Finite Element Method and Computational Structural Dynamics

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Lecture - 08
Mathematical Modelling and Approximate Solutions - I

Hello, so, after having discussed the basics of floating point arithmetic, the nuances of doing scientific computation on digital computers we have covered a little bit of elementary background on linear algebra for vector spaces.

So, those are the basic building blocks on which we will now begin to build our development of finite element method, which is a very powerful technique for solution of differential equations. A very popular book by Robert D. Cook about finite element method mentions in its introductory chapter that finite element method is a very powerful tool which can make a good engineer better, but a bad engineer dangerous.

The second part of the sentence is very important to note. And, that is why it is important to understand how things can go wrong. After all it is a numerical method, it is an approximate solution.

So, it is important for us to figure out the quality of approximation when we are developing the solution and also try to find out ways to judge which constitutes a better approximation or how to build a better approximation. So that our purpose - coming up with good engineering designs and products - are adequately met.

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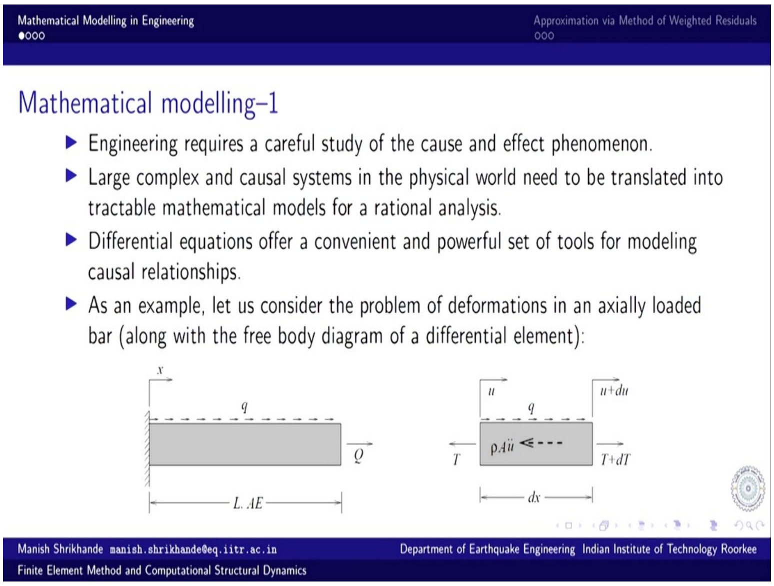

To start with the problem of the process of Mathematical Modelling is the first step of any engineering analysis. And for any engineering analysis we require a very careful study of the influencing environment. The system per se that we are looking to design and what are the operating conditions of the system, what are the operating parameters of the system and so on.

Usually it is a very complex system with lots of influencing parameters and there are cause and effect relationships between the system output and the input that it takes from the environment conditions or the operating conditions.

So, obviously, this complex physical model is a very large system and needs to be transformed into a mathematical model with suitable assumptions within a suitable assumed framework of certain idealizations such that we retain what are the significant influencing parameters and what are the significant variables of the problem that will adequately capture the basic essence of the problem.

We never try to model everything in absolute detail. A very common quote that is often attributed to Einstein (not very sure about the authenticity) - “a model should be as simple as possible, but no simpler”. So, we need to have all the influencing parameters all the significant parameters that have considerable influence on the operating conditions and the output of the system and construct as simple model as possible using all these influencing parameters.

So, the first step towards engineering analysis is to convert a large complex and causal system in the physical world to a set of tractable mathematical models for a rational analysis. Mathematical models are amenable to our basic tools of mathematics and of course, amenable to treatment via digital computers. Mathematical model allows itself to be studied under the influence of various operating conditions which may not be possible to do in case of physical world.

A physical experiment is very difficult to perform and time consuming. It is very expensive and sometimes it may not even be possible. For example, it is almost impossible to test a complete aircraft prototype in a laboratory.

But if we develop a mathematical model then it is possible to see what are the influencing parameters. How the complete system interacts with the environment that it has to operate on and draw inferences, learn from the results, study the system behaviour and then take appropriate call on modification of design if required.

So, the essential ingredient of this development of mathematical model is a differential equation. Differential equations offer a convenient and powerful set of tools for modeling causal relationship if something is known to be dependent on certain things certain variables so, we have a system of dependent variable and independent variable and we study and establish the rate of changes and that effectively gives us the basic differential equation of the phenomena.

As an example let us look at the very simple problem in mechanics that is the axial deformations of a bar as we see in this figure. This can represent several things. It is a simple bar element or truss element or member of the truss that you see.

For example, you might have traveled through several rail bridges. Each member of that bridge is modeled as a truss element and that is a bar under axial deformation.

Another situation is in this particular example that refers to end bearing piles; so, the problem of driving a pile through the pile foundation. All of them are essentially problems of single element, single prismatic member which is under the influence of axial loads.

On the right hand side you see a differential element and its free body diagram.

So, if we consider the equilibrium of all these forces, axial thrust is the stress resultant for axial deformation and as you can see axial thrust is essentially force multiplied by direct stress that is in the body and across the section there is of course, going to be change in the axial thrust because of change in surface traction q.

This q is the surface traction, which in case of a pile driving problem refers to the skin friction. That is what gives the resistance for the pile. So, significant part of resistance offered by pile is coming from skin friction between pile surface and surrounding soil.

Because of this skin friction there will be a difference in the axial thrust and of course, if the deformations are time variant, then we need to look at the basic Newtonian mechanics - Newton laws of motion. The law of momentum which states that "all forces in the body are equal to rate of change of momentum of the body". So, this is $\rho A \ddot{u}$ that is essentially rate of change of momentum and in this case particularly it is shown as a dotted line as a fictitious inertia force.

So, inertia force nomenclature is very common parlance in structural dynamics. Here parlance refer to inertial term as a force term because in the equilibrium equation, on the left hand side you have the force terms and on the right hand side you have rate of change of momentum.

So, it is dimensionally consistent. Hence it is equal to the total applied forces, but still it is not a force. It is a separate law; it is a law of momentum. So, all applied forces on the body are equal to the rate of change of momentum.

Now, why is it not a force? Because this particular rate of change of momentum that we have, happens to have the units of force and that is why its left hand side is equal to right hand side in the Newton's second law, but it violates Newton's third law as there is no reaction to this force.

That is why it is often referred to as a force, but it is always made with a distinction. It is drawn as a dotted line just to indicate that it is a fictitious force and we are talking about an equivalent representation of force equilibrium, instantaneous equilibrium by D'Alembert's principle and we can treat that as a force and just treat it as a static equilibrium holding at each instant of time.

So, using this rate of change of momentum as a fictitious inertia force and that works because in this particular case $\rho A \ddot{u}$ is time variant for most of the systems except for aeronautical systems where significant amount of mass comes from the fuel and fuel is burning at a rapid rate to give the thrust that it needs. Therefore, rate of change of momentum is actually dependent on the rate of change of mass as well.

So, for aeronautical systems situation is little more complex than what is presented here there will be one more term that is dependent on the rate of change of mass. But, let us remain anchored to the ground and for most of the problems that we deal with in Civil Engineering and Mechanical Engineering systems the mass is time invariant and the rate of change of momentum essentially refers to mass times acceleration.

So, if we establish this equilibrium consider this equilibrium and collect the terms together then we can develop the equilibrium equation.

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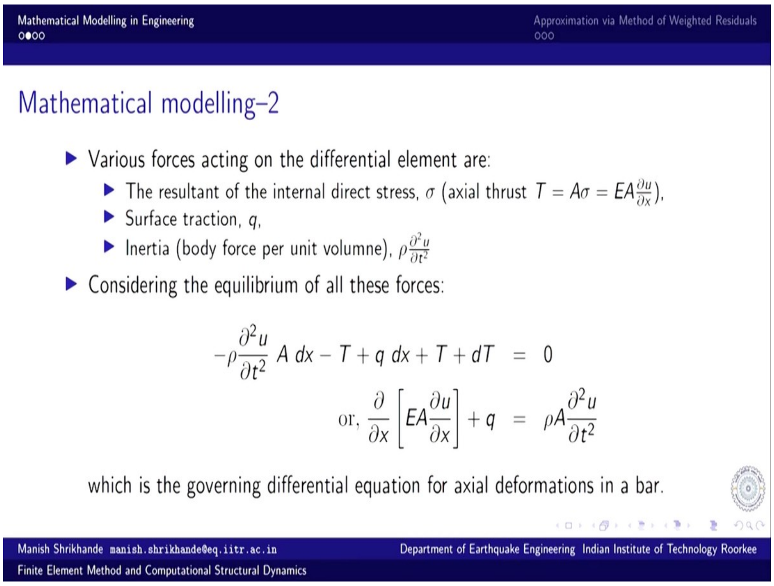

You can see the various forces acting on this differential element. So, the axial thrust is area times the direct stress and that is of course, equal to by making use of Young's modulus and Hooke's law.

$$\text{Axial thrust}\,(T) = A\sigma = AE\frac{\partial u}{\partial x}$$

Then there is a surface traction q distributed on the periphery of the body and inertia. And the body force per unit volume is $\rho\frac{\partial^2 u}{\partial t^2}$.

Considering the D'Alembert's principle the fictitious inertial force acts in the direction opposite to the direction of motion.

So, considering the instantaneous equilibrium of forces and then dividing through by the length differential element dx we arrive at this governing differential equation.

$$\frac{\partial}{\partial x}AE\frac{\partial u}{\partial x}+q=\rho A\frac{\partial^2 u}{\partial t^2}$$

That is the governing differential equation of the axial deformations in the bar and as you can see this is a second order differential equation. The highest order of derivative that you see is two with both respect to spatial derivatives as well as with respect to time derivatives.

Now, there are two independent variables here spatial variable x and time variable t. That is a complication that we would like to avoid getting into at the moment. We will deal with time dimension subsequently, but to keep matters simple let us consider the situation in which the deformations vary slowly with respect to time. So, if the deformations do not change very rapidly with respect to time, then the acceleration term is going to be negligible.

So, $\frac{\partial^2 u}{\partial t^2}$ is going to be a very small quantity and therefore, we can ignore that term and we are left with an ordinary differential equation.

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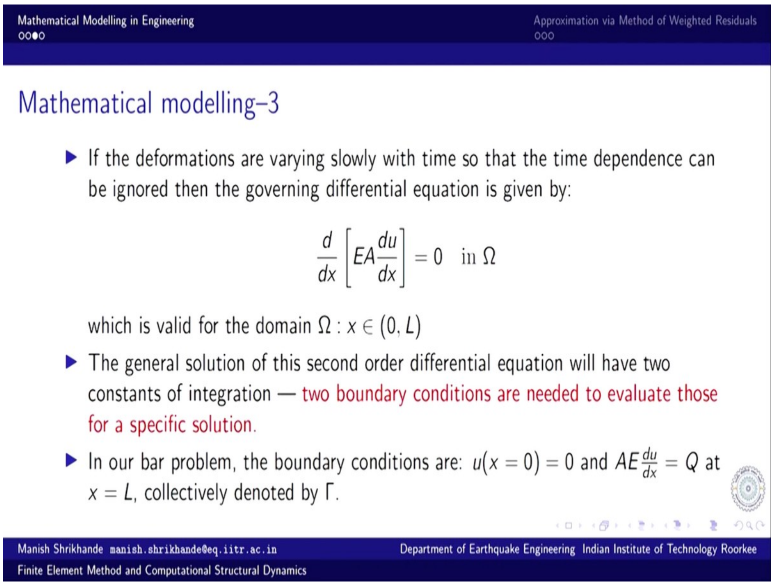

So, the governing differential equation now reduces to an ordinary differential equation because there is only one independent variable now.

$$\frac{d}{dx} AE \frac{du}{dx} + q = 0 \; ; \quad \text{in} \quad \Omega$$

Now, there is an interesting addition to the whole thing.

See the problem is this differential equation that we define, where does this apply in the infinite space? or what is the extent of the system that is described by this differential equation? So, that is called a domain of the problem. In this particular case the domain of the problem is the length of the bar starting from x = 0 to x = l.

So, that is the domain of the problem. That is the domain over which this particular differential equation holds. So, every differential equation that we develop has to have a particular domain of applicability and that has to be specified along with the differential equation. A differential equation has no meaning unless we specify what is the domain of its application.

Now, this is as I said it is a second order differential equation and obviously, it will have two constants of integration. The general solution of the second order differential equation will have two constants of integration. And, obviously, we will need two boundary conditions, two additional conditions to determine specific solution to the problem.

What do I mean by specific solution? For example, let me go back to the equation. Solution will be different if I removed the fixed boundary. If this fixed boundary was removed and made a free boundary that is a different problem than what is presented here.

Secondly, if I remove the force q and apply a point load at one end, that is a different problem. If I apply a constraint at the other end with some specified value of displacement or make it some other non-zero value will also make it a completely different problem.

So, for a differential equation we can have a general solution, but for each of this specific problems that can arise because of the peculiarities of the situation, there has to be specific solution for each of these peculiarities. Now, those specific solutions are captured by appropriate evaluation of the unknown coefficients of or unknown constants of integration.

So, two constants of integration and those constants of integration will be evaluated by suitable boundary conditions. Governing differential equation is generic. But the boundary conditions define the specific problem that we need to solve.

In our bar problem, we have the boundary conditions as u(x=0) = 0. So, at this particular point (x=0) the deformation is constrained to be 0. So, that is the one boundary condition. u - the axial deformation of the bar, is the basic unknown of the problem.

The axial thrust, $\left(AE\frac{du}{dx}\right)_{x=L} = Q$ at the free end should be equal to Q to ensure equilibrium condition otherwise the system will not be in equilibrium.

These are the two boundary conditions and that will define the specific solution to this problem. So, a general solution to the second order differential equation will be made specific to this particular problem by evaluating the unknown coefficients of the problem (constants of integration) which will suit these two boundary conditions.

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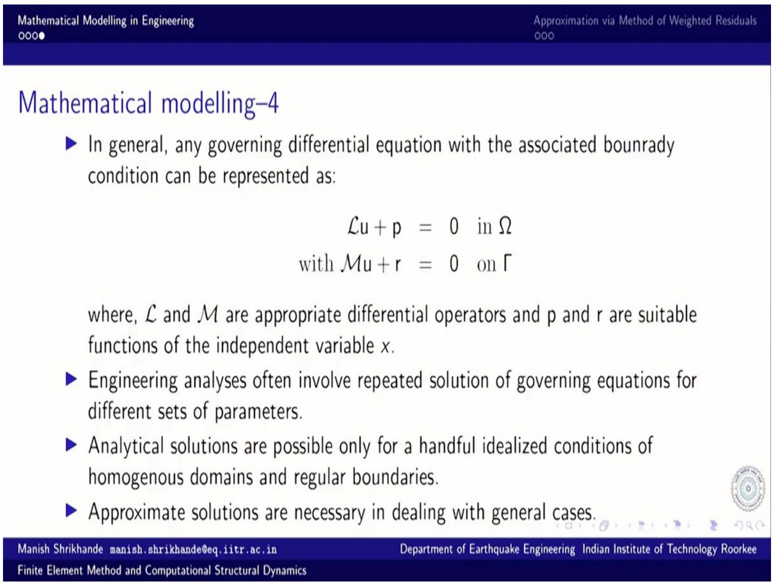

So, what we discussed was with specific reference to a bar problem. Similarly, we can develop similar formulation for any problem and in general it will be associated with some differential operator let us say L. In this particular case our differential operator was second order differential operator.

$L = \frac{d}{dx} AE \frac{d}{dx}$ is the differential operator and p can be anything. It it is a function of independent variable x. So, that is how the problem is defined and that is equal to 0 in Ω .

So, this differential equation

$$Lu + p = 0 \quad \text{in} \quad \Omega$$

is the governing equation of any problem. It can come from anywhere. It can come from mechanics; it can come from fluid mechanics, structural mechanics, heat conduction, electromagnetics, anything.

And, then obviously, as we discussed differential equation will have a general solution, but the solution specific to the problem at hand will be defined by the boundary conditions. So, associated boundary conditions are M u + r = 0 on Γ . So, we have this situation where domain is defined by Ω and its boundary by Γ .

In case of bar problem gamma constitutes two points two end points x = 0 and x = L in between x is equal to 0 and L we have the domain of the problem. So, that is one dimensional problem. In case of two dimensions Γ will be a curve which will enclose the area and the domain becomes the area between this bounding curve. In three dimensions we have a surface boundary and the volume enclosed by that surface becomes the domain of the problem. So, conceptually it translates and scales very well from one dimension to three dimensions and the similar construction holds. Only the tools get little more tedious as the dimensions of the problem increases, but conceptually it remains exactly identical.

So, we have a governing differential equation defined over a domain and subjected to certain boundary conditions on the boundary enclosing that domain. Now, engineering analysis often involve repeated solution of governing differential equation. We need to do lots of analysis for different influencing parameters before we come up with a suitable design or valid design or valid solution to the problem.

And, analytical solutions are only possible for a handful idealized conditions of homogeneous domains and regular boundaries. Homogeneous domains means the properties are uniform over the entire domain, it does not change. So, approximation does not really has to account for these point to point changes in the material properties or the geometrical properties.

For example, in this particular situation the bar problem we discussed, we considered a uniform bar with constant cross section and same material throughout. If the problem was little different for example, it was an irregular cross section and the materials were different over different portions then situation would have been much more difficult.

It becomes very difficult to solve analytically this particular kind of problem with different or heterogeneous medium. It is still possible but becomes a little tedious and difficult to solve analytically.

So, we can develop analytical solutions only for a handful of idealized conditions of homogeneous domains and regular very well defined boundaries. The moment any of these idealized conditions are violated, analytical solutions are no longer possible.

It is not easy to develop analytical solutions and hence approximate solutions are necessary in dealing with such cases. And, that is what we will discuss in our next lecture. How do we deal with general solution of partial differential equations using approximate method? The method is called method of weighted residuals which we will discuss in our next lecture.

Thank you.