Finite Element Method and Computational Structural Dynamics
Prof. Manish Shrikhande
Department of Earthquake Engineering
Indian Institute of Technology, Roorkee

Lecture - 55
Discrete Fourier Transform - II

Hello friends, we have seen the discussion I mean we have been discussing about Fourier series and how periodic functions can be represented as a summation infinite series, involving terms from cosine and sine series, harmonic functions. Now, we extend that explanation I mean we extend that formulation the expansion of any function any periodic function in terms of sine and cosine functions by using Euler's formula.

I mean as we know all know that complex exponentials can be represented as sum of cosine and sine functions. So, if we use substitute these Euler formula then can we I mean the Fourier series of course involves a sine and cosine term. So, we can combine sine and cosine terms and try to develop complex exponential form of the Fourier series.

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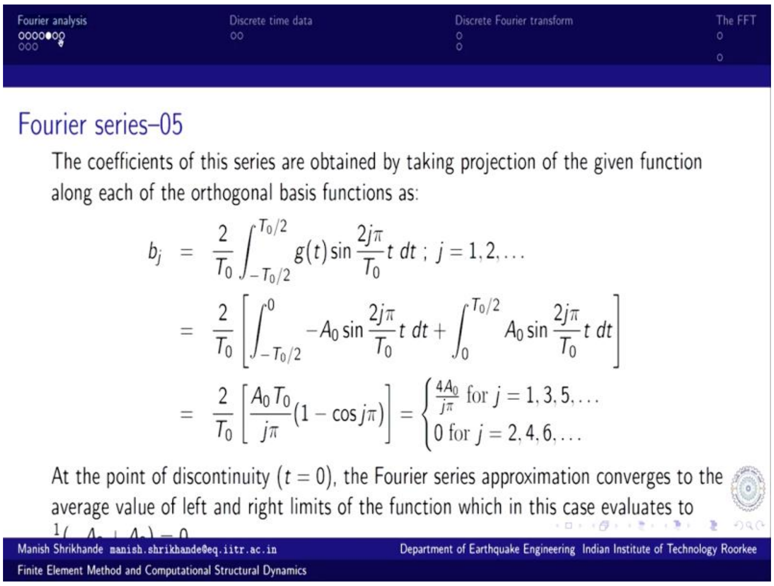

So, in this case we have been looking at this square waveform the Fourier series expansion of square wave form and we found that it is an odd function and it involves only cosine terms and the coefficients of the cosine term b j they can be given as 4 times A_0 / j pi for odd number of harmonics odd or odd harmonics j is equal to 1, 3, 5 and they are 0 for even harmonics j is equal to 2, 4, 6 etcetera. And at the point of discontinuity the Fourier series approximation converges to the average value of the left and right limit.

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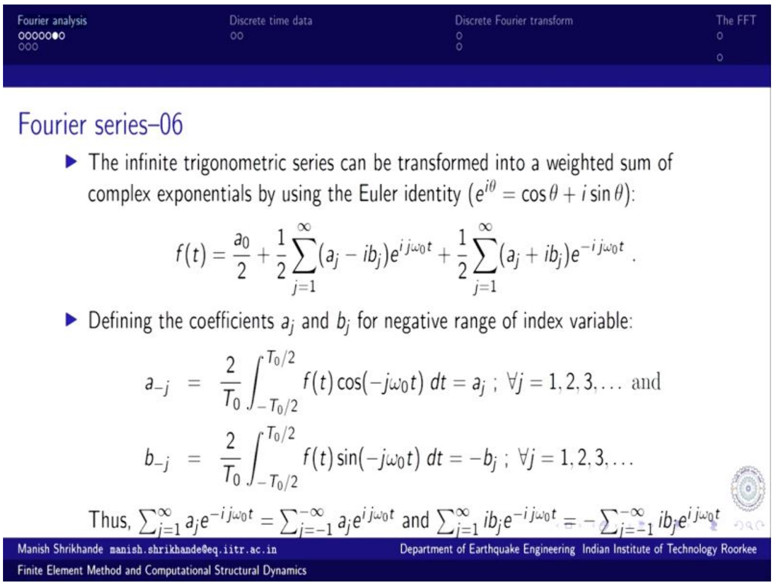

Now, let us look at that expansion into the transformation into the complex form by using Euler identity that is $e^{i\Theta} = \cos\Theta + i\sin\Theta$, then we can actually combine the terms of the Fourier series.

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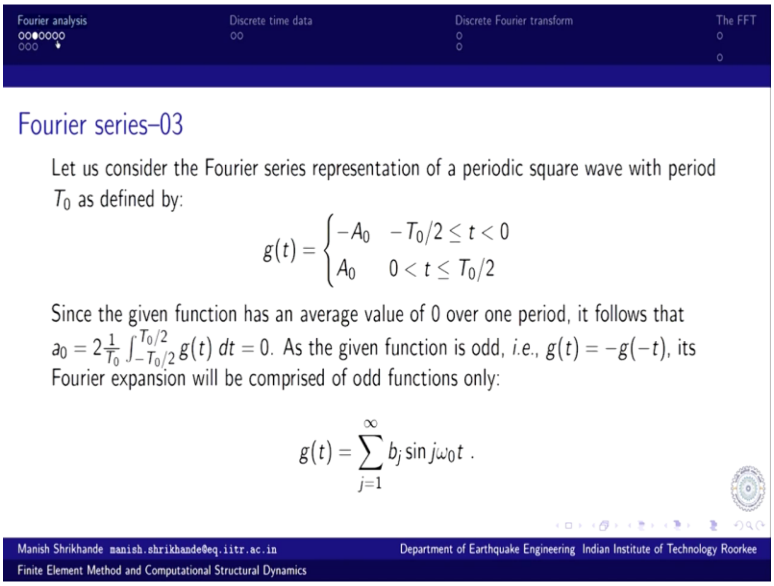

If you look at it the Fourier series expansion.

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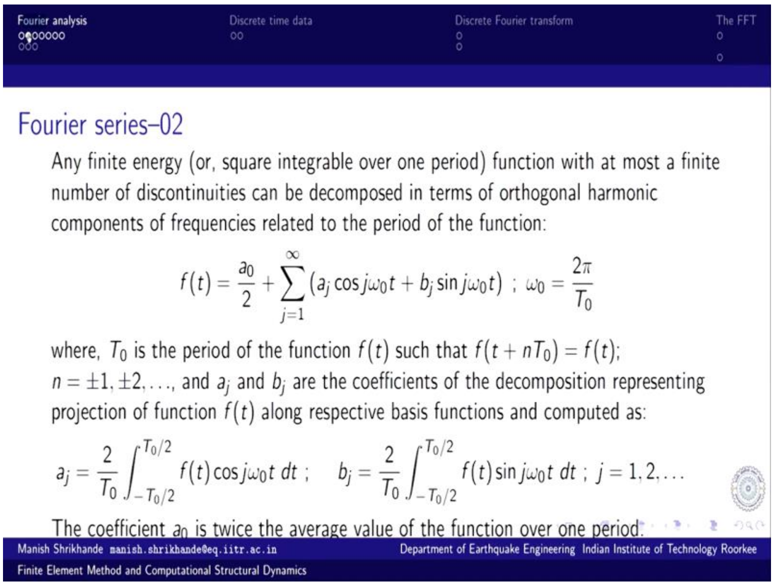

So, these cos terms and sin terms they can be combined together by using Euler formula, I mean cos j ω_0 t can be represented as average of 2 complex harmonics complex exponentials that is e^i j cos j ω t and plus e^{-i} times j ω_0t and average of that would be cos j ω_0t. And similarly we can have the representation for complex I means difference of complex these complex exponentials complex conjugate exponentials that results in expansion for representation for sin ω t term. So, we can combine these so representation for cosΘ and sin Θ

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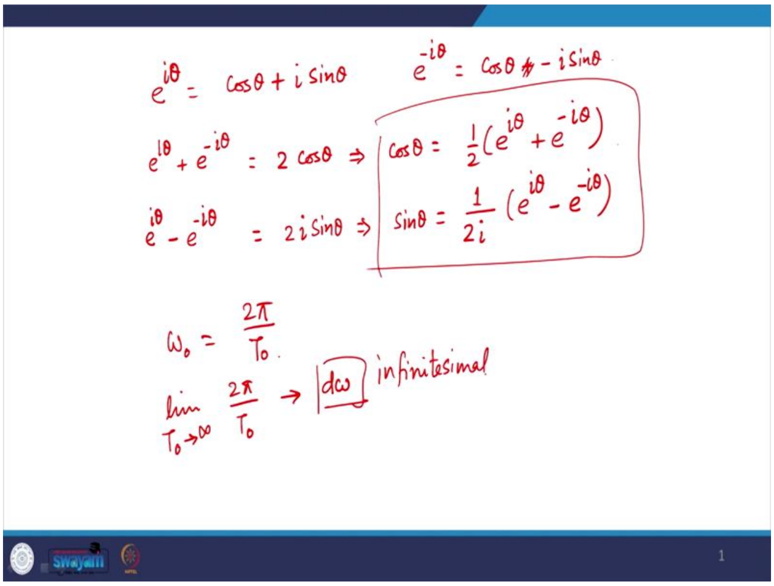

So, I use these two expressions in the Fourier series expansion replacing Θ by j ω_0t and I have this complex representation and that I use substitute in the Fourier series and that gives me that brings us to this for particular formulation. So, that will be from j is equal to 1 to infinity, so this is of course the constant term. So, all terms involving cosine and sine they can be represented as positive exponential and negative exponential and terms a and b can be combined.

Now little bit of arithmetic and we can actually now look at it that this can be represented as e^{-j} w_0 can be thought of as e^{ij} ω_0 t with j range extending to minus range with the j index going into the negative range. And once we accept that then this brings us to the next formulation. So, we can substitute this Euler identity and coefficients for a cosine and sine terms in terms of sum of complex exponentials and difference of complex exponentials and that Fourier series can be represented in this form.

Now, if we look at it these are two terms of complex exponentials one involving positive power and other involving negative power and j is an index. So, this e^{-ij} ω_o t that can be considered as belonging to negative values for index j. And with that interaction with that interpretation this entire thing can be combined together and this summation can now extend from minus infinity to plus infinity with j = 0 accounting for this constant term as well.

So, with that so these are the coefficients for negative range of index variable they can be evaluated. And therefore, this can be represented as negative powers of negative values for index j for these terms and with this substitution this entire expansion these 2 one half summation for j is equal to 1 to infinity and j is equal to 1 to infinity they can be merged with the with these with the help of these results.

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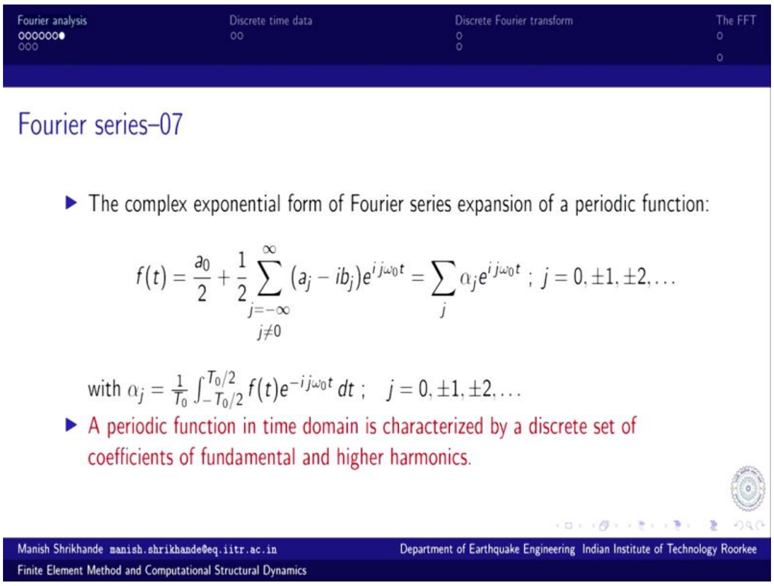

And we arrive at a very simple expression, so in this case j extends from minus infinity to plus infinity and j is still not equal to 0, because j is equal to 0 term is taken care of by this constant term. So, if we combine that and introduce bring that in this fold then we define this a_j – I b_j as a complex number alpha j. So, that would be given as a representative of Fourier I mean function approximation.

So, this entire series can be represented as summation over j of alpha j multiplied by $e^{i j \omega_0 t}$. And now j index varies from 0 and then it takes on value plus or minus 1. So, entire range of integers from minus infinity to plus infinity and that is again the same I mean it is exactly same as our standard Fourier series expansion in terms of sine and cosine.

Except that we now write it in terms of complex exponentials and obviously, since the multiplier this basis is complex exponential the coefficient is also going to be complex such that the result of this summation is a real function f t. If f t is real to begin with, then now alpha j can be computed using again from the original function.

So, instead of computing different coefficients a_j and b_j separately and then combining them together the complex exponential coefficient α_j can be directly computed by using by not taking this e raised to the power I mean the multiplying this α_j term multiplied by $e^{i j \omega_0 t}$. So, that will cancel out this term particular one particular harmonic and we will get the j^{th} coefficient of the series and that will fetch the desired coefficient averaged over the period T naught of the periodic wave.

So, integration be over one period f t multiplied by e^{-ij} ω_0 t and this changes this holds for all values of j, j ranging from minus infinity to plus infinity and we will have appropriate value of coefficient alpha j. So, the bottom line the key take away from this discussion and preceding discussion is that a periodic function in time domain.

If a function is periodic in time domain, so far we have always we have been discussing only periodic functions and a periodic function in time domain is characterized by a discrete set of coefficients or a discrete sum of different frequencies and higher of fundamental harmonic and higher harmonic.

So, these are all for j is equal to the Fourier transform this alpha j is defined for j is equal to some integer multiple of ω_0, that is the basic harmonic that depends on the period of oscillation period of the function. So, 2 pi over T_0 that is the fundamental harmonic ω_0 and these are the discrete of harmonics ω_0, 2 ω_0, 3 ω_0 and so on, positive negative all.

So, these are the discrete frequencies and these only these frequencies will have composition and that are an important take away. That any periodic function in time is going to have a discrete representation discrete frequency in frequency domain or Fourier domain.

So, that is one key aspect and that we will come back to we will come back to this particular result again after some time. Now, let us look at the extension of this particular definition, Fourier series as we have seen is it is defined for periodic function which is finite and it is it has a finite energy signal and such that it has only a countable finite number of discontinuities. And the series converges to the average value of the discontinuity within the period.

Now, let us take the extension I mean little limiting process that if the periodic function is defined by a period p after which it recurs after which it recurs repeats itself. Then we can define an a periodic function or a transient as a periodic function with infinite period. So, the period I mean the function actually never repeats or it is period is infinite. So, then this entire formulation of Fourier series can be extended to cover the a periodic functions as well in the limit as the period tends to infinity. So, let us now look at how it pans out.

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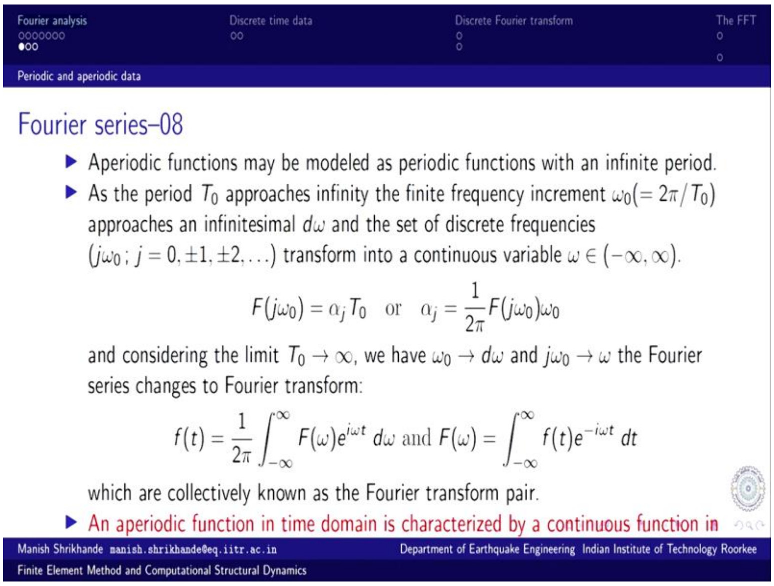

So, a periodic functions may be modeled as periodic functions with an infinite period. So, it is essentially saying that this function never repeats, so whatever we have seen that is it there is no way it is going to repeat again. So, as period T_0 approaches infinity the finite frequency increment that is ω_0 the frequency of fundamental frequency that was given as we computed it as defined it as 2 pi over T_0 that approaches an infinite decimal let us say d ω, because as T naught tends to infinity.

So, as ω_0 is defined as 2 pi over T_0. So, as limit T_0 tends to infinity. So, 2 pi / T_0 approaches infinite decimal d ω. So, this is an infinite decimal number and this is what we refer to so, these multiples of ω_0, so j ω_0 for j ranging from minus infinity to plus infinity. So, and multiples of infinite decimal; so d ω adding on to itself repeatedly.

So this transforms into a continuous variable. So, instead of having discrete frequencies here we now have ω ranging from minus infinity to plus infinity. So, it is a continuous variable over the entire range. So, instead of discrete frequencies we now have it defined over the complete range.

And we also have Fourier transforms that are f at j ω_0 is equal to α j times T_0. So, α_j is again given as 1 / 2 pi f j $_0$ and that is the definition that we have from the previous derivation of our α_j complex interval. And considering the limit T naught I mean the period extending to approaching infinity, then this omega naught the fundamental harmonic it approaches d omega and j ω_0 approaches continuous variable ω and then the summation Fourier series converges to or approaches the integral form.

The summation approaches the integral form and that is what we call as Fourier transform. So, f of t function of time t, so f t is an a periodic function is related to its Fourier transform f ω as shown in slide. And the inverse Fourier transform is given by F(ω) is defined as integral from - infinity to infinity f t e $^{i\omega t}$ d t and these 2 are collectively referred to as Fourier transform pair.

So, essentially we approached the Fourier transform definition I was starting from the Fourier series representation and then defining casting Fourier series for periodic function in terms of the complex exponentials. And then relating that complex exponential and through this limiting operation that as the period tends to infinity then the discrete frequencies they merge into a continuum and the summation approaches at an integral term.

And that is how I mean this is the most appealing way of deriving Fourier transforms. Of course, there are more rigorous ways of doing that, but coming from Fourier series has its specific appeal that starting from something that we know, we can extend that concept to through process of standard process of mathematical limits.

So now, in this particular result we now have another key takeaway and a periodic function in time domain is characterized by continuous function in frequency domain. So, f t is an a periodic function and it is represented in frequency domain as a continuous function of frequency f ω, f ω is a continuum and these two this as well as the previous one that for discrete frequencies we need a periodic function and any a periodic function has to have a continuous Fourier transform.

So now, how does Fourier domain representation of an a periodic function compare with that of an analogous periodic function. So for example, if we take this same example as that of square wave, but except that we now consider only one period of the wave and outside one period the waveform is essentially 0.

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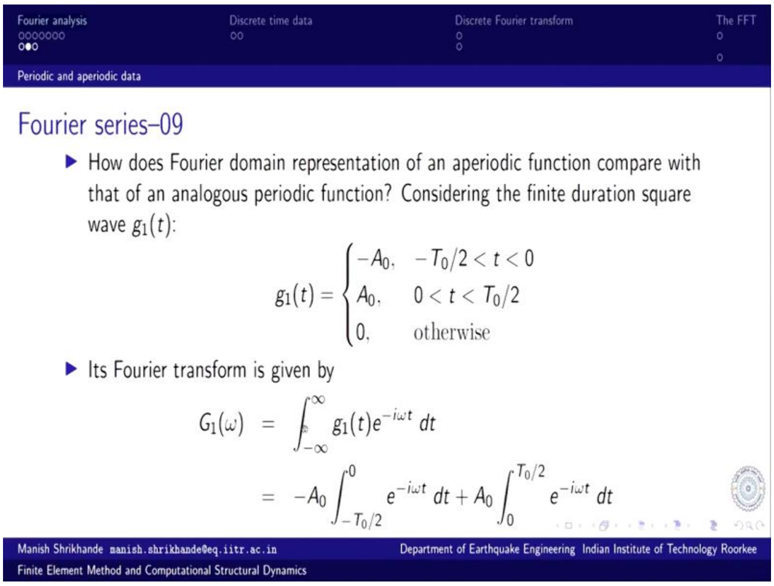

So, g_1 (t) so initial while we were discussing earlier the periodic function we had this pattern A_0 and - A_0 this pattern repeats itself after the period of T_0. So, it is an infinite periodic wave form square wave form, but now we only look at one period and outside this period the function is identically 0 everywhere for all times t.

So, it is Fourier transform we can compute by this standard definition, we can straight away take g_1 t multiplied by $e^{-i \omega t}$ dt integrate it from minus infinity to plus infinity. So, outside this domain of course this integral is 0 and that brings us to this particular integral interval. So, - T_0 to 2 T_0 / 2 to 0 and 0 to T_0 by 2 and we can evaluate this integral.

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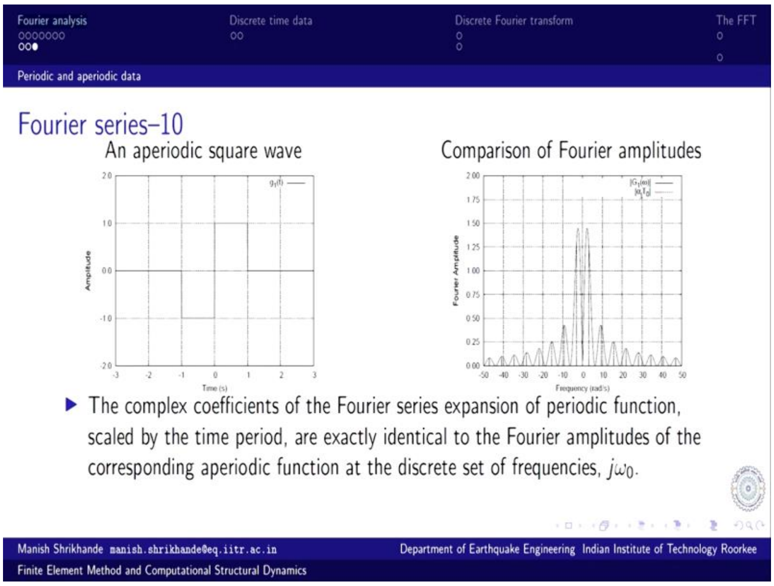

And we can find this these are the Fourier amplitudes that we have. So, we have this we know that this amplitude Fourier transform. So, we are plotting here only the modulus of the Fourier transform complex number with respect to frequencies. Now, as we know from the previous result that for a periodic function it is going to be continuous function and this is what that continuous function is so the Fourier amplitudes.

But what is interesting is I also have plotted these constant discrete values of frequencies different frequencies of the corresponding periodic function. So, Fourier amplitudes the complex exponential the modulus of complex exponentials multiplied by the period. So, it is scaled by the period of the wave form. So, alpha j multiplied by t_0. So, that is the period so complex exponential multiplied by modulus of complex exponential multiplied by the period. So, that I plot and interestingly the most that is a very revealing result the Fourier transform of this a periodic wave form. So, this is the a periodic wave form, so this is 0 all over it goes to minus 1 stays minus 1 in this range, then goes to plus 1 and again goes to 0 at t is equal to 1. So, from minus 1 second to plus t 1 second it has this one period of the waveform and we consider the amplitude as unity here.

So, for this waveform we compute the Fourier transform and also we superimpose this Fourier series expansion the complex exponential form the amplitudes of that. And we find that the amplitudes of the complex exponential form of the periodic wave form they agree exactly with the Fourier transform of this a periodic waveform.

Of course this complex exponential has to be scaled with scaled by the period of the waveform. So, α_j times T_0 the period of the waveform agrees with the Fourier transform of the a periodic wave form same wave form. But, whose period I mean which is a periodic and that agrees exactly at discrete frequencies now that is a very important result.

So, complex coefficients of the Fourier series expansion of periodic function scaled by the time period are exactly identical to the Fourier amplitudes of the corresponding a periodic function at discrete set of frequencies j ω_0. And this leads us to a very interesting result that for the I can possibly characterize this continuous frequency Fourier transform by these scaled complex exponentials of the periodic waveform corresponding periodic waveform.

And that is the key to development of what we call as discrete Fourier transform, under what conditions that holds and what are the specific issues that we will be dealing with that we will soon see. But nevertheless suffices it suffices to say that this equality is really remarkable and we can go ahead, we can develop a suitable algorithm and procedure to represent these continuous frequency waveform by this discrete wave frequency equivalence.

And it is exactly similar to what we do in case of normal digital discrete normal processing data processing using digital computers. For example, all waveforms are continuous time waveforms most of the almost always, most of the physical waveforms, physical signals that we pick up or that we observe they are continuous time signals.

But, by virtue of the design of digital computers the values that we need they need to be in discrete time samples, we need to have quantization and we need to have sampling at different instance of time. So, we cannot have I mean we cannot a digital computer cannot deal with a continuous function, it can only deal with a set of data or the vector of vector or matrices of discrete data set.

So, we need to approximate the continuous waveform by using discrete time samples and that particular aspect discrete time samples that brings us another dimension completely different dimension to this whole discussion of Fourier series. As it is we know we have converged to a great extent find out what is I mean for periodic function, it is in for Fourier domain it is going to be discrete frequencies.

And for a periodic function in time domain in frequency domain it is going to be continuous function. But those continuous function values can be approximated or they agree very well exactly with the values corresponding values obtained from the periodic function, complex exponentials for periodic function scaled with respect by the factor of time period.

So, this is pretty much settled and this is fine, but the problem is this waveform that we have continuous time waveform. This is not available, so for computation what we have instead is discrete values at some delta t some interval apart. So, what do we do with that and what are the implications of that and how do we first generate the those discrete samples.

So, these are some of the aspects which have their roots I mean lot of time and effort has gone into study of these phenomena apparently very innocuous operation of instead of having this continuous time waveform by replacing them by closely spaced samples. Just the data points we do that normally during I mean of practical I mean any kind of observation while we were preparing graphs and plots that is what we do. We observe for certain x value we observe corresponding y value and then mark the point and then join the curve and that curve is a continuous curve.

So, this is something that we routinely do. But on a digital for processing on a digital computer this kind of sampling at discrete time instance has it is own what should I say vagaries or peculiarities and that in turn has implication on this Fourier transform calculation that we have actually arrived at.

For example, this Fourier transform that we had defined here again we need to go back to the summation, because there is no way this g_1 t is not defined as a continuous function of time. It is only defined as discrete samples, so these integrals will again have to be computed as a accumulation, so kind of like quadrature. So, this is essentially same as quadrature definite integral. So, we need to again go back to our numerical integration and discrete summation, accumulation of function values summation and that will have its own consequences as we will discuss in our next lecture.

Thank you.