. The marginal totals give probability of y is equal to y and probability of x is equal to x respectively, that is this gives probability of y is equal to y and this row here gives probability of x is equal to x notice that the sum of these must add up to one, because you are talking about the probabilities. Probability mass function of x is here. So, the sum must be equal to 1.

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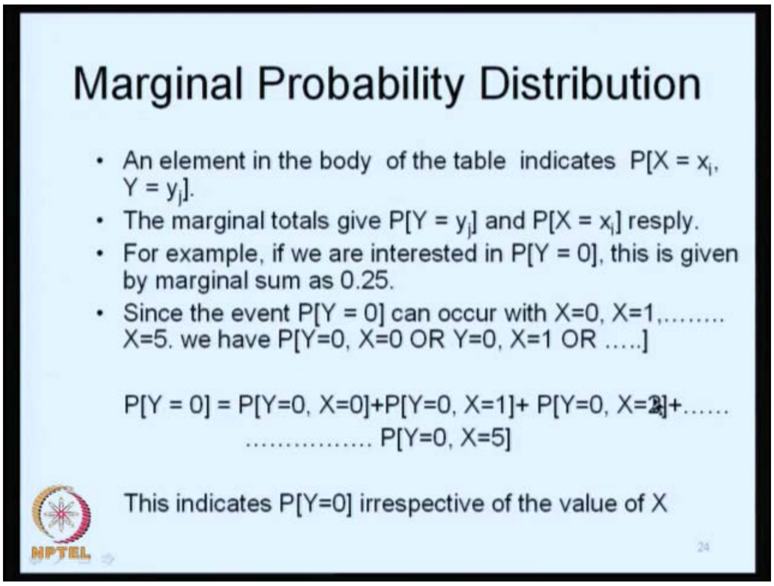

Let see you are talking about probability of y is equal to 0, this can occur with x is equal to 0, x is equal to 1, etc, x is equal to 5. So, we define this as probability of y is equal to 0 is equal to probability of y is equal to 0 and x equal to 0 plus probability of y is equal to 0 and x is equal to 1 and so on. So, this indicates as I said probability of y is equal to 0, irrespective of the value that the random variable x takes.

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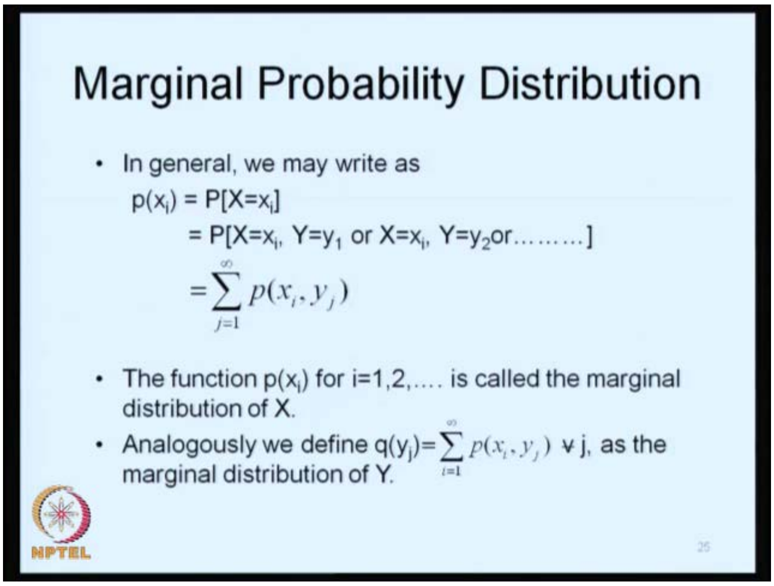

So, in general we write this as probability of x i, which is a marginal probability is equal to probability of x is equal to x I. x takes on a particular value x I, and that we write it as probability of x is equal to x i, y is equal to y i or x is equal to x i, y is equal to y 2 and so on which is simply written as sum over all possible j of p of x i, y j now, the function p of x i for i is equal to 1, 2 etc is called the marginal distribution of x. Analogously we also define the marginal distribution of y which is simply q of y j is equal to sum over all possible values of x, p of x i, y j. So, to obtain the marginal density marginal distribution for x you sum over all possible values of y marginal distribution of y, you sum over all possible values of x.

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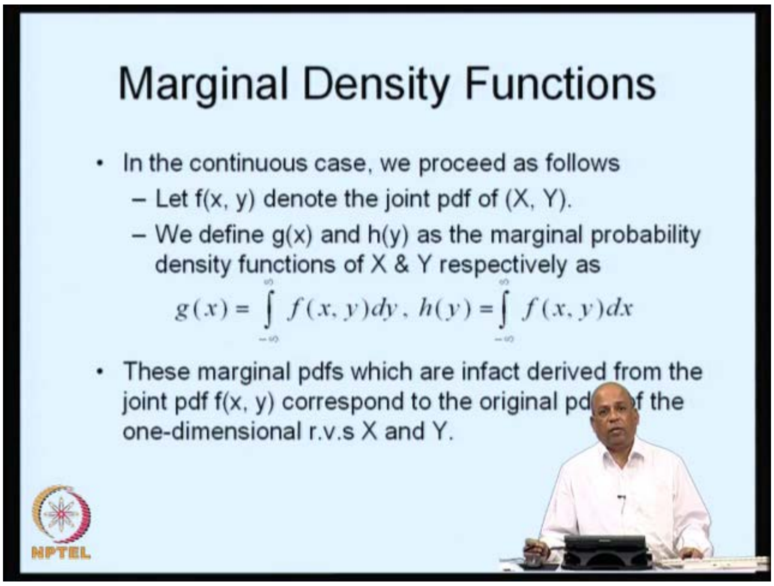

Similarly we go to the continuous case, and denote we define the marginal densities for x as g of x what we in the case of discrete distributions we summed over all possible values of y to get the marginal distribution of x. So, the marginal density of x is obtained by integrating over the entire region of y the joint pdf f of x y. Similarly the joint density h of y is obtained by integrating over x the joint pdf f of x y, now, these marginal densities are derived from the joint densities, but we also can see that these marginal densities are in fact, the original distributions of x original densities of x and y themselves.

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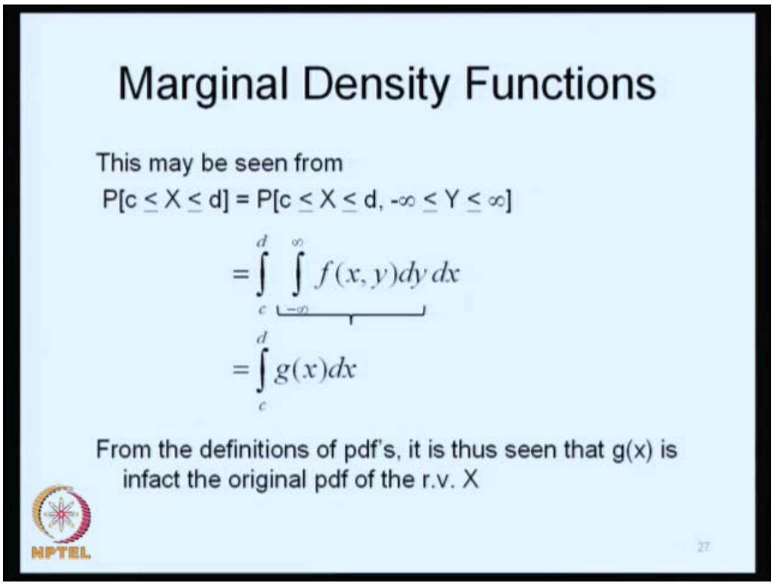

This may be seen for example, if we are interested in probability of x lying between c and d this is obtained as probability that x lies between c and d, and y lies between minus infinity to plus infinity; that means you are talking about probability of x taking on a certain values irrespective of where y lies now, that is given by first two integrate with respect to y to get the probability that y lies between minus infinity to plus infinity. So, f of x y, dy and then you integrate with respect to x for getting the probability that x lies between c and d. Now, by definition of our marginal density integral minus infinity to plus infinity f of x y, dy is in fact, the marginal density g of x and therefore, we write this as integral c to d of g of x d x.

How would we have obtained the probability of c the probability that x takes on values between c and d, if we had the original density function f of x we would have simply integrated the density function f of x in the range c to d with respect to x. So, from this it is obvious that the marginal densities as we obtained from the joint densities are in fact, the original probability density functions of the two random variables x and y.

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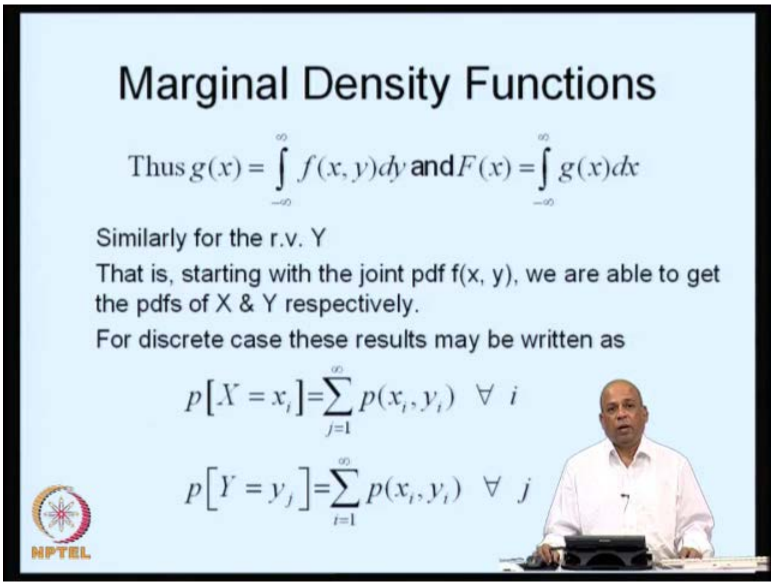

We thus summarize g of x we define as minus infinity to plus infinity f of x y, dy we integrate over by, and then we obtained the associated cdf as minus infinity to x this has to be x g of x dx. So, given the joint cdf joint pdf, we can obtain the marginal pdf of both x and y, and then start talking about the cdf associated cdf. Similarly we can do it for the random variable y now, in the case of discrete random variables, this results can be summarized as follows as we have discussed probability of x is equal to x i, you sum over all possible values of y similarly y is equal to y j you sum over all possible values of x i.

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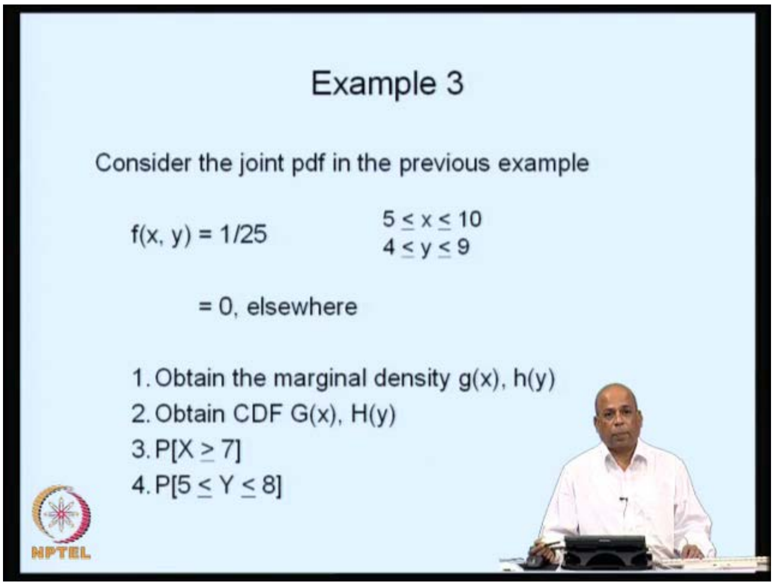

Let us now consider one of examples, let say your f of x y is defined as 1 over 25 the same example, that we talked of earlier where x ranges between 5 and 10 and y goes between 4 and 9 and it is 0 elsewhere. So, first let us get the marginal densities g of x and h of y we will also obtain the associated cdf, the G denotes the cdf of f x and H denotes the cdf of y, and then from this we should be able to get probabilities such as probability as probability of x being greater than or equal to 7 and probability of y lying between the certain range 5 to 8.

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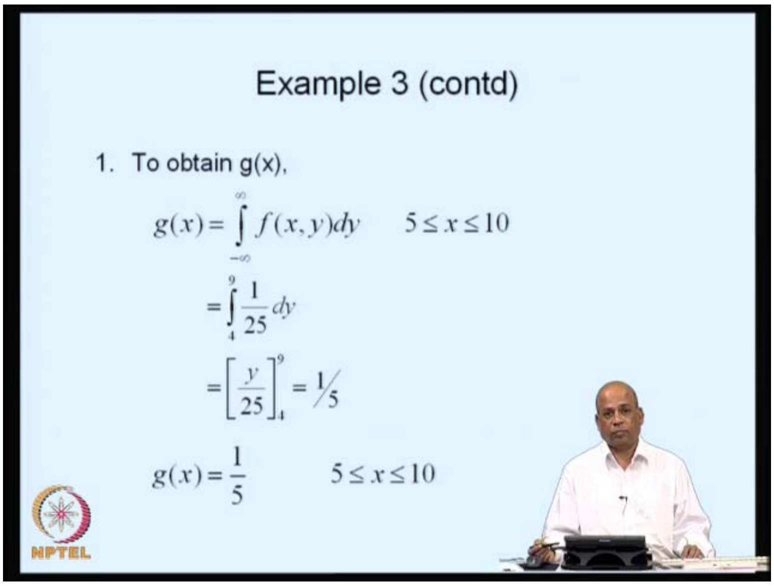

First to obtain g of x we integrate with respect to y for the entire region so, we integrate from 4 to 9 with respective y. So, you get g of x as 1 by 5 in this particular case it turns out to be a constant, but in general g of x will be a function of x alone and h of y will be a function of y alone so, we get g of x is equal to 1 by 5 for the region x lying between 5 and 10.

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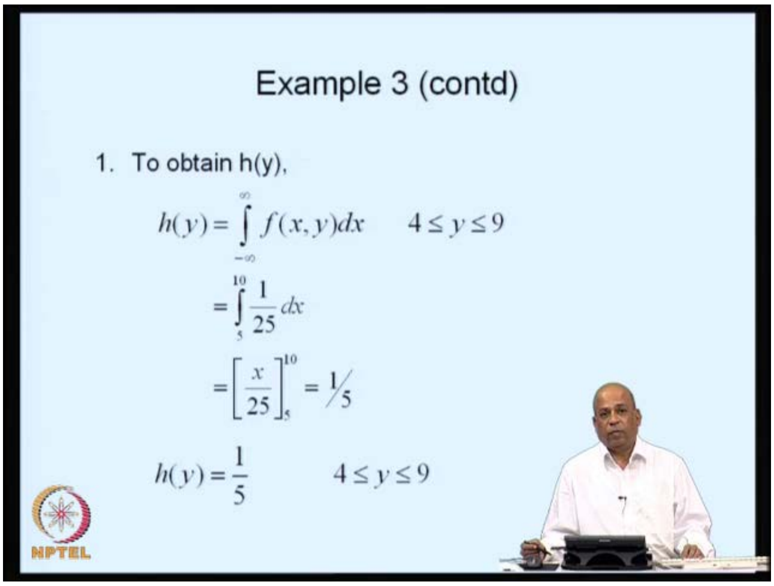

Similarly we get h of y by integrating the joint pdf over the entire region x, and we again obtain this as 1 by 5 for the region y lying between 4 and 9. Then we obtain G of x which is the cdf of x from the pdf we obtain this as x minus 5 by 5 in the region x lying between 5 and 10, and then we obtain h of y as y minus 4 by 5 for the region y lying between 4 and 9.

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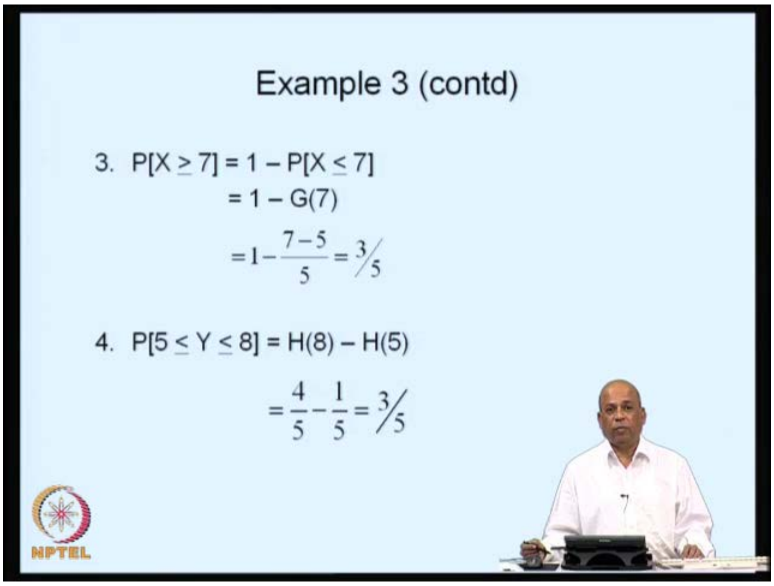

Form these we get the probabilities, probability of x being greater than or equal to 7 as 1 minus probability of x being less than or equal to 7, which is 1 minus G of 7, which turns out to be 3 by 5, similarly probability of y taking on values between 5 and 8 turns out to be 3 by 5.

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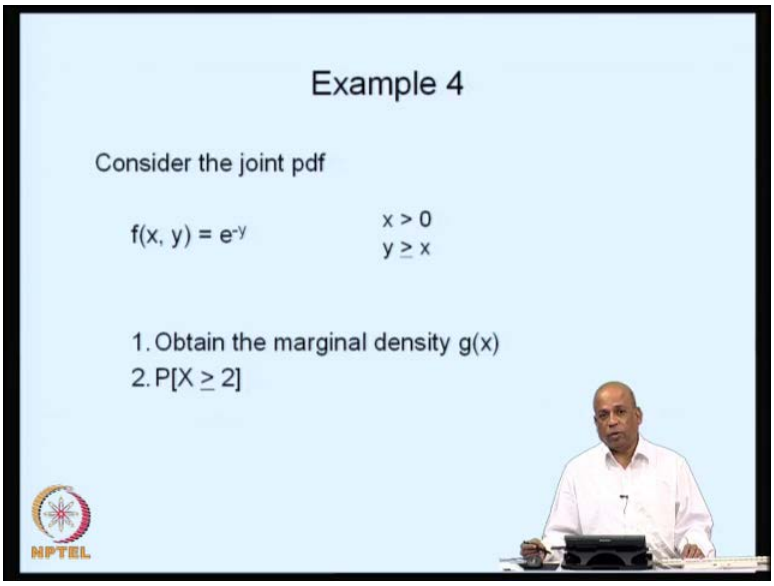

We will again do another example, so, that the use of the joint pdf to obtain the marginal densities is clear. So, f of x y is equal to e to the power minus y, x is greater than 0 and y is greater than or equal to x, and we will be interested in getting the marginal density of x as well as probability, that x is greater than or equal to 2 from the marginal density.

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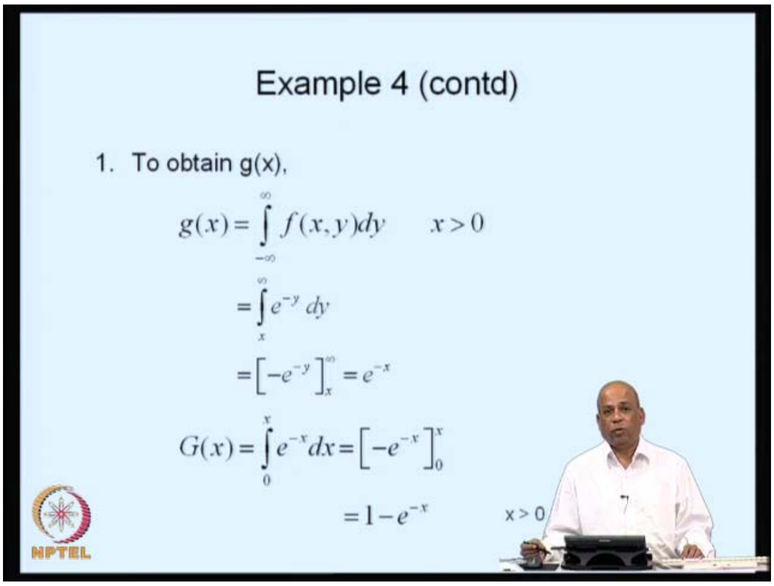

So, to obtain g of x, what we do is we integrate over the enter region y, because y is greater than or equal to x the limits for y turn out to be x to infinity. So, from which we obtain g of x as e to the power minus x, and from the pdf this is the marginal density function of x from the marginal density, we obtain the cdf of x that turns out to be 1 minus e to the power minus x which is valid for x greater than 0.

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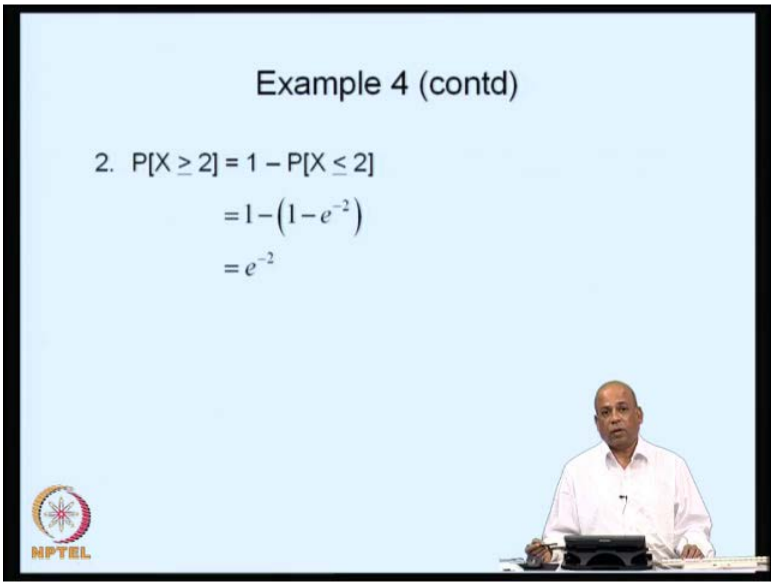

Once we get the marginal cdf, you can obtain any probabilities associated with that so, probability of x being greater than or equal to 2 turns out to be e to the power minus 2 as shown here.

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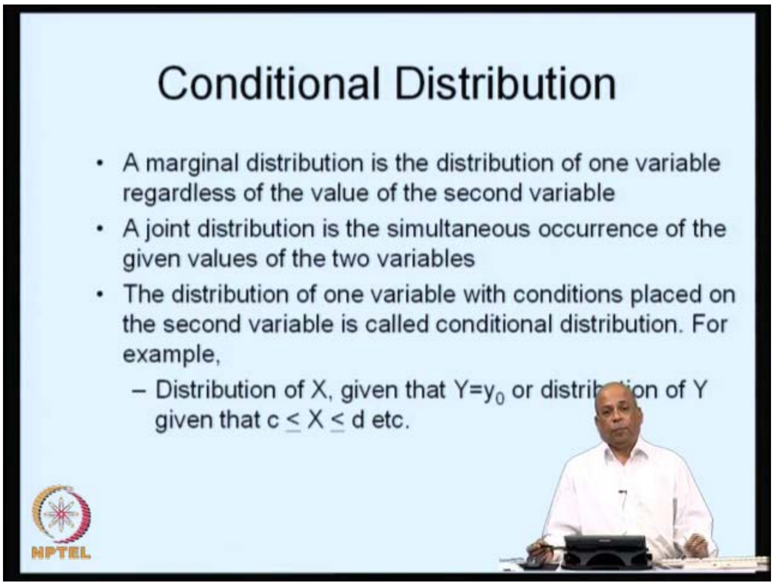

So, we talked about joint pdf which is the joint density function of the two dimensional random variable x, y, which provides the simultaneous behavior of the two random variables x and y then, we talked about the marginal density functions which provides us the distribution of one of the variables irrespective of what values the other random variable takes irrespective of the values that the other random variable takes. Now, we start talking about distribution of one of the variables subject to certain conditions placed on the other random variable for example, we may be interested in what is the distribution of random variable? x given that y has taken a certain value y, y naught or what is the distribution of x on the line y is equal to y naught. So, these questions are answered by conditional distributions so, we will now introduce the concept of the conditional distributions.

So, we define the conditional distribution as I just said the distribution of one variable with conditions placed on the second variable is called the conditional distribution. For in the case of two random variables for example, distribution of x given that y is equal to y naught or distribution of y given that x lies in a certain region between c and d. So, you may place conditions on one of the random variables and we would be interested in getting the distributions of the other random variables.

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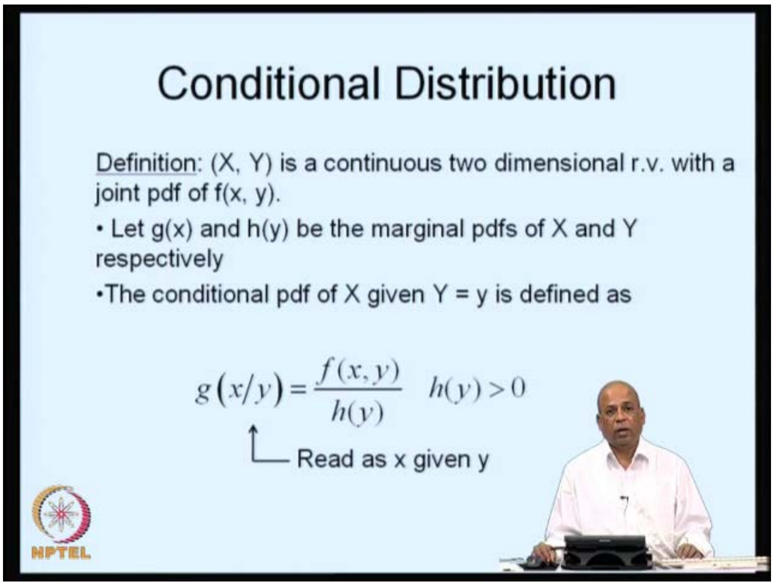

So, in the case of continuous two dimensional random variables with a joint pdf of f of x, y we have the marginal densities marginal pdf g of x, and h of y then the condition pdf of x given y is equal to y is defined as G of x given y that line there is read as x given y is equal to f of x, y that is the joint density function divided by the marginal density of y, h of y this is defined for strictly positive values of h of y. So, this is the definition, that is the density of x given y is equal to y is equal to f of x y by h of y for h of y strictly positive.

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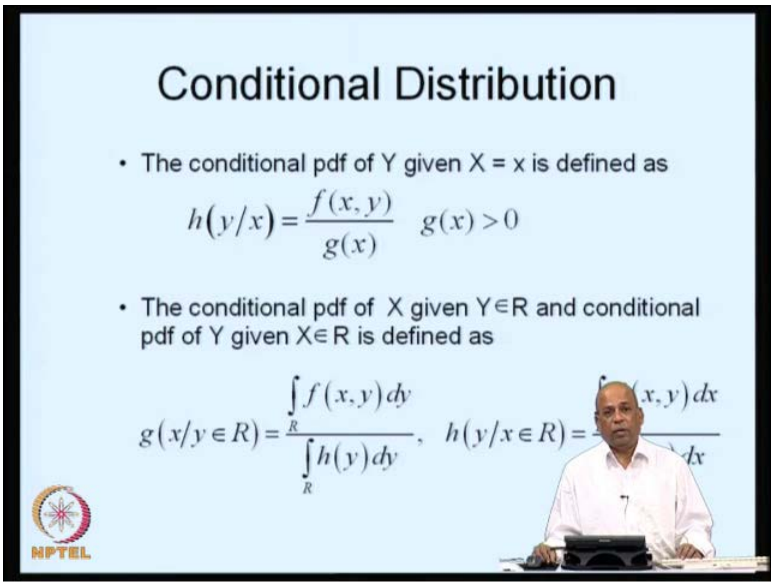

Similarly the conditional pdf of y for given x is equal to x is defined as h of y given x is equal to f of x, y that is a joint density function divided by the marginal density of x for g of x greater than 0 remember this definition that way just introduced is for y is equal to y now, we may place conditions on y or one of the variables not taking on exactly a given value, but belong into a certain region. In which case for example, we may be talking about the density of x given that y belongs to a certain region R then the definition can be shown to be integral of f of x y with respect to y over the entire region r to which the variable y belongs divided by the integral over the same region of the marginal density of y with respect to y.

Similarly, for h of y given x belonging to a region so, when we are talking about the conditions place on one of the variables as belong in to a certain region R then we integrate both f of x y as well as the marginal density over that particular region with respect to the second the variable on which the conditions are placed.

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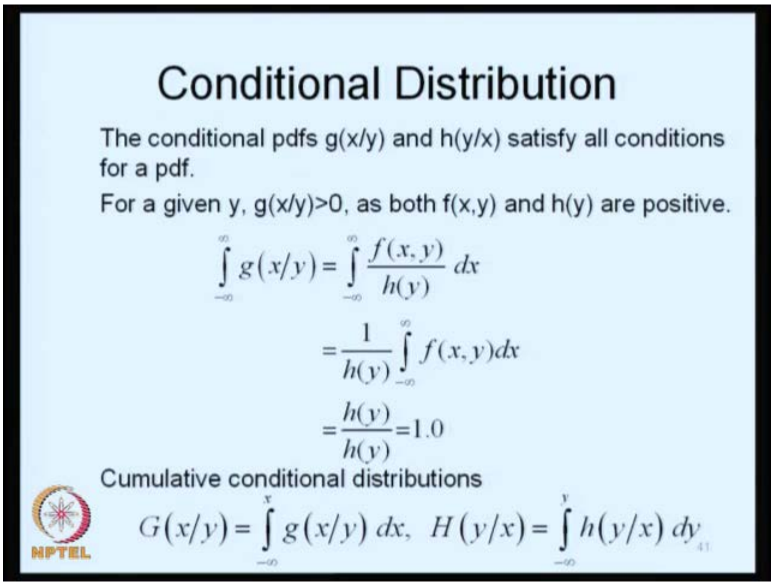

Now, once we get the conditional density functions, because these happen to be the pdf they have to satisfy the conditions of the pdf for example, g of x given y must be greater than or equal to 0 now, this is obvious, because g of x given y is equal to f of x y divided by h of y and f of x y is nonnegative, and the definition is for strictly positive values of h of y and therefore, g of x given y is nonnegative, then the second condition is minus infinity to plus infinity g of x given y with respect to x this must be equal to 1.

So, we use the definition g of x given y as f of x y by h of y now, h of y being a function of y alone comes out of the integral and then we are talking about minus infinity to plus infinity f of x y dx and by the definition of the marginal density of y this term here is in fact, the marginal density of y, h of y therefore, the integral minus infinity to plus infinity g of x given y dx is equal to 1, once we get the conditional density functions we can talk about the associated cumulative distribution functions. So, we talk about g the G of x given y as integral minus infinity to x g of x given y dx, and similarly the conditional cumulative distribution function of y, h of y given x has minus infinity to y, h of y given x   dy from these we should be able to talk about probabilities such that, such as probability that x takes on a certain values given that y is equal to y, y is y has taken on a certain value y.

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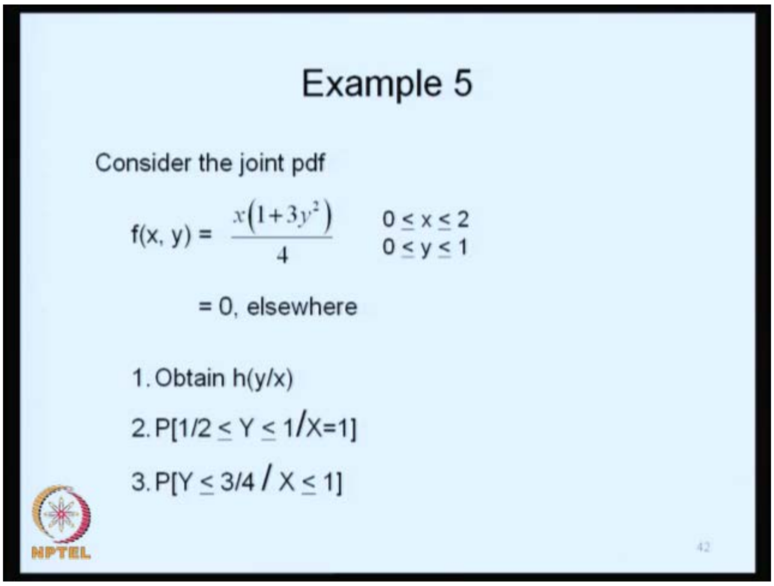

So, let us see how we apply these let us take the joint pdf, f of x given y given by this expression here, and defined over this region let us obtain first h of y given x; that means, the conditional probability of y given x, and then from that we should be able to talk about probability that y takes on certain values in this region y being lying in the region 1 by 2 to 1 given that x is equal to 1. We will also see how we obtain probability of y being less than or equal to 3 by 4 given that x belongs to a region define by x is less than or equal to 1.

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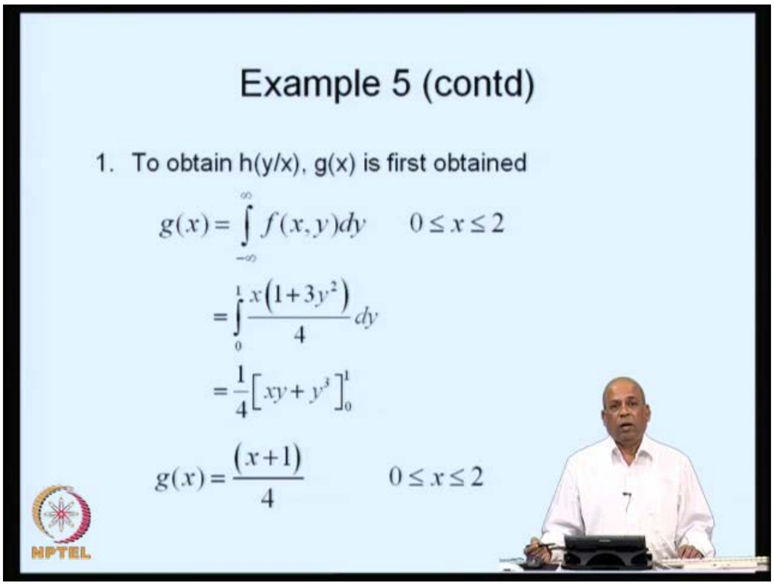

First to obtain h of y given x recall, that the definition of h of y given x is f of x, y divided by g of x for g of x strictly positive. So, first we need g of x so, we obtain g of x from the definition minus infinity to plus infinity we integrate over by the joint density function and we obtain g of x as x plus 1 dived by 4 over this region.

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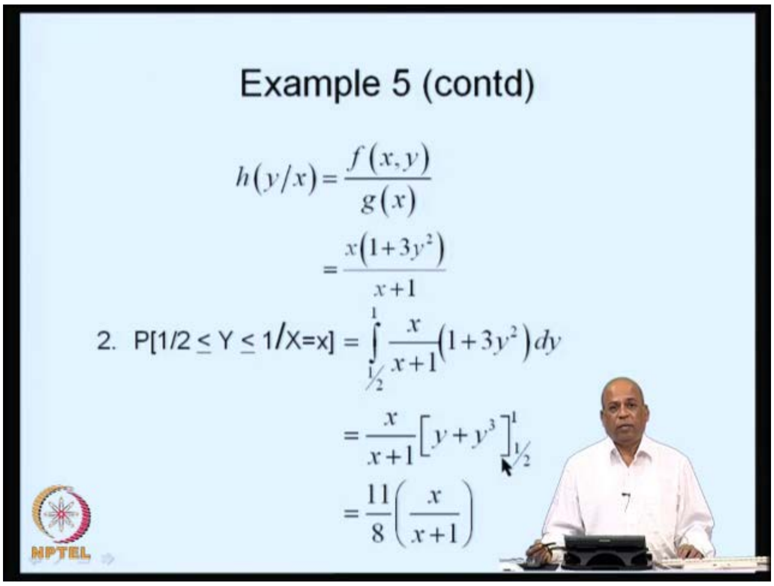

And then we get h of y given x as f of x, y over g of x so, this is h of y given x from this we should be able to talk about probability of y lying in a certain region, for a given x by simply integrating the associated pdf. So, we integrate the associated pdf over the region y is equal to half to y is equal to 1, and get the probability that y lies in this region for a given value of x. So, this will be a function of x 11 by 8 x divided by x plus 1 now we are interested in the probability that x y lies in this particular region.

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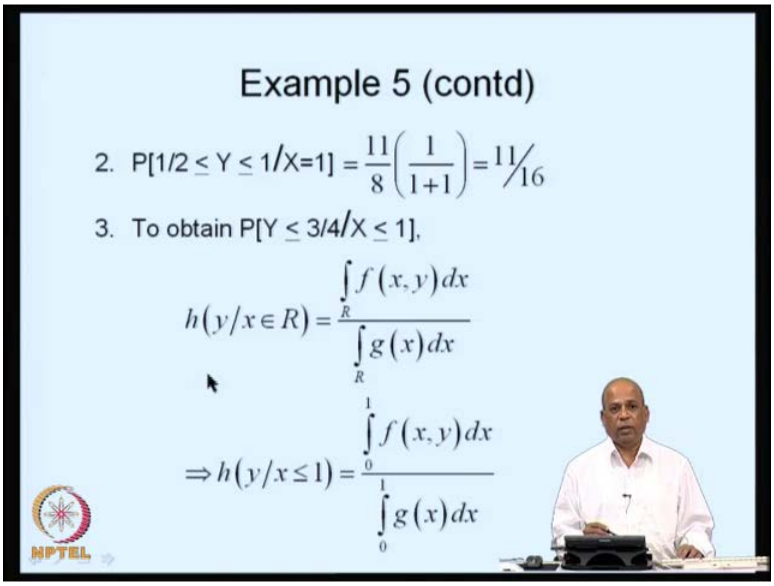

Given that x takes on a value of 1 so, in this we substitute x is equal to 1, and obtain the probability as 11 by 16. Now, we move on to the next type of problems where x instead of taking a specific value of x has it did it in the previous example, it takes the condition is that it takes values in a certain region x is less than or equal to 1, then by definition we will integrate the joint pdf over that particular region similarly we integrate the marginal pdf of x over that region. So, we obtain h of y given x is less than or equal to 1 as integral over 0 to 1 f of x y dx, and integral of g of x or 0 to 1 with respect to x.

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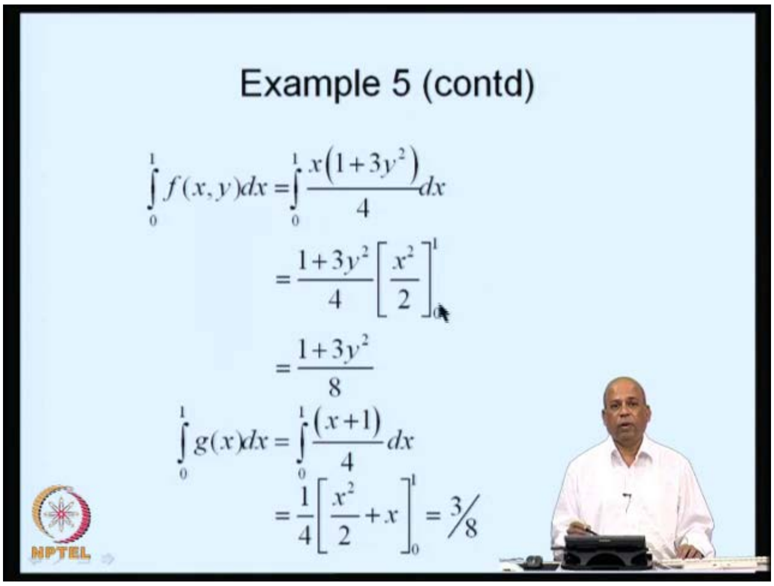

When we do that we get remember here we are talking about h of y given x is less than or equal to 1 so, x has taken certain values in this region so, this is h of y. So, we get this as 1 plus 3 y square divided by 8 once we get this, then we are also talking about we need the marginal density integral of the marginal density over the same region. So, we obtain this integral as 3 by 8.

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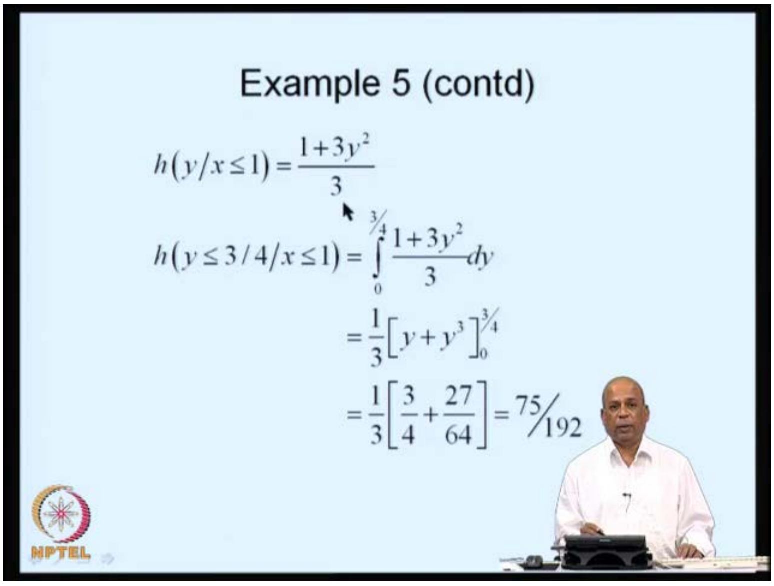

From this we get the conditional density here h of y conditioned on x being less or equal 1 as 1 plus 3 y square divided by 3. So, from this we should be able to get any probability associated with y conditioned on x being less than or equal to 1 so, we integrate in this region 0 to 3 by 4, the conditional density h of y given x less than or equal to 1, we obtain this as 75 by 192 the point to be remembered here is that given any density function. If you integrate the density function over a certain region you get the probabilities of that particular region, let say we are talking about f of x and you integrate over a certain region for x belonging to certain region, then you get the probability of x taking on certain values, the values in the particular region.

We are talking about conditional densities g of x given y, now you integrate this with respect to x, remember when we say g of x given y we are talking about density of x. So, you have to integrate with respect to x, and you are placing the conditions on y. So, this fundamental points we should remember to use the conditional densities and the joint densities for obtaining the associated probabilities. Now, in the next class we will solve one more example, and then introduce the concept of independent random variables, and then move on to functions of two random variables. So, in this particular class what we have covered is we have introduce the concept of bivariate random variables, and then we talk about the joint density functions, and joint probability mass functions, and the associated cumulative distribution functions, and then we introduce the concept of conditional probabilities, and the conditional density functions, and the conditional distribution functions. Thank you very much for your attention.