Finite Element Method and Computational Structural Dynamics

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Lecture - 09
Mathematical Modelling and Approximate Solutions - II

Hello. So, as we discussed the need for developing approximate solutions to the governing differential equation, the method of weighted residuals is the most commonly used technique. The basic idea behind method of weighted residuals is based on the basic idea of linear algebra that any finite function over a finite domain can be represented by a linear combination of linearly independent basis functions as long as that family of basis functions constitute a complete basis. So the basis should span the complete function space or vector space.

So, as long as we are assured of spanning the complete space it should be possible to construct an approximation for any finite function over a finite region by suitable linear combination of those basis functions. If the basis functions are polynomials, then that particular case is covered by Weierstrass approximation theorem that we had discussed during interpolation theory.

So, the basic idea is, we seek an approximate solution. So, if $u(x)$ is the true solution, we do not look to find $u(x)$ exactly. Instead we try to find a function $\hat{u}(x)$ as an approximation to the unknown function $u(x)$ in terms of a series using a linear combination of basis functions (also referred to as trial functions) with unknown coefficients. The chosen set of functions should be linearly independent and they should represent a complete family of basis functions. That is, all the basis functions taken together should span the complete space, function space or vector space.

So, a polynomial for example is an infinite family. We can keep on increasing the degree of polynomial and they are linearly independent. There is no way a constant term can represent what is represented by a linear trend. There is no way a constant and linear term can represent what is contained in a quadratic function and so on.

So, a linear combination of increasing degree of polynomials constitutes a complete family and we can possibly construct any approximation which will in principle should

be able to approximate or should be able to capture the sufficient details of any finite function over any finite region.

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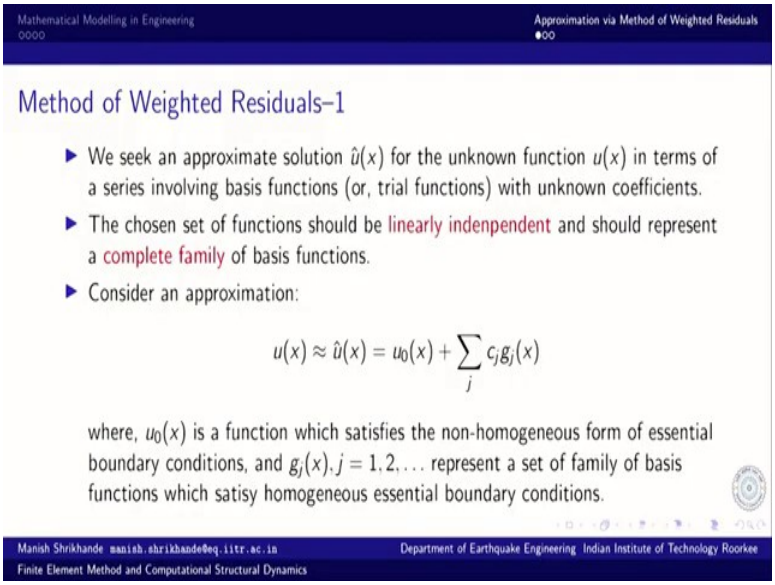

So, we consider the approximation $\hat{u}(x)$ as an approximation to $u(x)$, and $\hat{u}(x)$ is given by

$$\hat{u}(x) = u_o(x) + \sum c_j g_j(x)$$

g_j are a set of linearly independent basis functions satisfying homogeneous essential boundary conditions. More on this essential boundary condition business little later kindly bear with me until that point. In this particular bar problem, let me just say that the essential boundary condition constitutes the boundary condition imposed on the displacement.

So, if you recall the bar problem that we had discussed; there were two boundary conditions; first at x = 0, the boundary condition was imposed on the axial deformation as 0 and the other condition at x = L, the force boundary condition i.e. the axial thrust was equal to the applied force. So that is called the force boundary condition or natural boundary condition.

In this particular case for the construction of approximation, it is important that the basis function we choose, should be of a form which satisfies the homogeneous essential boundary conditions all over the problem.

And $u_0(x)$ is a function, which satisfies non-homogeneous form of essential boundary conditions. For example, in the bar problem, if I had specified deformation to be non-zero at x = 0, then that is a non homogeneous boundary condition. So, the deformation at x = 0 is, let us say, equal to some constant δ .

So, then that is a non homogeneous boundary condition and that has to be specified. So, if it is a constant then $u_0(x)$ is simply that constant δ that is equal to the non homogeneous boundary condition of the problem. And, the series solution that you see $c_j g_j$, just operate on top of this non-homogeneous boundary condition function and together complete summation gives us an approximation to the solution.

c_j , are the unknown coefficients that are not yet known. So, we have to find a way to determine these coefficients and that is why the name weighted residual comes. So, that mystery behind the name will be clear just in a while.

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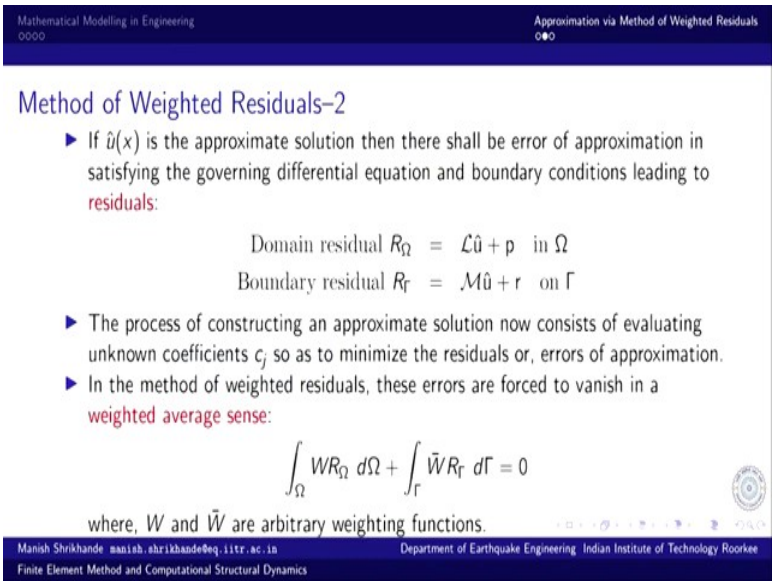

So naturally, if we had true solution $u(x)$, then it will satisfy both the governing differential equation and the boundary conditions exactly and there will not be any error. The very fact that we are admitting the possibility that we are not looking for exact solution. We will be satisfied with an approximate solution as long as it is approximation is good enough.

So, the moment I say approximate solution, the inherent implication is there are errors in satisfying the governing differential equation and possibly the boundary conditions. So, the governing differential equation $Lu+p$ need not be equal to 0 all over the domain.

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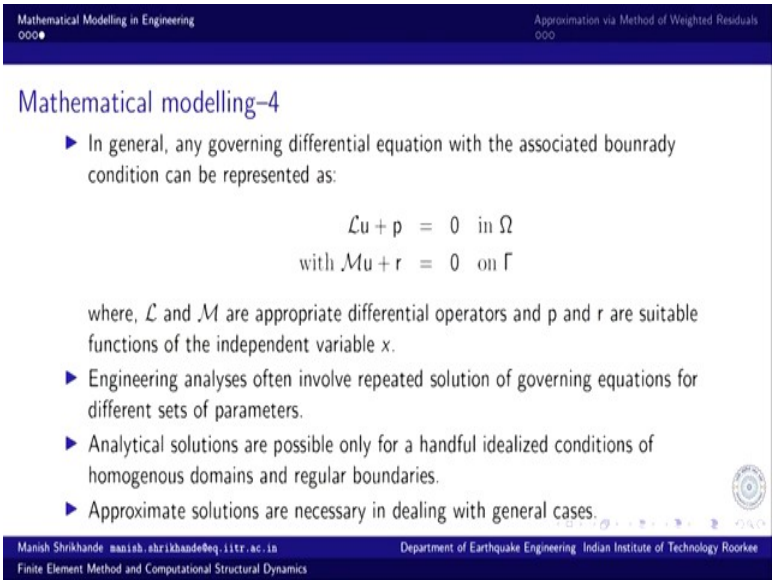

So, if I substitute approximate solution $\hat{u}(x)$ in place of $u(x)$, then it will not hold all over the domain. There will be an error because it is an approximate solution. So, this governing differential equation may not be satisfied exactly.

Similarly, for the boundary condition, the boundary conditions may not be satisfied exactly on all the boundaries. So, there may be error in satisfying boundary conditions. So, if I substitute the approximate solution $\hat{u}(x)$ in the governing differential equation and the boundary conditions then there will be an error of approximation and these errors are referred to as residuals.

We will have two types of residuals; one is domain residual coming from the governing differential equation that is $R_\Omega = L\hat{u} + p$ in Ω. And the second residual is the boundary residual $R_\Gamma = M\hat{u} + r$ that comes from the boundary conditions.

The quality of approximation; depends on how do we minimize these errors (residuals). So that provides us the required conditions to evaluate the coefficients c_j.

So, $u_0(x)$ is known here because that comes from the boundary conditions of the problem, g_j are the basis functions that we have assumed, c_j are the unknown coefficients of this summation. The task is to find these coefficients c_j somehow so at to minimize the error in approximation.

So, in the method of weighted residuals, we emphasize that we do not look at these errors individually, rather we look to make the errors vanish in a weighted average sense over the entire domain. Let W and $\bar{W}$ be the weighting functions, the weighted residual statement becomes,

$$\int_{\Omega} W(L\hat{u}+p)+\int_{\Gamma} \bar{W}(M\hat{u}+r)=0$$

Now, let us delve upon this. It is a very straightforward equation, but it has significant implications and very important results are derived from this. First of all, the exact solution is contained in this statement. This is called the statement of weighted residuals.

Now, if I knew the exact solution $u(x)$ of the governing differential equation then; the domain residual R_{Ω} and the boundary residual R_{Γ} would vanish and this weighted residual statement would hold irrespective of what weighting functions I choose. So, for any arbitrary weighting function, W and $\bar{W}$, if I know the exact solution then R_{Ω} is identically equal to 0 all over the domain, R_{Γ} is identically equal to 0 all over the boundary and this weighted sum of a residual would be 0.

So the statement of weighted residual would be identically satisfied for any arbitrary function W and $\bar{W}$. So, that is a reassuring thought that the exact solution is contained in this particular weighted residuals statement.

Now, think over this a little bit more. For any arbitrary choice of W and $\bar{W}$, this weighted residual statement should hold if it was a true solution. Now, if I try to satisfy the statement of weighted residual over a large number of W and $\bar{W}$, then in a way I am approaching the true solution. If I consider arbitrary function pairs, let us say $(W_1,\bar{W}_1),(W_2,\bar{W}_2),(W_3,\bar{W}_3),(W_4,\bar{W}_4),\ldots$ and so on and try to enforce this statement of weighted residual over these functions as :

$\int W_1 R_\Omega \, d\Omega + \int \bar{W}_1 R_\Gamma \, d\Gamma = 0$ for first pair of arbitrary functions

$\int W_2 R_\Omega \, d\Omega + \int \bar{W}_2 R_\Gamma \, d\Gamma = 0$ for second pair and so on...

As we increase these number of weighted residual statements, we find that we will be approaching the condition imposed by true solution. That, no matter what weighting function we choose the weighted residual statement holds. So, if I consider a large number of these weighting functions and enforce this condition then I am in a way approaching the true solution.

Obviously, I cannot approach infinity on the digital computers or for any finite operations, so it remains an approximation, but an approximation that approaches the true solution as the number of weighting functions for which these weighted residual statement holds will increase the quality of approximation.

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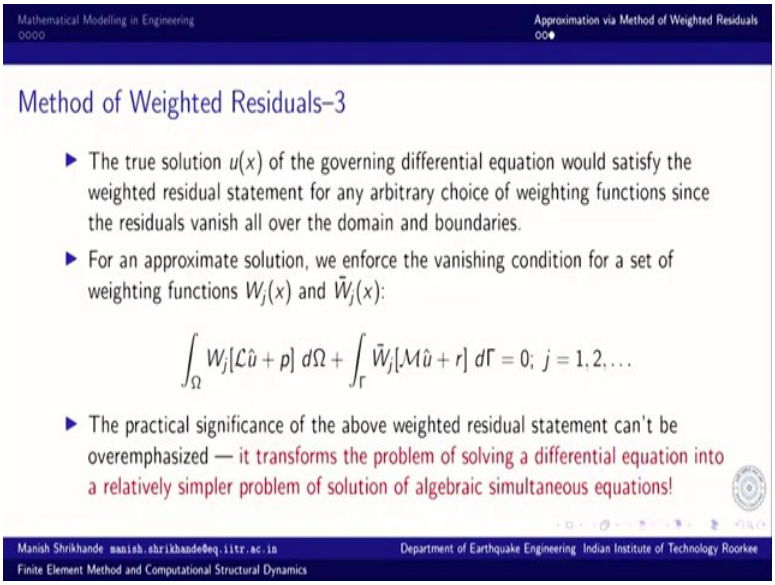

So, as I said true solution of governing differential equation would satisfy the weighted residual statement for any arbitrary choice of weighting functions since the residuals vanish all over the domain and the boundaries. Now, for an approximate solution, we enforce the vanishing condition for a set of weighting functions W_j and $\bar{W}_j$ where j can range from 1 to a large number.

So, the statement now becomes following integral where we represent R_Ω by substituting the approximate solution $\hat{u}$ there.

$$\int W_j \left[L\hat{u} + p \right] d\Omega + \int \bar{W}_j \left[M\hat{u} + r \right] d\Gamma = 0 \quad \forall \quad j = 1,2,3,4,\ldots.$$

So, how many terms we need? how far do we go? Obviously, we have some unknown coefficients to evaluate, and those unknown coefficients can be evaluated only when we have some conditions on those unknown coefficients and that is what we will try to do. We will use as many functions as there are unknown coefficients.

If there is a three term approximation involving three coefficient c_1, c_2, c_3 , we will have a set of these three equations for $j = 1,2,3$. We evaluate three statements of weighted residuals and to give us some relationships in terms of c_1, c_2 and c_3 .

We solve those three equations simultaneously to get c_1, c_2, c_3 and substitute in our basic approximation to have a working approximation to the governing differential equation. This in a sense is the method of weighted residuals. Now, this has a very profound practical significance and that cannot be over emphasized. What it does? What this method of weighted residuals does is to convert the problem of solving a differential equation into a much simpler problem of solution of algebraic simultaneous equations. Because after these evaluation of integrals we will only have simple algebraic equations.

These simultaneous equations can be solved using any usual linear equation solver. You can use Cholesky method, LU decomposition or even Gaussian elimination to evaluate unknown coefficients c_j . Once we have these coefficients those can be plugged back into $\hat{u}(x)$ to provide the basic approximation.

The quality of approximation can of course be improved by increasing the number of terms in the series, which will reduce error. But the exact solution is unknown so the error of approximation can only be gauged by considering the convergence by successive addition of the terms in the series.

You might have discussed series convergence earlier in your mathematics course, but we will discuss it briefly to assess the convergence of this series approximation in solution of the governing differential equation.

So, this is the basic idea of the approximate methods of solution. We are essentially trying to solve governing differential equation by transforming it into a set of algebraic equations and then solve the set of algebraic equations for the unknown coefficients. This is a very straightforward and robust procedure which can handle many complex problems for which analytical solutions may not exist, and we can still work out a very decent workable approximate solutions that can be used for our subsequent engineering analysis. So, we will continue on this with more examples. There are different versions of method of weighted residuals depending on how we choose these weighting functions.

So we will see how these weighting functions are chosen or what are the different variations on method of weighted residuals and then let us get our hands dirty. Let us work out some problems. I will discuss couple of cases and then see how the solution varies, so that you get some idea of the working of the method of weighted residual and gain some confidence.

It does give us something meaningful, something workable to base our engineering judgment upon. We will continue with the Variations on Method of Weighted Residuals in our next lecture.

Thank you.