Finite Element Method and Computational Structural Dynamics
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Lecture - 52
The Time Marching - II

Hello friends. So, we have seen the solution of ordinary differential equation in time the initial value problem featuring the first order differential equations or first rate of change of gradient function and function value, given as a gradient function and evaluating it for the initial condition with the specified initial condition for function value at time T is equal to 0.

And we could develop highly accurate techniques and Runge-Kutta fourth order and fifth order accurate. They are very efficient and very accurate techniques and which can serve the purpose and then, we also developed some techniques based on the interpolation of the gradient function over the in a duration of interval of integration time marching interval.

And that is based on Adam's techniques. So, series of predictor character explicit and implicit schemes that we discussed. Now, for structural dynamics applications, we often deal with second order differential equations with two initial conditions that is displacements and the velocity at time step zero.

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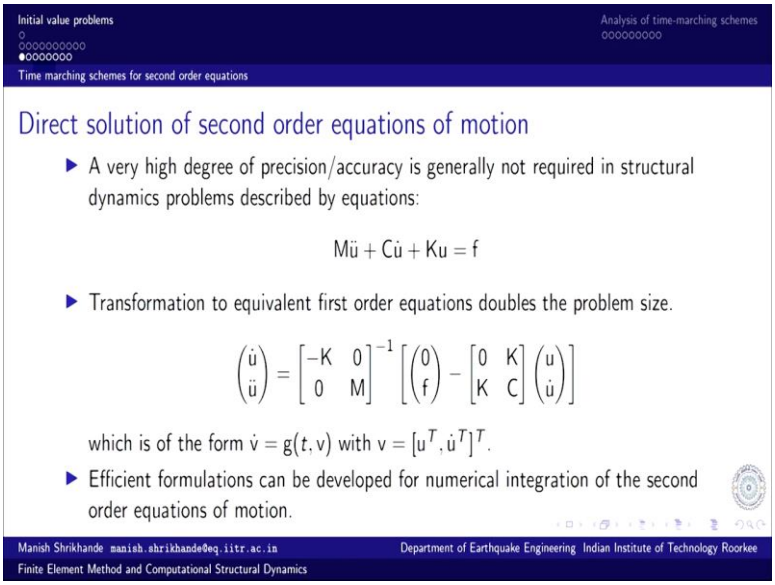

So, we look at the direct solution of second order differential equations that we generally do not require very high degree of accuracy for structural dynamics problems. I mean our tolerance level is generally, not v very precise. So, we can do with reasonably accurate solution I mean even with the second order accurate techniques, they would serve as well.

So, the equations; equation of motion is defined as as shown in slide, f is in general function of time the forcing function, K is the stiffness matrix, u displacement and we assume velocity proportional damping and M is the inertia term and u at double dot, they represent the generalized accelerations.

Now, this second order differential equation as I mentioned earlier, these can be converted into equivalent first order equation by using the state space approach. And we can transform these into equivalent first order equation just by saying state vector v as u and $\dot{u}$. So, this $\dot{u}$.and $\ddot{u}$represents $\dot{v}$. And $\dot{v}$.is given by this is the gradient function minus K M inverse of 0 and f. These are the right hand side vector and minus this term that involves the state vector v. So, u and $\dot{u}$..

So, essentially second row if you look at it the second row of the equation is what the governing differential equation is and the first row is merely statement of the identity that is K $\dot{u}$- K $\dot{u}$= 0. That is all. So, that is an identity statement. So, that is an auxiliary equation. It does not add any more information, anything extra information.

It just allows us to recast the equation into an equivalent first order equation. So, second order equation can be converted into an equivalent first order equation. Of course, the size of the problem increases. So, that is the penalty that we have to pay. So, second order equation converted to first order equation of double the size.

So, m by m is a n by n matrix matrix M inertia at matrix is m n by n and stiffness matrix is another n by n. So, this becomes 2n by 2n. So, this entire variable there, these are 2n variables. So, and these can be evaluated using any of the techniques that we have already discussed for first order equation.

So, however we need not go into this formulation unless we are looking at a highly non-linear system and K itself is a function of u and u dot for non-linear system, then of course we may need to go for very accurate analysis and this kind of first order approximation, first order derivative conversion might be appropriate in that case, but

most of the time for the accuracy that we are dealing with or that is acceptable direct analysis of the second order equation can be good enough. So, as long as the accuracy is guaranteed to have a reasonable degree of accuracy at least first order or second order accuracy is achieved. So, this direct solution is of course is always beneficial because it reduces the problem size and it also simplifies the memory requirement and computational resources are not taxed heavily.

So, first scheme one of the earlier schemes to directly integrate the second order differential equation is based on finite difference approximation of the derivatives and that is called Houbolt scheme.

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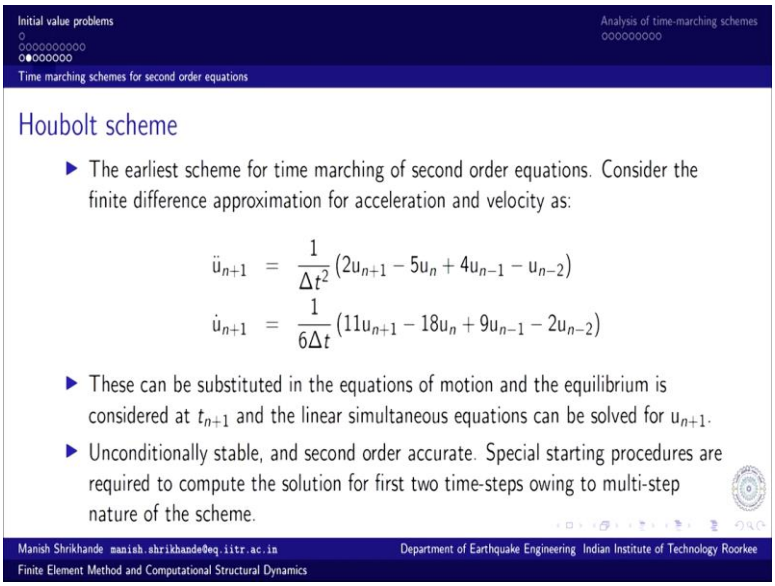

So, derivatives are obtained as shown in slide. At next time instant can be obtained as this kind of derivative I mean finite difference approximation. So, for acceleration we have this and for velocities we have this kind of expression. So, all of these terms are displacement terms and denominator is 2 Δ t^2 and here all of these are displacement terms and denominator is Δ t.

So, it is a velocity term and once we substitute these terms in the statement of equilibrium governing equilibrium equation, so the only unknown would be u_{n+1}. So, the entire system can be compressed and transformed into an equivalent stiffness matrix multiplied by the displacements at n $+1^{th}$ time instant and then, the system can be solved as quasi static problem at that particular time instant linear simultaneous equations. So,

these can be substituted in the equilibrium statement of equations of motion and the equilibrium is considered at time instant t of n + 1 and the resulting linear simultaneous equations can be solved for displacement u at n +1th time instant.

Now, this is fine because u_n is of course we may know the initial condition, but what to do about u_n -1 and u_n - 2 that say obviously this technique Houbolt scheme is a multi-step procedure. So, some kind of special starting conditions are required for using this Houbolt scheme. So, maybe two steps of either backward Euler or Runge-Kutta method, they are required before we can start using Houbolt scheme for computation or next times instance.

And this is unconditionally stable and second order accurate and special starting procedures are required to compute the solution for its first two time steps going to multi step nature because it goes back in three steps back in time. And that requires special starting techniques or we need to compute the solution for at least for two steps. We need to compute solution using some other technique. Let us say I mean for example backward Euler or Runge-Kutta.

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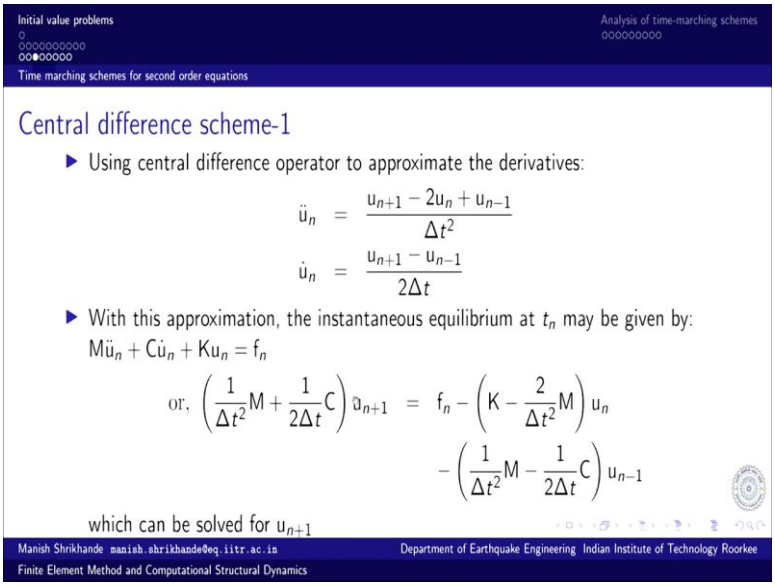

Next, again finite difference operator we can use central difference scheme to define approximate the accelerations at a time step nth time, step acceleration at nth time instant.can be given as finite difference operator and involves function values at u_n +1

and u_n and $u_n - 1$. And similarly the velocities can be given over this time interval twice $\Delta t\ u_n + 1$ and $u_n - 1$.

And we can substitute this and equation equilibrium at t_n. The time step nth time instant can be imposed and the unknown the equations can be segregated, terms can be segregated and the unknown displacement at n + 1 term can be separated out and the solution can be computed subsequently. And this can be solved as a linear simultaneous equation for at the current time instant to find out what is going to be the displacement at n + 1th time instant.

And again this method is explicit. It involves only the information about current and previous time instant and this solution can proceed in time very quickly and this is particularly very advantageous, very special. If the damping is very small; such that it can be ignored in comparison to the mass term and if the inertia term can be modeled as a lumped mass model using some kind of lumping procedure.

And it is a if it is a diagonal matrix, then this solution system becomes very trivial and then the because it is explicit, so very small time steps are of course required and that small time step choice of small time step is offset by the efficiency in computation. If damping is small and the mass matrix or inertia matrix is diagonal, then the computations can be implemented very quickly and efficiently.

And central different scheme is in fact one of the popular schemes for using control systems or modelling control systems for many of the experimental systems pseudo dynamic control systems. So, central scheme; central difference scheme is the technique used for their used in that those applications because of these simplicity of calculation. So, to start the process again, it is one step back information is required. So, special initial conditions can be derived.

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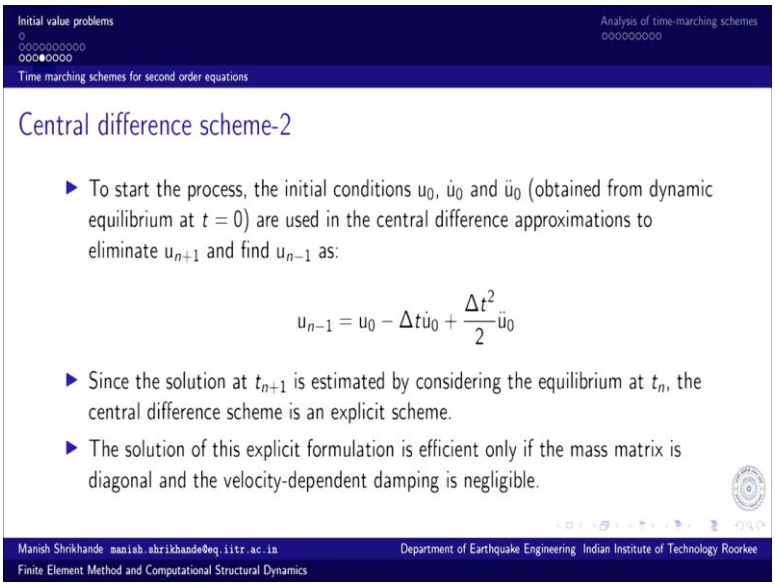

So, initial conditions we know what is the displacement and what is the velocity. At time instant t = 0 and therefore, we can find out what is the acceleration at time step 0 that can be obtained from the equilibrium at t = 0 and this can be used to eliminate u n plus 1 and find u $_n$ minus 1.

So, central difference approximation we can find out what is the value for previous time instant and that can be used for solution. The solution at time instant t $_n$ + 1 is estimated by considering the equilibrium at previous time instant t $_n$. The central difference scheme is obviously an explicit scheme and it is for the special case when the damping is very small or it is undamped system that we are looking at and for lumped inertia matrix, the system, the equations can be solved.

This scheme can be can produce very efficient solutions even for very small time steps which is necessary by virtue of conditional stability by because of it is explicit nature. So, solution of this explicit formulation is efficient only if the mass matrix is diagonal and the velocity dependent damping is negligible.

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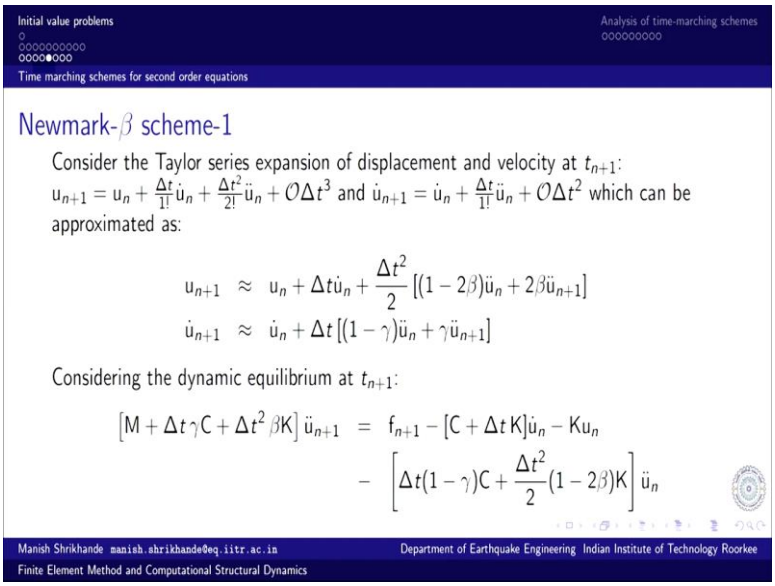

So, next in the scheme of things is very popular family of methods that are now popularly known as Newmark-Beta scheme and this can be derived there are several ways to derive these techniques Newmark-Beta scheme which can be derived using finite difference approximation. It can be derived using Taylor series approximation.

So, let us look at the displacement and velocity at t $_n$ + 1. So, expanding it around the nth time instant, we can have it in terms of velocity and acceleration at n^{th} time instant and similarly, the velocity can be expanded in around nth time instant and by ignoring the higher order term, the displacement to the third order term and for velocity beyond the acceleration term.

If we ignore those terms, then the displacements and the velocity at the next time instant can be obtained as linear combination of I mean this is kind of relaxation parameter that we had encountered earlier using while using Jacobi iterative solver. So, successive over relaxation, so, it is something like that the weighted sum of previous time step and the current time step value.

So, linear combination of them and then when we consider the equilibrium at impose, the equilibrium at n + 1th time instant. And then collecting the terms unknown, what is the acceleration at n + 1th time instant. That is the only unknown.

Rest of the things on the left hand side, they are all known quantities at nth time instant. And then this system of equations can be solved for these accelerations at n + 1th time instant and then, knowing these accelerations we can go back and find out what is the displacement and velocity from these approximations that were initially substituted.

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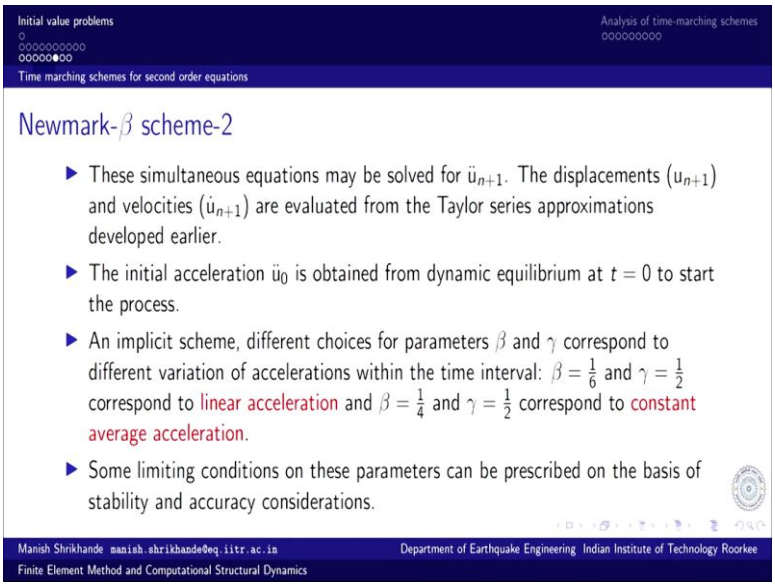

So, the simultaneous equations can be solved for acceleration at n + 1th time instant. The displacements and velocities at n + 1th time instants are obtained from the Taylor series approximation that have been that were developed earlier. And initial acceleration u $_0$ is obtained from dynamic equilibrium at t is equal to 0 to start the process and subsequently the solution marches ahead in time at one step at a time.

So, this is an implicit scheme and different choices for the parameters beta and gamma that are used for linear combination approximation for the displacement and velocities in terms of combining the acceleration at beginning and end of the time step. So, different choices lead to different variation of accelerations within the interval and certain choices.

For example, choice of beta is equal to 1 over 6 and gamma is equal to 1 / 2 corresponds to assumption of acceleration varying linearly and the choice of beta is equal to 1 / 4 and gamma is equal to 1 / 2 corresponds to constant average acceleration. So, the acceleration being maintained at a average of the two values at the in the interval between two time instants.

So, constant average acceleration and if you recall the quadrature rule, we had found that trapezoidal error in trapezoidal is somewhat more than trapezoidal quadrature is somewhat more than the rectangle rule and similar conditions apply here. So, linear acceleration is similar to trapezoidal rule, constant average acceleration is similar to rectangle rule and constant average acceleration generally, has a better numerical performance than linear acceleration technique.

So, some limiting conditions on these parameters can be prescribed on the basis of stability and accuracy considerations. So, how these are to guarantee that the scheme the Newmark-Beta scheme, the method actually converges or does not diverge.

There are certain limits on the parameters beta and gamma. The limiting parameters can be derived on the basis of stability of the scheme. What are the conditions under which the solution scheme will provide stable results, stable and accurate results? So, we can examine those constraints and we will see these stability considerations in our next lecture.

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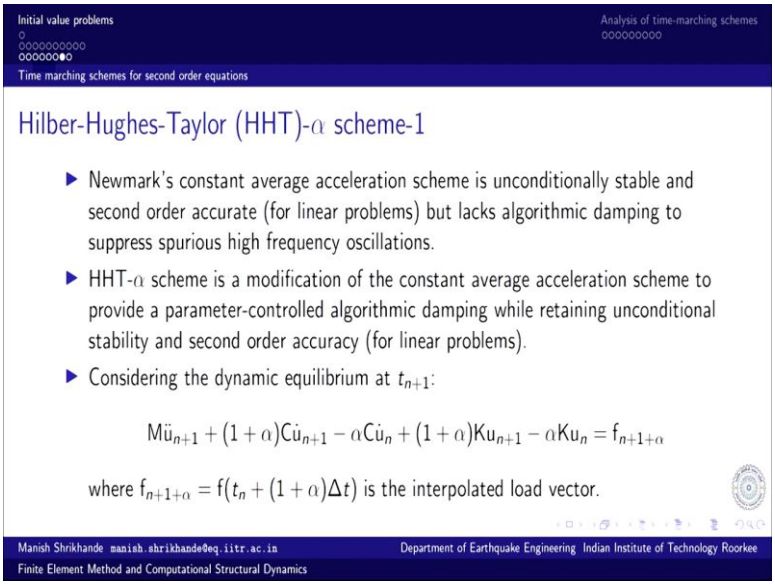

So, Newmark's constant average acceleration scheme is unconditionally stable as I said it has better numerical properties and it is also second order accurate for linear problems, but it lags algorithmic damping that is, in case of some higher modes if they participate some kind of high as frequency oscillations are there which seem to be which may be

completely numerical in nature and the these need to be suppressed I mean there is no basis for that.

So, some kind of higher mode participation is there and if those have to be controlled, then those can be controlled by using I mean we need to have that is a desirable property for any time marching scheme or numerical integration scheme to have some kind of damping parameter, which can be controlled or parametric damping effect should be present in the numerical integration schemes and Newmark constant average acceleration does not have any algorithmic damping. We will see what we mean by algorithmic damping. It is essentially that the method has some kind of lost mechanism in its own and structure. I mean in order to make the method stable, we need to impose certain constraints, which makes the solution to decay. Even if the true solution does not decay over a period of time, but the stability requirement would make the solutions decay and that is purely because of the numerical nature numerical structure of the scheme of integration. And that is called the algorithmic damping.

Sometimes it is good and it is desirable as long as it is controlled. So, it can be controlled using some parameter. So, those algorithmic damping is desirable and Newmark Beta constant average acceleration does not have any algorithmic damping.

So, to introduce this algorithmic damping in Newmark-Beta, so we now have a new technique which is more commonly used in all computation structural dynamics codes that is called HST alpha or after the names of it is developers Hilber-Hughes and Taylor. And that is essentially one more term addition of one more term modification of Newmark-Beta scheme by to incorporate this parameter controlled algorithmic damping while retaining the desirable property of unconditional stability of the system of I mean the solution scheme and this is known as I mean similarly it is again the construction alpha parameter is again similar to the over relaxation scheme.

So, based on the linear combination of damping terms at n plus 1th time instant and nth time instant and the displacement terms at n + 1th time instant and nth time instant and again this forcing function is interpolated load vector. So, this is equation of equilibrium is dynamic equation is changed to this form and for suitable choice of alpha, we can solve the system of equations.

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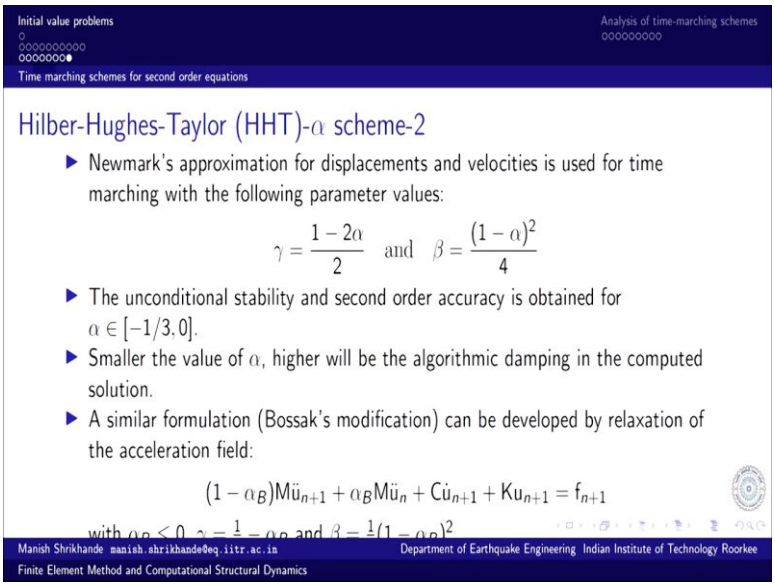

And Newmark's approximation for displacement and velocities is used for time marching along with the following parameter values. So, gamma parameter in Newmark scheme is related to alpha like this and beta parameter is related to alpha like this and for alpha is equal to 0, they reduce to constant average acceleration.

So, unconditional stability and second order accuracy is obtained when alpha parameter is between minus 1 by 3 to 0. Smaller the value of alpha, higher will be the algorithmic damping in the computed solution. And similar formulation that is Bossak's modification can also be developed by using relaxation of the acceleration field. In this case, we use relaxation of velocity and acceleration field.

Bossak's modification is based on relaxation of acceleration field, but the effect is more or less similar. So, with this we come to the end of numerical evaluation of time marching schemes or solution of equation of motion and in our next lecture, we will start discuss the analysis of these methods, these techniques stability and accuracy aspects.

Thank you.