Finite Element Method and Computational Structural Dynamics
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Lecture - 60
Discrete Fourier Transform-VII

Hello friends. So, we have seen one important application of DFT in structural dynamics that is the process of convolution and deconvolution in frequency domain, which can facilitate several analytical procedures and allow us for computing certain results very quickly and very efficiently.

Now we also look at another aspect that can be useful or that in which DFT can be of great help and that is the vibration data processing. As we all may be aware you all may be aware that high frequency data high frequency vibrations they are often picked up by accelerometers.

So, it is the acceleration at the base that is often recorded and while acceleration that seem so pretty simple and seems to be having I mean symmetric kind of vibration signature on the plot, but one crucial thing that remains is we do not I mean acceleration remains a derived parameter the basic parameter is of course, the displacement.

So, we need to compute the displacement and which are useful for several purposes. And conversely certain other seismometers etcetera they may record velocity signals velocity traces and it may be required to compute acceleration or displacement from that. So, essentially we need displacements and velocities for computation.

And in frequency domain if we perform the complete analysis in frequency domain then we can compute the displacement response and from having computed displacement response how to compute the velocity and acceleration if required.

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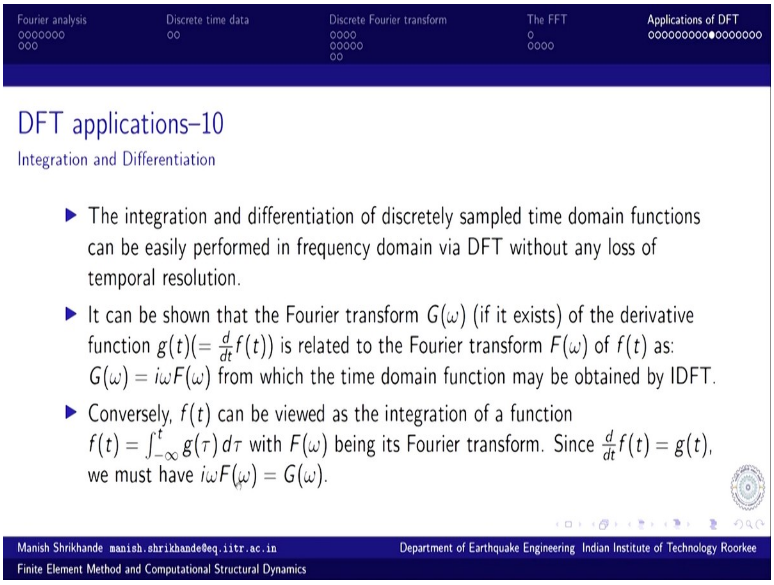

So, we consider that so, integration and differentiation of discretely sample time domain functions. We all know that differentiation can be done by using finite difference approximation integration can be done by using quadrature a numerical quadrature, but in both these cases there would be some loss of resolution.

And while if we do this in frequency domain then there is no loss of(t)emporal resolution, whatever is the resolution of original data we retain the same resolution in the derived quantity or the integral integration that we do or differentiation that we do. So, that there is no difference in the sampling interval subsequently.

So, it can be shown we can analytically verify this that Fourier transform $G(\omega)$ if it exist because that existence conditions have to be satisfied. So, of the derivative function let us say g(t) is given as a time derivative of the function of some function f(t)and f(t) has a Fourier transform of $F(\omega)$. So, Fourier transform of g(t)that is G(%omega) is related to the Fourier transform $F(\omega)$ of f(t)is given as $G(\omega) = i\omega \times F(\omega)$ so, very simple.

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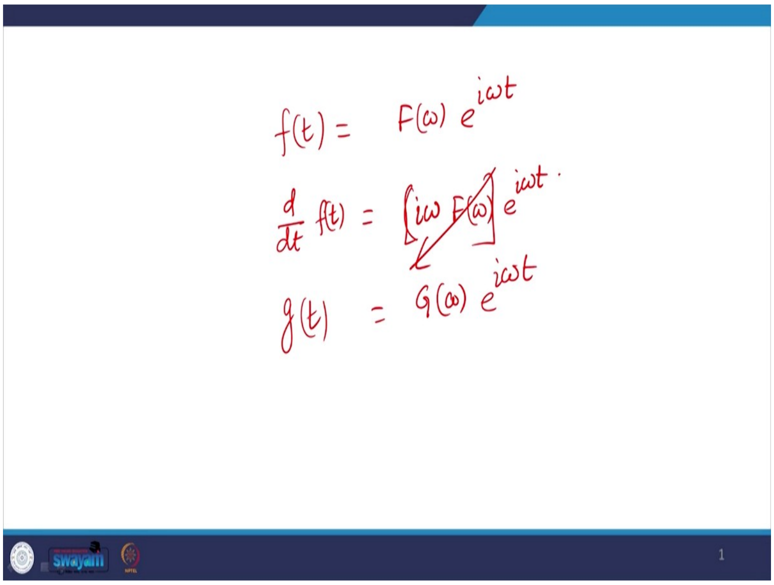

And that actually follows from if I simply say that f(t) = if I go by harmonic representation $i\omega t$ then the only function time domain function is this exponential function. So, that gives us the result $i\omega F(\omega)e^{i\omega t}$. So, that replies that implies that this is the desired function, Fourier transform function, Fourier transform of g(t).

And that is what allows us I mean if we are of course, this is very what should I say engineering mathematics kind of proof. So, intuitively appealing and we can derive this result of using very rigorous Fourier transform relations as well, but essentially all that we need to do is integration or differentiation is multiplication by $i\omega$. And if differentiation is multiplication by $i\omega$ what can the integration be integration has to be the reverse operation of this.

So, integration can often can be modeled as division by $i\omega$ and that is what we do conversely. f(t) can be viewed as integration of the function $g(\tau)d\tau$ integrated from -infinity to t and being its Fourier transform $F(\omega)$ is the Fourier transform of f(t). And since first derivative of f(t) is g(t) then we must have $i\omega F(\omega)$ as $G(\omega)$ and therefore, $F(\omega)$ can be written as $\frac{G(\omega)}{i\omega}$.

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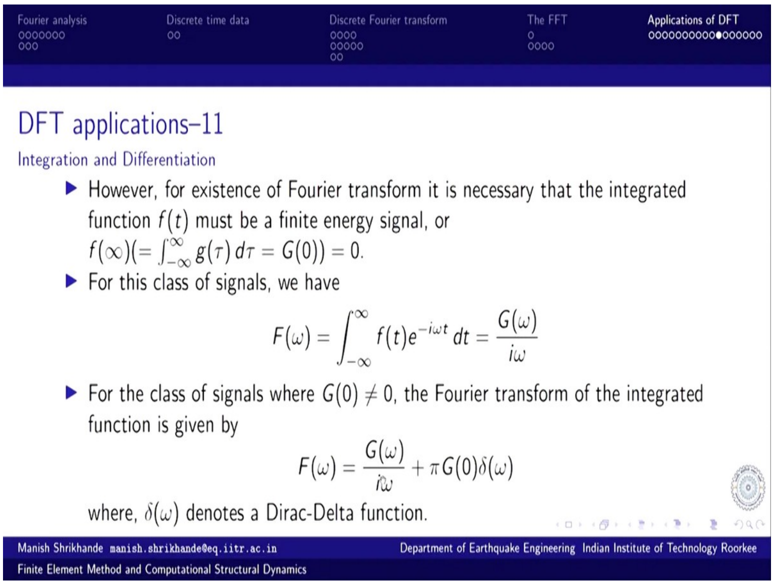

So, integration existence of Fourier transform it is necessary that the integrant that is the f(t) integrated function f(t) must be a finite energy signal I mean the Fourier transform existence of Fourier transform it is essential that the time domain signal has to have finite energy and that is what is done by using this integral. So, f of infinities that will evaluate the complete integral of $g(\tau)d\tau$ and that essentially replies as implements as g evaluated at 0th frequency so, that is 0.

So, as long as the mean value the function has 0 mean value then it should be fine. So, for this class of signals $F(\omega)$ is given as $\frac{G(\omega)}{i\omega}$ and if g(t) does not have 0 mean value. So, for the class of signal where g at 0 frequency is not 0 then the Fourier transform of integrated safe function is given by $\frac{G(\omega)}{i\omega} + \pi \times G(0)\Delta\omega$. So, that is the Dirac delta function at 0th 0 frequency so, that models the non 0 mean value constant mean value.

So, this way we can very quickly estimate the integrated waveform or the integration of the waveform or differentiation of the waveform and that can be done again very very quickly. So, the problem with time domain operations of finite difference approximation for derivatives or quadrature approximation for integration is that not only they will affect the temporal resolution they will also affect.

Because if i do the finite difference approximation then the availability of data and the original data and the differentiation derivative process time samples they would be slightly offset. And they also require a large number of floating point multiplications and floating point in general large number of floating point operations. And all those floating point operations needlessly I mean not only they are time consuming they are also error prone.

The more floating point operations I perform more chances of errors creeping because of round of error possible round of error. And DFT allows us to have a very seamless process for integration and differentiation as long as we take care of this peculiar problem of 0 frequency because this ω frequency term goes in the denominator so one has to be really careful about dealing with 0 frequency component.

And as long as we take care of that then this entire process can be computed very quickly and at the same time as we compute the response displacement response in the same way. Immediately we can compute the velocity response and acceleration response directly at the just sequential calls for inverse DFT functions and we have the complete results available very very quickly. I mean the computation Fourier transform discrete Fourier transform either forward or backward does not really take much time.

So, the results computation is very quick and processed very efficiently. Of course, one has to be careful about this manipulation of divided multiplying by i %omega and dividing by i %omega. Because often this Fourier transform that we are mentioning here these are complex numbers, but in the actual computation of DFT they are all dealt with as a real array of real numbers.

First half is stored normally first half is used for storing real part and other half is used for storing the imaginary part or another compact representation spacing representation could be the real and imaginary parts are stored sequentially.

So, first two elements would correspond to real and imaginary part of the first a component of Fourier transform and third and fourth elements of the real array would correspond to the real and imaginary part of second component of second frequency component of Fourier transform and so on.

So, different algorithms FFT algorithms that are implemented on these software's they have their own encoding scheme and one needs to be careful and one needs to get familiar with that encoding scheme before this before trying out this kind of operation to get consistent results.

So, this modification multiplication or differentiation in frequency domain and integration in frequency domain that can be effected very easily as long as we are familiar and we take care of consistency with respect to the encoding that is used for compact representation of complex numbers in using real arrays, in different Fourier transform FFT implementations. Now, we look at another application area that is of spectrum estimation.

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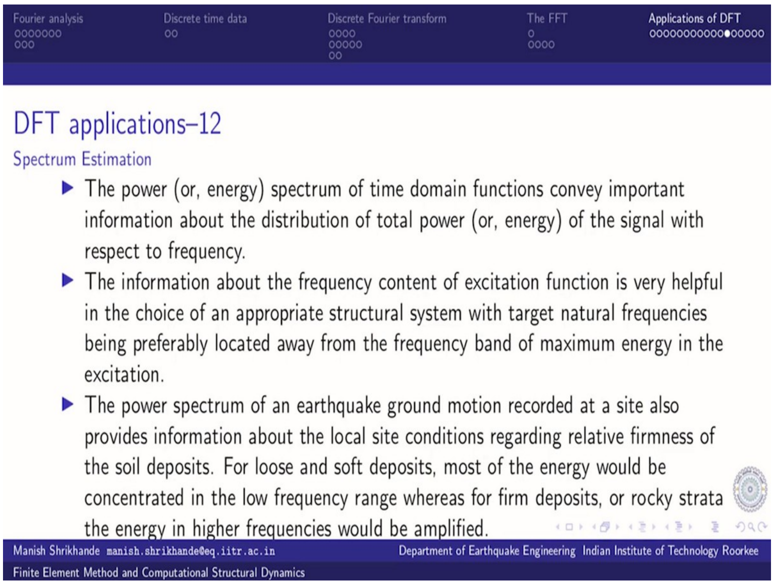

So, power or energy spectrum of the time domain functions, they carry important information about the distribution of power or energy of the signal with respect to frequency. And this information is generally not visible or not available in the time domain representation of the waveform if the waveform is complex then there is no wave we can decipher that if the waveform is very simple like a harmonic waveform then of course, it is easily discernable you can we can identify what is the frequency of this waveform.

But for a complex waveform it is not possible to identify the distribution of frequencies or which frequencies are present in the waveform from looking at the time domain signal. And for that we need to have spectrum we need to look at this spectrum either power spectrum or energy spectrum of this time domain signal.

And this information about the frequency content of excitation. So, for any system that needs to be designed for with standing any effect of any excitation if we know what are the frequency characteristics of the excitation then it provides very useful input for design.

In the sense that if we know what are the dominant frequencies in the excitation we can choose a design of the, system that we had looking at or that we are trying to design by avoiding that those dominant frequencies as the natural frequency of the system.

We will try to develop a system design a system such that its natural frequencies are away from the dominant frequency band of the excitation. So, that way the dynamic excitation or large response problems can be avoided. So, the power spectrum of an earthquake ground motion recorded at a site also provides information about the local site condition.

So, very useful information about the site characterization often involves just recording the motion and looking at this spectrum of the ground motion that is recorded there and that provides us very useful information about the character of the soil deposit or character of the site what kind of frequencies are being admitted and what kind of frequencies are being amplified.

So, it provides us information about local site conditions regarding relative firmness of the soil deposits for loose and soft deposits we may see amplification of lower frequencies in the lower frequency band whereas, if it is a firm or rocky strata then the energy amplification would be observed in higher frequencies and the that is the rough idea about what kind of strata that we are looking at. So, very quick estimation that can happen by just looking at the spectrum of the recorded motion.

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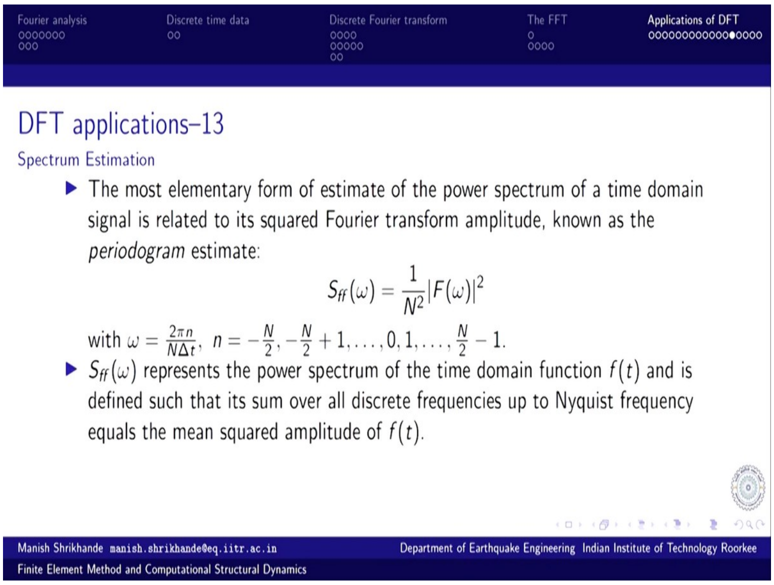

And the most elementary form we can find out the power spectrum the most elementary form of a power spectrum is obtained by using Fourier transform amplitude of the waveform and that is called the periodogram estimate. So, S_ff that is the auto power spectrum of any function to waveform f(t) is simply given as 1 over N square. So, that is N is the number of data points. So, 1 over N square modulus of F(%omega) squared and for different frequencies for and ω is defined as $\frac{2\pi n}{N\Delta t}$.

So, that is the frequency range complete frequency range and N ranging from $\frac{-N}{2}$ to $\frac{+N}{2}-1$. So, total N number of data points and $S_{ff}(\omega)$ represents the power spectrum of the time domain function f(t) and is defined such that the sum over all discrete frequencies up to the Nyquist frequency equals the main the mean squared amplitude of f(t).

So, the power that is what the power spectrum refers to, the raw periodogram estimate that we obtain from Fourier amplitude spectrum is; however, very zigzag I mean very high variance. So, and that high variance does not reduce even if we consider a long stretch of record so periodogram has that problem that it may look very very raw and very very jagged look.

And there are need to be the often is need to smoothen it and the way to do that is to reduce the variance is to look at periodogram estimate of different segments of data. And when we look at different segments of data estimate periodogram and then average that periodogram estimate. So, that will reduce the variance and providers a smoother estimate of the power spectrum.

So, one way to reduce the variance of the estimate is to divide the time domain sequence into several parts and then average the periodogram estimates for each of these parts at each frequency.

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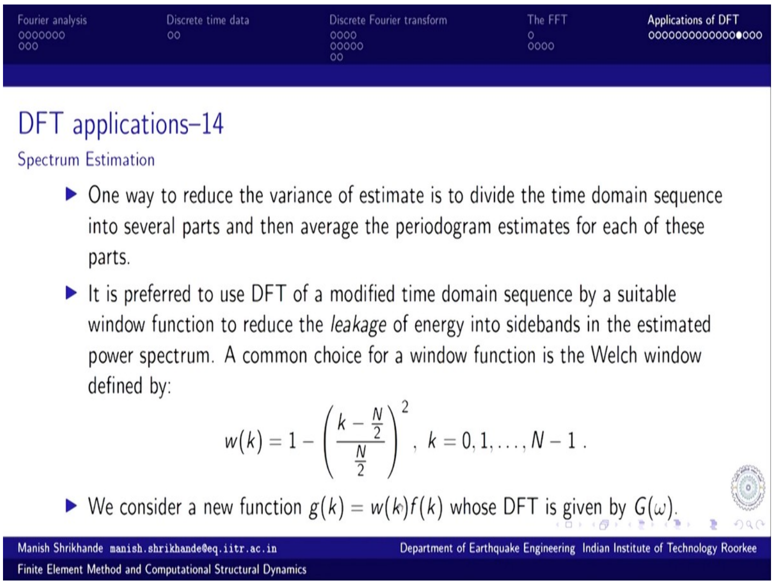

And it will also important it is also preferred to use DFT Discrete Fourier Transform of a modified time domain sequence by a suitable window function. So, window function that we use normally whatever we have been discussing so far we always assume that its a rectangular window that we are looking at one period of the waveform.

So, that rectangular window has very undesirable side effects in frequency domain when we use this rectangular window to extract one period of the waveform. In the scenes that there are side lobes which leak energy into frequencies beyond Nyquist frequency.

So, to reduce those leakage energy leakage through side lobes it is often preferred to have a smooth window that may that avoids having those sharp cuts that we have transition phases of rectangular window where it suddenly stops and suddenly starts. So, instead of having those sudden starts in sudden stops we can have a smooth window and that smoothness in time domain will result into reduced spilling over of energy through side lobes.

But of course, it has another side effect that the Fourier spectrum that we compute that is little smeared. So, we do not have as fine resolution of Fourier spectrum as earlier. So, it always everything all these fixes they have certain side effects unintended side effect. So, one solution can never be prescribed for coverage covering all sorts of problems. So, one has to be really careful and pick up appropriate solution depending on the problem that one is facing.

So, there are several windows window functions that are available like a Welch window or Hanning window; Hanning window is particularly very common that is based on use of a cosine wave. And Welch window is a periodic waveform a parabolic function of the sample number. And shape wise both Welch window and Hanning window they actually look very much same except that the there might be small differences in the values of the ordinate function values.

So, we consider a new function that is g(k)that is given as window function w(k) multiplied by f (k), k represents the index or the sample number discrete samples from 0 to N - 1. And discrete Fourier transform of g(k) is given by $G(\omega)$ that we can compute.

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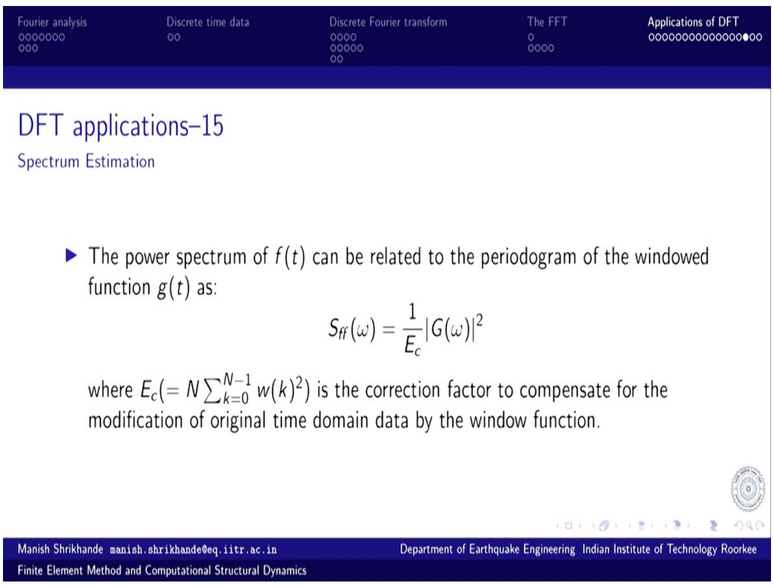

Now, knowing g(k) , $G(\omega)$ and this window parameters we can now estimate what should have been the Fourier transform of f(k) or the power spectrum of f(t) can be estimated by using the periodogram estimate of g(t) $|G(\omega)|^2$ and divided by certain error correction certain error correction factor that is based on the weighting of window function. To compensate the modification of original time domain data by the window function.

Now, this estimate is of course, going to be much smoother than the raw periodogram estimate. And it is often used for analysis and interpretation of results. Another related concept is that of coherence of two time domain sequences let us say f_1(t) and f_2(t)

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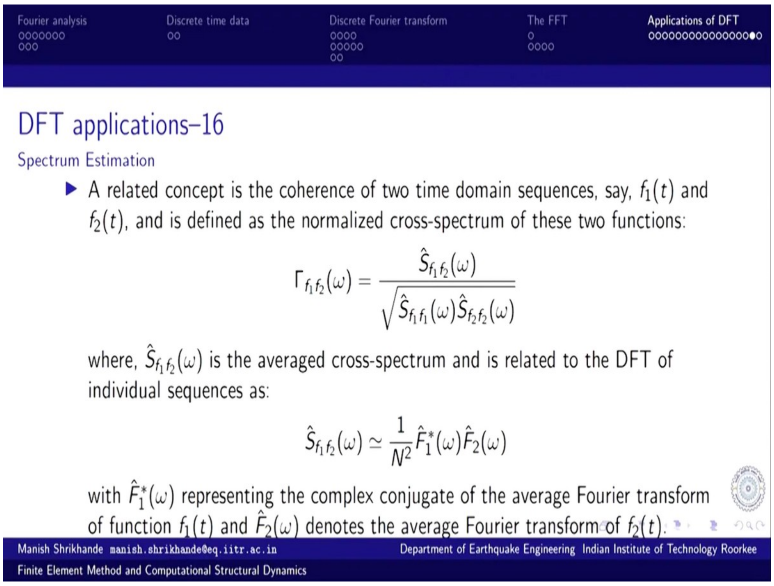

And their coherence can be defined as a normalized cross spectrum between these two functions and individual cross spectrum are related true through their periodogram estimate. And using this periodogram estimate the coherence function can be formed. I mean the importance of coherence function is in the sense that it measures the linear dependence of between the two time sequences or two waveforms.

So, if we have two waveforms we can find out what is the linear relationship between the two whether they are to what extent they are linearly dependent on each other what is the extents extent of dependence. So, magnitude of the coherence function describes whether the frequency components of one signal are associated with large or small amplitudes at same frequency in the second signal.

While the phase of coherence function indicates the phase lag or lead of one signal with respect to the other signal second signal for a given frequency component. And these kind of studies this kind of representation coherence analysis coherence function they are often used for modeling waveforms which are recorded at different points in space.

So, for example, if we are looking for analysis of a bridge or long pipeline subjected to earthquake excitation or earthquake ground motion then; obviously, earthquake ground motion undergoes several changes as it propagates through the soil medium earth layered earth. So, the motion at different points of supports which are supporting the bridges long span structures like bridges or pipelines they may be subjected to different motions.

So, the coherence model is very convenient tool to define the motion or spatial variation of such waveforms with respect to distance and frequencies. So, magnitude of coherence function is a measure of linear dependence between the two signals for a given frequency component and coherence function is often used in modeling of spatial variation of earthquake ground motion which is an important consideration in the analysis of spatially extended facilities.

The instantaneous differential motion of ground motion differential ground motion I mean one motion moves support moves like this other motion may go down. So, there may be out of phase motion at different points. So, this differential motion can have just by virtue of moving differently the structure the pipeline may experience quasi static deformation.

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And in addition to the dynamic effects and that may be significant addition and that may be important those additional stresses that are developed because of quasi static deformation that may be important for consideration for the design calculations. And for that purpose these coherence modeling they are often used and they can have several different application domains for example, in pipelines and different motions being supported I mean different points undergoing different motion.

And for example, industrial structures I mean there can be pipes running all around and supported at different points and different points may undergo different motion. So, that different motion differential motion can actually add to quasi static deformation which may not have been anticipated or may not have been accounted for in design and it is important to do so to avoid any unpleasant incidents.

Now, there can be several I mean there is no end to applications of DFT I have tried to present a very brief spectrum of the issues associated with Fourier transforms and discrete Fourier transform and fast Fourier transform algorithm, and how it is used in analysis and how it makes our life much easier for computation of dynamic phenomena computation with dynamic phenomena.

So, with that we come to the end of this course we have I mean as I hope I have been able to give you a broad overview. I mean in a course of this limited duration it is not possible to go into very great philosophical details about everything, but I have tried to cover a broad spectrum and given you enough exposure to the basic thinking fundamental thinking in the development of various processes and various formulations.

And that should help you in exploring the subject on your own and try to pick up the threads and iron out a few wrinkles wherever they may be. So, we started with I mean as you might have appreciate you might appreciate now that finite element method is completely useless without digital computers being available.

And that is why the while the something similar to finite element concept was known at the time of gauss, but it did not pick up because it did not catch the fancy of users or professionals in practitioners of the field because there were no computational tools adequate computational tools available for the purpose.

And when we use computational tools there are certain rules of the game that needs to be adhere to and that you need to know and that is what we started this discussion with, because digital computer is such an indispensable tool for effective use of finite element method. We discussed the digital computer architecture and more importantly how real numbers of floating point numbers floating point numbers are represented on a digital computer and how floating point operations take place.

And that gives us an appreciation of the possible sources of errors and how to minimize the possible sources of errors. And with that we develop the basic formulation for vector algebra linear algebra the vector spaces the linear basis. How we can basically construct approximation any kind of function can be expressed as a sum of linear basis function linear combination of basis vectors or basis functions in a given function space or vector space.

And then how to evolve how to develop how to develop suitable approximations interpolation and then for numerical calculation how the integrals are evaluated. And using these basic constructs, we use these basic constructs that we discussed earlier in the development of finite element model their the method of weighted residual we discuss the classical method of weighted residual that was known for a long time.

But we adopted method of weighted residual in the familiar notion of part to whole and then whole to I mean whole to part and then part to whole dividing the sub domain and looking at approximation over smaller sub domains using the constructs that we had already discussed beforehand the interpolation theory.

And using those interpolation models over smaller and regular domain we constructed the approximation over smaller domains and then the process of assembly that we discussed. And it that process of assembly followed by imposition of boundary conditions allows us to have simultaneous equation and this process remains the same different problems different governed by different differential equations they only differ in the choice of interpolation model.

And with the suitable choice of interpolation model plugged into the basic finite element procedure rest of the finite element steps of the finite element method that remains the same. And at the end of the complete finite element modeling we are left with either with a simultaneous algebraic equation k u = f or if it is a time dependent problem then we end up with a second order differential equation with constant coefficients with certain initial conditions.

And those problems the time domain problems they are solved either by using time marching scheme that we discussed starting with simple converting second order differential equation into first order differential equation equivalent first order differential equation and then using either Euler scheme or Runge Kutta integration time merging.

Or we can use some of the methods that are directly applicable for solving second order differential equation like Houbolt method or central difference method or Newmark beta or the modern day modification of Newmark beta that is HHT alpha Hughes Hilber and Taylor alpha method.

And then we looked at the other aspects the solution of simultaneous equation algebraic solution and the eigen value problems how those are important in the sense that natural frequencies are identified as positive square roots of the eigen values of that is presented by the free vibration problem of the equation of motion.

And finally, we ended up with discussion of the most powerful and most very exciting technique that is discrete Fourier transform and the fast Fourier transform which allowed us to deal with this dynamic problems from a different perspective altogether.

And with that we cover complete 360 degree coverage of the dynamic phenomena we look at the waveform or we look at the information in time domain we look at the information in frequency domain and both of these views they give us complementary information. And both of these sets of information can be very valuable in taking a resent decision in design process designing any engineering system.

I hope that you have enjoyed this journey of finite element method and computational structural dynamics with me. I am always available at this email address that you can see manish dot shrikhande at eq dot iitr dot ac dot in, feel free to write to me for any queries any questions any discussion.

Thank you.