Finite Element Method & Computational Structural Dynamics
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Lecture - 41
The Time Dimension and Dynamic Effects - I

Hello friends. So, we have seen, using finite element method we can convert partial differential equation into simultaneous algebraic equations, if the loading conditions are not very rapidly changing with time. Nothing is constant with time everything has to start at some point and all the loading loads will be removed at some point in time. So, nothing is purely static, but the time scales can be very different.

So, if the time scale of fluctuation of loading condition is very large, I mean it takes a very slowly varying loads, then the dynamic effects are negligible and we can ignore the dynamic term. And that is what we have been doing all this along. We had been ignoring the domain integral involving acceleration terms.

In certain cases, in many important cases for some load conditions for example, wind excited loading, or earthquake excitation, or even machine induced vibrations, or jet exhaust and the loading out of that, and then, the bluff body eddies flow past bluff bodies and the eddies forming from that. So, those are all dynamic loading conditions which can have a very important influence on the response of the structure. And if we ignore, if we do not account for the time varying nature the resulting calculations might be erroneous.

So, first thing that is to appreciate is under what conditions a problem may be treated as a static problem and under what conditions the problem may be dynamic in nature and it has to be, I mean time dimension has to be accounted for. So, just to recollect what we are referring to is an integral of this type.

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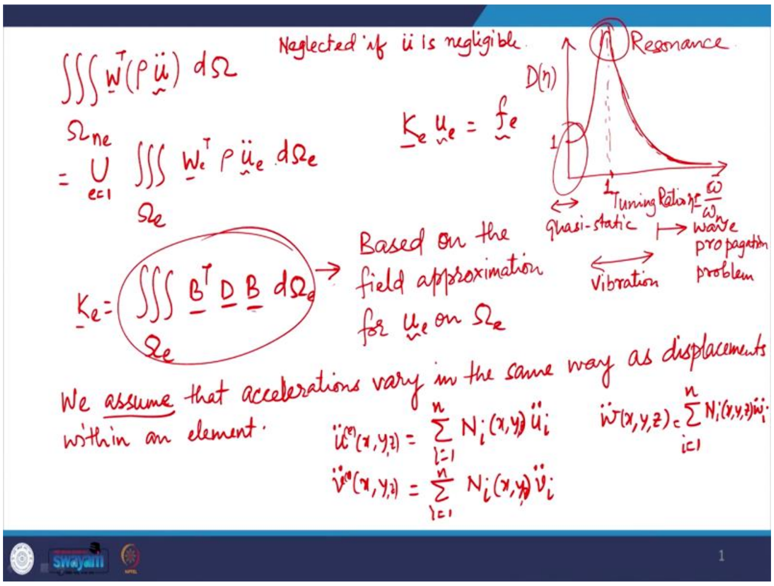

So, these are the domain integrals. So, W is the weight function, matrix of weight function. So, $\ddot{u}$ acceleration along x direction, $\ddot{v}$ acceleration along y direction, and $\ddot{w}$ acceleration along the z direction. And this W is the, capital W is the shape function matrix of all the, I mean weighting function diagonal matrix of weighting function and that is what is used for evaluation of the weighted residual statement.

So, this is a domain integral and this term is neglected, if $\ddot{u}$ is negligible. So, that is the accelerations are very small, very small as compared to the other terms in the equation. So, under those conditions, it is acceptable to model the problem as a quasi-static problem, and we may ignore this dynamic term that is important contribution and that, important element of the governing differential equation of motion.

So, under what conditions is this term negligible? So, let us look at. On the x axis, we have this ratio that is called tuning ratio, that is the ratio of ω, that is $\bar{\omega}$ that is excitation frequency. Let us say a harmonic forcing function of frequency $\bar{\omega}$ is applied to any system whose natural frequency is ω_n. So, ω_n is the natural frequency of the structure or any system on which is excited by the harmonic excitation of frequency $\bar{\omega}$.

So, if $\bar{\omega}$ is very small in comparison with ω_n, then this tuning ratio will be close to origin. So, in that case, the problem is essentially close to static. So, this is what we call as dynamic magnification factor or the dynamic amplification factor. So, that is the that actually measures how much is the response different from static response; if the same amp force of a given amplitude was applied statically then what difference does the dynamic application make.

So, this tuning ratio of 1, that is when harmonic frequency excitation is tuned with the natural frequency of the excitation of the system. So, this starts with 1, at a dynamic magnification factor starts with 1, at a tuning ratio of 0. So, 0 exciting frequency, so that means, the static purely static loading condition the loads are not cyclic. So, it is a static load.

So, typically, it is something like this. The response dynamic amplification is vary, something like this. So, it is an exponential decay. So, at the tuning ratio of 1, it is close to, it is very high amplification. So, that is what we call as the condition of resonance. All of us are familiar with this phenomenon of resonance.

This response is close to static behaviour. So, this range is known as quasi-static. And that is when the dynamic effects can be ignored and problem may be analyzed more or less as a static problem without bothering about dynamic effects at all safely. So, this is the very low frequencies. When the excitation frequencies are very small in comparison with the natural frequency, then this problem can be dealt with as a static problem as we have been doing so far.

Beyond that it is a dynamic problem. And there are different categorization for some range when the excitation frequency is of the same order of magnitude as the natural frequency, that is called vibration problem. And beyond that, if the excitation frequency is very large compared to the natural frequency of the system, that becomes, it is a still dynamic problem, but it becomes a wave propagation problem.

So, equation of motion still remains the same. It involves second derivatives in time. And, but it is the interplay between the frequency of the excitation or the time scale of the excitation and the natural frequency or time period of the structural system that is withstanding that excitation.

And depending on this interplay the problem may be modeled as a quasi-static problem or it may be modeled as a it may have to be modeled as a vibration problem or it may have to be modeled as a wave propagation problem depending on what is the range of frequencies that we are dealing with. And appropriately the solution techniques and the interpretation of results also varies.

So, quasi-static that is what we have been doing so far. For the vibration problem for dynamic problem for vibration as well as wave propagation problem this term cannot be ignored. The vibration, the acceleration terms are no longer negligible and those need to be accounted for in the equilibrium equation. I mean that Newton's second law of motion comes into picture which says that all the forces acting on the body, on any body.

So, net resultant of all forces acting on the body is equal to the rate of change of momentum of the body. So, if the rate of change of momentum if the mass is invariant. For example, if we are we assume the mass of the system does not change with time, then all that rate of change of momentum refers to is just the acceleration, what is the it is proportional to acceleration.

If the mass changes, as it happens in the case of aeronautical system where the fuel is being burnt at a very high rate, and that is what produces thrust for getting the escape velocity for the rockets. So, then it in those condition the mass is obviously not invariant and this rate of change of momentum includes two terms.

One is mass multiplied by acceleration and second one is rate of change of mass multiplied by the velocity. So, those two terms contribute to the rate of change of momentum term in the formulation of equation of motion that follows Newton's second law. So, in our formulation, finite element formulation we have been talking about I mean this in domain integral, they have to be replaced by integrals over individual elements, right.

So, domain integral over individual element. And then it is all assembled together. So, element number 1 to element number total number of elements in the domain, so this total integral of over the total domain, problem domain is represented as integral, this integral evaluated at individual elements separately and then assembling all the results together.

So, the same way as we do it for stiffness matrix, this is what I mean the other domain integral involving strains and derivatives for primary variable spatial derivatives, this term involves the time derivative of the primary variable. And for the special case of that is what we will be dealing with, mass invariant system. So, it refers to acceleration variation.

So, in order to evaluate this within the domain, what we did in the case of other integral that was stiffness matrix was defined as strain displacement matrix transpose, constitutive relation matrix multiplied by strain displacement matrix. And this entire product is evaluated over the integrated over the element domain.

Now, this can be done either using isoparametric formulation, regular element, whichever formulation we use. But essentially this is the integral that needs to be evaluated and this is what is the element stiffness matrix. Now, this the term in the equation is of course, with respect to there is a the equation that we eventually get is stiffness matrix of the element, this matrix, multiplied by nodal variables of the primary variable, nodal values of the primary variable. And this is equal to the equivalent nodal forces that are applied into the on the structure or the element. So, this includes, I mean this single vector includes contribution coming from body forces as well as from the surface traction whatever those are if element boundary coincides with the problem boundary, domain boundary.

So, all these I mean this strain displacement matrix is of course, based on interpolation of displacement. How we interpolate displacements? So, this is based on the field approximation for u_e / ω_e. So, for the particular element ω_e, element domain sub domain ω_e that is a particular finite element, we approximate the variation of primary variables that is deformations, displacement along x, displacement along y, displacement along z.

And we develop suitable polynomial interpolation depending on what it the required degree of continuity in the weak form of weighted residual statement. And from that approximation, we compute the derivative operate the differential operator on that to evaluate the strain field, and from strain field we get the stresses and this is how we get the strain energy term or the way we can substitute this displacement approximation in the weak form and we get the same variation if we use Galerkin approach for choosing the weighting function. So, we get arrive at this statement.

So, now, this integral is possible to evaluate because we have certain variation available for displacement over the element. Now, how do we evaluate this integral involving acceleration? We have variation defined for displacement, but we do not know how accelerations vary over the element. Nobody knows how the accelerations vary over the element, over individual finite element.

So, in case of complete absence of any clue, what to do? We do what is the easiest thing to do, that is we assume accelerations the acceleration field to be interpolated in the same way as the displacement field. So, we assume that accelerations vary in the same way as displacements within an element.

So, these are the displacements which are interpolated in terms of nodal values of these primary variables at the nodes of the element. And these are interpolated. So, this displacement field along x, so displacement along x the field variation within the element domain is given as interpolation between the values of displacement u along on the nodes of the element whatever the element geometry be and similarly, for displacement component along y and displacement component along z.

So, we extend the same variation we assume that the accelerations also vary in the same form same way as the displacement. So, all that I do is I assume accelerations to vary in the same way as the displacement. So, the nodal values of acceleration, they are interpolated in the same way as the displacement to derive the acceleration field.

So, once I have acceleration field defined over, the element this is a definite integral which can be, now we have variation defined for the acceleration field and this can be evaluated in a standard procedure without any difficulty. W will be given by the interpolation matrix.

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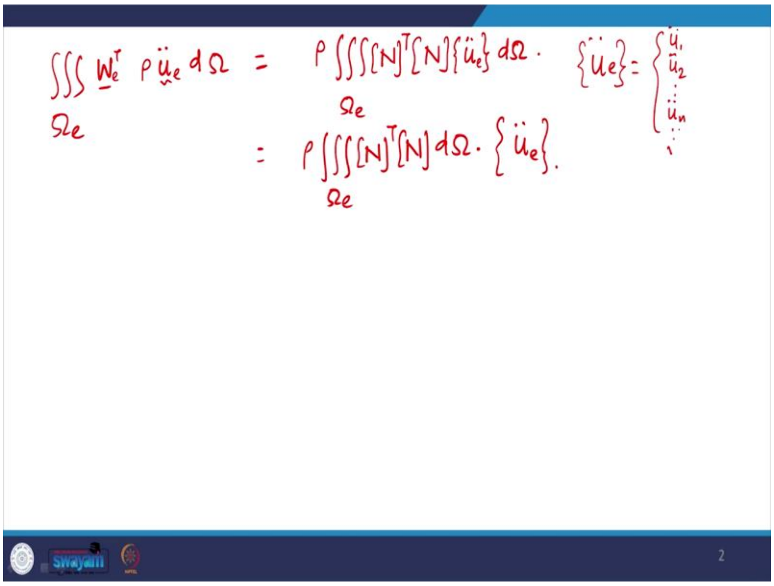

So, in that case, this becomes this entire thing now can be written as. Since this nodal values of the acceleration, this is of course, not dependent on the coordinates within the element. So, these are nodal values defined at different nodes. So, these can be taken out of the integral.

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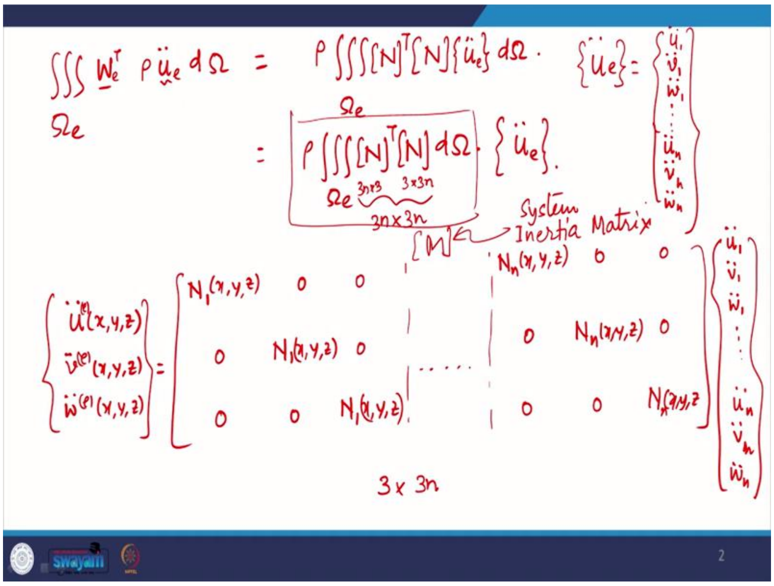

And this matrix will be this N is of course, 3 by n, n is the number of nodes. So, there are 3 displacement components. So, there are 3 accelerations, 3 acceleration components to be interpolated at n number of nodes. So, this N will be a matrix of 3 multiplied by 3 n actually, because there are 3 variables at each node. So, essentially this will be like u acceleration as a function of x, y, z will be given by N_1 (x, y, z).

So, this corresponds to node 1, this corresponds to node n. So, in all there will be 3 n number of columns. So, if n is the number of nodes, then there will be 3 n number of columns, and since there are 3 components of acceleration, so this is going to be 3 times 3 n matrix.

And this is what shape function matrix or interpolation function matrix is. And this n transpose, obviously, is then going to be 3 n multiplied by 3, so together this becomes 3 n multiplied by 3 n. And that is this entire integral is referred to as system mass matrix or inertia matrix.

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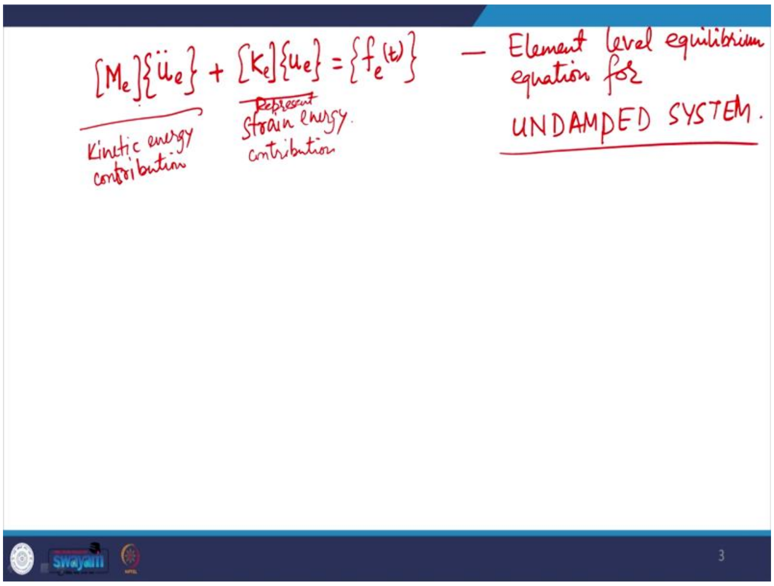

So, when we add this, then the governing differential equation, the loading function equivalent nodal loads will also be in general function of time. So, this is the equivalent, I mean the element level equilibrium equation for undamped system.

What do we mean by undamped system? We are not modeling any energy loss here, any dissipation in the system. So, this is an ideal system which is fully conservative. So, this term represents strain energy. And this term would represent, this represents contribution strain energy contribution, and this represents kinetic energy contribution.

The stiffness term, the restoring force or the elastic term this represents, the strain energy or the strain energy potential energy storage mechanism and this second term the dynamic term that represents the kinetic energy storage mechanism.

And energy keeps on transforming between kinetic energy to potential energy and vice versa. And this happens in this particular system, this happens with 100 percent efficiency because there is no dissipative system that has been modeled in the system in this equation. So, there is no loss of energy whatsoever.

And as we all know there is no physical system, which is 100 percent efficient. Every energy transformation must be associated with some kind of loss mechanism. Any energy transformation cannot happen in a physical system cannot happen with 100 percent efficiency, and there has to be a loss mechanism for any physically realizable system. So, we will see how that happens. And that is what will be referred to as damped system or we introduce damping in the system. We will discuss that in our next lecture.

Thank you.