Finite Element Method & Computational Structural Dynamics
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Lecture - 43
Solution of Linear Simultaneous Equations - I

Hello friends, we have completed our discussion on Finite Element Method formulation, the basic approximation of finite element method and we have gone through the process, the fairly automatic process of discretization of the domain using finite elements and construction of approximation functions within the element domain, then constructing equilibrium equations for individual elements and then, assembling those equilibrium equations into global system of equations.

Subsequently, the essential boundary conditions are imposed; boundary conditions prescribed on the primary variable and that leads us to the solution of simultaneous equation either for a static problem, finite element method formulation directly will lead to the simultaneous equation stage or if it is a time dependent problem, then the time marching schemes that we will discuss subsequently, which are used for integration of equation of motion in, that is ordinary differential equation and that also leads to simultaneous equation; solution of simultaneous equation at every time step.

So, it is a time marching solution of solution is advanced one step at a time. So, at every time step, we need to solve the system of algebraic system of equation simultaneous equations. So, we start our discussion. Let us explore this simultaneous equation, how we do it on a digital computer. Of course, this process is familiar to all of us.

This is something we all studied in our basic mathematics course. I mean probably middle school, when we first were introduced to the concept of algebra and elimination of variables and that is how we learnt our in the initial exposure to the problem of simultaneous equation, solution of simultaneous equations.

So, we have been doing that, but let us in order to explore, because the thrust of I mean one of the points that I mentioned in this course would be; we I mean computational methods are full of potential mine fields and we one has to be very careful to avoid those mine fields; otherwise, you never know when the solution will blow up in your face and you one is never sure whether the results that we are looking at, do they make sense; I mean is it correct or it is trash that we are looking at. So in order to understand the basic issues involved in the simultaneous equation, the solution of simultaneous equation using digital computers, let us look at a simple two variable problem.

So, whatever it has I mean the it encompasses, encapsulates the basic requirement of simultaneous equation. There are two variables and there are two equations to be solved and it is simple enough to be visualized. So, we know what is going on right and therein, lies several clues to what happens when something goes wrong or how to identify that something might go wrong and some careful approach or careful handling is required in the case. So we look at it through the prism of geometry. The problem of simultaneous equation, we look at it a view through the geometric prism.

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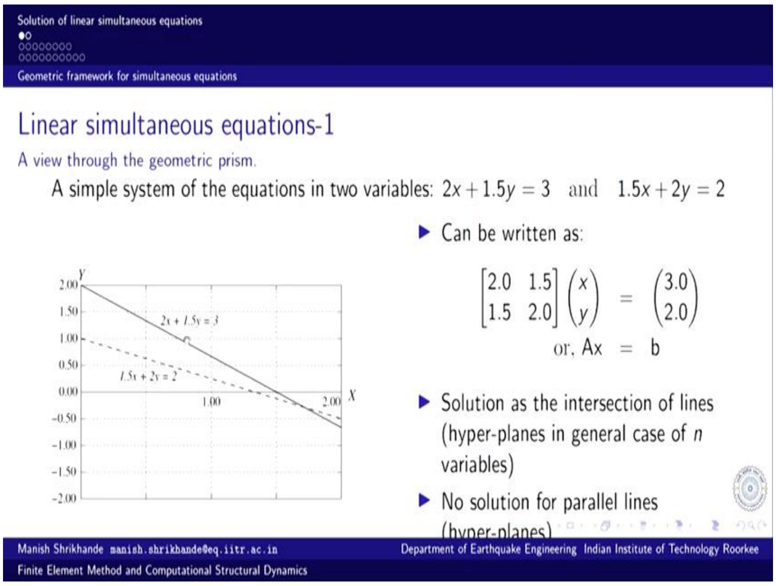

So, a simple system of two equations in two variables. So, 2 x + 1.5 y = 3 and 1.5 x+ 2 y = 2. I mean just some arbitrary random numbers that I could first came to my mind. So, these are the two equations. In linear equations, I mean x and y both appear in the first powers and two variables and these can be solved for unknowns x and y.

So, what do we mean by solution for x and y? What we all know from the geometry or from our high school or middle school physics, basic mathematics training that when we solve the system of equations, what we are looking at is intersection between these lines or planes. If it is two variables, then it is lines; if it is three variables, then those are the planes and for more than three variables, it is hyper plane. So, we are looking at intersection of those geometric entities, relevant geometric entities and the solution is the common point of the intersection.

So, the existence, the existence of solution is defined by whether the these geometric entities lines, planes or hyper planes whether they have a common intersection point or not and as you can see this per these two lines, I have plotted this on x y plane, x y graph and they do have an intersection point here. So, that is the solution of these two equations.

So, this is the graphical method of solution that we all have studied earlier. So, and the equation itself can be written as a we can separate and arrange it in a form that is suitable for digital computers, we can store the coefficients in array and the unknown variables can be represented as a vector.

So, 2, 1.5, 1.5, 2 and x y, these are the variables and right hand side is again another vector. So, conceptually, we can write this as a simultaneous equation as A x =b; A is the coefficient matrix, square coefficient matrix; x is the variable, a vector of variables and b is the right hand side vector. So, solution x can be theoretically computed as pre-multiplication by A inverse. So, if I pre multiplied by a inverse of A on both sides. So, that would be x is equal to A inverse b.

But inverse of a matrix is never computed. That is almost a rule in computational mechanics, whenever we see whenever we come across the expression A inverse b, what is implied is solution of simultaneous equation. It is never intended that one should compute the inverse of the matrix and pre multiply it.

Inverse of a matrix is a very expensive operation numerically and once, we have numerical too many numerical operations happening, it is an n cube operation. If n by n is the size of the matrix, then we also studied, we also we have also seen that every floating point operation, any floating point arithmetic is a potential source of round off errors.

So, if we are doing so many floating point operation in computation of inverse, then that has a potential of incurring huge round off error in addition to being expensive in its own right. So, there is a penalty even after spending so much of computational effort the result is not likely to be any accurate. So, it will actually be worse than what we can possibly compute with much less computational effort as we will see very soon. So, solution as the intersection of lines or hyper planes in the general case of n number of variables and if those are parallel, then of course, there is no intersection and there is no solution; there is no unique solution that exists.

So, for two variables problem it that would be represented as these two lines being parallel to each other. So, if these two lines are parallel, then obviously, they do not meet; they do not intersect and there will not be any solution. Now, we can also look at it through the prism of vector spaces because this geometric interpretation, this holds I mean this can work for small number of variables. I mean two variables and three variables at the most; beyond that, it is beyond visualization, we cannot visualize what is the geometry like beyond three-dimensions, beyond three variables.

So, vector space formulation is very general and the dimensions can increase to whatever length and the conclusions, the interpretations, they hold equally valid and they can be extended seamlessly. So, how does this look like? What does what do we mean by solution of simultaneous equation as looked through vector space prism?

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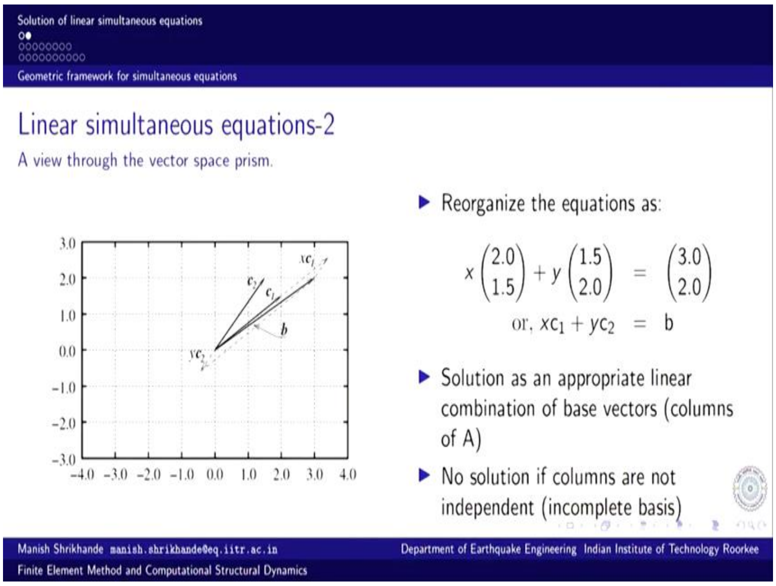

And that is we look at it as solution, right hand side vector b is if we go back to this problem this equation. So, what essentially this means is we can actually represent this as this is first column and this is second column of the coefficient matrix.

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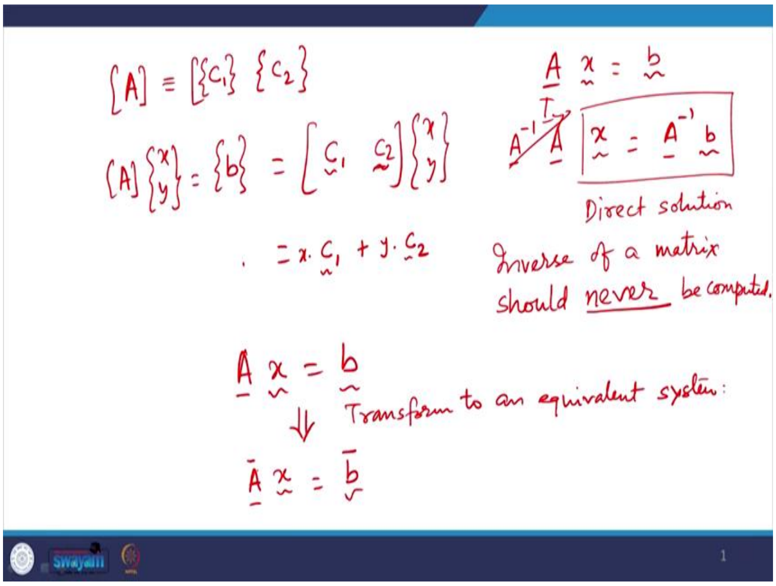

So, I can write this as x times first column plus y times second column is equal to b. Is not it? So, I can write this as A, I define A as c 1 and c 2.

So, second column. So, essentially, what we are looking at here is b, the right hand side vector b is given or we need to find a linear combination of the columns of vector, columns of the coefficient matrix A. So, columns of vector space, I mean they form the vector space and b should span the vector space defined by the columns of matrix b matrix A.

So, that is what we are looking at it. So, c 1 and c 2, these are the two vectors that I draw, the I can draw using these coefficients x and y plots. So, these are the vectors and this is the vector right hand side vector b.

So, solution is to be found within the space of as an algebraic summation of c 1 and c 2 such that linear combination of c 1 and c 2 should be equal to b and that is what we have represented here. So, c 1 is scaled by factor x, so that comes here and c 2 is scaled by factor y and that is projected here and this parallelogram of consider constituted by c x times c 1 and y times c 2, the resultant of this parallelogram, these two vectors is what b is; right hand side vector b is.

So, that is what the re-organization of the vector system of equations is, linear combination of the column space. So, c 1 is the first column, c 2 is second column. So, x times c 1 and plus y times c 2 is equal to the right hand side vector and we represent, we say that the solution desired solution that is what should be the value of coefficient x and what should be the value of coefficient y, such that this linear combination holds. So, solution is an appropriate linear combination of the base vectors and base vectors are defined by the columns of matrix A and this is fine.

So, for two-dimensional problem, we need two base vectors and those two vectors of the columns matrix A as long as they are linearly independent, they form sufficient basis for representing any two-dimensional vector and that is what is sort for. So, and there is no solution, if the columns are not independent.

So, then, then, there is no in incomplete basis. So, there is no I mean, we do not have a complete system of base vectors to represent the any arbitrary vector in that space. So, that is the basic idea of this vector space organization. This is the basic idea behind our discussion that we will follow because we can extend this analogy to any dimension; I mean any size of vectors.

So, if it is an n by n matrix with n number of variables, then what we are looking for is a right hand side vector should be given as a linear combination of n number of column vectors of the coefficient matrix. So, n number of column vectors, they should form the complete basis and that is only possible when they are linearly independent. So, they are all independent of each other. So, how do we solve the system of equations? What are the different ways of solving systems of equations?

There are two methods, two ways of solving. One of them is similar to what we already did what we already know from our high school mathematics or middle school mathematics that is we keep on eliminating variables one after the other and we get the solution, complete solution; I mean the entire process.

Once we complete the entire process, we have the solution to the system of equations as one go. So, those are called direct solvers. So, once there is a well-defined algorithm, there is a well-defined procedure which if followed will provide us the solution for unknown of the problem, unknown variables of the simultaneous equations and that is a direct solution.

On the other hand, we have iterative schemes, where we start with an approximate solution initial guess and then, we gradually try to improve on the solution and as the approximation, as the iterations proceed we keep on refining solutions gradually and so, that is a step I mean very incremental nudging towards the exact solution or as and we can nudge it, nudge the computed solution as closely to the true solution as required during based on the acceptable tolerance.

So, we talk about we start our discussion for simultaneous equation with the discussion on direct solvers and we will briefly discuss direct solvers and then, we will move on to iterative schemes. Because iterative schemes are very attractive, we will see why.

So, direct solvers I mean A x is equal to b, as I said simple the most direct way to solve that is to pre-multiply by A inverse. So, A x = b. So, this would be pre multiplied by A inverse and this would be of obviously, identity matrix. So, x is equal to; this is the solution and this is what we have as direct solution. But as I said, inverse of a matrix should never be computed. That is almost a rule in a computational algebra. What? Whenever you need multiplication by A inverse, there is always a more efficient and more reliable, more robust method, more robust technique available to do the job.

So, A inverse b that is only to be used in that is only used in mathematical literature or numerical techniques literature as a symbol; symbolism that this involves there is a simultaneous equation being solved here. That is all. It is never intended that inverse of the matrix should be computed and the multiplication followed by a matrix multiplication.

So, what is done in direct solver is of course, we can eliminate variables and try to find some of these I mean this is as I said inverse, if it is available. We it can be considered as a direct solver, I mean we can call it as a direct solver; but it is very expensive and erroneous operation.

So, what is done is we try to transform the system of equations; we have an original system of equations. So, this is an original system of equations A x = b. This is transformed into $\bar{A}x = \bar{b}$ equivalent system right. So, this is transform to an equivalent system. So, what is an equivalent system? It is a different algebraic system of equations which has the same solution. So, the solution that we will compute for $\bar{A}x = \bar{b}$ that solution is same as what shot in the original problem and that is our end objective is. We need to find what is the solution of A x = b.

So, what we try to do is we try to transform this problem into an equivalent system of equations, which has the same solution as the original; but with the caveat that this should be easier to solve in some way. That is the whole point of transformation; otherwise, if there is no benefit, then why would I bother about transforming.

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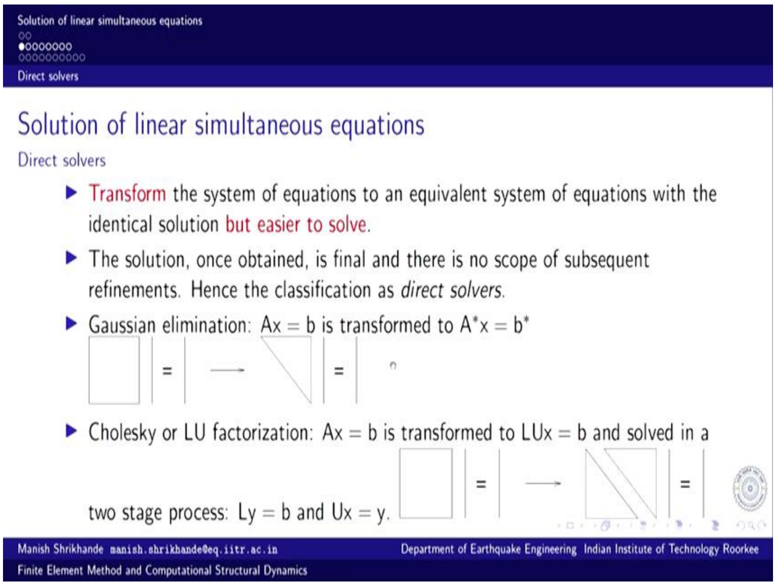

So, that is the essential keyword. So, transform the system of equations to an equivalent system of equations with the identical solution; but it is easier to solve. The solution once obtained is final and there is no scope of subsequent refinements and that is what is called as direct solver.

So, we have a complete set of process and at the end of that complete procedure, we have the solution as the procedure provides. There is no further I mean it is all I mean it is like take it or leave it. This is all that the algorithm can provide, there is no further scope for refinement. If it is good, it good; if it is bad, it is bad whatever.

So, what is done is Gaussian elimination, in this case Gaussian elimination A x = b is transformed into another system of equation let us say A^* x = b^* and what is done is essentially this square matrix is transformed into upper triangular matrix. If you recall, if you look at it very closely, then you will realize that this is essentially I mean successive elimination of variables from equations, below the first equation and that is why it keeps on getting reduced number of variables, keep on getting reduced.

At the end, we have just one equation of one variable unknown in this equation and which can be solved and obviously, when we perform this elimination, right hand side also gets changed up suitably and that is what is indicated in b^*. So, right hand side also gets modified as the coefficient matrix gets modified to facilitate this elimination of variable successive elimination of variables. So, eventually, this square matrix is transformed into this upper triangular matrix and that is followed by a backward substitution.

Now, we compute start computing solution from the nth variable onwards. So, x_n, so A_{nn} multiplied by x_n is equal to b_n and that provides us solution for x_n and then, we go to next previous value n minus 1th rho. So, there are two variables; x_n - 1 and x_n. x_n we have already computed. So, once we knowing x_n, again there is only one variable that is to be computed and we solve for that unknown variable. So, exactly what we do in the case of elimination of variables approach that we all know very well.

Another way is I mean this of course, is little tedious in the sense that if we need to solve for multiple right hand side vectors which is very often the case for multiple load cases, if analysis has to be performed for multiple load cases, then either this information has to be preserved; how this transformation has been carried out. Because that right hand side vector would have to be suitably modified for all cases or the other way approach is matrix A is factorized into a product of lower triangular and upper triangular matrix.

So, this square matrix is transformed as a product of lower triangular and upper triangular matrix and then, there is this desired solution is obtained as a suitable I mean two-step process.

So, essentially A_x = b is transformed as LU x = b, then we define U x as another variable y right. So, unknown as of now. So, lets substitute L U x as y and solve the system of equations L_y is equal to b. So, it is a triangular system and this can be solved by using forward substitution. So, y_1 computed first; then, knowing y_1, y_2 can be computed; knowing y_1 and y_2, y_3 can be computed and so on.

Then, knowing y, we can go back to this original substitution U x = y and then, compute the desired vector x from the system of equations. So, this is what the direct solvers, two basic direct solvers that we do employ in finite element analysis for we are fortunate in the case of solid mechanics or structural mechanics problems, the coefficient matrix is always positive definite.

So, the system of equations is always well-conditioned. Although, we will examine what are the conditions of numerical ill-conditioning, how to identify that and how to improve on that, we will discuss those issues during the process of solution.

Because this quality of computed solution that depends on how stable or how well-conditioned the system of equations are. When we say well-conditioned that implies in the sense of intersection, how sharp is the intersection between lines or planes or hyper planes. In case of vector space, how orthogonal, how verbs, how much independent different columns are in the vector space.

So, if the columns are very close to each other in the vector space, not very different from each other, then there might be kind of numerical instability creeping into the process and that will be reflected in what is known as condition number of the matrix. So, we will discuss all that in our next lecture.

Thank you.