Finite Element Method and Computational Structural Dynamics
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Lecture - 59
Discrete Fourier Transform-VI

Hello friends. So, we have seen the DFT the process of DFT and how DFT is implemented very efficiently using a fast Fourier transform algorithm. We discussed briefly the basis and genesis of radix 2 algorithm wherein, the data length is considered to be integer power of 2. And that allows us to repeatedly divide the problem by a factor of 2 and until we end up with two pairs of data one pair of one set of two data points.

And Fourier transform of two data points is only given as sum of those two data and difference of these two data. And subsequently, this is these are again scrambled back the original full length waveform. So, and efficient as we discussed I mentioned earlier that efficient implementations of fast Fourier transform are already available. The main emphasis of our discussion is to understand the structure or the relationship between discrete Fourier transform and the continuous Fourier transforms.

Under what conditions discrete Fourier transforms can be a good approximation to continuous Fourier transform. Because in our theoretical developments we often talk about continuous domain the time domain is also in as a continuous time domain waveform and corresponding transform frequency domain representation is also considered to be continuous function of frequency.

Now, for numerical computation those these are of course, represented by discretely sampled time data points and also discrete frequency components. And these discrete time samples and discrete frequency components they are they form a set of periodic, infinite periodic functions.

And as long as we look at one period of these waveforms in time domain or in frequency domain within that one period they correspond to representative values at the those discrete points of the continuous waveform and continuous Fourier transform. Provided that certain care is taken about the parameters of transform.

So, now we come to the application of DFT. So, far we have been discussing about the theoretical aspects what how DFT and what how to interpret DFT results. But coming to theoretical applications where do we use Fourier transform in engineering analysis and why it is such an important development, such a major development in engineering analysis that actually revolutionalize the complete, revolutionize the complete field I mean several new applications and new products came into being just because there was a new algorithm fast Fourier transform which allowed very efficient computation of Fourier transform on a digital computer.

So, let us look at the applications discuss about the applications. And the first one that is the most commonly most widely used application as far as vibration analysis or structural dynamics applications are concerned is the evaluation of convolution integral. Convolution integral often is also referred to as the Duhamel integral in theory of vibration or structural dynamics literature. And it is the basic way to compute the dynamic response to any arbitrary excitation of a linear time invariant system which is at rest initially.

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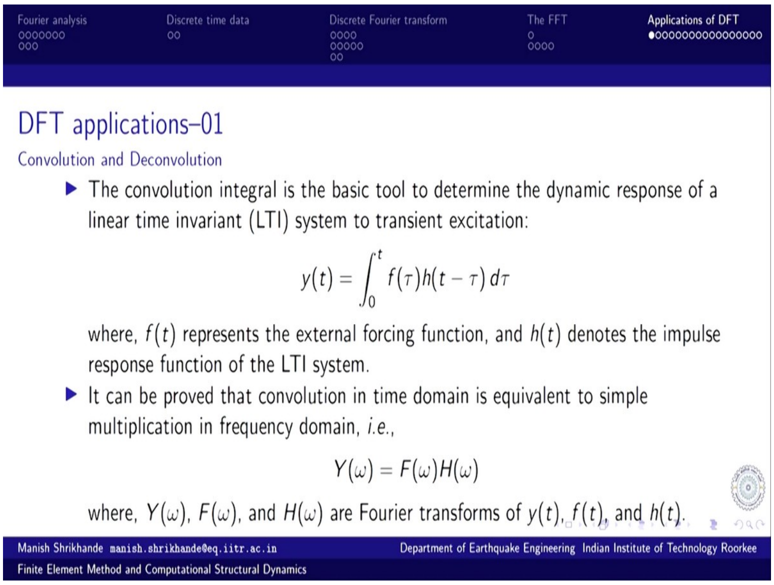

So, theoretically it is given as y(t) that is the response of the system is given as convolution between excitation $f(\tau)$ and multiplied by convolved with the impulse response function $h(t-\tau)d\tau$. Now, this is seemingly very simple expression, but this appearances can be deceptive and this is a very very expensive operation to do on a computer, let me show you what are the aspects involved.

First of all this the variable of integration is τ and $h(t-\tau)$ actually involves three operations. First it is the we call it $h(-\tau)$, so that is actually the folding operation. So, we look at the mirror image. So, whatever is the impulse response function h of as a function of τ impulse response function of course, it has to be for a causal system it can happen only after the application of the impulse.

So, if we assume that then impulse, unit impulse is applied at time t = τ = 0 then the response will always happen only for positive values subsequent to the application of the impulse. So, that impulse response function is defined only for positive values of τ. So, what we are doing here is we are first we will first fold the impulse response function about the origin.

So, that gives us $h(\tau)$ is mapped to the mirror image we look at the mirror image. So, that gives us $h(-\tau)$ as a negative function, negative value of τ and then it is shifted then it is shifted by parameter t. So, whatever time instant we need to compute the response we shift it by that much amount.

And then these two products are calculated $f(\tau)$ and $h(t-\tau)$ this product is calculated and then the integral is evaluated. So, all these operations of course, continuous time it can be done I mean analytical expression, the analytical integrals can always be compute evaluated if $f(\tau)$ functional form is known.

But it is often done numerically using by replacing this integral by a summation. So, that sequence of operations are done as folding, shifting, multiplication and then accumulation of the product terms area under the curve. So, that is the total process that it takes for calculation of response. Now this; obviously, requires lot of lot and lot of floating point multiplications and that consumes quite a large amount of time and effort.

Now, this convolution in time domain it can be converted, it can be shown, it can be proved very easily that if we take the Fourier transform of both sides then it essentially this convolution in time domain is replaced by simple term by term multiplication of Fourier transforms of individual components.

So, in frequency domain Y %omega the response of the system is simply given as F %omega that is the Fourier transform of $f(\tau)$ and $H(\omega)$ that is the Fourier transform of impulse response function. So, $H(\omega)$ is known as frequency response function that is Fourier transform pair of impulse, unit impulse response function. And $f(t)$ and $F(\omega)$ they are they form the Fourier transform pairs of excitation and its corresponding Fourier transform.

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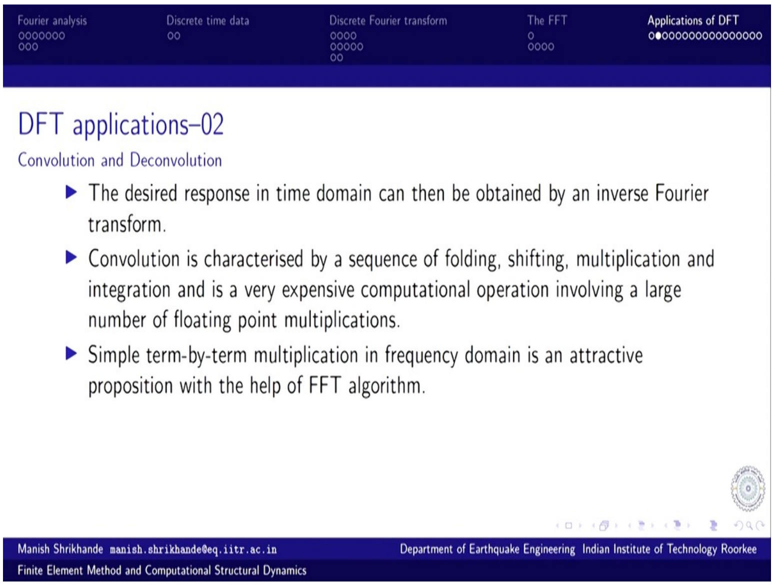

So, desired response in time domain can be obtained by inverse Fourier transform of $Y(\omega)$ since $Y(\omega)$ can be computed simply by considering the product of $F(\omega)$ and $H(\omega)$ that is very easily computed. And then Fourier inverse Fourier transform can be done can be computed for $Y(\omega)$ and we have response of y(t) that is desired in time domain.

Because time domain response is important because we can extract the maximum value the critical value for that is that may be important for design calculations. That is available only from time domain representation that maximum value cannot be extracted from frequency domain representation. So, convolution is characterized by a sequence of folding, shifting, multiplication and integration and it is a very very expensive computational operation involving a large number of floating point multiplications.

So, simple term by term multiplication in frequency domain is an attractive proposition particularly with the help of FFT algorithm which allows calculation of discrete Fourier transform very efficiently and very quickly and very accurately if we are careful. Convolution of the continuous time domain waveform and its continuous frequency transform can be approximated reasonably well by one period of discrete time domain samples and discrete Fourier transforms over fundamental range of - %pi to + pi.

If the period, the time domain period and the sampling interval and the period of the frequency domain representation if that is chosen carefully. So, if we are not careful in this selection then we will see how things can go horribly wrong.

Fortunately, the need for having data length of as integer power of 2 comes to our rescue even if somebody or a user is not aware of the pitfalls of these periodicity and associated complications of discrete Fourier transform he may he or she may not realize that problem because most of these issues are taken care of by the process of extending the data length. So, that the data that we handle is suitable for use with radix two algorithm.

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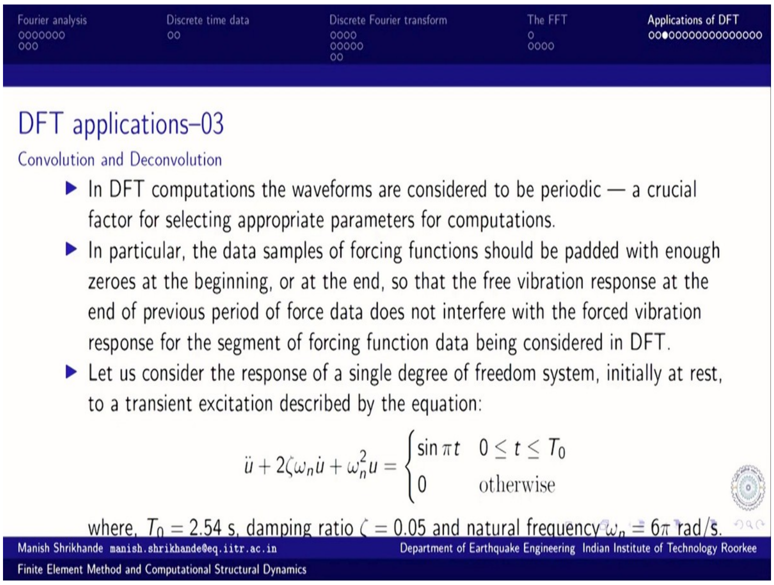

So, in DFT computations the waveforms are considered to be periodic a crucial factor for selecting appropriate parameters for computations. In particular the data samples of forcing function should be padded with enough zeros at the beginning, or at the end, or at both ends whichever is preferred for individual liking.

So, that the free vibration response at the end of previous period of force data does not interfere with the forced vibration response for the segment of forcing function data being considered in the DFT. So, we will see this the implication of this somewhat long winded statement in a example that we do.

So, let us consider the response of a single degree of freedom system, very simple excitation I mean. So, one degree of freedom only one response and that too excited by a sine wave, so fraction of a sine wave. So, sin %pi t and we consider T_0 as a some duration which is intentionally we choose that which is not an integer number of the period of the wave.

So, T_0 we consider as 2.54 second and damping ratio critical damping ratio is 0.05 and natural frequency of the oscillator is ω_n is considered to be 6π radian per second. So, the excitation is $\sin \pi t$ as for 0 to as t ranging from 0 to T_0 and outside this range it is 0. So, let us see how it looks like, the analytical solution for this can be computed using analytical response I mean using two eras, forced vibration era.

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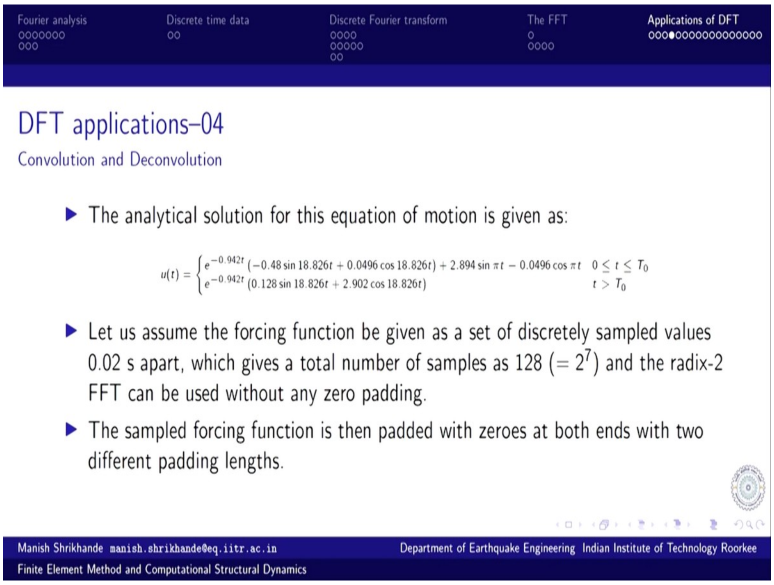

It is going to be sum of this transient part which is because of the initial conditions and this is the forced vibration response that comes that is as you can see it is defined by $\sin \pi t$ and $\cos \pi t$ terms. So, these are the forced vibration response and this is the transient response which will decay after some time as t increases and this is computed just, so as to have zero initial conditions. So, displacement and velocity are 0, so the system starts at rest from at rest condition.

And after T_0 once the excitation has been removed the system, the oscillator will execute simple harmonic motion because of the initial whatever is the velocity and displacement at the instant of removal of the excitation that is at $T_{0.}$ So, at that particular instant whatever is the displacement and velocity those will become the initial conditions for free vibration response.

And this second solution is for that free vibration era and for times greater than T_0 . So, let us assume I mean this is the analytical solution of course, we can it is a simple problem and we can have solve this differential equation analytically and now let us try to compute the solution by using Fourier discrete Fourier transform.

So, let us for that let us assume the forcing function be given as a set of discretely sampled values 0.02 second apart. So, that gives us a total number of samples as 128 which is 2 rise to the power 7. And I mean, because it is integer power of 2, so radix 2 algorithm FFT algorithm can be used theoretically I mean nothing prevents me from using it without any zero padding.

And that is the reason I took this sampling rate of 0.02 sampling interval because the number of sampling total number of samples is integer power of 2. So, that integer power of 2 is of course, necessary ingredient, but that still does not preclude or does not relieve us from the consideration of zero padding as we will see in this example. So, zero padding serves two purposes one is of course, to allow radix 2 algorithm and for that we need total data length to be equal to integer power of 2.

But it also requires that zero padding also help us in taking care of the issues arising because of the periodic nature of waveforms in DFT calculations. The sampled forcing function is then padded with zeros at both ends with two different padding lengths.

So, once case we take it as without any zero padding because total length of data points is integer power of 2, second case we consider as we add some zero at both ends. I mean, I use both ends because its more conscious it is visible physic that the effect of periodicity can happen. Response of the equation of motion is obtained by convolution via DFT.

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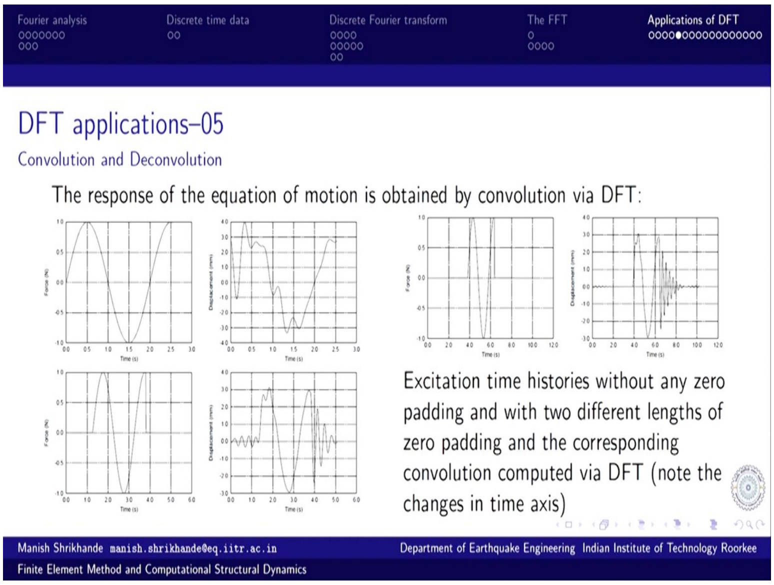

So, the same process I mean $Y(\omega)$ computed as $F(\omega)$ multiplied by $H(\omega)$. And this is the response, this is the excitation as you can see the sine wave and 2.54 second is that is the duration of this excitation. And there is no zero padding as you can see this is we are using this excitation as it is.

And the on the right hand side we have corresponding displacement solution y as a function of time t. So, y as a function of time t again given for up to that 2.54 second as simple process compute the Fourier transform of this DFT of this time domain waveform multiply it by the impulse response function frequency response function in discrete frequencies.

And then compute the inverse Fourier transform, inverse discrete Fourier transform to give us this waveform. Now, first thing I mean we cannot say anything from this does not look anything unless we alarming I mean, it looks like a waveform. But one peculiar thing should be noted that I mentioned we mentioned that the problem this system starts at rest condition. And the initial conditions here in this case are I mean; obviously, the problem does not satisfy the initial conditions, the solution does not satisfy the initial conditions of the problem.

So, the initial displacement is somewhere around 2.8 mm and the velocity is also very high negative velocity in the at time t = 0. So; obviously, this is something is wrong somewhere and, what happens? So, if I pad some zeros let us say add some zeros at in the front end, some zeros at the back of the excitation. So, this excitation looks like sudden start of this sine wave pulse and then sudden withdrawal and after that it is 0.

So, if I use this excitation and again compute DFT and try to find what is the inverse discrete Fourier transform $Y(\omega)$ that is computed using DFT and we get something like this. So, this is of course, the response is little different I mean significantly different from what we had in the earlier case there is some ripple here again this is erroneous I mean, there cannot be any response before the excitation is applied. So, excitation comes at about in this case about 1.25 second.

So, this is this part of the response seems to be fine that it starts from this point, but in this portion these oscillations they are of course, spurious, because there is no excitation at this point. So, how can there be any response, the system is supposed to be at rest here. And after the withdrawal of the excitation then of course, it is the free vibration that we can see the familiar free vibration curve, so that seems fine.

So, little improvement that it the initial condition at the start of the excitations does appear to be close to the at rest condition, but we still have this problem of these ripples in the beginning portion where there was no excitation. So, we extend the zero pad further, so the time axis in all these cases is different. So, first set of first pair of graph the time axis ranges from 0 to 3 in the second pair of graphs the time axis ranges from 0 to 6 seconds.

Now, we go back go to another case of zero padding, now the time axis goes from 0 to 12 seconds. So, zero padding is increased the waveform that we have is still 2.54 second long. So, rest of these is all zero padding and once we do that then again you we see that the ripples are there, but ripples are reduced almost to negligible extent.

And particularly at the close of the beginning of the excitation, two excitation it is almost imperceptible. And the solution does start from at rest condition and we have this response and followed by at the time of withdrawal of the excitation the free vibration response follows as it should correctly.

So, what is happening here? The problem here is that the DFT, for DFT when we compute it is computing the response of a periodic response to a periodic excitation with this particular whatever excitation we are looking at that repeating itself. So, when this period once this excitation is ends this is the excitation that we look at 2.54 and then this pattern repeats.

Then what happens is the initial conditions or whatever is at the end of the excitation the free vibration response, at the end of this excitation is added to the vibration response of this individual length of the excitation. And that is what does not allow this waveform this response to start from 0 initial condition.

Because the we already have some kind of response because of the previous period previous segment or data segment that we will have, because the this free vibration response will of course, be there after the excitation is removed. So, it will always be there and on top of that we add this force vibration response again, so this kind of error will always be there.

And this can be eliminated only when the padding length is sufficient for this free vibration response to die down. So, if this free vibration response dies down considerably then it has no effect whatsoever on the computed response and the response computation would be more or less representative of the true response.

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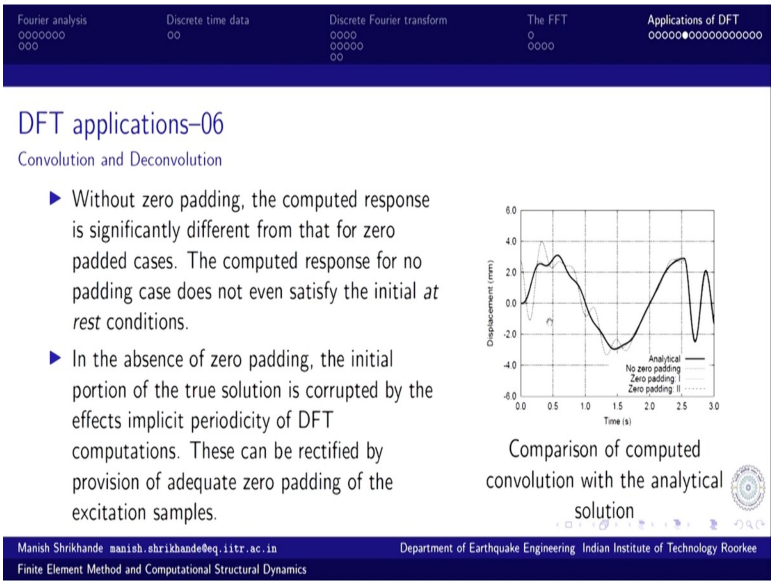

So, without zero padding the computed response is significantly different from that for zero padded cases. The computed response for no padding case does not even satisfy the initial at rest conditions. In the absence of zero padding the initial portion of the solution is corrupted by the implicit periodicity effects of DFT computations. These can be rectified by the provision of adequate zero padding of the excitation samples.

So, we compare these three cases for the removing after the calculation after we remove the zero padding, so that we can plot the all of them on the same common time axis. And you can see that the dark solid line is the analytical response that we computed from the theoretical solution. And this dashed response is what we have from without any zero padding.

So, to begin with it is oscillating about the true solution and gradually it merges with the true solution towards the end of the excitation. And with the zero padding the solution actually matches very well with the true solution, analytical solution. For small length of padding there are some ripples over the peaks, but for a larger zero padding the it is almost indistinguishable from the analytical solution.

So, the whole point of this example was to demonstrate the importance of zero padding the zeros that you see in that time domain signal discrete sample data those are important and those should not be discarded those should not be discarded as useless data points. They do serve some purpose and they help us in controlling the transients that are originating that are that will creep into the solution computed solution because of the hidden periodicities of DFT computations.

So, that is the convolution, so computing the response. Now, we come to the other aspect the deconvolution, deconvolution is the process of reconstructing input signal from the response by reversing the effect of convolution on input signal.

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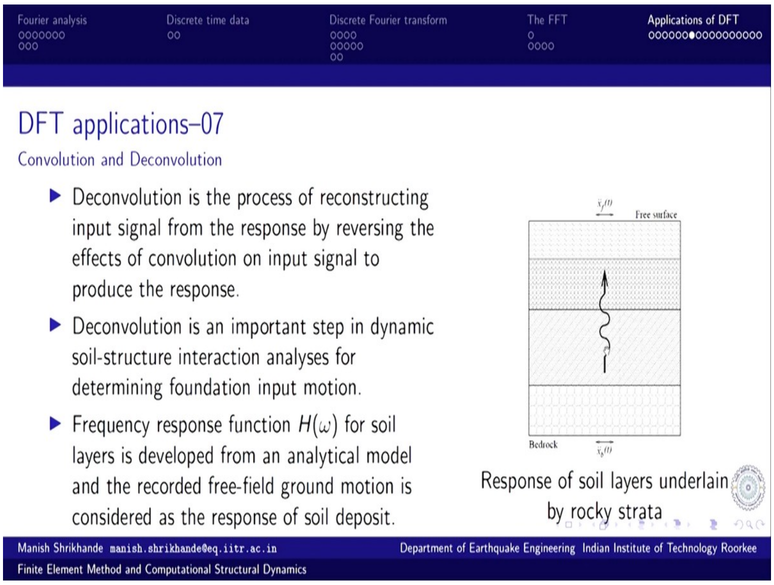

So, again if we do the time domain analysis the it is almost impossible to deconvolute anything, but infrequency domain because its only term by term multiplication. So, deconvolution can be performed very easily infrequency domain and this is often done. Because most of the time in earthquake engineering we have recordings at the free field surfaces, ground surface and we do we can have earthquake ground motion recorded at the free field.

But we know that the this motion has been modified in moving up in propagating upwards from the bedrock to the soil surface. And usually the foundations are laid at the firm strata or at the bedrock level. So, what is the excitation to the foundation to the building or whatever structure, so that should be the motion that is being experienced by the foundation at the base of the foundation. So, that foundation input motion can be computed by using deconvolution process.

So, deconvolution is an important step in dynamic soil structure interaction analysis for determining foundation input motion. The frequency response function H %omega for soil layers is developed from an analytical model and the recorded free field ground motion is considered as the response of soil deposit. So, this is like the schematic the response of soil layers underlain by rocky strata.

So, this is bedrock and the base motion the ground acceleration the because of earthquake there is an acceleration and this leads to elastic waves propagating through the soil layer. So, they are they depending on different types of soil layer there would be different elastic properties and accordingly the motion will be modified and eventually we record the surface motion.

So, this surface motion if I apply the surface motion at the bedrock level for the excitation that may not be correct. Because it may erroneously be using large amplification of certain frequencies amplified by these soil layers and it may deamplify certain frequencies which are not transmitted by these soil layers.

So, the input using free surface record as an input at the foundation input motion is a error in input is not the correct input to provide. So, what is done is we can compute this free field response and use the mass characteristics or frequency response function of this soil strata and find out what is the input motion at the base by deconvolution, how to do that?

But before we do that there are certain basic observations, it has been observed it has been observed that about 75 percent of the power, about 87 percent of the amplitude in a free surface motion can be attributed to vertically propagating shear waves for frequencies up to 15 hertz. So, before we perform any deconvolution exercise what we do is it is usual practice to use a filtered free field motion for deconvolution.

So, typically we use a low pass filter with cut of frequency at 15 hertz to eliminate any spurious amplification of high frequencies, because this deconvolution will involve amplification of high frequencies. So, if there are even small traces of high frequency energy is there that will be amplified considerably during the process of deconvolution. And that may lead to unreasonably large accelerations at depth, there is no evidence of having such large acceleration at deeper strata.

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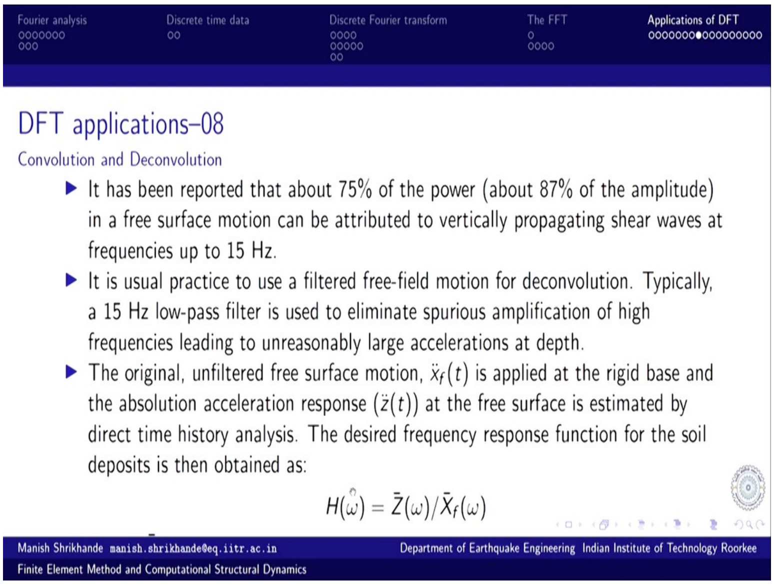

So, the process that we use adopt in deconvolution is the original unfiltered free surface motion $\ddot{x}_f$ is applied at the rigid base. So, the base motion we apply this recorded motion at the rigid base and we can have finite element module, plane strain model for the soil strata and we can develop the analytical model.

And we can compute the response at the free field because of this input motion by using time merging scheme or even infrequency domain we can do that whatever. So, if time merging is comfortable then we can do the time merging and compute the response at the free surface.

Now, knowing the total absolute, total acceleration response, so total acceleration response at the free surface. So, knowing total acceleration at the free surface and the acceleration at the bottom we can now relate these two we can take the Fourier transform and take the ratio response divided by input and that gives us the input motion.

So, $H(\omega)$ that is the frequency response function that we are looking for that is given as Fourier transform or discrete Fourier transform of total acceleration at the top and divided by discrete Fourier transform of the input motion that we used as the recorded motion acceleration at the free surface. And this is term by term frequency at each frequency it is a term by term division and that is always defined.

And once we have this term by term definition for each frequency we have the estimate of discrete values of frequency response function. And once we have this frequency response function this can be used to deconvolve the free field surface motion to find out what is the response.

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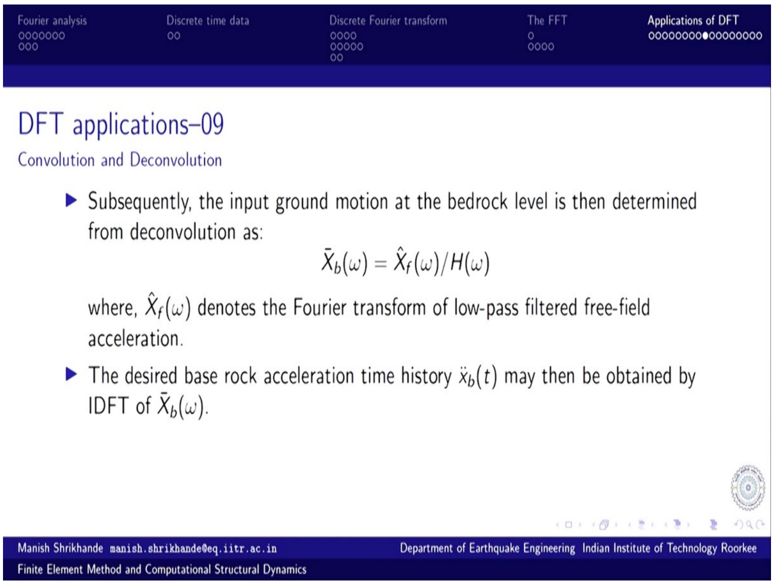

So, the base motion the Fourier transform of the base motion can be obtained as $X_f(\omega)$ that is the Fourier transform of the low pass filtered free field acceleration divided by $H(\omega)$ that we have just computed earlier. And once we obtain this Fourier transform of the base acceleration that is we convert it to time domain by using inverse discrete Fourier transform. So, desired base rock acceleration time history $\ddot{x}_b$ may then be obtained as inverse discrete Fourier transform of $X_b(\omega)$.

It is often necessary because the soil can have strain dependent properties. So, it is often necessary to iterate this process, this entire process is based on linear theory. So, it may be important that for each process we may need to evaluate what is the strain level and for that strain level use the appropriate model appropriate material properties and then carry out the analysis and find out the motion at the base. So, this kind of iteration with strain dependent constitutive properties of soil deposit may be required before a reasonable estimate of deconvolved bedrock motion can be obtained.

Having said that it is of course, it needs to be appreciated that this estimation of bedrock motion is a numerical estimation and it should any numerical estimation should always be treated with caution, adequate caution and care. So, that completes our discussion on first major application of DFT, we now discuss another application area of DFT in signal processing, vibration data processing in our next lecture.

Thank you.