Finite Element Method & Computational Structural Dynamics
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Lecture - 42
The Time Dimension and Dynamic Effects - II

Hello friends. So, we have seen how to model dynamic effects in finite element analysis. We assume accelerations to be interpolated in the same way as the displacement field is interpolated or approximated within the element, and then the domain integrals are evaluated. And we end up with matrix, so inertia system inertia matrix and that is referred to as consistent mass matrix.

So, this word consistent is nothing consistent about this evaluation system of evaluation, it only refers to that accelerations are interpolated consistently with that for the displacement. So, the interpolation model for acceleration is consistent with the interpolation model for displacement that is the only interpretation.

There is not nothing I mean as I said earlier nobody knows how the accelerations vary, but this simple construct of using the same interpolation model as displacement for approximation of acceleration agrees with the, I mean the results agree with the observations.

And moreover that is because of the fact that over no matter what the variation actual variation might be, over a small enough region any curve, any variation can be reasonably approximated by a lower degree polynomial, and that is how this simple approximation for displacement being extended to interpolate the acceleration actually works in practice. So, we have consistent mass matrix.

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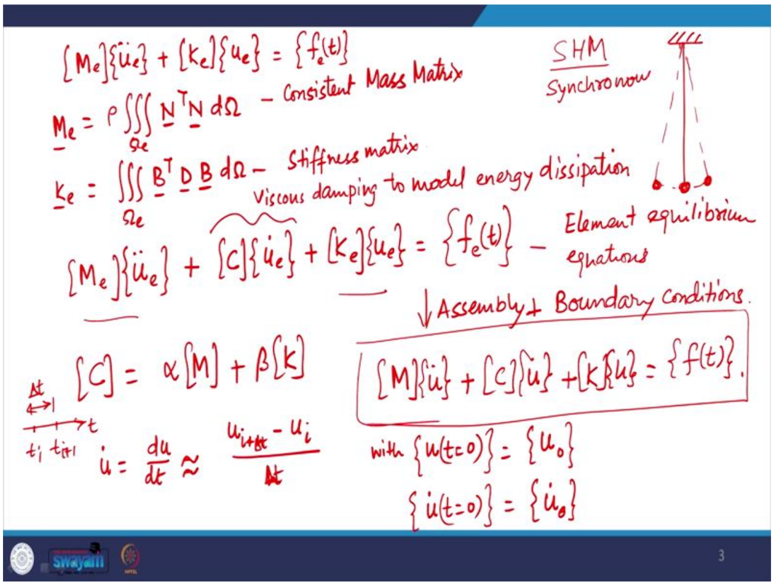

And we saw that the conservative system the equation of motion can be given as for the element level, it has mass times nodal accelerations plus stiffness times nodal displacements, and this is equal to the nodal equivalent time dependent forces. So, this is what we call as system inertia mass, and that can be evaluated using interpolation of the displacement N^T N integrated over the domain multiplied by the density.

And that leads to this consistent mass matrix and K is of course given by the domain integral involving strain displacement matrix.

So, B^T D, D B, this of course works very well for continuum elements, interpretation of B varies from B and D the strain displacement and continuum constitutive relation that varies from application to application for example the strain displacement matrix that we are referring to that refers to the curvature in case of beam problem, and constitutive element here D in that case refers to the flexural rigidity of the problem.

And similarly for plate bending problem, also we have similar kind of curvatures and flexural stiffness, flexural rigidity coming in picture coming into the analysis. So, with appropriate interpretation, this generic expression for element stiffness matrix that is B^T D B, where B is the strain field relation I mean relation matrix relating the relevant strain field strain or curvature fields to the nodal values of primary variables. And D is the matrix or the element which transforms which relates the strain field or curvature field to this appropriate stress or stress resultants.

And this element mass matrix is of course $N^T N$, so that is consistent mass matrix. And using these, we can have the basic constituents of any dynamic system. So, this simplest dynamic system would involve some mechanism to store potential energy which is represented by this stiffness matrix or the stiffness of the element, and some mechanism to store kinetic energy which is represented by these inertia terms.

And with these, the oscillation once the system is set in motion it will continue to oscillate infinitely it will never come to rest that is what the conservation I mean the transformation of energy will do. The potential energy will come keep on converting to kinetic energy; kinetic energy will keep on converting to potential energy just as in the case of motion of simple pendulum.

Simple pendulum oscillates from mean position to extreme position, and to the other extreme position. So, this happens by virtue of constant transfer of energy from potential energy to kinetic energy, from kinetic energy to potential energy. At this point extreme position, the system has maximum potential energy. And once it is and it is at rest. Once it is released from this position of rest, the it begins to move downward, and the potential energy is gradually converted into kinetic energy until it reaches mean position.

When the pendulum reaches this mean position, all energy has been converted into kinetic energy and by virtue of its momentum it moves forward. And again the energy conversion from kinetic energy to potential energy happens until the point when all kinetic energy has been converted to potential energy, and the motion stops because there is no further momentum the velocity is 0.

Then it begins downward motion. And again this cycle of this constant interchange of energy keeps on happening. And this motion, this motion will keep on happening indefinitely because there is no loss in this particular in this equation that we have there is no mechanism by which some part of energy or is I mean loss in energy during transformation is modeled.

So, if there is no loss during this transformation, then this keeps this oscillations will continue forever. And obviously that does not happen, and it remains any physical system because by the that is by the principles of thermodynamics it is impossible to have any physical system which will not involve any transformation without any loss of energy right. So, any physical system, energy transformation system has to be accompanied by some loss without that it is not physically realizable.

So, when we do that, so some loss mechanism has to be incorporated of course, there are the energy can be lost in several ways, but it so happens that most of the time the energy lost in engineering systems is very, very small in per cycle. For example, the simple pendulum experiment if you recall, it keeps on and doing infinite very, very long. I mean if we leave it for on it is own it will continue to oscillate for a very long time and only after very long time, it will the pendulum will come to rest on its own. So, during every cycle of oscillation the amount of energy that is lost is very small. So, while there can be several mechanism, for example, energy can be lost during friction, energy can be lost due to a viscosity or the viscous drag and so on.

So, there are different mechanisms, but we also need to look at what is convenient for us for analysis. It so happens that any energy loss mechanism other than velocity proportional damping viscous damping that is what we have during in a some form of viscous damping. Of course, it can also have different powers of velocity. So, we only restrict our attention to viscous damping which is proportional to the linear velocity term.

So, when we do that, then we have very interesting expression. So, this is mass multiplied by acceleration. So, then we can have some velocity proportional damping, and then displacement proportional elastic force restoring forces, and then the nodal equivalent of forces.

So, this term is of course, loss mechanism viscous damping to model energy dissipation. Only this form of energy dissipation gives us a linear differential equation. This is still a differential equation, second order derivatives in time, first order derivative in time velocity, and this is displacement.

So, this is still a differential equation, ordinary differential equation, but if we choose any other dissipation energy dissipation mechanism this equation will be a non-linear equation. This is the only variation, this is the only energy dissipation form which provides us which yields a linear equation.

And therefore, we continue with this form because as it is compare to these two terms inertia term and the elastic force term, restoring force term, this term the energy dissipation term is generally very small. So, it really does not is not very significant or very dominant term in the equation, but we need to model, we need to have some mechanism of energy dissipation. So, we include this in the form of a linear viscous damping term.

And then what should be the damping coefficient? Again we can adopt possibly the same formulation as that for the mass matrix or inertia matrix. We can assume velocities are also interpolated in the same way as the displacements. And some velocity proportional coefficient, viscous damping coefficient and we integrated over the domain, and we get we can get the damping matrix C.

So that would again if you look at it that would again be proportional to mass matrix because that will again involve N^T N integral over the entire domain, so that would be proportional to mass proportional damping as it is called so that can be directly derived from the mass matrix that we already have.

Other equation can be, there are Rayleigh damping models so called Rayleigh damping model. So, what is done is matrix C is considered to be proportional to mass and stiffness. So, α_M times β_K and alpha and beta coefficients are to be evaluated by some condition.

So, if user specified, if the user specified certain condition, then alpha and beta can be evaluated. And with that matrix C can be determined. If matrix C needs to be characterized in total, we often do not need matrix C to be mentioned to be defined explicitly. We will see why.

So, once we have this element level equilibrium, so total element level equilibrium including dissipation, this can be assembled, this can be taken to assembly, process of assembly and plus boundary conditions, and that gives us the final global system of equations.

So, essentially what we have done here is up to this point by using finite element modeling, we have transformed a partial differential equation which involve derivatives with respect to space coordinates as well as derivatives with respect to time. So, we have used finite element modeling within in the spatial domain, and converted that partial differential equation into an ordinary differential equation.

Now, if the dynamic terms were not there, then it would be simply quasi static problem. But if the dynamic terms are important, if the frequency of excitation is reasonably close to frequency of natural frequency of the structure or frequency of excitation is very high compare to natural frequency of the structure, then the dynamic effects are important and the complete system needs to be modeled.

So, once we do that, then we have this ordinary differential equation with constant coefficients with certain initial conditions. Now, boundary conditions have been incorporated in the problem. And there would be some initial conditions with u at time t = 0, there would be some vector initial displacements. And similarly $\dot{u}$ at time t is equal to 0 that would be initial displacements and initial velocity. Because it is a second order differential equation we will need two initial conditions to determine the constants of integration. And those are determined those are given by the specified initial displacements and initial velocity.

And this system of equations can now be solved using suitable techniques for integrating this ordinary differential equation. Now, we can solve this problem using finite element again I mean because it is a time is also a dimension. So, we could have in principle, we could have possibly modeled entire system using four dimensional finite elements, three spatial dimensions and one time dimension.

But as you can appreciate the problem size increases exponentially from one dimension to two dimension, there is a steep rise in the problem size, number of variables. From two dimension to three dimension, it is almost with respect to N cube. And from three dimension to four dimensions including time also in the same finite element formulation, then the number of variables become huge. And practically I mean almost impractical problem to solve.

So, what is done is we decouple these systems. And we first develop I mean it is a variable separable form. We distinguish this space variation with respect to space coordinates separate from variation with respect to time coordinate. And we model finite element use finite element to model the spatial variation.

And for the and then we end up with this ordinary differential equation in time. And now this ordinary differential equation in time can also be used solved again using one-dimensional finite element in time dimension. Because there is only one dimension left now in time so that can be those kind of finite element formulations can also be developed.

But it so happens that one-dimensional finite element formulation in time provides results which are in many cases identical with those that we arrive by approximation of these time derivatives by finite differences. Finite differences, if you look at it, those are based on some simple application of Taylor series.

So, at two different instants of time, so or it could be t and delta t. So, i, so this is the time dimension. So, this is t_i, and this is $t_i + 1$ distance between them spaced delta t apart. So, what it means is $t_i + 1$, so the value two different values are two adjacent time instants divided by the time interval between them, so that finite difference is an approximation to the derivative of the function.

So, this actually comes from after truncating the Taylor series expansion at expanded around time t is equal to t_i. Now, it can be done I mean this is of course, forward difference, we can have a backward difference, we can have central difference and so on. So, first order derivatives can be approximated, second order derivatives can be approximated. And then once we do that, then we find that these are derivatives would be replaced by displacements.

So, the acceleration terms would be replaced by displacement components and delta t terms the time interval. So, eventually this differential equation will again be converted into an algebraic simultaneous equation which can be solved. So, we will come to that this is what we call as time marching scheme. We go from one time step to another time step and predicting solution starting from initial condition that is at time t is equal to 0, we keep on predicting motion at next time steps, so that is a what we call as time marching schemes.

And we will discuss those numerical integration schemes in for the time dimension in separate lecture. But point here is, there is another thing that we need to analyze here. We have been referring to the importance of frequency or when it is important to model dynamic effects, and this concept of natural frequency has been recurring very often repeatedly.

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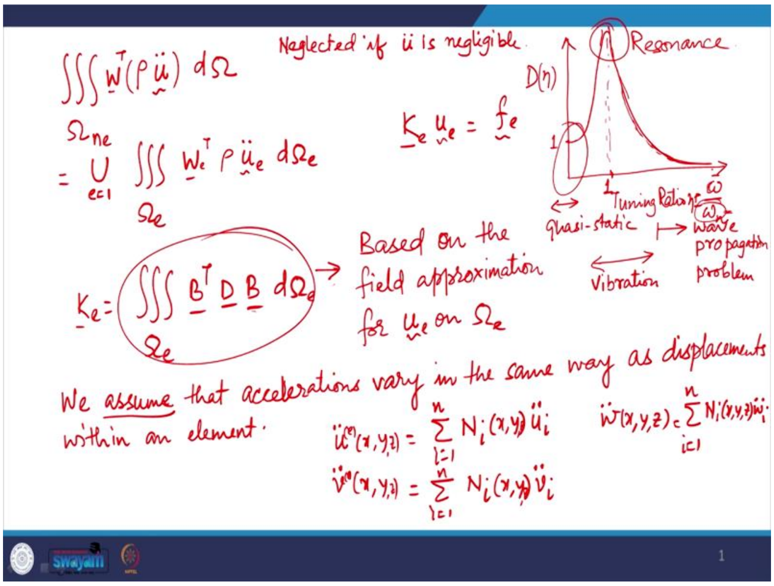

So, all that if you recall the basic formulation that we had, so $\overline{\omega}$ over ω_n. So, this is omega n that is the natural frequency of this structure; omega bar is the excitation frequency. So, how do I find or what is this what do we mean by natural frequency of structure? And how can we find it, what is the, what is the way to determine it?

One way is of course simple pendulum if you recall that experiment in physics, simple experiment in physics, we actually compute the I mean measure the time taken for complete oscillation, and that becomes the natural period. And inverse of that is the natural frequency that is how we calculate experimentally.

But that kind of experiment cannot be performed for every kind of structural system. So, what is it or under what conditions do we have those natural conditions or what we call as natural frequency. So, how do we determine natural frequency? One thing that we observe from this simple pendulum experiment is that this motion of the pendulum is a very special kind of motion.

It is called of course Simple Harmonic Motion, SHM, but it is not just this motion of this bob. Every single point in this simple pendulum system executes simple harmonic motion of the same frequency, because every point on this thread as well as bob reaches the extreme position simultaneously it will cross the mean position at the same time, and it will reach the every single point will reach the other extreme position at the same time.

So, this is what we have it is not just SHM, it is synchronous SHM. Every single point executes simple harmonic motion of the same frequency in phase right, synchronous simple harmonic motion. So, we try to ask ourselves this question, is it possible for the system to execute synchronous simple harmonic motion? And for that, we ignore the damping term just to make it simpler as it is. As I said damping is a very, very negligible very, very small quantity, so that can be neglected.

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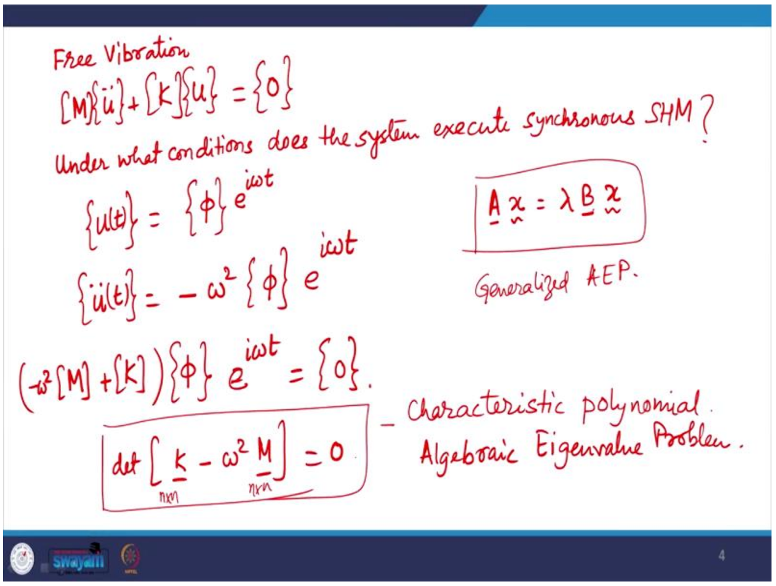

So, we look at what we call as free vibration problem because in the simple pendulum oscillation, there is no external force acting on it. It is just initial disturbance. And after that, the pendulum oscillates, so that is a free vibration. And during the free vibration, it is like natural instincts, I mean, if you are left free, then your natural tendencies come out.

And it is the same thing in the vibration. So, if the structure is allowed to vibrate freely on its own accord, it reveals its character. And that character is what we refer to as natural frequencies and how it vibrates. So, free vibration, so free vibration problem, we define it as M. So, there is no excitation force. So, there would be initial disturbance and that is all. After that the system will be allowed to vibrate on its own.

So, we examine under what conditions does the system execute synchronous simple harmonic motion. So, what is synchronous simple harmonic motion? If u (t) that is the displacement nodal values of displacement, so I define this as so let us say some amplitudes because in the simple pendulum experiment also if you see every point execute simple harmonic motion, but the amplitude is different.

So, this ϕ is the vector of amplitude of different points different nodes in this structure may execute simple harmonic motion with different amplitudes, but their time variation would be of the same frequency ω. So, e^{iw_t}, so that is a harmonic motion. So, it is an harmonic of omega.

So, under what condition will this admit? So, it can be either sin ω_t or cos $\omega_t$whatever. Time t is variable from 0 to infinity, it can be anything.

So, this term obviously cannot vanish. ϕ is an arbitrary vector, I mean if it ϕ is a absolutely if ϕ is null vector, then again it represents trivial solution that is no motion is happening nothing is moving at all system is at rest, so that is not of interest to us. We are looking for oscillatory motion response, so ϕ cannot be 0. So, the only way this equation can hold is when this becomes singular, coefficient matrix becomes singular, and that is when determinant of K - ω^2M is 0.

And this is what we call as characteristic polynomial because characteristic polynomial in ω^2, $\omega^2$is a variable. So, this will be I mean if this is n/ n if there are total number of n number of degrees of freedom or the size is let us say matrix K is n by n, M is n by n, then this is going to be nth degree polynomial in ω^2 this determinant is equal to 0. So, this is going to be nth degree polynomial in ω^2. And the roots of this polynomial are referred to as the eigen values.

Eigen is the German word for character. So, those are the eigen values, and the positive square root of those eigen values that will be the natural frequencies of the system. So, the idea is the solution, the answer to this question is under what conditions does the system execute synchronous simple harmonic motion?

The answer is there are a few discrete frequencies which are roots of the characteristic polynomial, positive square roots of the roots of characteristic polynomial at which such synchronous simple harmonic motion is possible. And for each frequency, there would be corresponding set of amplitudes. Now, that would be proportional.

So, if ϕ is an amplitude vector, corresponding amplitude vector of simple harmonic motion, any constant multiple of phi is also an admissible amplitude vector because they all move proportionately. So, this is referred to as solution of this is referred to as eigen value problem or the algebraic eigen value problem. And this has tremendous applications in the different fields of science and engineering, solution of algebraic eigenvalue problem.

This is if I write in this form general form this is equal to A x = λB x. So, here lambda is referring to ω^2, and x is referring to this vector ϕ right. So, this is what we call as generalized algebraic eigenvalue problem. The standard form is just A x is equal to λ x, when matrix B is identity matrix.

So, in this, in the structural dynamics problem, we have stiffness matrix as one matrix and mass matrix as another matrix. So, together they form a generalized eigen value problem. And positive square root of the roots of these positive square roots of the eigen values are the natural frequencies they correspond to the natural frequencies of the system at which synchronous simple harmonic motion is possible.

And those are also the frequencies at which resonance can happen. So, the excitation frequencies that is a very important criteria for dynamic design for dynamic effects. Try to design a system such that natural frequencies are as far as possible detuned or they are far removed from the excitation frequencies. If we can do that that will lead to considerable savings in engineering design.

So, we stop here and we will discuss more on how to solve this algebraic eigen value problem. The problem itself is a very vast and very vast problem, and different types of eigen value problems and there are different algorithms for doing that. And we will try to understand it more how to solve this eigen value problem numerically and in an efficient manner more efficient methods of solving algebraic eigen value problem.

One thing is with any numerical technique the definition of a concept is almost never a good algorithm to compute. So, this eigen value computation is never done by trying to find the roots of this characteristic polynomial. This is only for the purpose of definition. The computation is entirely different ball game. We will deal with it in our next, I mean after a couple of lectures.

Thank you.