Finite Element Method and Computational Structural Dynamics

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Lecture - 53
The Time Marching - III

Hello friends. So, we have seen how to solve the initial value problems first by transforming into first order differential equation by suitable addition of auxiliary equations. And first order derivatives I mean first order equations can be solved using several techniques and we discussed Euler, forward Euler, backward Euler and then Runge Kutta and then Adam's family of methods.

And they are in general I mean Runge Kutta particularly fourth order method is very accurate and very efficient technique. And is the go to algorithm in case of very severe nonlinearities and chaotic system analysis. But for normal vibration analysis with not very strong nonlinearities then second order differential equation can be solved directly and generally second order accuracy is still available.

And that is good enough for engineering purposes. And particularly the advantage being we need to solve almost the problem size becomes half because if we need to convert a second order differential equation into an equivalent first order differential equation the size of the problem increases by a factor of 2. So, that way it is advantageous to solve directly the second order differential equation.

So, we discussed several of the schemes starting from Houbolt method and central difference method and then Newmark beta and then recent additions of HHT alpha Hilber Hughes and Taylor alpha method and then Bashrc modification based on successive over relaxation on Newmark's algorithm. So, now in during the discussion of the, development of those methods we had mentioned that some of the parameters are determined based on the stability constraint.

We did talk about unconditional stability of constant average acceleration method and conditional stability of central difference method. So, what do we mean by this concept of stability of numerical scheme or time marching scheme? And how do we assess them and how what is the impact or what does it convey the notion of stability of time

marching scheme? So, we in this lecture we will discuss about the analysis of this time marching schemes how do we judge the quality of these algorithms or the parameters of the algorithm.

And if we are required to choose suitable parameters of the algorithm of a given algorithm for example, beta gamma of Newmark beta, Newmark algorithm then how do we choose beta gamma other than linear acceleration and constant average acceleration choices. So, whether any other variation and leading to different values of beta and gamma whether they would be admissible for acceptable accuracy do they guarantee admissible or reasonable accuracy for computation and usability of results? So, that is what we discuss in this lecture today analysis of time marching schemes.

And as I have been emphasizing again and again all numerical computations are at best approximations. So, there is it is a myth to talk about exact solution for anything coming out of digital computer. Digital computers cannot provide us exact solution all of all solutions all numbers returned by the computer they are approximate numbers.

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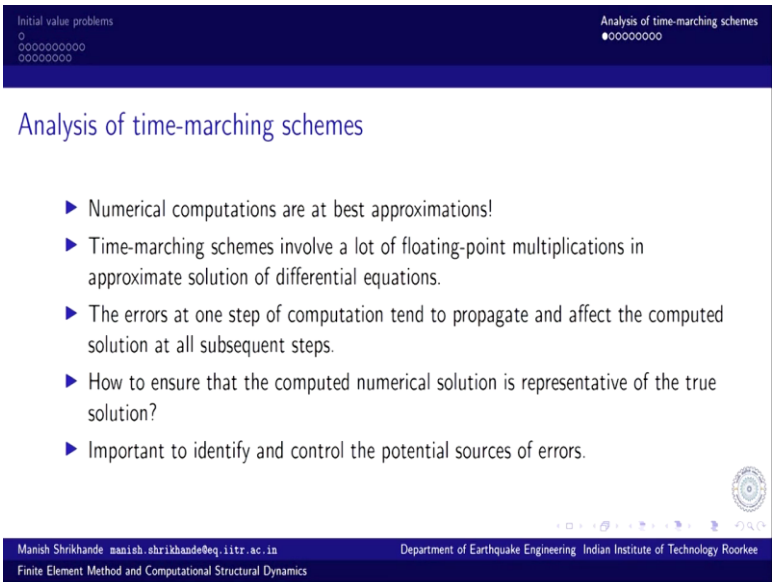

So, time marching schemes they involve a lot of floating point multiplications in approximate solution of differential equations. So, if you recall the entire development of floating point time marching scheme is solution of linear simultaneous equation at every instant of time. The second order schemes that we developed I mean whether it be

Houbolt method, Houbolt scheme or central difference scheme or Newmark beta scheme HHT alpha method.

So, all of them they essentially involve solution of simultaneous equations at each instant of time. And then knowing the values at each instant at a given instant of time all displacements, velocities and acceleration at an instant of time then we use this for advancing the solution in the next to the next time step. So, these repeated floating point operations every as I keep on saying every floating point operation is a source of error.

There are round of errors that are incurred in every floating point operation in addition to of course, multiplication is the expensive most expensive operation I mean second only to division; division is even more expensive than multiplication slightly more expensive. So, the natural question comes, if I compute the solution at nth time instant there are lots of floating point operations happening at nth time instant and the solution that we compute at nth time instant that is an approximate solution full of round of errors at every stage.

And I use this information to compute the solution at the next time instant n plus 1th time instant. So, I am essentially piling up errors round off errors on top of round off errors incurred in all previous time instants. And that is a if you think about it then it is a cause of alarm I mean till the end I mean I am keeping up adding error on top of earlier errors. So, how long can this go and how long before the results become useless if they are to become useless that is. So, that is what the errors at one step of computation tend to propagate and affect the computed solution at all subsequent steps. So, the pollution I mean the errors incurred at one step it will tend to pollute the solution computed at all other subsequent steps. So, how to ensure that the computed numerical solution is representative of the true solution, it is a representative of the true solution we can never hope to get the arrive at the true solution. But what we can do and we can achieve that it is very much within the domain I mean our capacity of numerical techniques.

To arrange our computations such that the result computed result gives a reasonable representation of the true solution. And for this purpose it is important to identify we need to know; know thy enemy that is the first rule of any combat. So, we need to identify and control the potential sources of errors in computations in this computations.

So, just as we identified the sources of error the main source of error of error propagation in Gaussian elimination that is of small pivot and we found a work around that we can use pivoting strategy and interchange the I mean reshuffle the order of equations for elimination of variables. And that actually eliminates the problem of error propagation in Gaussian elimination.

And we do some the similar kind of exercise we try to find a systematic way of tracking down I mean to analyze to identify if there is any potential error, potential sources of error or potential problems with the parameters of time marching scheme that we are using. So, three terms are important in this context and this is a very useful theorem we will see why it is so useful and so often referred to Lax Richtmyer equivalence theorem and it says for consistent numerical approximation.

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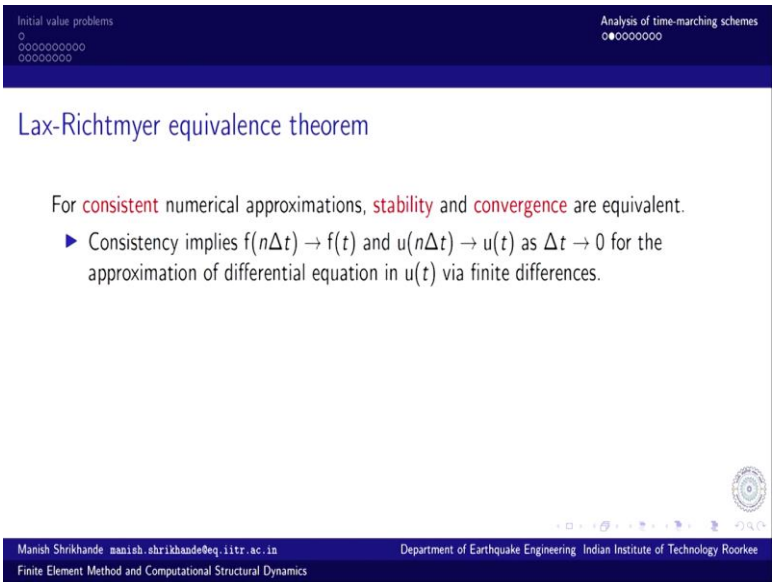

For consistent numerical approximation stability the concept of stability and the concept of convergence they are equivalent. So, the idea is if the approximation is consistent we will come to the definition of consistency little in a little while. So, if the numerical approximations are consistent then if we can prove stability then the convergence is implied or if we can prove convergence then stability of computation is implied we need not prove stability and convergence both.

In this convergence is associated with accuracy, I mean; we converge to the true solution or accurate solution which is representative of the true solution. And consistency implies

that I mean it is a discretely sampled data. So, we are not looking we are looking at discrete time steps if delta t is the interval of time step then all we are looking at is solution at discrete intervals.

So, we evaluate the motion I mean equation of motion is we know the initial conditions at t is equal to 0 next from starting from t is equal to 0 we evaluate the solution at t is equal to Δ t. Whatever happens between 0 and Δ t we do not know we only have we only impose the equilibrium at Δ t and knowing the solution computing the solution at Δ t we then predict the solution estimate the solution at 2 Δ t.

So, the point is we are looking at discretely sample data and consistency implies that this discretely sample data is actually the continuously sample data it will reduce to continuously sample data or it will it tends to continuously sample data in the limit as Δ t becomes 0. So, as delta t I mean if we reduce the time step smaller and smaller I mean to smaller and smaller so it becomes a continuum.

So, that is what continuous consistency implies so all these discretely sample data they are essentially samples of the values of the continuous wave form as Δ t tends to 0 it will converge to the continuous waveform. So, f_n times Δ t approaches f of t as the continuous variable time and displacement u of at n times Δ t converges to u of t, as Δ t approaches 0 as an approximation for difference equations in u t via finite differences.

So, all these second order schemes that we had discussed they all are derived or they can be derived by using finite difference approximations of derivatives. So, this is consistency that finite differences that we have and the discreetly sampled values they would represent they would approach the continuously sampled waveform as the interval or the difference interval of time delta t approaches 0. So, if the approximation this is of course, the approximation nature of approximation.

So, if this nature of approximation is consistent so numerical approximation that we are making if this consistency property is adhered to. Then we need to look at stability and stability implies I mean the notion of stability is very we all understand intuitively what is stability it only means that to any system if the input is bounded the response of the system is going to be bounded. It is impossible for any system to have infinite or unbounded response for a finite input if the system is stable.

If the system is of course, unstable then even for small nudge it can actually build up response to very large level. And just as we mentioned for an consistent approximation stability of computation implies convergence. So, if the approximation is consistent and we can prove that the computations are arranged in a stable manner then convergence is ensured and that is the importance of this theorem Lax Richtmyer theorem.

Because we will see stability of a computational algorithm is often easier to prove than convergence because in order to prove convergence we need to know what is true solution, and in this in most of the times, most of the times we do not know what the true solution is. Had we known true solution then there was no need for any computation subsequently.

So, convergence is very difficult to achieve at best we can look at convergence in Cauchy sense for finite element solution by different mesh sizes and we can look at the convergence of those results, but in the case of time domain that is not really a very convenient proposition because that would involve repeating the calculations for different time steps. So, I do it for Δt, I do the same analysis for $\Delta t / 2$ I do the same analysis for $\Delta t /4$ and then see if the results are converging.

And most of the time the excitation itself is not defined at such fine intervals. So, Δt, $\Delta t / 2$, $\Delta t / 4$ the excitation may be available only at maybe samples of Δt interval. So, what do I do between $\Delta t / 2$ and $\Delta t / 4$. So, by approximating we can of course, use linear interpolation to generate data, but that is another approximation that we are introducing another source of error that we are introducing.

Because linear interpolation does not really satisfy the does not agree with the rules of discrete discretely sample data in the frequency content. So, that is it violates the Nyquist frequency criteria it will generate frequencies beyond that and it will lead to it can lead to other issues, but let us not get into that. Suffice it to say that it is much easier and much more convenient to analyze the stability of algorithm.

And if the approximation is consistent and we can prove the stability of algorithm for whatever operating parameters of the scheme time marching scheme. Then the convergence of the computed solution or the time marching scheme is guaranteed and that is the reason for taking up this analysis of time marching scheme. So, that we can be

sure under what conditions the results of time marching of any particular time marching scheme can be relied upon.

So, while stability of a continuation computational algorithm is easily examined, it is much more difficult to establish convergence because of the reasons I mentioned, convergence would require study for different step sizes and that is much difficult job than studying the stability of a computational algorithm. Stability of a computational algorithm can be done on the analytical using certain analytical principles and no further computation is generally required for that purpose.

So, eventually all time marching schemes whether it is first order techniques or second order techniques eventually all time marching schemes can be arranged in the form that the solution at n + 1 th time instant they are given as they can be represented as some amplification some effect of the solution at the previous time instant so nth time instant displacement and velocity. And some effect of the forces or external loads at the current time instant and the previous time instant.

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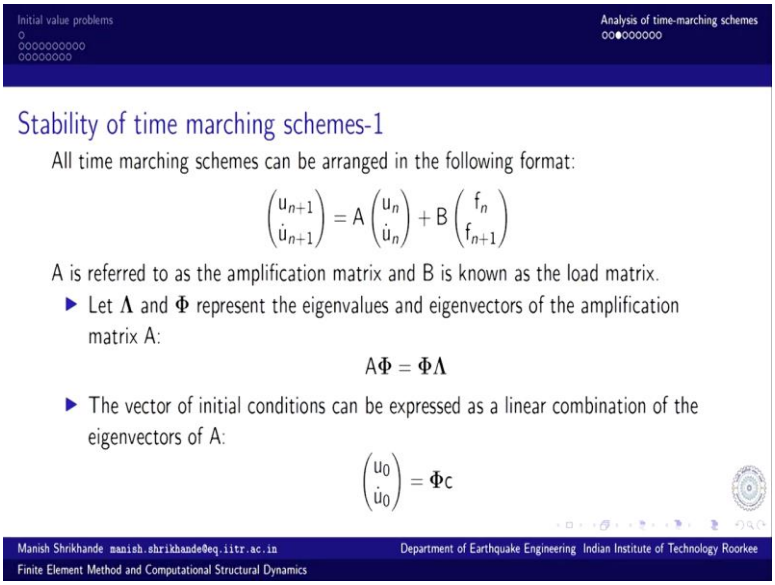

So, the at n + 1 th time instant so the force at nth time instant in and n + 1 time instant. So, matrix a here that is referred to as amplification matrix because it amplifies the effect of initial condition so to say, because in the beginning at time t is equal to zero this will these will be the initial conditions. And these will carry propagate through all through the subsequent analysis in gradually. And b which gives the contribution controls the

contribution of these external forces at two time instants n th time instant and n + 1 th time instant this is known as the load matrix.

Now, let us assume λ and ϕ they represent the eigenvalues and eigenvectors of the amplification matrix a right. So, we look at this amplification matrix a and it is a matrix square matrix so it relates 2 by 2 to 2 by 1 vector to another 2 by 1 vector. For this is for single degree freedom system otherwise it will be 2 n by 1 relating to 2 n by 1. So, it becomes 2 n by 2 n matrix where, n is the number of degrees of freedom of the system being analyzed.

So, if λ and ϕ they represent the eigenvalues and eigenvectors of the amplification matrix then by definition of eigenvalue problem standard eigenvalue problem a ϕ is equal to ϕ λ, λ is of course, the diagonal matrix of eigenvalues. And by virtue of orthogonality of eigenvectors the initial condition vector of initial conditions u $_0$ and $\dot{u}_0$ it is a vector in the same vector space and that can be represented as a linear combination of the eigenvectors of matrix a.

So, I can represent u_0 and $\dot{u}_0$ as a linear combination of eigenvector. So, c $_1$ ϕ $_1$ + c $_2$ ϕ $_2$ + c $_3$ ϕ $_3$ and so on as many terms as the dimension of the system. So, far so good so, this is based on the orthogonality property and that allows us to use eigenvectors as the basis of expansion, we can expand this vector of initial values into a linear combination of the eigenvectors.

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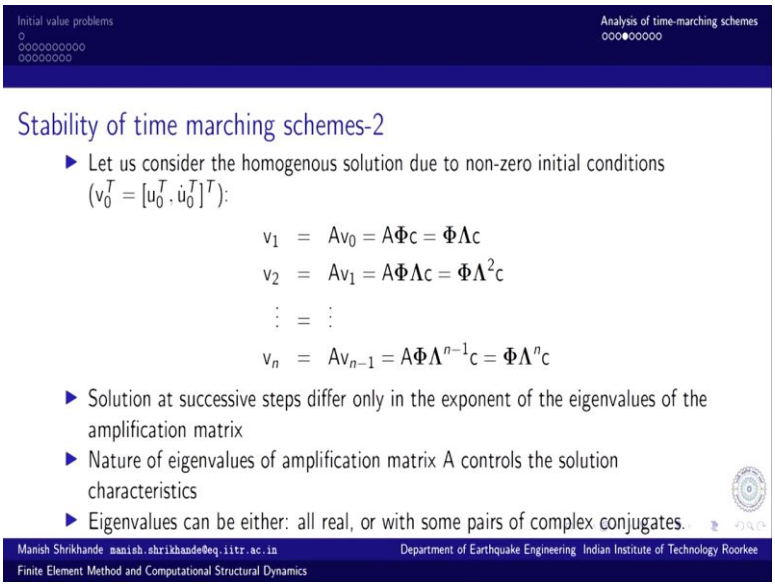

So, let us consider the homogeneous solution due to non zero initial conditions so let us for the time being let us forget about the loading condition. So, how the system how the analysis techniques the time marching schemes they would predict the free vibration solution. That is how the system will oscillate or vibrate if it is disturbed from its initial position by for some nonzero initial conditions.

So, the non zero initial conditions they are given as u_0 and $\dot{u}_0$ and if we multiply I mean v_1 as a one step amplification. So, at first time step it will be A times initial conditions at time t = 0 so that would be a times v_0. So, v_0 is the now state vector so u_0 and $\dot{u}_0$ comprising of u_0 and $\dot{u}_0$. So, we have A times and we already defined v_0 or u_0, $\dot{u}_0$ as a linear combination of eigenvectors of matrix a amplification matrix. So, we can replace u_0 and $\dot{u}_0$vector of u_0 $\dot{u}_0$as ϕ_c. So, this ϕ_c represents u_0 $\dot{u}_0$.

And ϕ being the eigenvectors matrix of eigenvectors of matrix A this is $A\phi = \phi\lambda$ by definition of eigenvalue problem. And the next step having computed this v_1 or the solution at free vibration solution at one step beyond the initial condition then we go to the second step so solution at twice Δt. So, v_2 = A times v_1 previously computed just previous time instant solution. So, again we have substituting for v 1 from here we get A times $\phi\lambda_c$ and again A_ϕ =to $\phi\lambda$ and $\lambda$is a diagonal matrix so it simply becomes λ^2 and so on.

So, at n^{th} time instant we will have the solution as A times $\phi\lambda$ rise to the power n-1 c and that will again a phi can be transformed to phi lambda so eventually we will have ϕtimes lambda rise to the power n c. So, the only thing that is changing I mean if you look at this trend the only thing that is changing here is the power of the eigenvalue of matrix A in each time step.

So, for the first time step it is first power of lambda second time step it is second power of lambda and for nth time step it is nth power of eigenvalues eigenvectors remain the same the coefficient vector c remains the same. So, the solution at different instants of time for free vibration that is only given by the powers of the eigenvalues of amplification matrix A now what does this mean.

So, solution at successive steps differ only in the exponent of eigenvalues of the amplification matrix. Now what are the implications for choices of eigenvalues. Nature

of eigenvalues of amplification matrix controls this solution characteristics; obviously, because that is the only thing that is changing phi is constant and c is constant. So, if anything has to happen though the solution it has to happen because of these eigenvalues.

So, nature of eigenvalue depends on or governs the solution characteristics. So, what are the nature; what will happen if the nature is? Let us look at the possibilities. So, if the solution there are several at least there are three possibilities. If all eigenvalues are real and positive right so if all lambda if they are real and positive then there is absolutely no scope of any change in sign in every in a different time step.

So, whatever is the sign of this solution at first time step that same sign will continue all over or whatever is the sign of the solution whatever is the nature of solution at the initial time step the same nature will continue all through, right it is not going to change sign at all. So, that implication is there is no oscillation. If λ is real and positive then it is all being multiplied again and again and again and again so, it is a positive number so it is never changing sign.

So, that is something that is very odd I mean how do, I have a vibration solution without ever changing sign at any time instant. So, and this is guaranteed if λ is positive real and positive then there is now there is no scope of any sign change at any time step time step can you can keep on computing at any large number of time steps and this will never change sign.

So; obviously, that is a problem and oscillatory motion cannot be captured. So, even if the problem is that of a vibrating system mechanical system undergoing free vibrations if the algorithm is such that the amplification matrix a has eigenvalues which are all real and positive all eigenvalues of the amplification matrix are real and positive. Then it cannot capture the oscillatory motion and; obviously, that is a problem.

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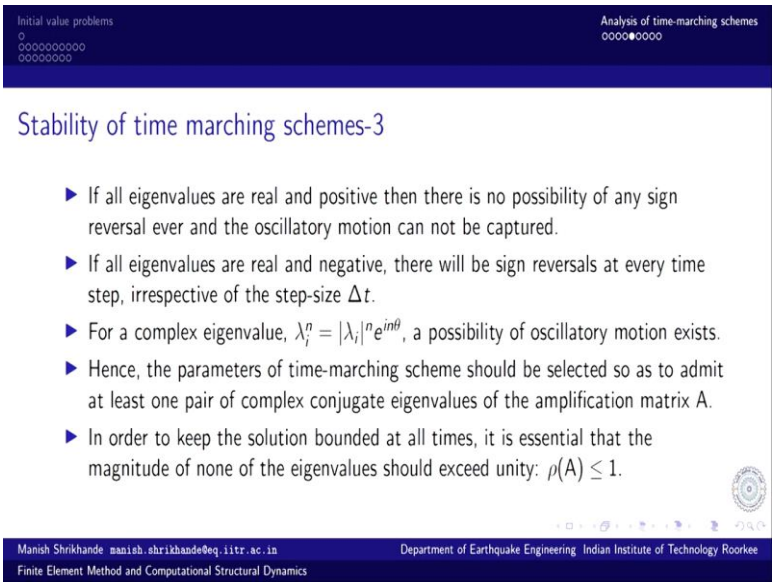

And any such algorithm any such choice of parameters which renders eigenvalues of the amplification matrix to be positive and real they are all eigenvalues some of them can be positive and real no harm, but all eigenvalues the emphasis is on all eigenvalues. So, if all eigenvalues are real and positive then we are looking at a problem. Now, look at the other part if all eigenvalues are real and negative.

So, we first case we dismissed it off as not admissible all eigenvalues real and positive now the other alternative is all eigenvalues are real and negative. In that case because the powers are I mean first is it is odd power then it becomes even power then it will be odd power then it becomes even power. So, if these eigenvalues all eigenvalues are negative then there is going to be a sign change force sign change at every time instant.

And that time instant is not related to I mean these are only referred to as 1 two and delta t might be changed might be different. So, different choices of delta t might actually give us different results for a given time instant so very untenable situation. So if all eigenvalues are real and negative there will be sign reversals at every time step irrespective of this step size Δ t.

So, whatever may be the Δ t so there is going to be sign change at every time step and that is a problem if I use this let us say for the choice of Δt, I have amplification matrix which has all eigenvalues as real and negative then let us say initial conditions are

positive then the solution predicted at Δ t is going to be of opposite sign. So, it will be negative.

Now, if I use the same amplification matrix, but I try to compute it at Δ t of is equal to now the is half use the half of Δ t. So, now, the next computation is one is at 0 and another one is at delta t by 2, now the negative sign is for Δt / 2 and at Δ t it becomes positive again, now that is a problem. Just because of by the order of the I mean the size now by changing the time step size of the time step the solution should not be so sensitive to that.

And now it is impossible to reconcile which one is the correct solution one solution gives me negative values another solution gives me positive values and if I choose any other Δ t I might get something else so it is a complete chaotic chaos there. So; obviously, this is also not tenable all eigenvalues cannot be real and negative. Now, let us come to another interesting possibility.

If the eigen value is complex let us say at least one eigenvalue is complex, forget about other eigenvalues at least one eigenvalue let us look at one eigenvalue. So, what happens to that complex eigenvalue ok, n th power of complex eigenvalue. So, n th power of complex eigenvalue I can represent lambda if lambda is a complex number I can represent it in this form polar format in the modulus and phase form.

So, nth power will increase the amplitude by nth power and it will also introduce a phase shift by n times right. So, everything gets so exponential it gets added up for every multiplication. So, it will be n times for n n times amplification there will be a phase shift of n times. And this is complex exponential so complex exponentials are essentially sinusoids sine waves and cosine waves. And now we have a hint of oscillatory motion.

So, there is a possibility of that oscillatory motion can exist and another thing is the amplitude is this. So, amplitude can either if λ amplitude of this eigenvalue if it is equal to 1 then no matter how many times I try to amplify it its amplitude does not change. And if its amplitude is greater than 1 even if it is slightly greater than 1 1.1 or 1.01 then this term λ I raise to absolute value of lambda I or modulus of lambda I raise to the power n will keep on increasing as the time step increases.

So, essentially the solution will start and it will begin to diverge gradually so that is an unstable phenomena. On the other hand if amplitude of λ is less than one then the solution will begin to decay and this decay by virtue of this lambda is what we call as algorithmic decay algorithmic damping this simulates the damping effect. And that is coming entirely purely because of the parameters of the time marching scheme.

So, these possibilities hence the parameters of time marching schemes should be selected so, as to admit at least one pair of complex conjugate eigenvalues of the amplification matrix. So, all the parameters for example, Newmark's scheme beta gamma and using HHT alpha the alpha parameter which is again controlling beta and gamma and again the Bosak's amplification Bosak's modification algorithm.

So, they all these parameters can be the bounds that we arrive at for example, alpha was found the bounds of for alpha are found to be between minus 1 /3 to 0. And it is said that if alpha in HHT alpha method if it alpha is in this bound then the algorithm is stable. And when we say stable it essentially means this that lambda value of lambda is going to be less than 1. If beyond this it may exceed 1 and even if the oscillatory motion is there the amplitudes may keep on increasing even if there is no reason physical reason to justify that amplification of amplitudes.

So, in order to keep the solution bounded at all times it is essential that the magnitude of none of the eigenvalues should exceed unity and; that means, spectral radius of the amplification matrix should be less than or equal to 1. So, spectral radius that is the largest eigenvalue of the matrix amplification matrix a should be less than or equal to 1. If it is less than one then it is going to experience decay at every time step and that decay is purely because of numerical nature because of the parameters of the time marching scheme. And if it is equal to 1 then there is not going to be any kind of algorithmic decay as what happens in the case of Newmark's constant average acceleration method.

So, Newmark's constant average acceleration method there is no algorithmic decay algorithmic damping. And finally, we also look at what is the time step that is required for spatial discretization because that is a information propagation. So, how the information propagates in space I mean the wave form I mean wave form has to propagate through the medium.

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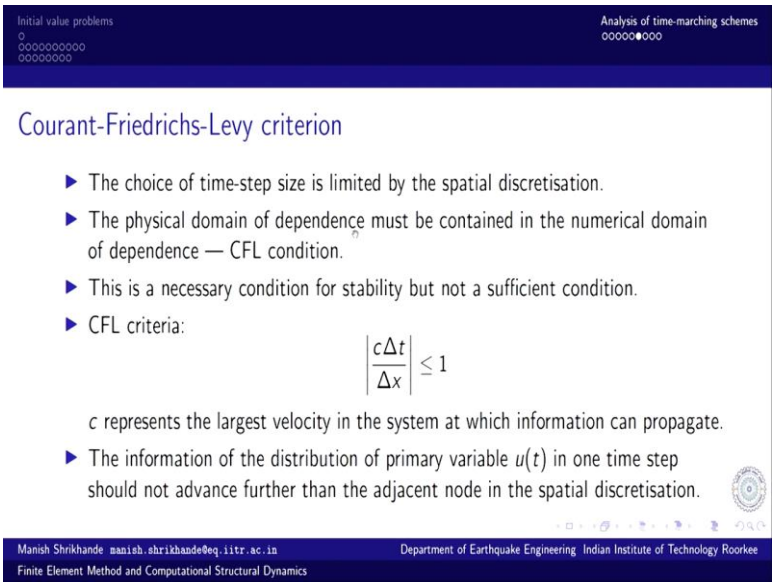

So, how do I define what time step is required for a given choice of spatial discretization finite element discretization? So, that is defined by CFL what we call as CFL criteria Courant Friedrichs and Lewy criteria. And that is governed by a physical domain of dependence must be contained in the numerical domain of dependence and that is called CFL condition and this is a necessary condition for stability, but not a sufficient condition.

And this criteria is defined as c times delta t divided by delta x should be less than 1 where, c represents the largest velocity in the system at which the information can propagate. So, whatever is the that depends on the properties of the medium that is being modeled. And Δ x is the spacing interval representative mesh size and Δ t is the time step that should be taken.

So, it is this choice of time step that is important what should be the time step that should be used for analysis purposes, but most of the time in practical problems we rarely need to look at this because the excitation is often sampled at time step that is way below the limit imposed by CFL criteria. So, information of the distribution of primary variables u t in one time step should not advance further than the adjacent node in the spatial discretization.

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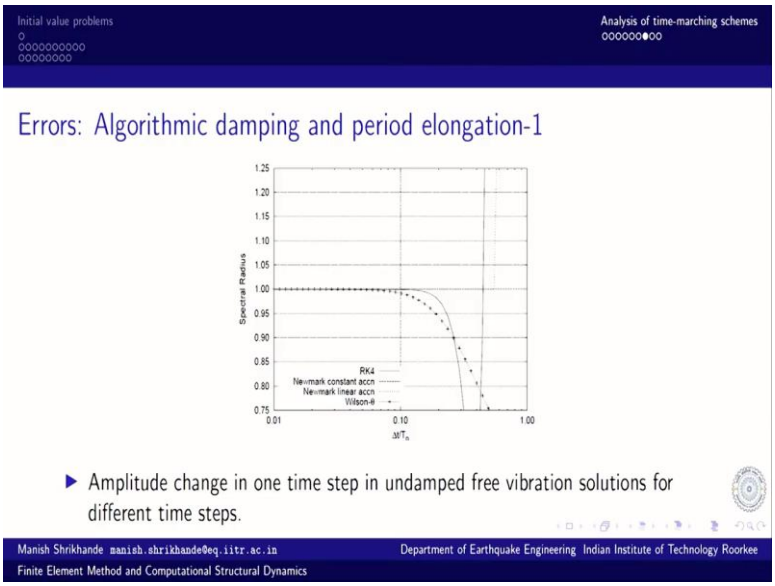

So, that is what this represents this means physical domain of dependence must be contained within the numerical domain of dependence. So, one time step should not go beyond information should not propagate beyond one adjacent node in this spatial discretization. So, again referring to algorithmic damping and period elongation, so these are some of the performance of different schemes

So, RK4 that is this solid line this represents the Runge Kutta fourth order tech method, then Newmark constant average acceleration. So, this is constant line that you see dotted line, Newmark linear acceleration so this is it is this particular line and Wilson theta is this cross dotted and with cross. So, as you can see the spectral radius it remains reasonably close to 1 for some choice of ratio of delta t versus delta natural period t n the natural period of the or the time period.

So, and that gives us an idea from this spectral radius context that we should choose Δ t I mean Δ t will affect the parameters of amplification matrix choice of Δ t. So, choice of Δ t should be often I mean if we look at these curves then it suggests that 10 percent of the natural period of the system is a reasonably good choice for delta t the parameter spectral radius is close to 1. Although for some of the algorithms it is slightly less than 1, but that is little bit of algorithm damping is always better than running into instabilities.

And this delta natural period that we are looking at that should correspond to the shortest natural period or the highest frequency that we want to represent in the solution. Not all

the natural period, not all modes are required in the computation or they are reliable. And therefore, this choice of t_n or the natural period depends on what is the highest mode that normal mode that needs to be represented in solution. And this is the phase change as we because if we look at the um the term involving complex eigenvalue.

So, at each instant of time there is a phase shift and this phase shift will push the solution ahead. And that makes it again phase distortion that actually makes it solution to move slightly away from what is the true solution. And this is what we call as phase distortion and every time step it will introduce an additional phase shift then what is actually there and this phase distortion is generally very small as for delta t for small time step. But it increases considerably as delta t increases beyond 10 percent of natural period.

So, again it is a fortuitous at I would say that phase distortion is under control for the choice of time step that is about 10 percent of the natural period of the highest mode for all lower periods this is of course, going to be much smaller. And this is what we mean by algorithmic damping and period elongation.

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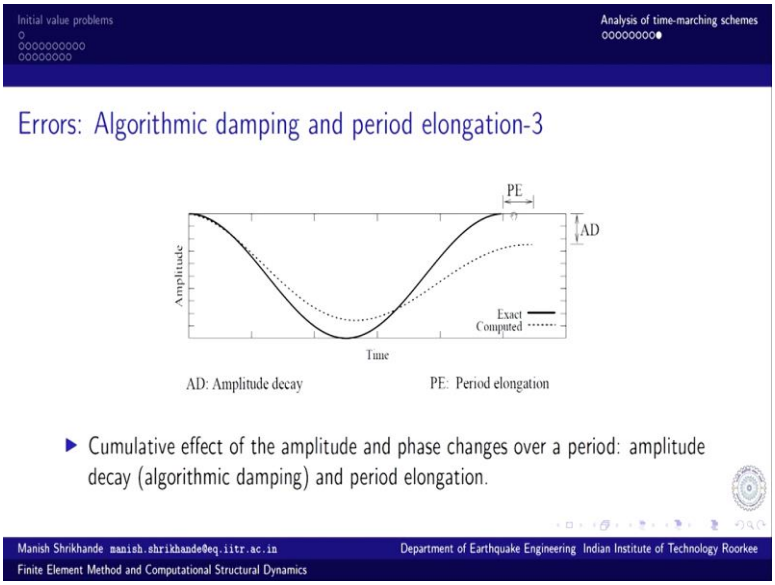

Period elongation is this is the true solution and computed solution might look like this and this is pre period elongation that we represent here PE that is entirely because of this phase shift that happens at every time step that we compute the solution. And this decay in amplification amplitudes this is entirely because that is called algorithmic damping,

and this is because the magnitude of the or the spectral radius of amplification matrix is less than 1.

And the even if the solution true solution does not have any damping the numerically computed value might indicate some damping. So, cumulative effect of amplitude and phase changes over a time period would show up as period elongation and algorithmic damping. So, these are the techniques these are the basic issues that we look at for during the analysis of time marching schemes. So, it is essentially the amplification matrix corresponding which affects which governs the how the initial conditions would be propagated over a period of time for free vibration analysis. And a simple technique that we use for testing the any time marching scheme parameters is to try to compute the free vibration solution for two sets of initial condition one for unit displacement and 0 velocity.

So, the computed solution should actually give us perfect sine wave that would be cosine wave essentially and the other one is initial displacement being 0, but the velocities are unity then the computed solution would correspond to a sine wave. And if the algorithm time marching scheme correctly propagates these pure cosine wave and pure sine wave for these two initial conditions. Then there is that is a reason enough to believe that the computed response for any transient analysis vibration analysis would be reasonably accurate.

So, with that we complete our discussion of time marching schemes and in next lecture we start with a very exciting topic. Although I get to spend very small amount of time correspond I mean in comparison to the significance of that topic but I hope I can generate enough excitement for you to follow it upon your own the Fourier transform and the fast Fourier transform algorithm and the analysis in frequency domain.

Thank you.