Finite Element Method and Computational Structural Dynamics
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Lecture - 04
Polynomial Interpolation and Numerical Quadrature-I

Hello friends. So, before we start the theoretical development of a finite element method, as we already discussed finite element method is an approximate solution method for solution of partial differential equations. Now, the basic idea behind finite element method is approximation of a function that will satisfy the governing partial differential equation of the problem that we wish to solve subject to appropriate boundary conditions and all.

Now, when we say approximate solution then; obviously, there is going to be an error. We are have been issued stating our position very clearly that we are not looking for an exact solution, and we are as long as the approximation is satisfactory we are good enough to live with a little bit of error as long as our approximate solution is adequate for our engineering purposes.

Now, the basic idea behind construction of finite element solution is the problem of I mean the approximation of the unknown function which is the solution and we do not know the exact solution of the governing differential equation. So we are trying to approximate that true solution by an approximate function.

Now, the function approximation is the basic idea behind the formulation of finite element method. And the way we go about it is by using a very familiar concept of interpolation. All of us handle the problem of interpolation routinely. The problem of interpolation is essentially, we are given a table of values, some sample independent variable x let us say ranging from x_1 up to x_n or x_n-1. And the value of the function $y = f(x)$ and sampled at these values at which the independent variable is defined, so $f(x_1)$, $f(x_2)$ and so on.

So, these are the table of values that are usually prescribed. And the common problem that we all dealt with is, what happens if we want to have a value somewhere let us say that falls between x_1 and x_2.

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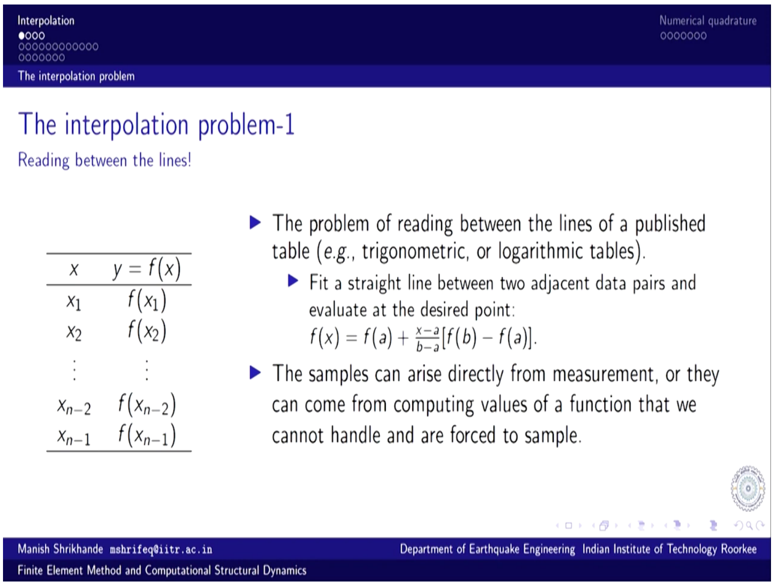

So, Function value y at x_1 is known, x_2 is known, but what if I want to find the value of y for some value of x that lies between x_1 and x_2. So, that is the problem of interpolation. So, that is the problem of reading between the lines of a published table. Now, that published table could be trigonometric table, could be logarithmic table, or could be any table any any set of values of independent variable and a dependent variable.

So, the standard process that we do, almost it comes without even a second thought that we fit a straight line if we know the values at two points, we can fit a unique straight line and find out the value. And then once we fit a straight line through these two points, then we can find out the function y at any point any value of x between x_1 and x_2.

So, this is basically the equation of straight line $y = (y_2 - y_1)(x - x_1) / (x_2 - x_1)$. So, that is the basic idea, fitting the straight line between two given points and sample the value at any two points in between, so that is what we naturally do. And the samples can arise from directly from measurement. So we can have tabular data of measurements or they can come by sampling the values and that is what this particular line shows. I would recommend that you keep that at the back of your mind that these values samples can come from computing values of a function that we cannot handle. And we are forced to sample that has a very important implication.

So, essentially what we are trying to do here is we are trying to I mean the interpolation per say is a sort of information regeneration rather than information retrieval. If I can find a function convenient functional form, then it encores information in a very concise form rather than trying to a publish a vast a set of tabulated values. And a reading a tabulated set of values is definitely not so appealing then evaluating a function. So, it is convenient for us to use function evaluation, then looking up a set of tabulated data points.

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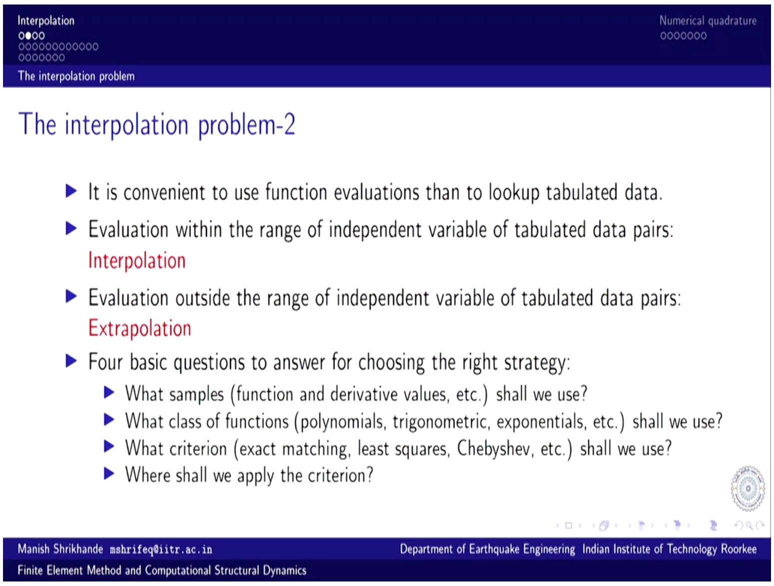

And evaluation within the range whatever data has been provided, If we evaluate the function in between the range whatever the sampling is done, then the problem is called interpolation. And if we try to evaluate the function outside the range of tabulated pair of data, then that process is called extrapolation. The function is still the same it is the fact that whether we are evaluating the function within the range of data that has been provided, then it is called the interpolation. And if we try to evaluate the function outside beyond the range of specified data, then it is called extrapolation. And obviously extrapolation is little bit of a problematic because there are no boundary conditions, there are no constraints. Constraints are already on the one side, and therefore, it is less reliable in comparison to the interpolation process.

So, four basic questions to answer to develop the right strategy for a function approximation. I mean essentially the interpolation, when we say interpolation it is we are trying to approximate the function that has been provided as a set of a tabulated data points. So, the four questions that we need to answer. What samples shall we use, I mean when we say samples are we going to rely only on the function values or we will also make use of derivative of the function right? So, not just the function values or whether we will have the derivative, and when we say derivative, we can go on. Whether it is all going to be a first derivative or whether it is going to be up to second derivative and so on. So, what degree of smoothness that function will have the approximation that we develop? So, that needs to be decided a priori and that will affect the way we develop approximation or the way we develop interpolation.

And what class of functions? What class of functions should be used? So, functions are functions I mean any function is fine we can use polynomials, we can use trigonometric functions, we can use exponentials. For example, all of us are familiar with a Fourier series expansion. Fourier series expansion is another way of function approximation, but in that the basic approximation function approximation is constructed in terms of trigonometric functions and exponentials. Well, we may consider trigonometric functions as complex exponentials, and there are other special cases where exponentials may also come in handy particularly in geophysics problems.

Then what criteria should be used? I mean when we say criteria, we talk about what is the degree of approximation, what constitutes a good approximation or acceptable approximation, or when we say that this is good enough for our purpose. So, do we insist on exact matching or are we looking at least squares fit or Chebyshev approximation or whatever? So, the criteria need to be defined a priori. And then based on those criteria we evaluate the quality of approximation.

And then where shall we apply? The criteria that is another problem that the criteria that we adopt may not be equally good at all places, all places over which the approximation has been developed. So, it has to be specified that how often that criteria is to be applied, and where it has to be applied that will improve or that will have a implication on the goodness of approximation that we construct.

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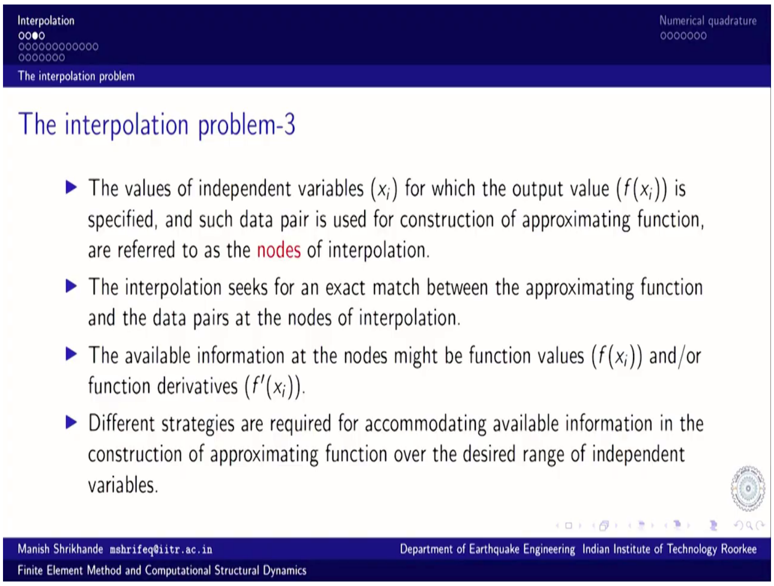

So, the values of independent variables for which the output variable is specified and such data pair is used for construction of approximation function, are referred to as the nodes of interpolation. So, the values of the independent variable, so x_1, x_2, x_3 at which the data value the function to be approximated is sampled or specified. So, those fixed points are known as the nodes of interpolation. Now, that has a very important usage, I mean this term node we will keep on repeating throughout this course.

So, the interpolation process seeks an exact match between the approximating function and the data pairs at the nodes of interpolation. Remember in the previous slide I mentioned about what is the criteria of approximation and where that criteria is to be imposed. So, when we talk of interpolation, we are looking at exact match between the approximating function. So, I construct an approximation function. So, I specify that at the nodes of interpolation, the approximation function has to agree exactly with the data that has been provided or the specified data. So, by definition interpolation requires that condition to be satisfied.

So, the available information at the nodes might be function values and or function derivative. So, I can have I mean different types of interpolation. And depending on whether I am looking at only the function values agreement only between function values and or if I am also looking at in addition to the agreement with specified function values whether I am also looking at agreement with the function derivatives.

So, different strategies are required for accommodating available information, whatever information is available and all that information is used for construction of an approximation function over the desired range of dependent independent variables. So, whatever is the range of independent variables, we construct an approximation based on whatever is the available information. We make use of, obviously, the more information we have; we have options to construct a better-quality approximation. And we will see how this pans out.

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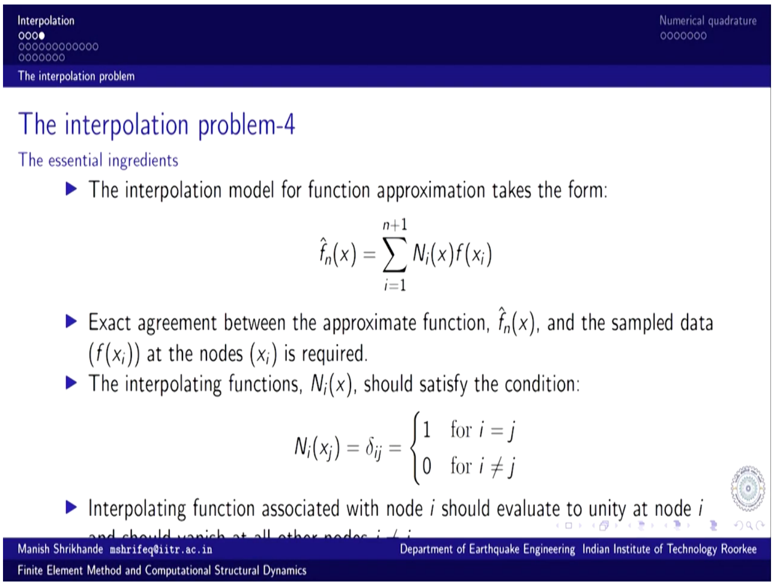

So, the interpolation model for any function approximation, it takes this form that so this term subscript n is actually n number of polynomial degrees. So, if there are n +1 points, then it is known that we can pass n^{th} degree of polynomial through n + 1 number of data points distinct data points. So, this summation as you can see extends from i = 1 to n + 1, so there is n plus 1 number of terms here. So, the function value f (x_i), so x_i ranges from x_1 to x_{n+1}, so there are n + 1 number of points. And for each of these points that is node there is a corresponding function that is designated by N_i. So, for each of these i^{th} nodes there is n function which we call as interpolation function N_i which is a function of independent variable.

And that when multiplied with the function value at that point, and then these terms are summed up, so that gives us what we call as interpolation model. Now, N_i can be any basis function, it can be polynomial, it can be trigonometric, it can be exponential whatever as long as all these functions are independent functions.

So, exact agreement between the approximate function and the sampled data $f(x_i)$ at the nodes x_i is required, so that is a requirement of interpolation. When I say that I am looking at interpolation model, then it means that the approximation that I am constructing has to agree exactly with the specified value at the nodes. The nodes are specified and the function value is also specified, and the approximate function has to exactly match with that tabulated value or given value. And this unique requirement can be met if the interpolating functions that I was referring to N_i, N_i as a function of x $N_i(x)$, these are called the interpolating functions for node I. And these interpolation functions should satisfy these basic criteria that N_i when evaluated at x_j other nodes of interpolation. So, x_j it is a node of interpolation. So, N_i at x_j should be equal to δ_{ij} .

So, δ_{ij} means it is it takes the value of unity only when i and j coincide. And if i and j are distinct indices then the value is 0. So, that means, for example, n_1 at x_1 is going to be 1; n_1 at x_2 is going to be 0. And similarly, n_2 at x_1 is going to be 0, and n_2 at x_2 is going to be 1, so that is the example, and that is the basis. And that is how we can if we satisfy this condition then naturally by use of that interpolation model that you see in that summation form, we can see that the basic requirement that approximate function and the sample data values at nodes x_i should exactly match will be easily satisfied. So, interpolation functions associated with node i should evaluate unity at node i and should vanish at all other nodes i not equal to j. So, that is the general theory of general basic idea about interpolation.

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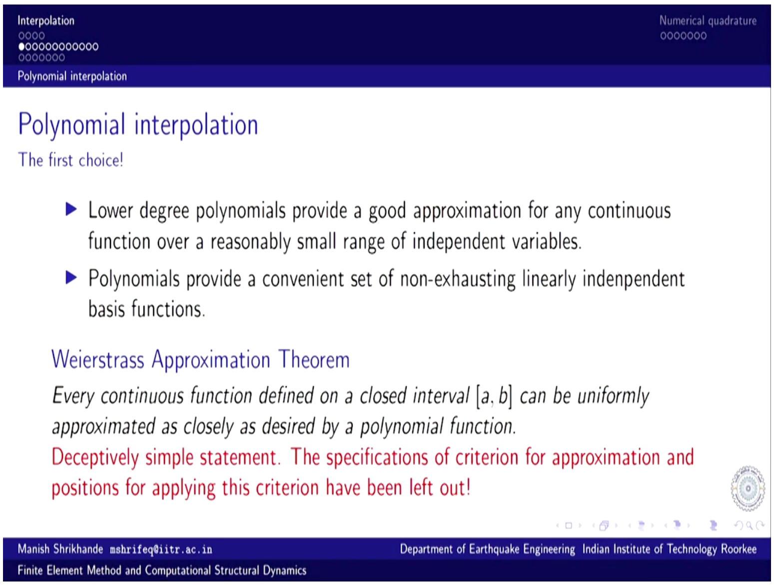

Now, we come to a very specific form of interpolation that is we call the polynomial interpolation and that I call it as the first choice. Polynomial interpolation is always the first choice. Why? Because it is very easy to create. Polynomials is an infinite family 0th degree polynomial a constant term first degree polynomial $a_0 + a_1 x$; second degree polynomial add one more term $a_0 + a_1 x + a_2 x^2$ and so on. So, we can keep on extending, we can keep on increasing the degree of polynomial, it is a never exhausting family you can keep on increasing the degree of polynomial as high as the need be, and it would never exhaust. And moreover, any complex curve no matter what is the nature of the function any function can be approximated by a lower degree polynomial. You can try it out, you can draw any function whatever the nature of function wherever it might be coming from. But if you look at a small enough location small enough of portion of the function, then it can be easily approximated by a lower degree polynomial. I do not even need to consider construct a very high degree polynomial approximation. So, a lower degree polynomial approximation provides a very good approximation for any continuous function over a finite range as long as that we are looking at a small enough range of variables.

So, as I was saying polynomials provide a convenient set of non-exhausting nearly linearly independent basis functions. So, constant term, linear term, second degree term, third degree term, they are all linearly independent. When I say linearly independent, I cannot get anything that is captured by a cubic variation by any linear combination of constant, linear and second-degree term. For capturing cubic variation, I will need to incorporate a cubic term in approximation; there is no other way to generate that, so that is a linear independence of all the terms in polynomial family.

And this preference for polynomial interpolation is also supported by a very famous theorem Weierstrass Approximation Theorem. I will state that. Every continuous function defined on a closed interval between a and b can be uniformly approximated as closely as desired by a polynomial function. I can choose a polynomial and I can approximate any function. So, whatever function original function might be, but a polynomial family is capable of approximating any arbitrary function over this finite range of independent variables. Now that is a deceptively simple statement. That of course tells us that polynomial family is capable of generating arbitrarily good approximation. I can choose the degree of approximation I can go as close as desired in principle that is what the theorem says. I can no matter what the function is like, but I can generate as close approximation as desired by using polynomial over a finite range. But what it leaves out in this statement is what is the criteria of approximation? When I say uniformly approximated as closely as desired, what is close approximation? That is not clearly stated in this theorem although we can hazard a guess and we can define our own criteria that is not too much of a problem. But it also does not specify where this criteria has to be imposed or where am I looking at the approximation. So, will come to this I mean why this thing is crucial we will see in our discussion that this specification is very crucial and that leaves a lot of scope for interpretation and developing suitable approximation.

So, the problem with Weierstrass theorem is we know and that comes from our basic understanding of linear algebra also that polynomial family is infinite family, so infinite basis. So, any function can be possibly approximated by a linear combination of polynomials. But the problem is how do I define the closeness of approximation. And how do I specify the conditions or optimal evaluation of coefficients of the polynomial, how to find out the optimal coefficients of the polynomial of whatever degree will provide the close enough approximation to a given function. So, those details are left out in Weierstrass theorem, but the theorem is of course true I mean theoretically in infinite family should be able to infinite basis should be able to approximate any function.

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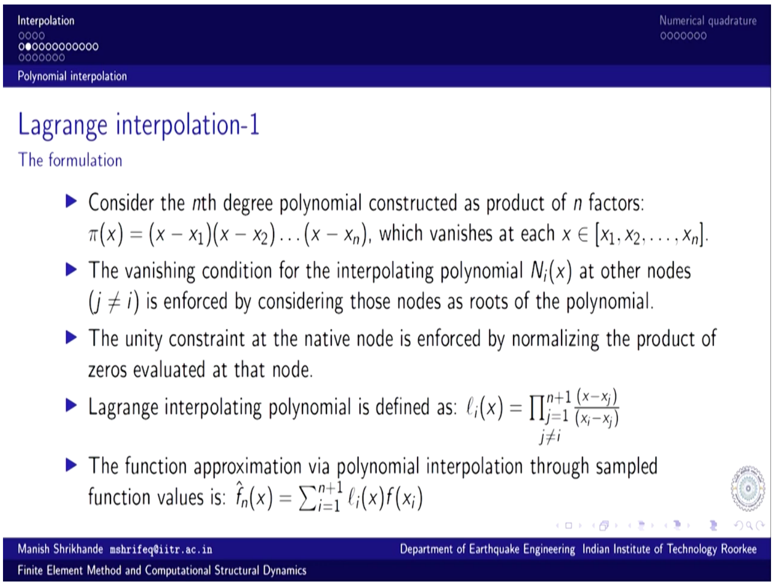

So, we consider first the simplest of all polynomial interpolation that is Lagrange interpolation. So, a Lagrange interpolation is simply I mean is very trivial I mean it is very simple to understand but numerically it is not so convenient to compute, but it is very easy to explain the process of interpolation or polynomial interpolation. So, and n th degree polynomial is constructed as a product of n number of linear terms.

So, here you see $\pi(x)$, pi is an approximation and n^{th} degree approximation that is constructed as a product of n number of terms that is $x - x_1$ multiplied by $x - x_2$ and so on up to $x - x_n$. So, you can see that this product vanishes at each of these points. if x= to x_1, $\pi(x)$ will vanish; if x= x_2, $\pi(x)$ will vanish and so on.

In between any value other than these nodes x_1, x_2, x_3 etc., this function will not vanish; it will have some value right. So, one criteria that we need in Lagrange in the polynomial interpolation or the basic definition of interpolation that the interpolating function should vanish at all other nodes except the node for which it is being defined. So, n_1 should vanish at all values of x_i except x_1, so that is how we construct I mean if I am constructing something for n_1, then the easiest thing to do would be to say n_1 of x should be equal to $x - x_2$ or should be proportional to $(x - x_2)$ $(x - x_3)$ $(x - x_4)$ and so on right of appropriate number of points, then the vanishing condition is obviously satisfied.

So, the basic idea here is the vanishing condition for the interpolating polynomial $N_i(x)$ at other nodes i not equal to j is enforced by considering those nodes as the roots of the polynomial. So, when I say roots of the polynomial I just multiply by zeros of those points. So, wherever the function polynomial, I mean the value is of course the root for as a zero of that polynomial linear function.

Now, the other constraint that is required is it should evaluate to unity at the native node. So, n one of x should evaluate to unity at x_1, and it should vanish at all other nodes. So, vanishing condition is enforced by making the other nodes as the roots of this approximation function and that is easily done by just multiplying the solution $(x - x_2)$ $(x - x_3)$ $(x - x_4)$…. and so on. So, x_2, x_3, x_4 etc. they are the roots of this polynomial. So, the function will vanish.

Now, in order to normalize it to unity at x 1, all that I need to do is divide this function by its value at x_1. So, I have this $\pi(x)$ that is constructed and I just evaluate it at x_1 and whatever node I want and I can get the basic interpolation polynomial. So, this interpolation polynomial Lagrangian interpolation form is defined as l_i, so that is the Lagrange interpolation for i^{th} node is defined as product of all terms. So, π summation going from j is equal to 1 to n+ 1, so total number of n+ 1 number of points. Now, j going from 1 to n +1 and j is not equal to i. So, numerator you can see these are the roots I, mean the basically zeros of at the nodes first degree polynomial first degree terms which define the respective nodes as the roots. And the denominator is the normalizing condition that these were numerator terms, they are evaluated at the respective node i and that ensures the value the l_i will evaluate to unity when you evaluated at x equal to x_i. So, the function approximation via polynomial interpolation through sampled function values is now easily constructed as $\widehat{f_n}$ the n^{th} degree approximation. So, $l_i(x)$ multiplied by specified function value at x_i.

So, we will continue with this interpolation discussion in greater detail in our next lecture.