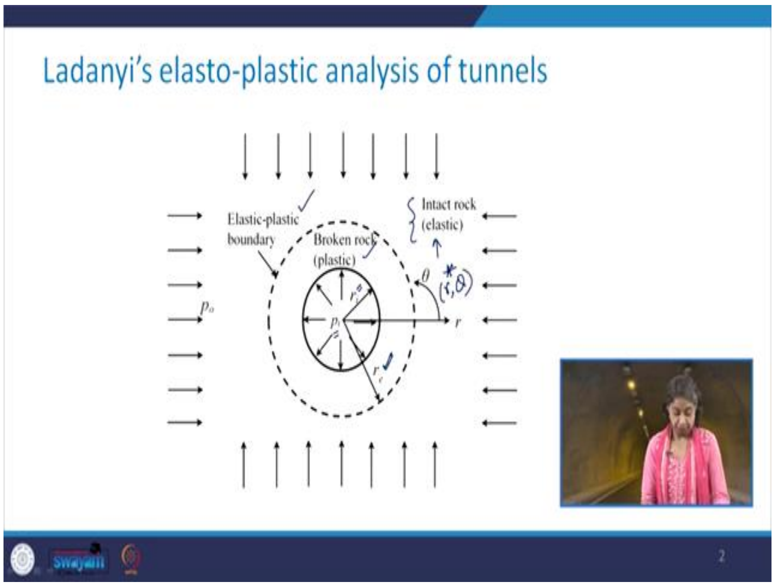
Underground Space Technology
Prof. Priti Maheshwari
Department of Civil Engineering
India Institute of Technology – Roorkee

Lecture – 44
Ladanyi's Elasto-Plastic Analysis of Tunnels: Analysis of Stresses and Deformations

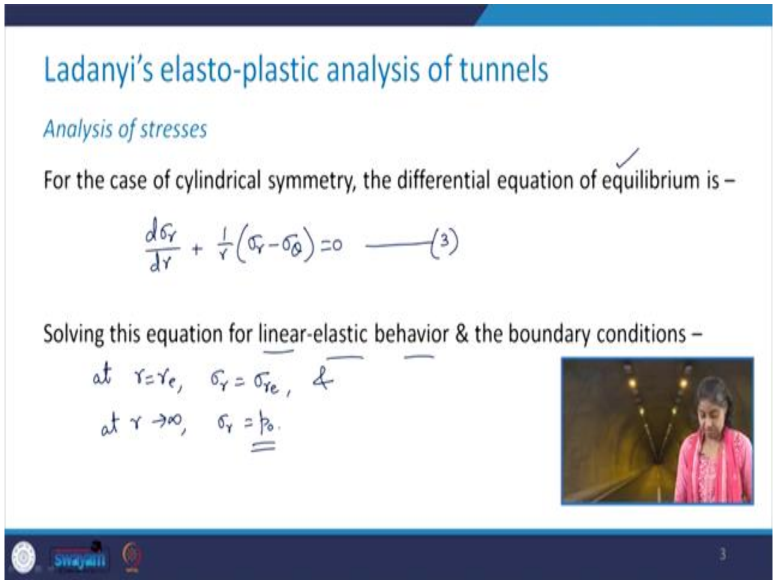
Hello everyone, in the previous class, we learnt about the factors influencing the rock mass tunnel support interaction phenomenon, and then we started our discussion with the Ladanyi's elasto-plastic analysis of tunnels. We discussed about the basic assumptions which are taken ah during the analysis of the tunnel using this method. So, today we will learn about the analysis of stresses and the deformation.

(Refer Slide Time: 01:01)



So, just to recall this is what is the geometry of the problem, where you have the excavation having the radius r_i and it is subjected to internal support pressure which is p_i, there is going to be the broken rock zone in the vicinity of the tunnel and then you will have the elasto-plastic boundary with the radius r_e. And beyond that the rock is considered to be intact rock which will behave in an elastic manner. Any point in the rock, rock mass here that will be represented by r, theta.

(Refer Slide Time: 01:50)



So, let us start with the analysis, you need to keep in mind that we have considered the problem to be symmetry. So, for the case of the cylindrical symmetry, the differential equation of equilibrium is written as:

$$\frac{d\sigma_r}{dr} + \frac{1}{r}(\sigma_r - \sigma_\theta) = 0$$

So, in continuation with the previous class, I am marking this equation as equation number 3. If we solve this equation for the linear elastic behaviour and along with the boundary conditions.

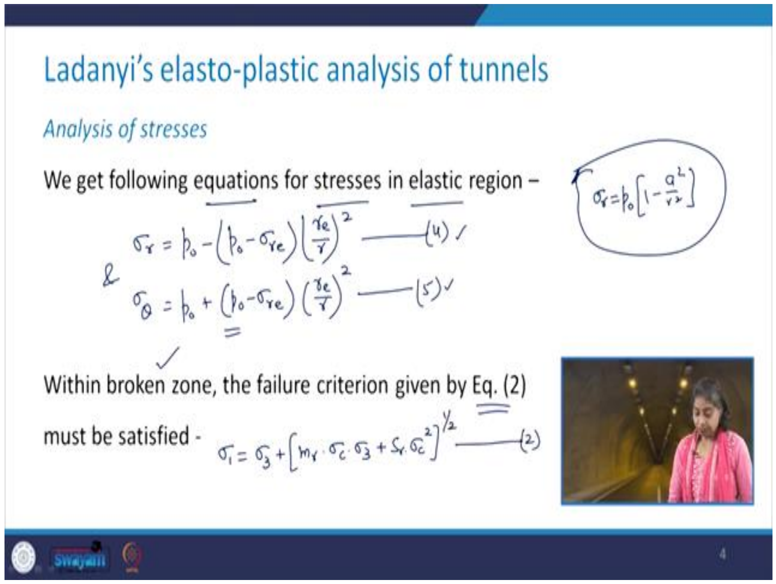
So, what are going to be the boundary conditions? So, we will have:

$$at\ r = r_e,\ \sigma_r = \sigma_{re}$$

$$at\ r \to \infty,\ \sigma_r = p_o$$

And if you recall, ah we assume that there is the hydrostatic state of stress.

(Refer Slide Time: 03:03)



So, we have these stresses in the elastic region, this we have seen earlier. That is:

$$\sigma_r = p_o - (p_o - \sigma_{re})\left(\frac{r_e}{r}\right)^2$$

$$\sigma_\theta = p_o + (p_o - \sigma_{re})\left(\frac{r_e}{r}\right)^2$$

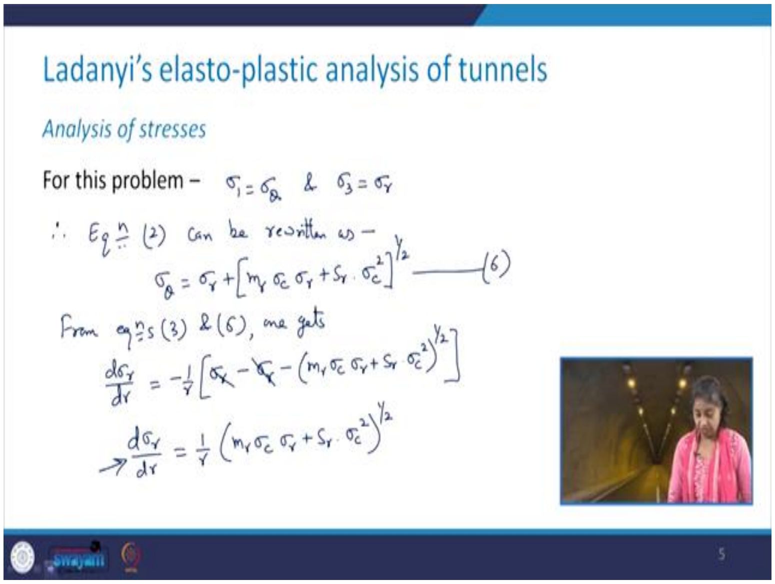
So, I will make this equation as 4 and this equation as 5. Now, can you recall we had this expression as σ_r as p_0 [1 - a^2 upon r^2] so, these equations 4 and 5 they are coming from this consideration only. That is in the elastic domain, this is how you are going to have the stresses?

So, the for the broken zone, the failure criterion that was given by equation number 2. It must be satisfied so what was that equation?

$$\sigma_1 = \sigma_3 + [m_r\sigma_c\sigma_3 + s_r{\sigma_c}^2]^{1/2}$$

Recall that this was our equation number 2.

(Refer Slide Time: 04:33)



So, in our case, we have sigma 1 = sigma theta and sigma 3 = sigma r so, we can write this equation 2. Equation 2 can be rewritten as:

$$\sigma_\theta = \sigma_r + [m_r \sigma_c \sigma_3 + s_r {\sigma_c}^2]^{1/2}$$

Make this equation as equation number 6. Now, if we take equation number 3 and equation 6 so, what we get?

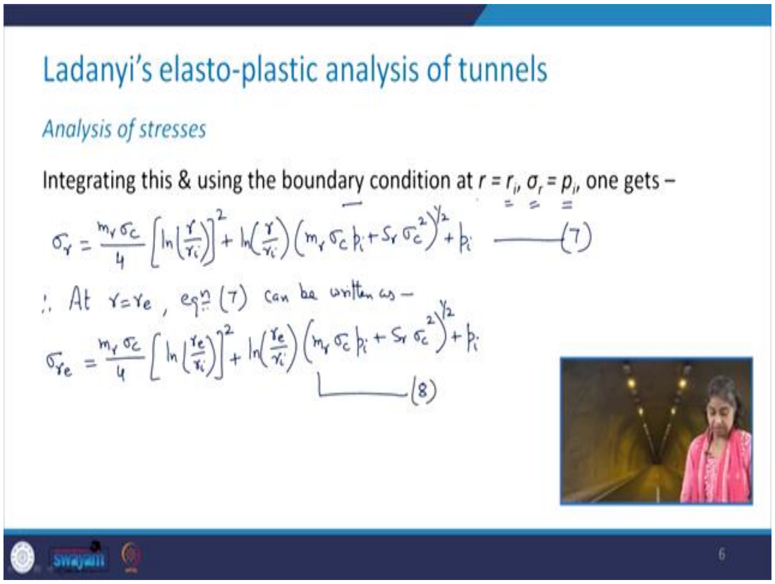
So, from equations 3 and 6, one gets:

$$\frac{d\sigma_r}{dr} = -\frac{1}{r}\left(\sigma_r - \sigma_r - [m_r \sigma_c \sigma_3 + s_r {\sigma_c}^2]^{1/2}\right)$$

$$\frac{d\sigma_r}{dr} = \frac{1}{r}\left([m_r \sigma_c \sigma_3 + s_r {\sigma_c}^2]^{1/2}\right)$$

So, this is what that we are going to get. Now, take a look at this equation, and if we integrate it with respect to r, we can get the expression for sigma r.

(Refer Slide Time: 06:31)



So, let us do that so, when you integrate this, you will get some constant to determine those constant, we need to use the boundary conditions. So, we are considering the boundary condition as $r = r_i$, we are going to have sigma r is equal to pi. So, what we have here as:

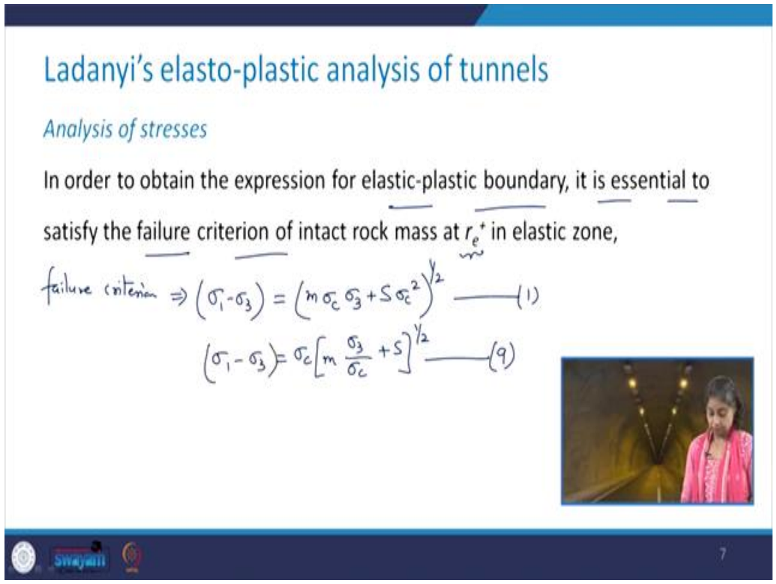
$$\sigma_r = \frac{m_r \sigma_c}{4}\left[ln\left(\frac{r_e}{r}\right)^2 + ln\left(\frac{r_e}{r}\right)\left(m_r \sigma_c p_i + s_r {\sigma_c}^2\right)^{1/2}\right] + p_i$$

This is equation number 7. Therefore, at $r = r_e$ we can write this equation 7 as:

$$\sigma_{re} = \frac{m_r \sigma_c}{4}\left[ln\left(\frac{r_e}{r_i}\right)^2 + ln\left(\frac{r_e}{r_i}\right)\left(m_r \sigma_c p_i + s_r {\sigma_c}^2\right)^{1/2}\right] + p_i$$

So, I will make this equation as equation number 8.

(Refer Slide Time: 08:30)



So, to obtain the expression for the elasto-plastic boundary that is re. It is essential for us to satisfy the failure criterion of the intact rock mass at r = r_e plus that means r_e plus delta re where delta re is tending to 0. So, what we are going to have is as the failure criterion? Will be:

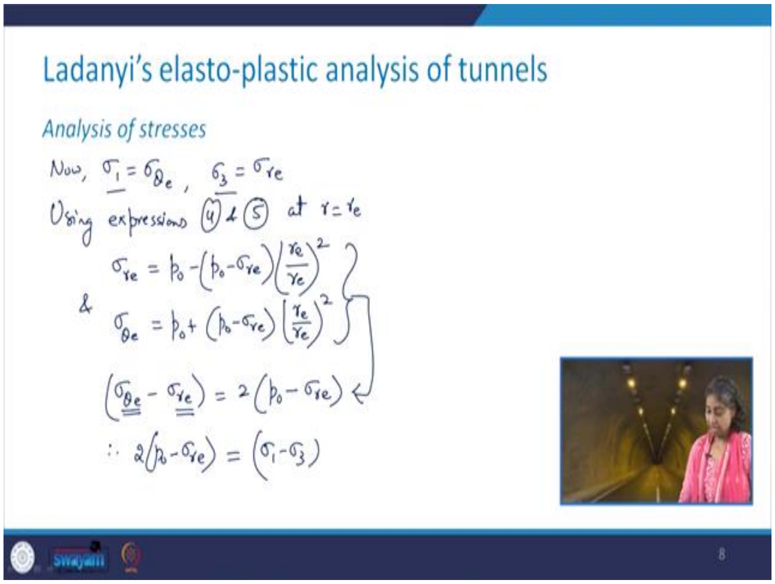
$$\sigma_1 - \sigma_3 = [m\sigma_c\sigma_3 + s{\sigma_c}^2]^{1/2}$$

This was our equation 1 if you recall that was for the original rock mass. So, this can also be written as:

$$\sigma_1 - \sigma_3 = \sigma_c\left[m\frac{\sigma_3}{\sigma_c} + s\right]^{1/2}$$

So, I am marking this equation as equation number 9 and on the left-hand side of course, you have sigma 1 minus sigma 3.

(Refer Slide Time: 09:44)



So, what we have here now is? $\sigma_1 = \sigma_{\theta e}$ and $\sigma_3 = \sigma_{re}$. So, we use the expressions 4 and 5 at $r = r_e$. So, using the expressions 4 and 5 at $r = r_e$. So, what we have is?

$$\sigma_{re} = p_o - (p_o - \sigma_{re})\left(\frac{r_e}{r_e}\right)^2$$

$$\sigma_{\theta e} = p_o + (p_o - \sigma_{re})\left(\frac{r_e}{r_e}\right)^2$$

So, I am just substituting $r = r_e$. So, this become 1 so, what we have is?

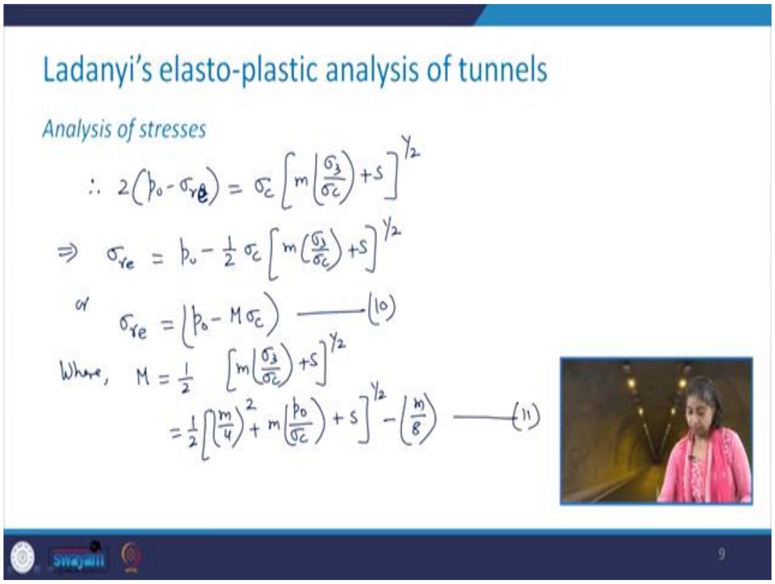
$$\sigma_{\theta e} - \sigma_{re} = 2(p_o - \sigma_{re})$$

So, from these two equations, I will be able to get these expressions. So, therefore, we can have:

$$2(p_o - \sigma_{re}) = \sigma_1 - \sigma_3$$

So, we can write them as sigma 1 minus sigma 3, and if we apply the material properties from the equation number 1.

(Refer Slide Time: 11:54)



Then sigma 1 minus sigma 3 can also be written as,

$$2(p_o - \sigma_{re}) = \sigma_c\left[m\frac{\sigma_3}{\sigma_c} + s\right]^{1/2}$$

So, from here I can find out sigma re as:

$$\sigma_{re} = p_o - \frac{1}{2}\sigma_c\left[m\frac{\sigma_3}{\sigma_c} + s\right]^{1/2}$$

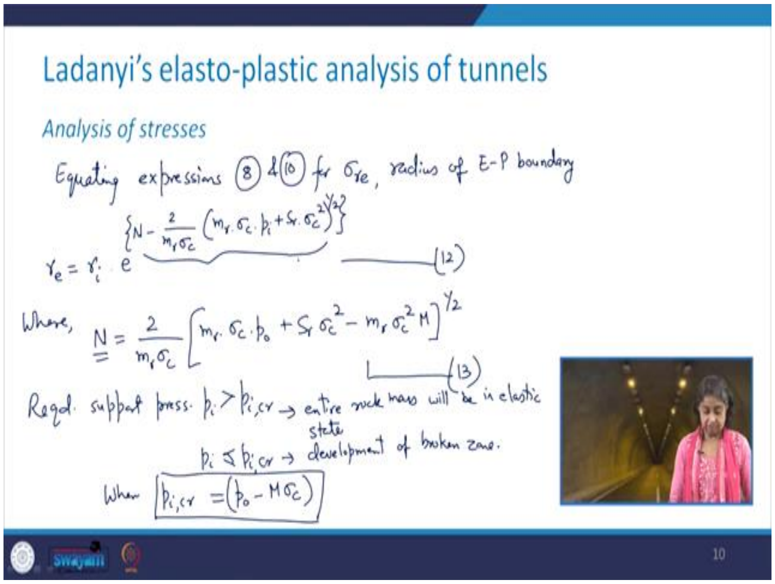
$$\sigma_{\mathrm{re}} = p_{\mathrm{o}} - \mathrm{M}\sigma_{\mathrm{c}}$$

I will make this equation as equation number 10. Where this capital M is written as:

$$\mathrm{M} = \frac{1}{2}[\mathrm{m}\frac{\sigma_3}{\sigma_c} + \mathrm{s}]^{1/2}$$

$$\mathrm{M} = \frac{1}{2}[(\frac{\mathrm{m}}{4})^2 + \mathrm{m}\frac{\mathrm{p}_0}{\sigma_{\mathrm{c}}} + \mathrm{s}]^{1/2} - \frac{\mathrm{m}}{8}$$

(Refer Slide Time: 13:26)



Then we equate the expressions 8 and 10 for sigma re, what is going to be the radius of elastic, plastic boundary? We will have it as r_e equal to,

$$r_e = r_i\, e^{\left\{N - \frac{2}{m_r \sigma_c}\left(m_r \sigma_c p_i + s_r {\sigma_c}^2\right)^{1/2}\right\}}$$

So, this whole expression is in exponential so, I will mark this equation as equation number 12. Where we have introduced another term N which is equal to,

$$N = \frac{2}{m_r \sigma_c}\left(m_r \sigma_c p_o + s_r {\sigma_c}^2 - m_r {\sigma_c}^2 M\right)^{1/2}$$

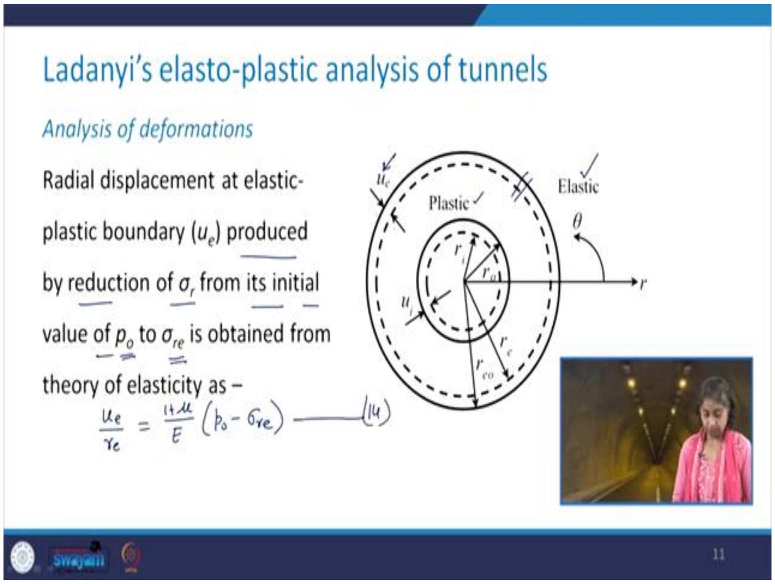
Ok. So, after having this expression of N, we can have the condition that is when we have the required support pressure pi greater than pi critical, then the entire rock mass will be in the elastic state.

But if we have this pi is less than or equal to pi critical, then there is going to be the development of the broken zone. Where what is this pi critical? That is equal to:

$$p_{i,cr} = p_o - M\sigma_c$$

Please remember this expression. So, this is how we can have the analysis of stresses.

(Refer Slide Time: 16:28)



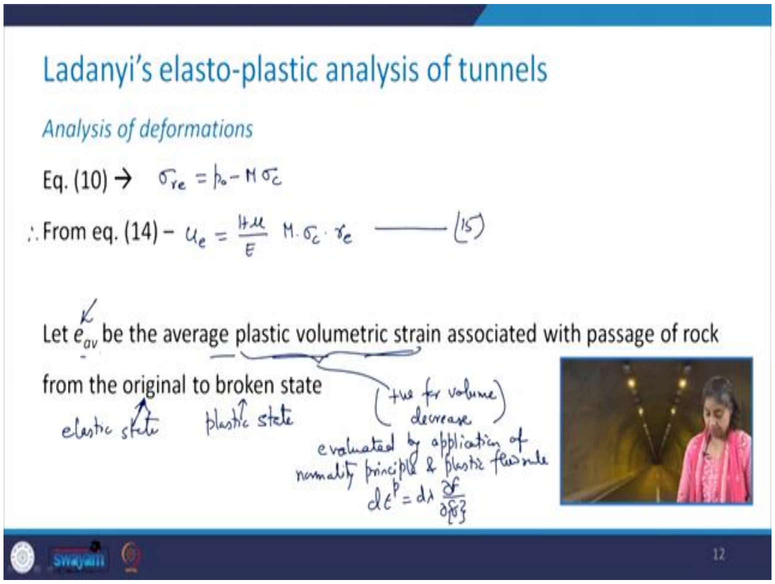
So, to start with the analysis of deformation again, let us take a look at this figure, so you have the excavated tunnel boundary that is r_o where, after the radial deformation u_i that becomes as r_i. And then this is the plastic zone, and beyond that again, you have the elastic zone. So, basically, this one is giving you the idea about the elasto-plastic boundary so the u_e is the radial displacement at the elastic-plastic boundary.

Which is produced by the reduction of sigma r from it is initial value of p_0, to sigma re and it can be obtained by using the theory of elasticity by the following expression. That is:

$$\frac{u_e}{r_e} = \frac{(1+\mu)}{E}(p_0 - \sigma_{re})$$

So, let us keep this equation as equation number 14.

(Refer Slide Time: 17:52)



So, your equation 10 was:

$$\sigma_{re} = p_o - M\sigma_c$$

So, from the therefore, from equation number 14 what you can get is? That u_e equal to,

$$u_e = \frac{(1+\mu)}{E} M\sigma_c r_e$$

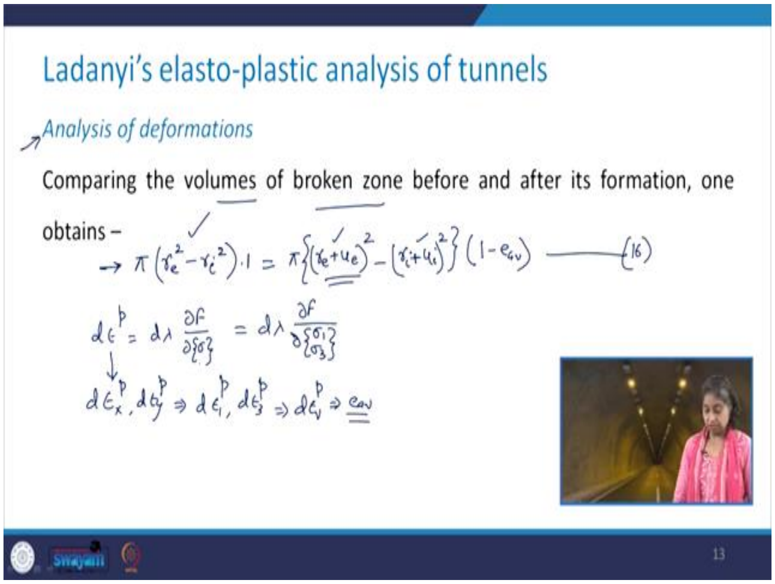
So, I will make this equation as equation number 15. Now, if e average be the average plastic volumetric strain which is associated with the passage of the rock from the original to the broken state.

So, here when we say that the original state means it is the elastic state that we are talking about. And when we talk about the broken state means we are in the plastic domain or the plastic state. So, this plastic volumetric strain this is going to be positive for the reduction in volume. And this can be evaluated by the application of normality principle and the plastic flow rule which was:

$$d \in_v{}^p = d\lambda \frac{\partial F}{\partial \{\sigma\}}$$

This we saw earlier.

(Refer Slide Time: 19:49)



Now, if we compare the volumes of the broken zone before and after it is formation, what we can get is? That it will be:

$$\pi(r_e^2 - r_i^2) \times 1 = \pi\{(r_e + u_e)^2 - (r_i + u_i)^2\}(1 - e_{av})$$

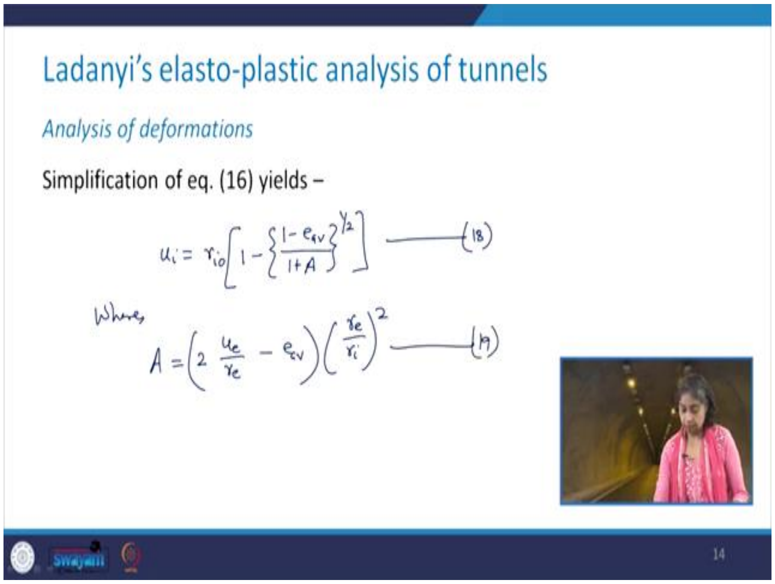
Take the reference of the figure that I showed you when we discussed about the analysis of deformation.

And you will realize that before the formation, it is this and after the formation since, there is some radial displacement so that we have to account for. So, this is how we can get the expression when we compare the volumes of the broken zone before and after it is formation. So, here we have this plastic flow rule which is:

$$d \in_v{}^p = d\lambda \frac{\partial F}{\partial\{\sigma\}} = d\lambda \frac{\partial F}{\partial \begin{Bmatrix}\sigma_1 \\ \sigma_3\end{Bmatrix}}$$

Because it is the stress vector. This one will have the components like this, epsilon p maybe this is d epsilon 1 plastic d epsilon 3 plastic, we will have the volumetric strain that will be in terms e average.

(Refer Slide Time: 22:08)



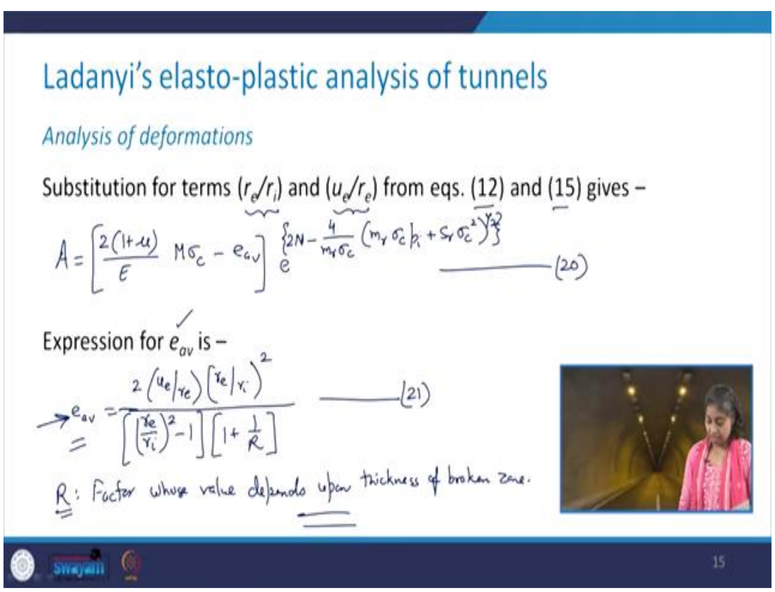
So, when we substitute this equation 16 that will give us u_i is

$$u_i = r_{io}\left[1 - \left\{\frac{1 - e_{av}}{1 + A}\right\}^{1/2}\right]$$

And this equation I will make it as equation number 18. Where this A is expressed as:

$$A = \left(\frac{2u_e}{r_e} - e_{av}\right)\left(\frac{r_e}{r_i}\right)^2$$

(Refer Slide Time: 22:56)



Now, if we substitute for terms re upon ri and ue upon re from the respective equations 12 and 15, what we get is? The expression for A as:

$$A = \left[\frac{(1+\mu)}{E} M\sigma_c - e_{av}\right] e^{\left\{2N - \frac{4}{m_r\sigma_c}\left(m_r\sigma_c p_i + s_r\sigma_c{}^2\right)^{1/2}\right\}}$$

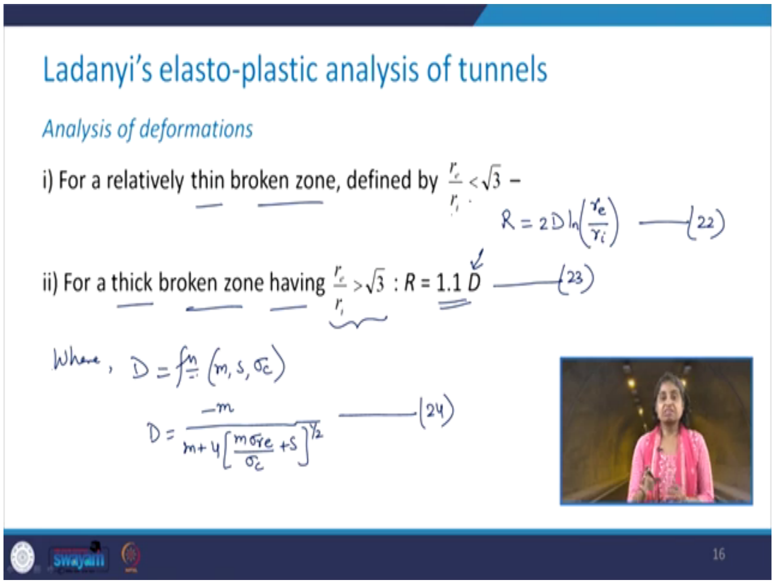
This is equation number 20. So basically, this complete expression is in exponential. The expression for e average can be written, therefore as e average as :

$$e_{av} = \frac{2(u_e/r_e)(r_e/r_i)^2}{\{(\frac{r_e}{r_i})^2 - 1\}\{1 + \frac{1}{R}\}}$$

Where this R is the factor whose value depends upon the thickness of the broken zone. So, basically, if you look at the original paper by Ladanyi so, he has given the detailed derivation for this expression.

But since these days we have the lot of computational facilities so, we do not need to really go into those details. But if you are interested, maybe you can refer back to that.

(Refer Slide Time: 25:24)



So, coming to this factor again, what should be it is value? So, in case, if there is a relatively thin broken zone which is defined by the expression that is:

$$\frac{r_e}{r_i} < \sqrt{3}, R = 2D \ln\frac{r_e}{r_i}$$

Equation number 22. And in case, if you have the thick broken zone represented by:

$$\frac{r_e}{r_i} > \sqrt{3}, \qquad R = 1.1\ D$$

Now, here comes another term which is D so, where we have this D as a function of m, s and sigma c and it is defined as:

$$D = \frac{-m}{m + 4[\frac{m\sigma_{re}}{\sigma_c} + s]^{1/2}}$$

So, let me make this as equation number 23 and this as equation number 24. So, this is how we can determine the analysis of stresses and the deformation. So, by following this analysis now we can find out that what will be the state of stresses, sigma r and sigma theta?

And also, we can find out what is going to be the deformations? So, ah like this, we can generate the ground response curve. So, in the next class, we will learn about the analysis with respect to the support system with reference to Ladanyi's elasto-plastic analysis of tunnels. Thank you very much.