Finite Element Method and Computational Structural Dynamics
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Lecture - 39
Finite Elements for Plates and Shells - II

Hello friends. So, we were discussing about the deformation and idealization for the deformation of plate elements plate plates and how we approximate the deformation within the plate by virtue of deformation of the mid surface plane and thereby reducing the 3 dimensional problem into a 2 dimensional problem by neglecting the variation of the deformation with respect to thickness normal deformation transverse deformation with respect to thickness direction.

So, the deformation I mean the Kirchhoff constraint, I mean vanishing shear strain constraint imposes that mandates the requirement that the plane section rotations they are same as the slope of the deflected slap deflected curve deflection curve normal deflection curve.

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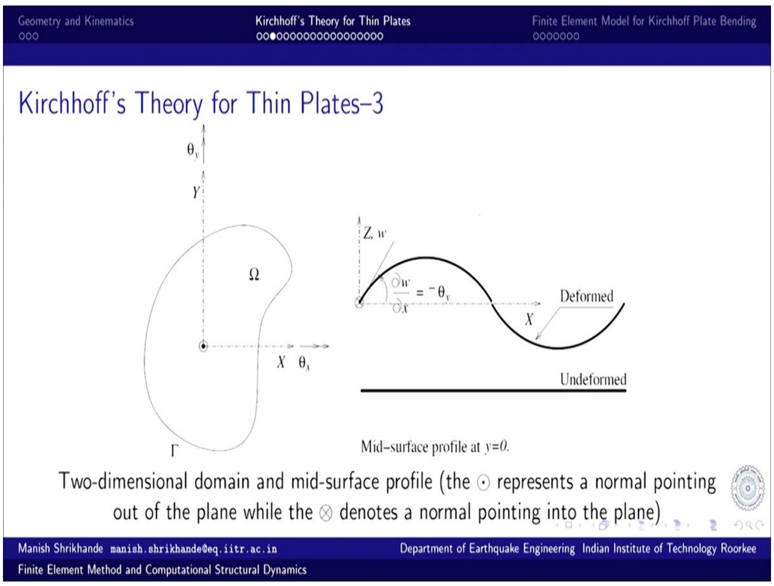

So, $\frac{\partial w}{\partial x} = -\theta_y$ that is rotation about the y axis and similarly $\frac{\partial w}{\partial y} = \theta_x$ that is the rotation about X axis. And that imposes I mean this equality imposes the constraint of vanishing shear strain.

So, once we have this vanishing shear strain then because of these deformations rotation of the about the axis section rotations about the individual axis x and y there can be axial deformations just as I mean theta y rotation about theta y will lead to deformation along stretch or compression along the x direction.

Similarly rotation about X axis will involve extension and compression about along y axis. And the extent of extension and compression will be proportional to the distance from its surface.

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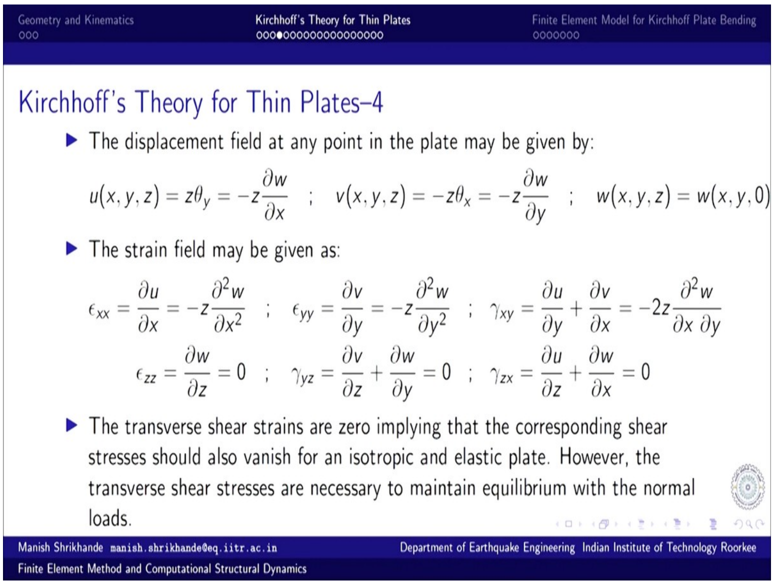

And that is what is given I mean this is similar to what we have in case of Euler Bernoulli beam, this axial strain is proportional to the distance from the neutral axis and we can have compression above neutral axis and tension below neutral axis in case of a sagging moment. And similar kind of geometry is defined for plate accept that it will be now in 2 dimensions.

So, we have $z \times \theta_y$ that gives the deformation along x axis that is u component and v as a function of x, y, z that will be $-z \times \theta_x$ rotation about the x axis. And if we substitute from the Kirchhoff constraint these section rotations can be represented in terms of derivative of the deflected shape.

And that is how we have this deflection uniform u component displacement proportional to derivative partial derivative of deflected shape and similarly v component displacement is proportional to partial derivative of w with respect to y. Now once we have this u and v displacement along x and y direction now; obviously, this can be this deformation field can be represented in terms of can be transformed into strains by standard definition.

So, ε_{xx} direct strain is given by partial derivative rate of change of u with respect to x and that gives us relationship I mean relates it to the curvature of the flexure of plate. And similarly ε_{yy} direct strain along y direction so that is also related to the curvature and then we have shear strain. So, that is $\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}$ and that is related to $2z\frac{\partial^2 w}{\partial x\partial y}$.

And then there are normal strain along z direction because the variation of w along the thickness direction has been assumed to be 0 so $\frac{\partial w}{\partial z}$ rate of change of w with respect to z is 0. And then the shear strain vanishing shear strain that directly follows from the Kirchhoff constraints. So, transverse shear strains are 0 implying that corresponding shear stresses should also vanish for an isotopic and elastic plate.

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Obviously shear strains are 0. So, shear stresses should also vanish; however, because there is transverse load on the plate. So, where the how does how do we get the equilibrium, what is there to balance this transverse load? So, there this transverse load on the normal surface has to be balanced by the shear forces shear stresses in the plate, that is how the vertical equilibrium will be held.

So, shear stresses cannot be 0 and this apparent dichotomy can be resolved by assuming by the limiting case that it is infinitely rigid in shear and that is again similar thing we also experience in case of Euler Bernoulli beam, there is shear strain there is no shear deformation yet we have shear forces there and that is resolved by considering the shear rigidity to be infinite.

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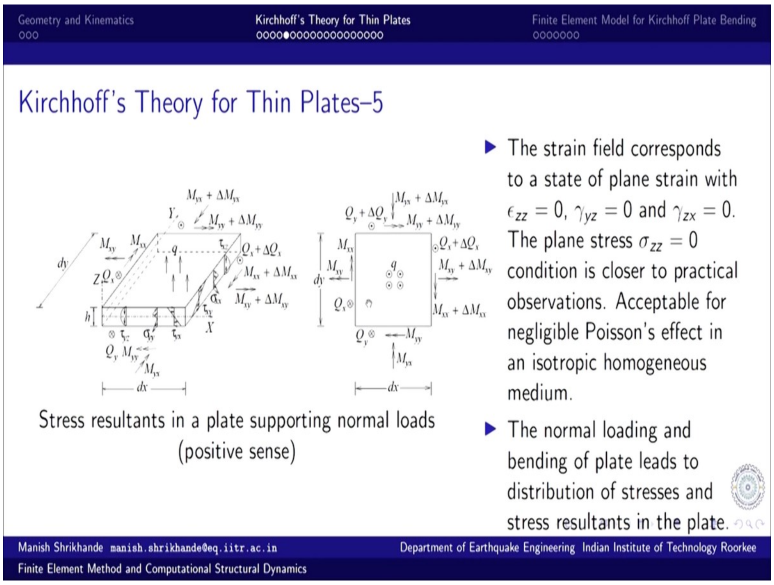

So, these are the stress resultants in a plate, simple I mean these are all the forces that can exist and based on these stress resultants we can impose by considering the equilibrium of forces. I mean force equilibrium and moment equilibrium we can develop the governing equation.

So, now, they we discussed that the strains are always in plane, out of plane strains are as 0 ε_{zz} is 0 and shear strains in yz and zx planes are 0. So, that corresponds to a state of plane strain; however, plane stress condition is more closer to observations. The results predicted by plane stress model is actually has a better agreement with what we observed.

But; obviously, it cannot be plane stress because there is the normal surface is not free of traction. So, normal to thickness direction we have forces here. So, it is not really plane stress condition ideally speaking. So, the problem is this is resolved by neglecting Poisson's effect and also assuming plane stress conditions to prevail over each laminar. So, each layer we consider the flip plane stress conditions prevail.

So, normal loading and distribution bending of plate leads to the distribution of stresses and stress resultants as we show here in this sketch and using these distribution of stress resultants we can now establish the equilibrium equations. So, the bending moment these moments they are related to the stresses here respective stress components.

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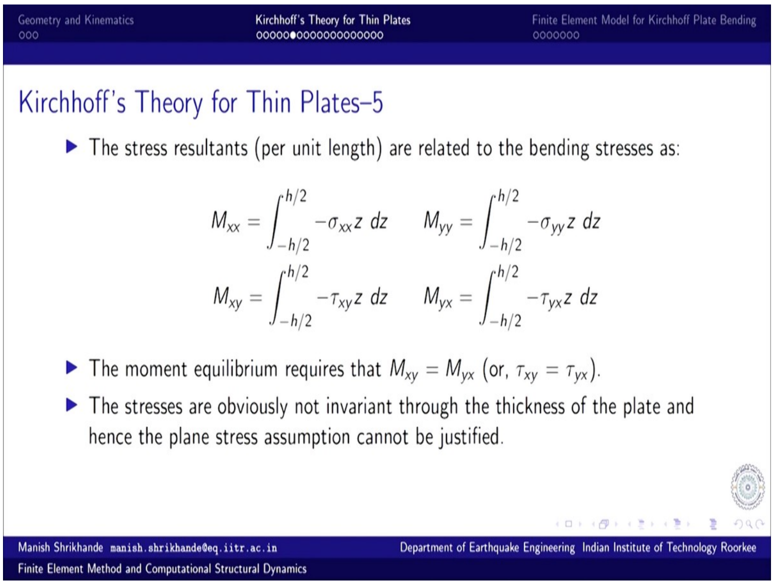

So, these are the stress stresses and these are the stress resultants that we have. So, bending moments stress resultants per unit length are obtained from the integration I mean the of the stress field with lever arm z and integrating over the thickness and we get the M_xx and M_yy and then twisting M_xy and M_yx.

And once we impose the moment equilibrium we will see that I mean that is from the standard mechanics continue mechanics point the complimentary shears should be equal. So, M_xy by corollary M_xy would be equal to M_yx because τ_{xy} is going to be equal to τ_{yx} .

So, although the deformation is considered to be invariant through the thickness w normal deformation of the plate the stresses; obviously, are not invariant through the thickness of the plate and therefore, the plane stress assumption is not exactly valid, but we resolve this problem by considering each lamina of the plate that is plane at constant value of z to be in a state of plane stress as I mentioned earlier.

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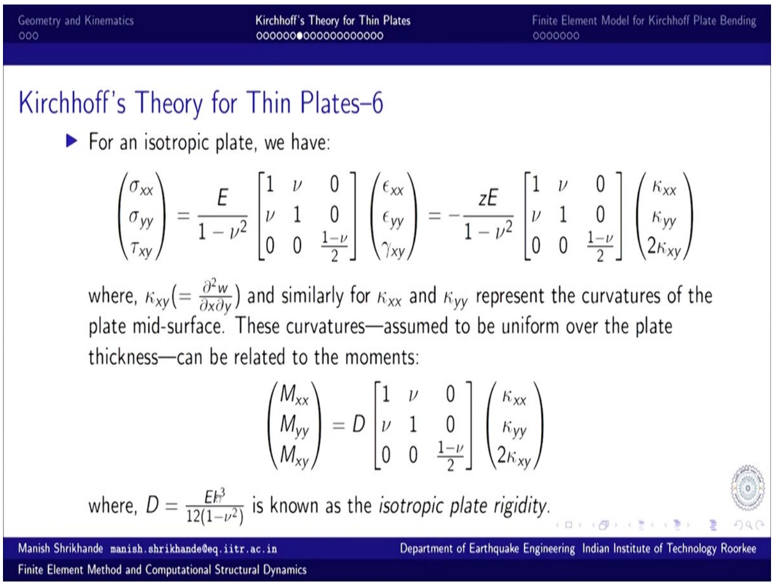

So, for an isotopic plate, isotropic medium we can relate stresses to strain using plane stress I mean so this is where plane stress assumption comes in for plane strain this matrix would be different. So, we as we said plane stress conditions are have a better agreement with what we observe in during experimental studies experimental observation and therefore, we stick to plane stress assumption.

So, this is the constitutive matrix for plane stress model and once we relate these strains to the curvatures. So, we can relate these stresses to curvature and then stresses can be transformed into moments as we discussed earlier there is a stresses relate to moments. So, we can have moment curvature relationship in terms of the constitutive matrix.

Where now this D is a isotropic plate rigidity so Eh cube. So, again here you see similar to what we have in the case of being flexural rigidity. So, that is depth appears as a third power. So, here also h appears as the thickness appears in the third power. So, and that gives us I mean thickness is important it has to have just having thickness dimension to be small very small compared to a 2 other dimension is not enough.

For a 2 dimensional entity to behave as a plate it is required that it should have sufficient flexural rigidity or plate rigidity and that is governed by this term Eh cube. So, there is some bit of thickness is required and of course, the Young's modulus.

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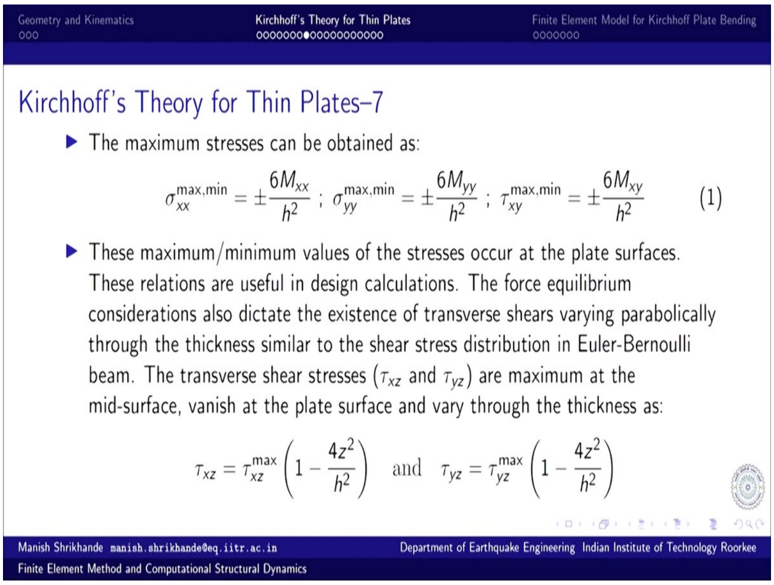

So, maximum stresses can be obtained by evaluating these at values of z = + or - h / 2 and they can be related to the moments. And maximum minimum values of stresses occur at the plate surfaces and these relations are of course, useful for design calculations. Once we have the moment then we can simply find out what is the maximum stress of respective stress I mean direct stresses and shear stresses.

So, transverse shear stresses they are maximum at the mid surface I mean in zy plane and zx plane and they vary through the thickness.

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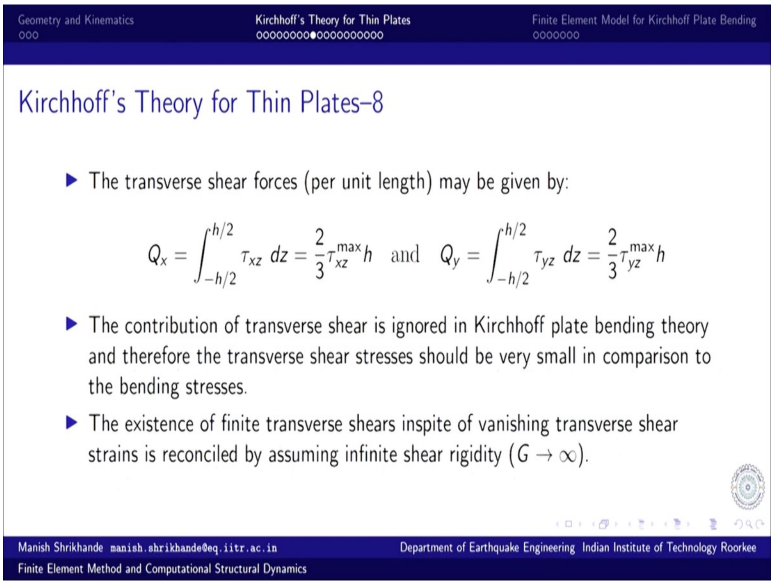

And once we integrate these transverse I mean we can find the resultant transverse shear forces and they can be represented as representative value of a scale down of maximum shear stress and that would be multiplied by thickness and two third of that; that is what it works out. So, these are again transverse shear forces that are going to be used for our equilibrium equations.

So, contribution of transverse shear is ignored in Kirchhoff plate bending theory and therefore, shear stresses transverse shear stresses should be very small in comparison with the bending stresses that actually govern the behaviour. So, existence of finite transverse shears in spite of vanishing transverse shear strain is reconciled by assuming infinite shear rigidity as I mentioned earlier.

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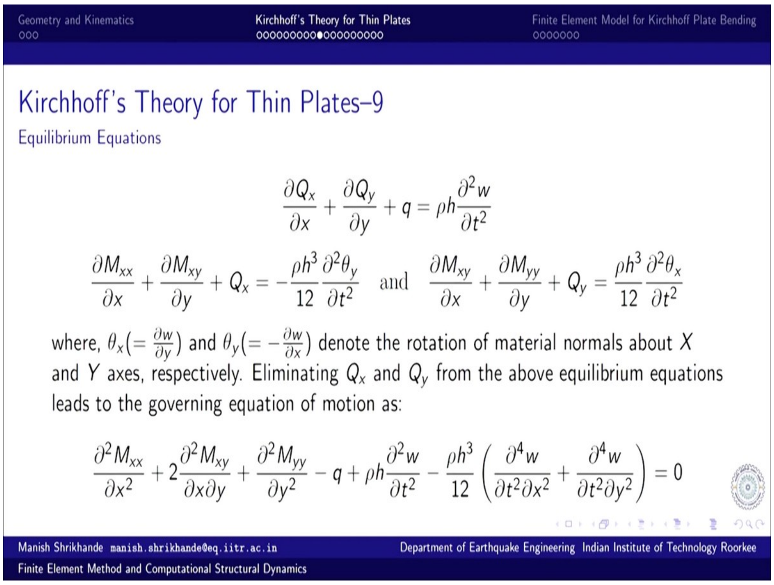

And then once we impose this vertical and vertical force equilibrium and moment equilibrium then we lead to we are led to 3 equations for these 3 vertical equilibrium and moment equilibrium. And then we can eliminate Qx and Qy from these equations and combine all of them to get one equation in terms of combining all moment components and including transverse load as well as the inertia term.

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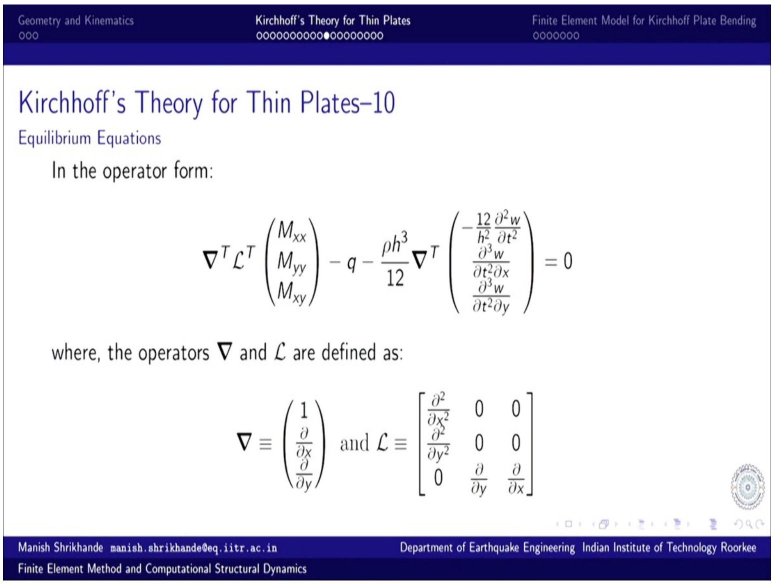

In operator form we can relate these two moment and curvatures moments and the applied forces, and where this nabla is the just the gradient vector and L is the differential operator involving curvatures defining curvatures.

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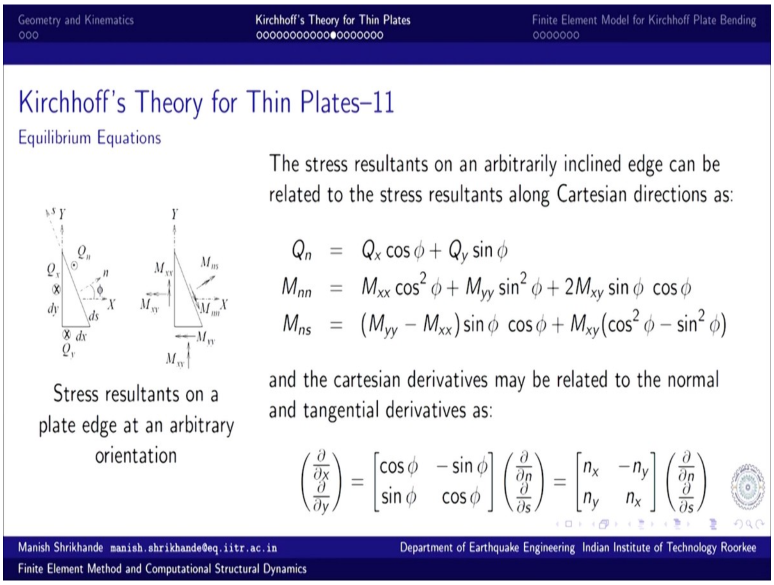

So, far we developed it for edges plate edges oriented along the frame of reference. So, x and y coordinate axis I mean if the plate dimension plate edge is oriented arbitrarily for any arbitrary orientation where the plate edge is defined by normal n and tangential direction s.

So, they can be related to Mx and the moments and shears along defined along x and y by suitable resolving and using the direction cosigns of these normal and the moments normal to the normal and torsional moments they can be related to the moments M x and Mx and Mxy. And similarly, Qn can be related to Qx and Qy using the direction cosines of the normal.

And the Cartesian derivatives can be related to similar to the Jacobian transformation, Jacobian of the transformation using that Jacobian of the transformation the Cartesian derivatives can be related to normal derivative and tangential derivatives.

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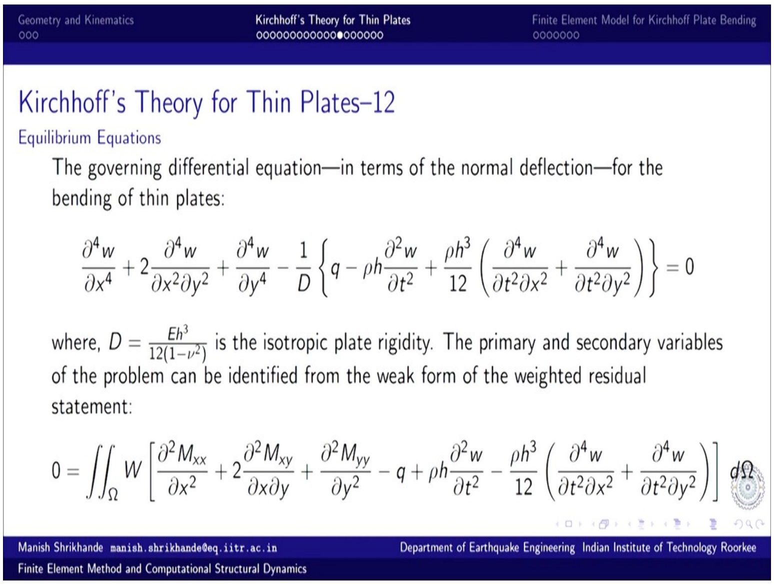

So, eventually we come with the governing differential equation in terms of normal deflection of the plate and that if we substitute for moments in the moment equilibrium equation then moments are proportional to the curvature. So, that is second order derivative of normal displacement or transverse displacement. So, that is so in terms of displacement it becomes a fourth order differential equation.

So, this is the governing differential equation and D is as I said it is the isotopic plate rigidity. Now, in order to define develop the I mean this is the finite equilibrium equation for the plate bending problem, in order to develop the finite element for finite element model for this problem it is first necessary to identify what are the primary variables and secondary variables and for that we develop the weak form of this governing differential equation.

So, it is as we know it is a 2 dimensional equation variations with respect to x and y. So, we define the weighted residual statement w is the weighting function and we take the weighted residual statement put it to 0 integrate it over the x y domain and again using divergence theorem it can be the boundary term and the differential equation can be transformed and a boundary term can be developed.

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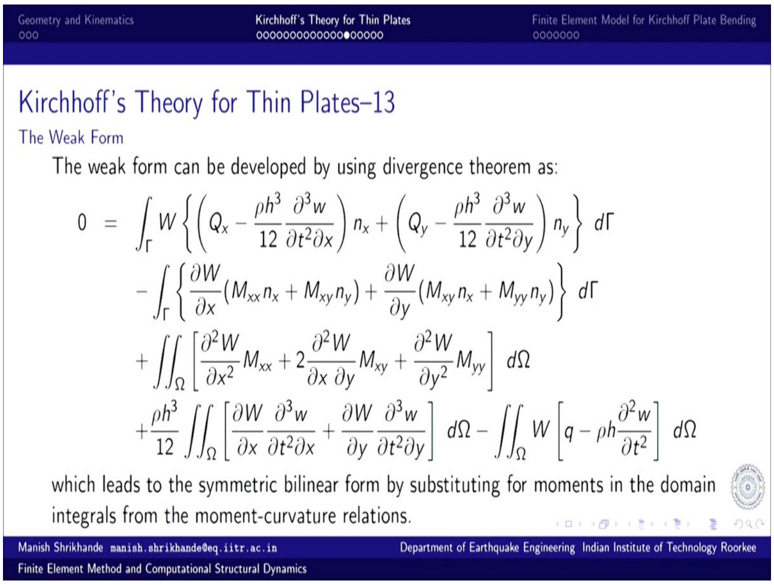

And the derivatives can be transferred. So, we have 2 boundary terms so that will have two derivatives being transferred to the waiting function terms. And eventually we will have only the highest order of spatial derivative as 2 in the entire equation in this weak form.

So, the highest order of spatial derivative is 2 Mx Mxy again it is the highest order is 2 Mxx is proportional to the curvature. So, that which is the second order derivative again. So, and then the boundary terms there are how many boundary terms. So, we can identify 3 primary variables here. So, $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ so these are the weighting function terms in the boundary term.

So, the appropriate primary variables would be the normal displacement along the z direction so that is $\frac{\partial w}{\partial x}$ so that is the deflection I mean slope of the deflected shape along x, then $\frac{\partial w}{\partial y}$ that is that will be slope of deflected shape along y so that would be.

So, these 3 are the primary variables that we can identify and corresponding to that we have secondary variable. So, this is the transverse shear and these are the moments corresponding moments corresponding to the respective rotations.

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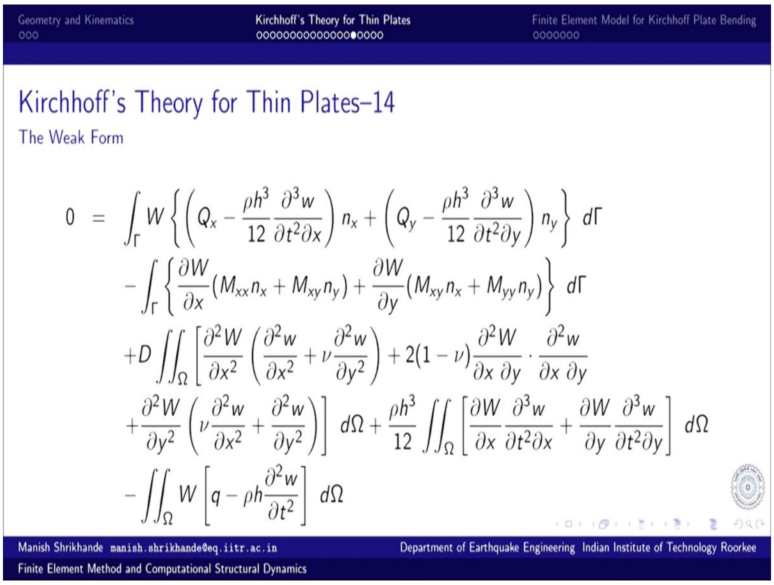

So, this is what eventually we have by substituting for that moment and in terms of moment curvature relationships, and we now have second order derivative spatial derivative that is highest.

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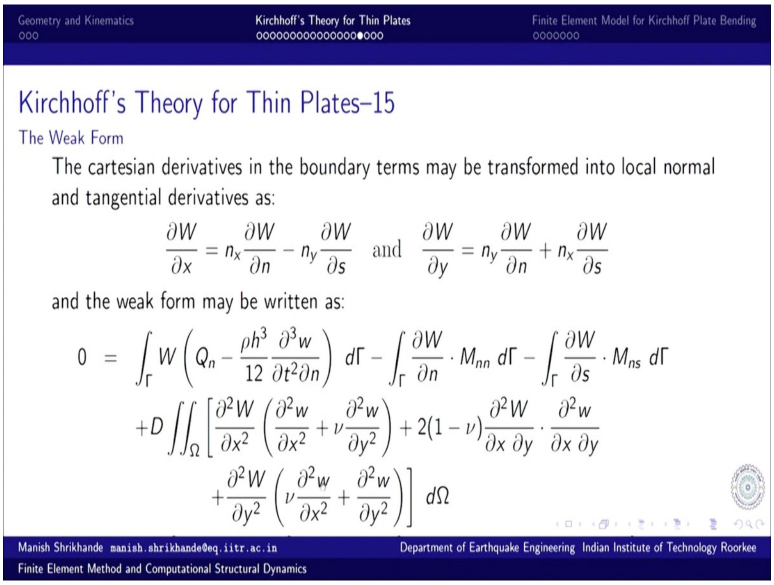

And we can identify we can again transform these into normal and tangential and combine the terms. So, we can have normal derivative and tangential derivative and Qx Qy can be combined to become Qn, Mx Mxy can be combined to become Mnn and similarly Mns. So, torsional and direct rotation.

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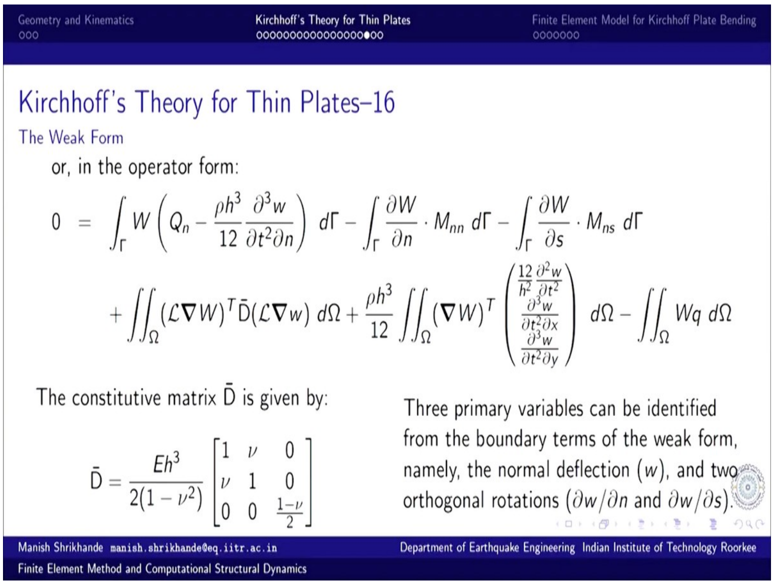

So, again we can use operator form to reduce for a compact representation of the this domain integral. So, that is the compact representation of domain integral and this is boundary integral and from boundary integral we can this is the constitutive matrix D is related to plate rigidity and plane stress model. So, three primary variables can be identified from the boundary term. So, that is the normal deflection w and 2 orthogonal rotations so that is $\frac{\partial w}{\partial n}$ and $\frac{\partial w}{\partial s}$.

So, either $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ in case the plate is oriented along x and y direction or in any arbitrary orientation it is better to refer to del n normal and tangential derivatives. So, essentially 3 primary variables and corresponding 3 secondary variables, but we should realize that it is a fourth order differential equation and a fourth order differential equation should not have 3 pairs of primary and secondary variables. The maximum primary variables and secondary variables should be only 2.

I mean you can relate it to what we have in case of Euler Bernoulli beam it is also a fourth order differential equation, but we only have 2 primary variables and 2 secondary variables. So, it so happens that these three conditions are not really independent so they can be reduced to just 2 independent conditions.

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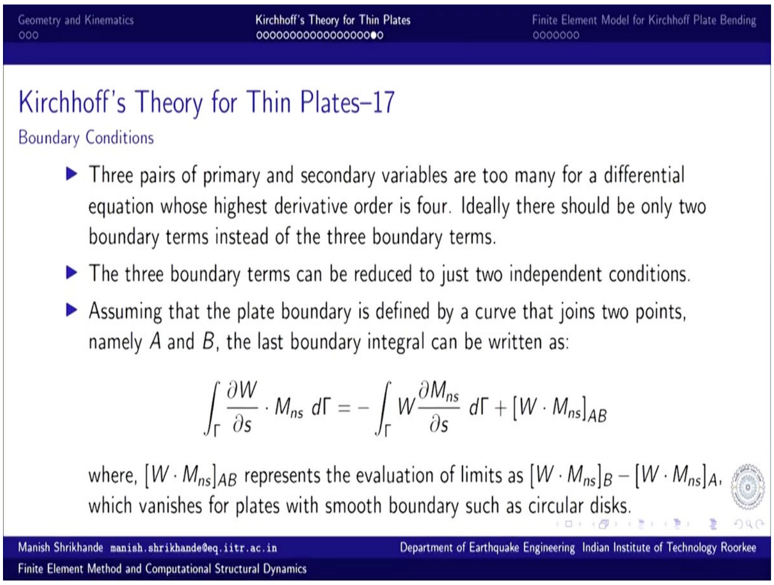

So, this integral can be defined as again integrating this in the boundary integral as assuming A and B are the points. So, if I integrate it along this boundary A to B then this is what it becomes by again using the divergence theorem, and then this is the boundary term and this is the evaluation of these limits.

And if it is a circular disk I mean smooth geometry then this evaluates to 0, it will be both the points will converge and they will the derivatives these normal Mns would be 0 at individual I mean this difference would not turn out to be cancel each other, but for plates with sharp corners this will not in general cancel out.

And that will lead to what we have as unbalanced shear force at the corners and that will lead to lifting up of the corners. So, in case of rectangular geometry plates if it is simply supported then we can see that knife edge support then we can actually see this in response to applied load in the middle the edges they tend to lift up. So, that is because of the unbalanced shear force that we have here at the boundaries and that is called free edge condition.

So, this is this problem or realization that these 3 boundary conditions that we have from the weak form they are not really 3 boundary conditions, they are not independent boundary conditions, but they are related to each other and they can be reduced to 2 independent boundary conditions and this is called Kirchhoff free edge condition and that is that results this dichotomy.

So, then using this constraints so the boundary conditions can be resolved into 2 parts and we have normal effective transverse shear and then effective moment.

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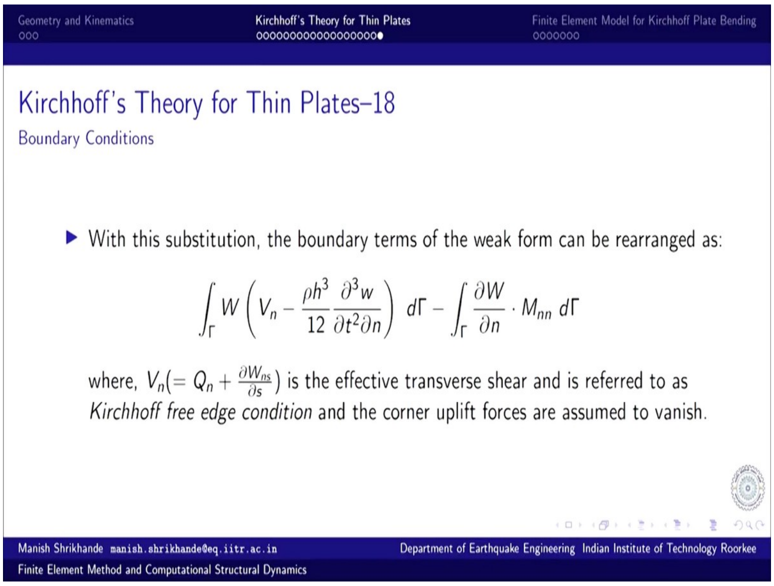

And then we have just two primary variables if you look at it in this form so transverse displacement and the rotation normal about the normal. So, that is about the theory about plate bending theory and now we look at what is the issue here. So, as it is the equations are rather involved and it becomes one thing that we have been consistently arguing all along that plate is essentially an extension of beam in another dimension I mean horizontal extension of beam.

So, a natural thought process would be probably it should be possible to derive finite element formulation in the way similar to Lagrangian for interpolation. We have interpolation model for one dimensional beam and let us say along x we can also have similar model for beam spanning in y direction and just have a product of these two terms and we should have interpolation model for xy plane flexural model for xy plane.

Just as we do for Lagrangian interpolation we have Lagrangian interpolation 1 dimensional Lagrangian interpolation in along x we have 1 dimensional Lagrangian interpolation along y and multiply these 2 and we get the desired Lagrangian interpolation for C^0 continuity for 2 dimension xy dimension. So, in this case of course, we need C^1 continuity because the primary variables are the derivatives first derivatives.

So, c 1 continuity we have beam elements cubic Hermation polynomials along x along y multiply them.

So, that was the first attempt at development of finite element model, but unfortunately that does not succeed, very soon it was realized that this is not possible because it cannot I mean this product term it cannot model constant curvature state. So, constant curvature that is second order derivative. So, that is what we have similar to the constant strain condition in case of flexure problem.

So, constant curvature state cannot be maintained by this product of cubic Hermite polynomials in 2 dimensions. So, cubic Hermite along x multiplied by cubic Hermite along y so this product function it does not ensure the constant curvature state. So, that way I mean if we cannot model constant curvature then; obviously, it is not suitable that kind of model is not suited for developing a finite element model.

So, how do we develop finite element model and what do we how do we go about it we will discuss in our next lecture.

Thank you.