Finite Element Method and Computational Structural Dynamics
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Lecture - 56
Discrete Fourier Transform-III

Hello friends. So, we were discussing about the effects of discretely sample data. How do we do that and how the Fourier Transforms that we have been computing, how that gets affected. And for discretely sample discrete time sample data very important thing that is to be understood is that the problem of under sampling. There is no problem with over sampling except probably it might be little expensive.

But under sampling will tend to corrupt the entire data beyond redemption and we will see why that such a strong statement about under sampling.

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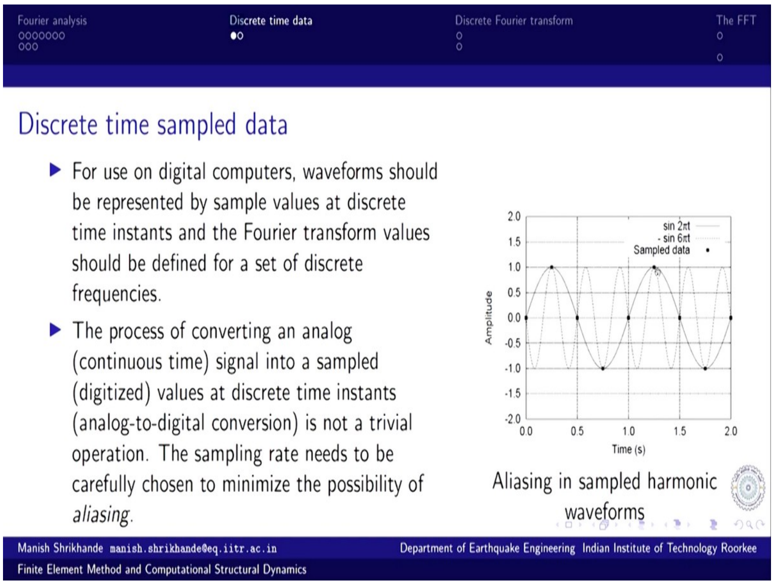

So, as I have been saying that for use on digital computers the wave forms should be represented by sample values at discrete time instants. And the Fourier Transform values should also be defined at a set of discrete frequencies. Only then I mean once unless they are at discrete values, they are discrete values they cannot be represented by a digital computer. I mean in computer we only get a series of a set of data, numerical data. So, we can have a one vector of numerical data getting transformed into another vector or matrix of numerical data that is all.

So, for processing it is important that the waveform data whether it is in time domain or infrequency domain both of them have to be in discrete form and that is a very special requirement because we have seen that for a continuous function, if the function is aperiodic then the wave form is the Fourier transform is going to be continuous and if the function is periodic, then the Fourier transform is going to be periodic in time.

Then the Fourier Transform in the frequency domain it is going to have definition only at discrete frequencies. But in case of nonperiodic data or aperiodic data irrespective of whether it is discrete or continuous time data, if it is aperiodic then the Fourier transform is going to be continuous variable continuously defined in over the variable frequency and that needs to be transformed.

So, there is a difference between discrete time Fourier transform and discrete Fourier transform because discrete time Fourier transform can have continuous Fourier transform in continuous frequency representation and that is not what we are looking at. We are looking at the situation in which both the time data is discretely sampled as well as the frequency data is discretely sampled.

So, how do we manage the two? So, first come coming to the problem of_sampling and the need of_sampling and the problems associated with that. So, process of_converting an analog continuous time signal into a sampled digitized value at discrete time instance. So, this is referred to as ADC or analog to digital conversion and this is in nontrivial operation; nontrivial because the sampling rate how fast do I do we acquire samples.

It needs to be carefully chosen to minimize the possibility of aliasing. Practically we can never eliminate aliasing completely, but we can keep the effects of aliasing to a minimum. So, that it does not corrupt the data that we are dealing with. So, what do we mean by aliasing? So, as the term suggests it looks it seems to convey the meaning that there is some kind of_some kind of possibility or case of mistaken identity.

So, one wave form may be mistaken for another wave form. That exactly is what we mean by aliasing. So, taking a simple case of two harmonic wave forms and see how it what aliasing means in the context of two harmonic wave forms. So, let us say this $\sin 2\pi t$ and $\sin 6\pi t$. These are the two data points. So, continuous wave form is $\sin 2\pi t$ and this dot dashed waveform is higher frequencies.

So, $\sin 6\pi t$ and the sample data let us say we sample this at this particular rate which is essentially I acquire I take 4 samples within 1 period of $\sin 2\pi t$ or 2π . So, 1 sample, 2 sample, 3rd sample, 4th sample. Now if you look at these samples then I mean I can shift the samples a little here bit all of them, but if you look at this, there is if I am sampling at these points then there is absolutely no possibility for me or for the sampler the basic measurement device to distinguish between these two wave forms.

So, whatever I get here that data point could be either coming from $\sin 2\pi t$ or it could be coming from $\sin 6\pi t$. So, whatever value we I get here that could have been contributed by either of the two wave forms or both of them who knows. So, now, because of this sampling there is no way to distinguish.

So, once I sample this data and then throughout the I mean the physical wave form away then there is just no way this information that can be retrieved. The only solution is I mean this problem can be solved if I had one more sampling in between here. So, in between these 2 samples if I had one more sampling here then that would have been enough to resolve this conflict.

So, these two data these two curves would have been easily distinguished by having one more sample in between this sampling rate. So, in this particular case the sampling rate is twice as high as it needs to be, 2 times the rate that it needs to be to resolve these two frequencies independently and that is a problem with aliasing. Now the problem the in the physical system what we have is continuous wave form. They are rich in all frequencies.

So, how do I decide what is the frequency? So, in order to uniquely identify the frequency it is any frequency component any harmonic of let us say frequency any particular frequency. We need at least 2 data points in ones cycle of that frequency. So, if we need. So, basically the idea is the sampling rate needs to be twice as high as the frequency.

So, if the frequency is some value let us say f_c then I need to sample the data at the rate of twice f_c . So, sampling rate f_s needs to be 2 times f_c . So, what this actually means that $f_s=\frac{1}{\Delta t}$.

So, Δt is the sampling interval. So, what it alternate representation is, if I am sampling data or time domain sampling is being sampled is being acquired at the interval of Δt here.

So, interval between these 2 sampling points time instant let us call it Δt then the highest frequency that can be resolved by samples sampling process at Δt seconds apart is $\frac{1}{2}\frac{1}{\Delta t}$.

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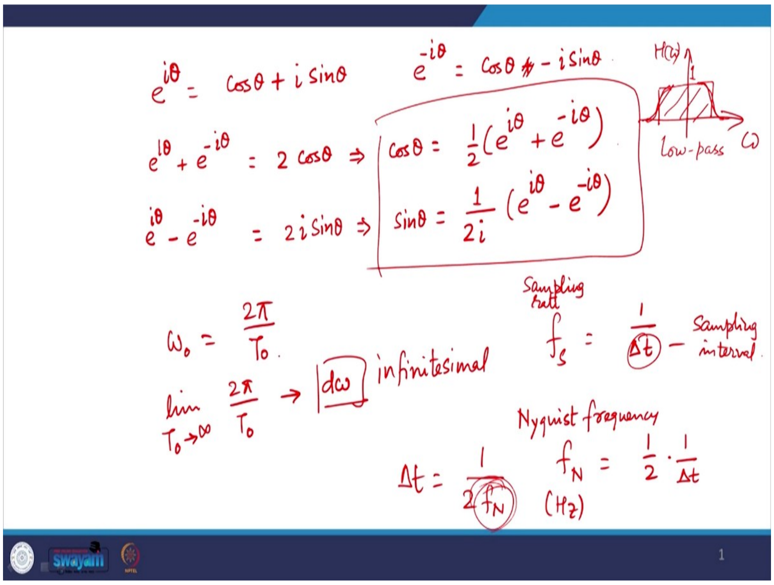

So, f_s is sampling rate. So, this is called sampling rate and Δt is the sampling interval so, the highest frequency that can be resolved. So, Nyquist frequency is half of ampling frequency. So, $\frac{1}{2\Delta t}$ is the Nyquist frequency that is the highest frequency that we can have in and that is in hertz cycles per second. So, this is the highest frequency and accordingly. So, alternately I can define. So, $\Delta t = \frac{1}{2 f_N}$.

So, if f_N I know what is the highest frequency I need to retain in the signal or what is the highest frequency present in the signal that gives me an upper bound on Δt . So, Δt cannot be higher than this. So, this is the theoretical limit practically Δt has to be let us say maybe 60 percent of this upper limit. So, if ΔT is kept may be 60 percent or even half of this limit then the frequency representation will be accurately captured in the sampling during the sampling process.

So, that is a problem and that is how we that is the major problem with analog to digital conversion and care has to be taken. Now how do we do this? How do we eliminate frequencies higher frequencies I mean it is a natural data that is coming in. So, it is a continuous time wave form. So, how do I filter it? So, there are ways of doing that.

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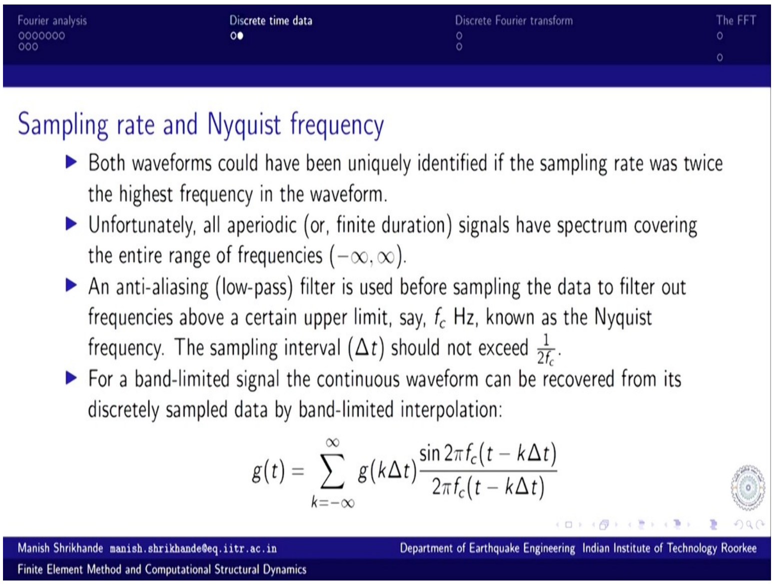

So, both wave forms could have been uniquely identified if the sampling rate was twice the highest frequency in the wave form. So, that is sampling frequency was twice the highest frequency in the wave form. So, that is what we saw. Unfortunately all wave period aperiodic or finite duration signals have spectrum covering the entire range of frequencies that goes from - infinity to + infinity.

So, the trick is to use an anti-aliasing which is essentially a low pass filter. So, if I use a an frequency band. So, if I use this frequency axis then I define a filter whose gain function is allows only frequencies in this range to pass through. So, this is called low pass filter. It allows low frequencies to pass through and the gain is of_course, maintained at unity tries. We try to maintain it at unity and this is an ideal filter and actual filter can never have this sharp drop.

So, that is a separate issue that is a matter for signal processing. So, ideal physical filter would have some taper here and then there would be some ripples depending on what how the filter is designed, but more or less that captures the basic idea that we need to eliminate high frequencies from the consideration and what is the high highest frequency that we need to retain that we have in the signal that will define how fast the signal should be sampled.

So, at least it has to be at least this value or Δt is $\frac{1}{2f_N}$, the highest frequency, but at most this is the at most value, it can be smaller than this if Δt is chosen to be lower than this then there is no issue. So, anti-aliasing filter is used before sampling the data to filter out frequencies above a certain upper limit let us say f_c in hertz Carnot frequency that is the final Carnot frequency of this spectrum.

And this is known as the Nyquist frequency. The sampling interval ΔT should not exceed $\frac{1}{2f_c}$. So, that is the upper limit if possible I have we would like to have ΔT smaller than this, as small as possible, but that would increase the problem size because number of samples would be very large. If I use too small an interval ΔT then that will actually lead to very very large amount of data to be processed.

And in principle this once we know that the signal this anti-aliasing filter has been applied, low pass filter has been applied. So, technically, theoretically we have a signal whose frequency content is known. It is called band pass filter. So, we know that it is a there is no energy beyond this frequency range.

So, for a band pass filter the complete wave form can be reconstructed. Time domain continuous waveform can be recovered from discretely sample data by what is known as Band Limited Interpolation. So, the function g(t) as a continuous wave form can be recovered from these discrete time sampled. So, these are discrete samples I mean integer multiple of ΔT .

So, g function varies from, the k varies from - infinity to + infinity and whatever is the sample discrete values that we have that can be multiplied with this interpolating function. Again depending on K for each term and the waveform can be regenerated from this discrete time wave form discrete time sample data. So, that is the theory in practice; obviously, this summation - infinity to + infinity never happens and secondly, this is too expensive an operation.

Although this is an ideal case band limited interpolation is the ideal way to interpolate or to regenerate the continuous waveform using this kind of interpolation function, but it is often most of the time people who use this do not under understand these issues of discretization and the effect on Nyquist frequency and the aliasing problem and most of the time if we need any value in between two discrete points we would simply go ahead and interpolate using simple linear Lagrangian Interpolation, Polynomial Interpolation.

y = y 1 + y 2 - y 1(x - x 1) / (x 2 - x 1) , standard equation of straight line and get on with it. Now this process of this linear interpolation if it is carried out too many times then the chances are whatever effect of low pass filtering had been imposed that all energy beyond f_c was chopped off. Those energies I mean the higher frequencies can again begin to have some representation.

Maybe those are spurious representation, but that is a fallout of the numerical process of interpolation that is being used. So, instead of using band limited interpolation which is the correct way to interpolate discretely sample data, if you use polynomial interpolation to interpolate between discretely sample data then that has its own problems and the aliasing problem can resurface if we are not cautious enough.

So, with that so, this is the problem with sampling rate and sampling frequency and discrete time data has to adhere to these basic notion, these basic ideas. So, with that we come to our next point of discussion; how do we mean once we have this discrete data. So, time domain data this g(t) is sampled at discrete time instants at interval of Δt . So, now, how do I convert it into frequency domain and vice versa and frequency domain it has to be in terms of again discrete frequencies.

So, those discrete frequencies how that maps on to this division. So, a very interesting we will just discuss the in principle of what are the issues involved, I mean just broad framework without getting into the formulation and development as of now. So, this is the complete process of discrete Fourier Transform.

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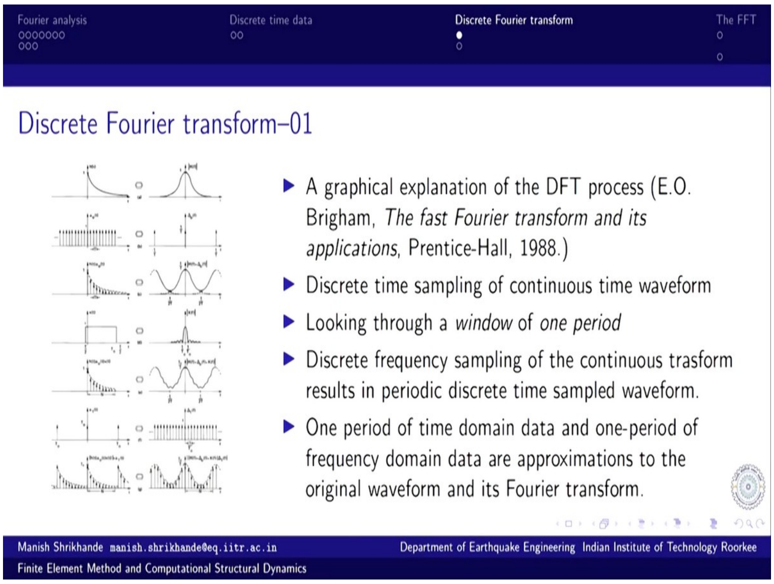

So, we have a time domain data and as you can see it is aperiodic data aperiodic wave form and it has Fourier transform which is continuous time Fourier Continuous Frequency. It is continuous in frequency domain right. So, these are these two are the Fourier transform pairs. Now I multiply this with this sampler discrete time sampler. So, that this actually picks up values only at discrete interval Δt . And that is what converts this. So, it is essentially what we call as sample and hold circuit sample circuit. So, we just pick up the value at I mean at every clock rate let us say Δt interval. So, whatever the function value is, we pick it up and store it at that particular index. So, this waveform is represented by this discrete sampled wave form at discrete times. So, at time T = 0, at time $T=\Delta t, 2\Delta t, 3\Delta t$ and so on.

Now the effect of this is the essentially it is a its Fourier transform this sampler that is sequence of impulse function, its Fourier transform is again impulse in frequency domain and now the process of this multiplication, this multiplication in time domain would be effected as convolution in frequency domain and the by virtue of this convolution what happens is this one spectrum that we had because this is convoluted with respect to with these impulse strain.

So, this spectrum keeps repeating itself after some time according to this again spacing. Now if this spacing is not enough not large enough and that spacing = $\frac{1}{\Delta t}$, this sampling interval. So, this spacing is actually 2 times $\frac{1}{\Delta t}$.

So, if this sampling is this spacing is not large enough then these spectrum that is going to be convolved I mean these impulse if these impulses were close enough then they would be overlapping and they will corrupt the solution, that is what aliasing is in frequency domain if we look at the frequency domain representation of aliasing. So, in order to avoid that we need to keep Δt as small as possible. So, that these impulses in frequency domain they are very far away.

So, that the convolution does not happen. So, this discrete time samples that we have that leads to a periodic spectrum in the frequency domain. So, frequency domain we have this spectrum which repeats itself after this interval in a frequency range. Now if I look at the problem that I need to look at one period or a finite duration of the data only one particular window.

So, like that problem that we discussed periodic wave form and then aperiodic wave form. So, if I want to look at only one period of the waveform then I just multiply it with an envelope function. So, this is like envelope function of one period and it goes to 0 outside that range. So, here it is shifted slightly from the origin towards half of Δt just to ensure that the ends points do not coincide with the period end or the beginning of the new wave form.

So, the duration is of course, one period. So, if I do that then the Fourier transform of this is of course, this sink function and when I multiply this, then this sink function gets convolved with this continuous and periodic spectrum in frequency domain and that bring some of its own peculiarity. Some these ripples come from these use of these window function.

So, choice of window function is very important in that case. If we choose this rectangular window function that is the easiest one to choose then that will have large number of ripples and that will have some effect on the resulting frequency spectrum.

Then as we said we need the frequency domain although the time domain is discretely sampled, but we need the frequency domain to be discretely sampled as well.

So, discrete frequency components. So, now, we use sampler in frequency domain and that is what we use this impulse strain in frequency domain. Now this impulse strain in frequency domain has Fourier transform of again impulses separated apart large distance in this time domain. So, once I multiply this sampler with this spectrum of a windowed transform sample sampled wave form then that gives me these sampled fourier amplitudes.

Again it is a periodic wave form periodic in frequency and by virtue of this multiplication, this needs to be convolved in time domain and by virtue of_convolution this waveform would now spread all over the impulses all over the time period. So, it again becomes a periodic.

So, a periodic wave form discrete time waveform results into a periodic frequency domain representation or discrete frequency representation and what is important in this context is to note that if we look at although this has if we look at the entire thing this has this; obviously, looks very different from what we had in the beginning, in the continuous time framework.

If we look at that, this is the true result true function time domain function and it is a Fourier transform and what we achieve after the process of discrete Fourier transform is a set of a periodic function discretely sampled periodic function and in time and in frequency domain again it is discreetly sampled periodic function in frequency domain, but what is important to note that if we look at one period of this time domain.

And if we look at one period of this frequency domain, they are very good approximation of the corresponding original continuous time and continuous frequency signals, continuous frequency representation. So, this one period of_sampled waveform is very good representation representative of the original continuous time wave form and this one period of the frequency domain representation is very good representation of this continuous frequency representation of the original wave form and its transform.

So, that serves our objective. We get what is the frequency representation in this time domain data by mapping a set of discrete time samples on to another set of discretely sampled values at frequencies, discrete frequencies. So, one set of numbers gets mapped onto another set of numbers and the result is meaningful as long as we associate this one period that whatever we are computing we understand we need to understand that whatever the numbers we are computing that corresponds to one period of infinite wave form, periodic waveform right and after that period it the waveform repeats itself.

So, we are looking at the snap shot. One window of only one period, one period in time domain and similar it gets mapped onto one period in frequency domain and those within that one period, the sample data is representative of or a very close approximation of the continuous time and continuous frequency data of the original waveform.

So, this graphical explanation of DFT process I got it from the very interesting book the first Fourier Transform and its applications by E O Brigham. If you have any chance ever get any chance to read this book please do it. It is very very interesting and very very lucid explanation and description of this very important algorithm.

So, as we say this is discrete time sampling of the continuous time wave form. So, original continuous time wave form and we sample it at discrete times using this discrete time sampler. Then looking through a window of one period then discrete sampling of the continuous wave transform.

So, this continuous transform that we get after looking through one period. So, this continuous transform can be sampled again and when we do that then it leads to a periodic representation of the time domain data. So, one period of time domain data and one period or one period of frequency domain data or one band of frequency domain data are approximations to the original wave form and its Fourier Transform.

So, that needs to be understood very very clearly that discrete Fourier transform is a numerical algorithm is a numerical process by which we try to approximate the continuous time transforms, but while computation if we only look at it comes across as digital values data, digital values one set of numbers gets mapped on to another set of numbers, but it is not the end of_story.

What we are looking at is only one period of the of this data, of this infinite data that we have, infinite in time domain as well as infinite in frequency domain whereas, only one period of time domain data is representative of the original time domain data and only one band of frequencies is representative of the original transform data.

So, as long as we understand this clearly and we understand what happens. We should be fine with our applications and using the results correctly. The problem happens once if we do not understand these then it becomes very difficult to interpret the data and interpret the results. We will see how it happens and what can happen if we do not appreciate this infinite waveform, I mean periodic way form of infinite duration that happens as a consequence of the DFT operation. So, we will discuss this in more detail in our next lecture.

Thank you.