Finite Element Method and Computational Structural Dynamics
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Lecture - 54
Discrete Fourier Transform - I

Hello friends. So, today we start a entirely new topic which is very different from what we have been discussing so far and that deals with one of the very powerful techniques of mathematical analysis that is a Fourier transform and more specifically Discrete Fourier Transform. Because we will be concentrating on the implementation of Fourier transform on digital computers which often obviously, deals with discrete time data.

And this process of discrete hand dealing with discrete time data brings in its own set of peculiarities which need to be understood. And of course, we will also be discussing the algorithm for computing of Fourier transform, why Fourier transforms have become such a powerful technique. It has happened only in the it is a relatively recent phenomenon only since 1965 when the first Fourier transform algorithm was discovered that Fourier analysis and Fourier transforms have really become a powerful tool.

Otherwise Fourier analysis and harmonic analysis as a mathematical tool they have been known for a very long time, but it is a practical application was not possible until the discovery of fast Fourier transform algorithm. Fast Fourier transform algorithm is essentially a very efficient way of computing discrete Fourier transforms. So, while there are many algorithms now available in public domain as well as in a proprietary format for computation of Fourier transforms fast Fourier transforms.

We will of course, discuss the basic idea behind first fast Fourier transform algorithm, but our main emphasis will be to understand that discrete Fourier transform. What are the peculiarities and water that happened when we deal with discrete time data and how it relates to the Fourier transforming Fourier series the its precursor Fourier series that it is based on. So, Fourier series is just another kind of Fourier transform we may say that.

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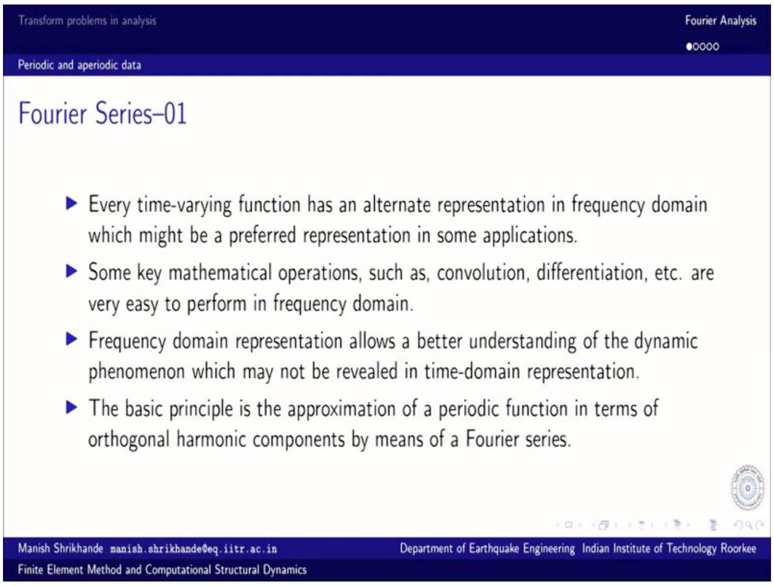

So, every time varying function has an alternate representation in frequency domain. So, the moment we talk about time variance then obviously, it is obvious that we are talking about some oscillations or some variation around some mean position. And that itself implies that we there is something to do with the frequency as a character, what frequencies or how fast the oscillations are happening across the mean position and so on.

So, that is a representation that how the rapidity of crossing the 0 line is what we are mean position refers to the frequency of the oscillation. And this frequency domain representation can be a preferred representation in some application.

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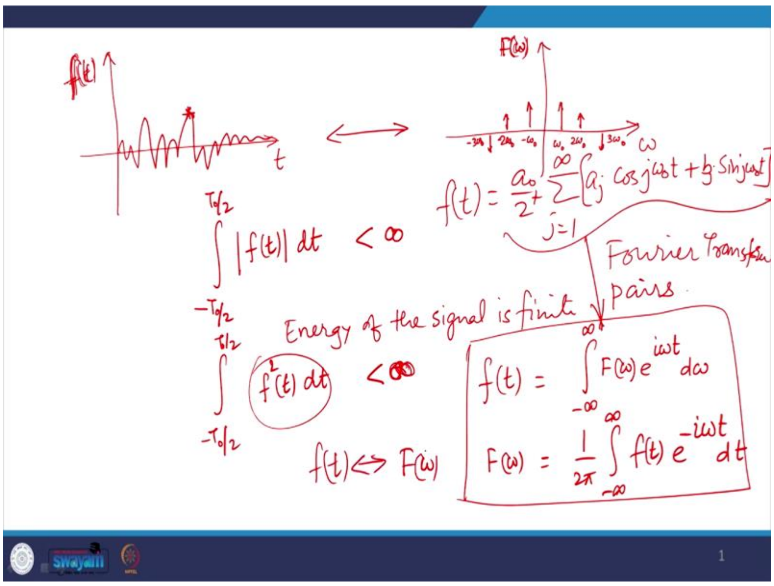

For example, we can have a time domain signal can be very varied very complex looking time domain signal and it will have some frequency domain representation analogues to. So, this is a very common representation. Lowercase alphabet is used to refer to a time domain signal and corresponding uppercase alphabet is used to refer to its a Fourier transform or frequency domain representation.

And there is a one to one correspondence. For every time domain signal there will be some frequency domain representation and so on and they may be interchangeable. So, these are all interchangeable representation. So, whatever is there in time domain the frequency domain representation is just another way of representing it. It is just another way of putting this information in another format.

But there are applications of both of them. For example, in frequency domain we can clearly while we can clearly identify which frequency components are present in the signal right. So, these are the frequencies that are present. And these this information the frequencies that are present in the signal that make up this constituent this time domain signal. This cannot be inferred from this time domain signature. While what is the maximum value of this time domain signature or the wave form which is what is often useful for a design calculation or for any engineering design and decision making. What is the maximum value, what is the critical worst case scenario, what is the worst possible maximum value that needs to be taken for I mean for capacity allocation? So, this maximum value while it can be extracted readily from a time domain representation we cannot really get this information from frequency domain representation. But this frequency domain representation allows me to tell that I am looking at a potential problem if the natural frequency of the system that I design that I am trying to design or trying to synthesize.

If the natural frequency somehow co insides or is in close vicinity with either ω_0 or 2 ω_0 or 3 ω_0 then I am looking at potential trouble spot. Because they may involve kind of resonant kind of condition and a dynamic response may actually build up to a very high order by a very high order of amplification with respect to what is expected for a static response.

So, in order to avoid this kind of error happening or this kind of catastrophe happening we can apriori look at what is the frequency content of the excitation and then take a suitable design call to avoid a natural frequency of the system to be designed to be anywhere in the vicinity of trouble spots in the frequency domain. And secondly, some of the key mathematical operators, for example, the we talked about a solution of equations of motion by time marching schemes.

So, those numerical integration as well as a numerical differentiation and a convolution the system response linear system response to dynamic excitation, so, that is convolution. And these operations in time domain they are very time consuming and very tedious and they become I mean these operations these mathematical operations they become very very simple and very trivial. They are reduced to trivial simple multiplication of two floating point numbers in case of frequency domain operations.

And that is why it is often possible and this actually falls in another arrow in our quiver of any person dealing with the dynamic quantities or time varying quantities. So, Fourier transform is a very important tool of the trade that can be used. I mean sometimes it becomes very simple or very efficient and very straightforward to analyze a system infrequency domain then it is in time domain.

And frequency domain representation allows a better understanding of the dynamic phenomenon which may not be revealed in time domain representation. As we discussed the frequency composition is not possible to be inferred from time domain representation. The basic principle is now we are going back to the concept of Fourier series that Fourier series is used to approximate any periodic function in terms of orthogonal harmonic components. So, it is an infinite series.

We all know we have all studied Fourier series and Fourier series is an infinite series involving sin and cosine terms. And any periodic function as long as it is a finite energy signal and with at most finite number of discontinuities within the period within a period, that can be expressed as a sum of sin and cosine; so, a Fourier series expansion. So, idea is any finite energy or square integrable. So, when I say finite energy so that means, - T_0 / 2 to T_0 /2, where T_0 is the period of the wave.

So, either I say absolute integrable if this is less than infinity or another way to say this is a square integrable that will take care of the negative sign and if this is less than infinity if this is finite. So, this is actually a square integrable. So, this is what we call as finite energy signal, energy of the signal is finite. So, any finite energy signal, so, this square integrable or finite energy or it can also be referred to or just taken as the absolute value function absolute value absolute integrable.

So, as long as the function has finite energy or absolute integrable over one period the function and with utmost finite number of discontinuities. It can be decomposed in terms of orthogonal harmonic components of frequencies and the frequencies that would be multiples of the period of the function. So, if T_0 is the period of the function the frequencies would be 2 pi over T_0 that is fundamental harmonic ω_0 and then the next harmonic would be 2 ω_0 next harmonic would be thrice ω_0 and so on.

So, integer multiple of the fundamental harmonic and that will constitute sin and cosine terms of these frequencies can be used to approximate the function.

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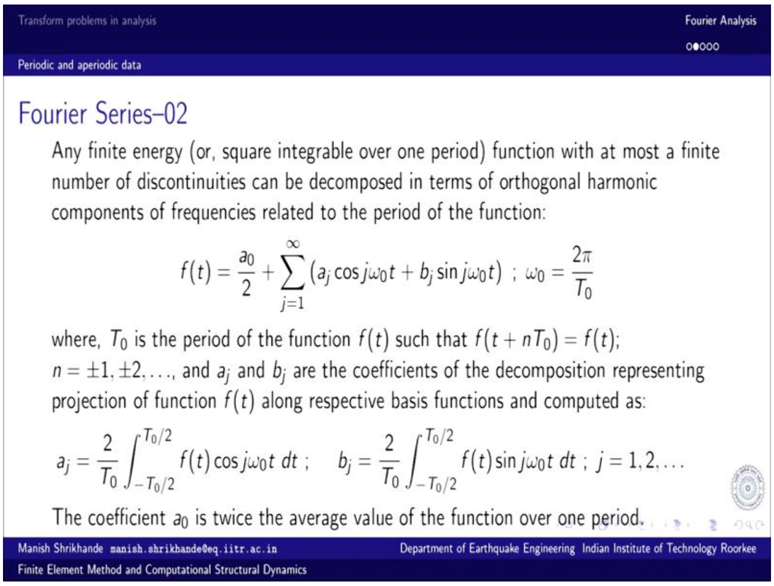

So, essentially the periodic function if f (t) is the periodic function of T_0 is the period then it can be represented as $a_0 / 2$. So, that is a constant term just to shift the mean position in case just in case the function has a non-zero mean. So, that can be modelled by using this constant term. And then there are sin and cosine terms of fundamental and a higher harmonics. So, j is the representation. So, multiple that represents the harmonic number.

So, which harmonic we are looking at. And function f (t) is such that it is periodic with period T_0 such that t + n T_0 is same as f (t) where n can be 1, 2 or 3 or anything integer.

And the coefficients a_j and b_j these are some like a generalized coefficient. They represent or they measure the representation or the extent to which individual term individual base function. So, this is so, Fourier series is just an extension of what we have been talking about as a vector space. So, this is in the limiting case as the dimension of the vector space tends to infinity. So, then we are looking at continuous function space and then these trigonometry functions they define a linear basis function for the function space. And this is the same extension just as we defined any arbitrary function as a linear combination of a basis function basis vectors in case of any arbitrary vector represented as a linear combination of base vectors and we are doing here the same.

Any arbitrary periodic function can be represented as a linear combination of base functions and base functions are trigonometric functions of fundamental harmonic or frequency ω_0 and integer multiples of these frequencies. And coefficients a_j and b_j they are the generalized coordinates and essentially the inner product of this function f (t) along these base functions.

So, essentially projection of function along the basis base function and that actually gives us a measure of how much a particular base function particular trigonometric function contributes to the making of this function f (t). So, a_j, and b_j are the coefficients of decomposition representing projection of function f (t) along the respective basis function and they are computed as inner product. So, that is how we take in case of vector spaces we took the dot product and in case of function space we just take the integral with of this inner product over the period of the function.

So, a_j are defined as f (t) multiplied by cos j ω_0 t and integrated over the period multiplied by 2 over T_0. And similarly b_j corresponds to the coefficient of a basis function sin functions and we take the projection along sine function and suitably a_j and b_j can be defined. And a_0 comes from the mean value theorem of the function a mean value theorem and it is twice the average value of the function over one period.

So, with this we complete the complete representation. So, f (t) if we can arbitrarily approximate to any arbitrary degree of accuracy, so, any periodic function of period T_0 having finite energy and finite number of at most finite number of discontinuities in within a period. They can be expanded as a summation of sin and cosine, linear combination of harmonic orthogonal harmonic functions.

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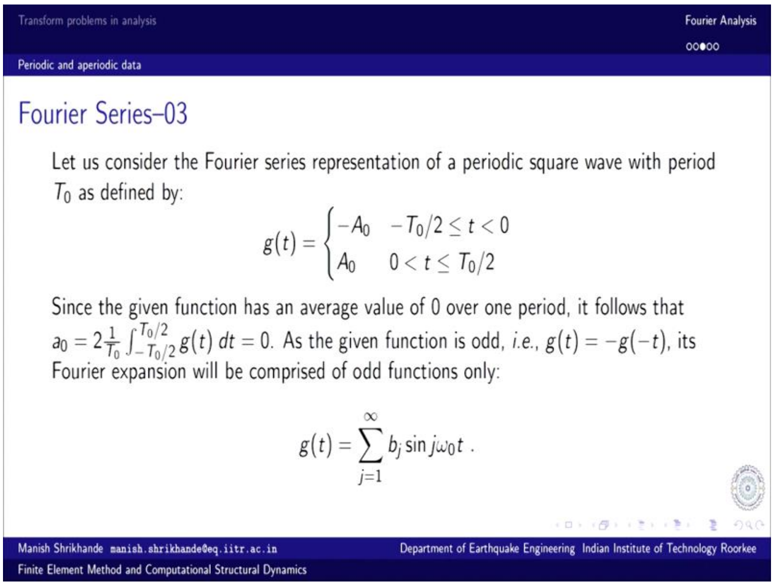

So, let us take a simple periodic square wave and let us say function g (t) is defined as – A_0 for a period ranging from – T_0 / 2 to t = 0 and for the other half for period 0 to T_0 t / 2 it is A_0 positive A_0. So, it is negative for one half of the period and positive for other half of the period amplitude being same, so, its square wave. So, for a square wave, so, the average value is of course, going to be 0 over one period. So, it follows that a_0 that is related to twice the average value of the function. So, a 0 is equal to 0.

And secondly, the function we can see that it is an odd function. So, odd function will have projection or we will have a component on only along the odd functions. So, odd basis functions. So, a cosine term which is any cosine term which is actually even function, so, they cannot represent any odd function. So, we can avoid unnecessary computation trying to find out composer components of A_1, A_2, A_3 and then evaluate something that is approximately equal to 0 rather than straightaway ignoring it by first principles argument. That any odd function cannot have any representation in terms of even function. So, now we will look at by these arguments we can represent g (t) as a Fourier series approximation as just sum of sine terms. So, j is equal to 1 to infinity b_j sin j times ω $(t)_j$, ω_0 t. Now, what are these functions coefficients b_j in this particular case?

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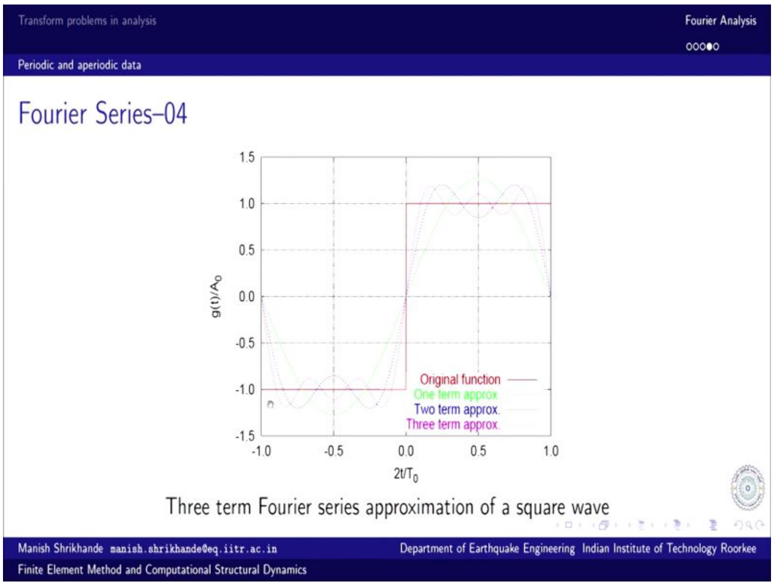

So, we can find allow the integration and work out the integrals and coefficients. So, this is the representation that we have successive approximation. So, original function that we have this square wave, so, - 1. So, we this is of course, scaled with respect to A_0. So, we consider a unit amplitude square wave. So, - 1 to + 1; so, for half of the period it is - 1 and for other half of the period it is + 1.

And if I use only one time approximation, so, this yellow wave, so, this is one sin wave and that approximates that is one time approximation for the square wave. And if I add one more term so, this becomes a blue curve that becomes the new approximation. So, as you can see the first term that gives us the basic trend of the wave form and second term that is added it adds the to the detail. It tries to fit the basic trend to the detail of the wave form or original wave form.

So, two component it tries to model the square wave pattern. And if I add one more term again it tries to model, I mean it tries to push the wave I mean broaden the peak and tries to model this flat portion of this square wave still further by pushing this peak to the either end. And this will keep on happening as I keep on adding more and more number of terms. So, this approximation of the wave form will become better and better.

And as we can all see that this is a discontinuous function. There is a discontinuity at time t = 0 and the Fourier series approximation converges at the average value of the discontinuity. At the point of discontinuity the Fourier series approximation will converge at the average value of the discontinuity and which suits us for this purpose. I mean average value is 0 and that is perfectly fine.

Another thing that we need to notice here this error maximum error in approximation this overshoot that happens that magnitude of overshoot never decreases. It is overshooting this amplitude by this amount that same the degree of overshoot remains the same for two component approximation. It is the same level for 3 component approximation, it is also the same level and this is called the Gibbs phenomena.

So, these overshooting will at the point of where that will happen wherever there is a point of discontinuity. So, at the point of discontinuity there will always be an overshooting and that is and that overshooting can never be removed irrespective of number of terms included in the approximation. But at other points, so, two things happen at the point of discontinuity here.

One the Fourier series converges to the midpoint of the average value of the discontinuity And secondly, at the point of discontinuity there is an I mean neighbourhood of discontinuity there is going to be an overshoot. And this is this degree of overshoot is not going to decrease by any increase in number of terms. So, that kind of error in the neighbourhood of discontinue discontinuity will always be there in case of Fourier series approximation for periodic functions with finite discontinuities.

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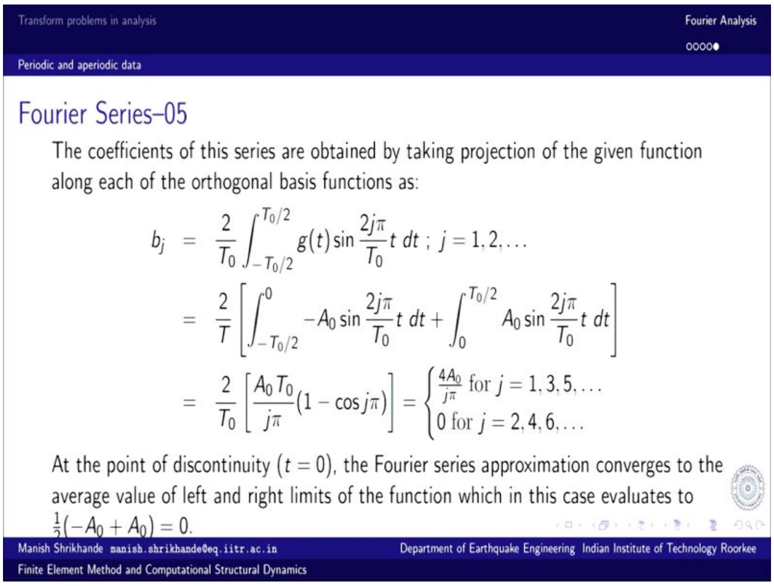

So, coefficients I mean these coefficients that we see of this 3 wave forms. So, the coefficients of these wave forms b_1, b_2, b_3 they can be evaluated by using that inner product g (t) times sin 2 j / t_0, j is the integer harmonic integer and they can be evaluated.

So, we can simply split it into two parts. I mean - T_0 / 2 to 0 and from 0 to T_0 / 2 and evaluate integral and that leads to this expression definite integral and after evaluation of limits we get this cosine in terms of cos of j times pi and that leads to simple expression. So, we have only odd number of terms.

So, odd number of harmonics are going to be present and even number harmonics will not be present. So that means, the function is of such form that it does not have any representation along the even harmonics. So, it is equal to if we can evaluate this it is equal to 4 A_0 over j pi, for j is equal to 1, 3, 5 etcetera and 0 for j is equal to 2, 4, 6.

Now, one thing that can be seen here is the relative magnitude of this coefficients b j. So, this is largest for j is equal to 1 because j is in the denominator, j figures in the denominator. So, it is largest for j is equal to 1. It becomes progressively smaller as j increases and that is a common theme for any representation generally for any kind of approximation.

So, the major lines contribution comes from capturing the trend. The basis function which captures the trend that has the maximum contribution, the rest of the functions they only add to the detail and they are lead they are only doing very fine work, fine work of modifying the local details they do not really contribute significant energy to the function itself. And that is why for engineering analysis we can get a reasonably good idea by looking at only a few number of terms. So, we may not be interested in a very accurate representation of the wave form as long as we can model the amount of energy that is going in and with a reasonable degree of accuracy then that is good enough for our purposes.

So, at the point of discontinuity t = 0, the Fourier series approximation converges to the average value of left and right limits of the function and in this particular case it evaluates to 0. And of course, in the neighbourhood of discontinuity there is going to be a ripple and a overshoot and that cannot be avoided irrespective of how many terms we include in this series.

So, this is what we did for periodic function and for periodic function I will again go back to the definition. So, for periodic function this is the definition of Fourier series. Now, what happens if I look at a function which is non periodic? The function that we looked at here, so, this is a non periodic function. So, how do I deal with a non periodic function? So, let us turn around the argument over its head and let us call a non periodic function such that it is a periodic function whose period is infinite. So, T_0 tends to infinity right.

So, if I look at this Fourier series definition and I treat this as a the function f (t) as a periodic function with whose period approach infinity, T naught approaches infinity then this ω_0 because T_0 approaches infinity then ω_0 becomes infinitesimal. So, it reduces to t ω, so, infinitesimal quantity. So, in the limit as T naught tends to infinity. So, 2 pi / T_0 approaches infinitesimal. So, it is a continuum. It becomes a continuous variable and this summation then converges to transforms to converts to the integral.

And then I have f (t) as shown in slide. And conversely I can define and these are called Fourier transform pairs. So, this transforms, so this Fourier series gets transform to an integral transformation.

And we can relate this F ω as a combination of a j and b j what we are computing as a j and b j and cos j ω t and sin j ω t they are of course, related to complex exponentials e^{iwt}. So, ω_0 becomes d ω. So, we will see how this transition happens and of course, this is what we call as a Fourier transform and this is how it is expressed. I mean of course, this is a continuous functions, f t is a continuous function of time, F ω is a continuous frequency function of circular frequency ω radians per second.

And they are referred to as Fourier transform pairs. f (t) can be transform to F (ω) I know F (ω) can be transform to f (t) as long as the functions exist as long as the Fourier transform is defined under the limiting conditions that we know, it has to have finite energy and finite number of discontinuities. So, we will now look at the issues, I mean now how Fourier series becomes a special case of Fourier transform or how we can derive Fourier transform as a limiting case of Fourier series as a taking a period t as approaching infinity.

And once we are comfortable with that then we will see how to adapt it for use on a digital computer. So, that is what we will be focus in our next lecture.

Thank you.