Finite Element Method and Computational Structural Dynamics
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Lecture - 38
Finite Elements for Plates and Shells - I

Hello friends. So, today we start a new topic on very interesting and exciting field for Development of Finite Element Method that is how to model the problems related to Plates and Shells. As you know, plates and shells, they form very very important parts of structural system ranging, I mean spanning across different disciplines starting from civil engineering systems, I mean roofs and shelters to the flanges of aircrafts and so on.

And these are all of course, have tremendous applications, I mean interest in industry. And not surprisingly this topic has experienced or has witnessed massive amount of research effort in trying to develop the theory for finite element analysis, finite element model for the solution of plates and shells. And one reason is of course that the problems are not so trivial or not so simple as earlier problems that we have been dealing with. And we will see why that is so.

And in the order of difficulty, plates have present their own set of difficulties of peculiarities in modeling that need to be considered. And shells add another dimension of complexity by virtue of its geometry, and that has its own set of peculiarities that need to be accounted for in the modeling.

And sometimes, it is possible to get reasonably good approximations reasonably good or acceptable degree of accuracy in the computed solution by using some of the ad hoc methods that we have seen to work very beautifully in the case of variational crimes. So, that we discussed in our earlier discussions.

So, before we begin with the finite element modeling of plates and shells problem, let us first discuss about the theory of plate bending, and shell behaviour, and what is the difference between plates and shells and how they are different from the other structural systems or other continuum mechanics models that we have discussed so far.

So, first as I said the geometry and kinematics, they play an important role in the theory of plates and shells. So, a simple example that is representative of what we are dealing with.

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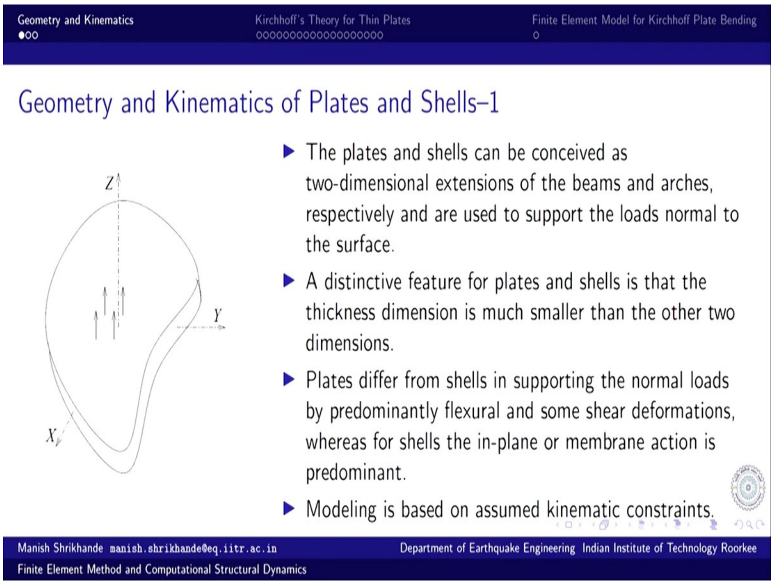

So, what do you see in this is that, it is a two-dimensional extent in X-Y plane, in this case. The dimensions are very large compared to the dimension along the thickness direction, and so, in that respect it seems more close to plane stress kind of situation that we had discussed earlier in continuum mechanics problem. Thickness direction is very small compared to the other in plane dimension in X-Y plane.

But, the difference from plane stress conditions is that the loading is normal to the plane whereas, in plane stress condition, the normal to surface loading was 0, the surfaces were free of traction. So, that is the main feature, distinguishing feature of this plane plates problem.

So, more formally the plates and shells can be conceived as two-dimensional extensions of beams and arches. So, in one-dimensional, what this purpose of beams is to withstand the normal loads or transfers to the axis, load supply transfer to the axis. They withstand that by virtue of primary predominantly by virtue of flexure.

And similarly, arches there are special forms which again withstand the loads normal to the span of the arches span direction. And the mechanics of resistance is not so much bending as it is through the axial thrust in the members.

So, these are all used to support loads normal to the surface. So, plates and shells, they can just stretch, I mean stretch beam to across the Y direction and stretch arches across the Y direction, and we get plates and shells. A distinctive feature of plates end shells is that the thickness dimension is much smaller than the other two-dimensions as we saw in during this discussion.

And the difference in plates and shells is that, in plates the resistance mechanism is primarily flexural and some shear deformation, whereas, for shells, it is primarily in plane or membrane action. Just as in arch, if it is an arch then this load is actually withstood by the axial thrust in the arch members.

So, it is the same thing as in the case of in the case of shells. The primary acts structural action is that of in plane or membrane action and that is what it distinguishes it from plates. But geometrically they all both of them share this common feature that thickness direction, dimension along thickness is very small compared to the other two-dimensions. And of course, shells have curvature, plates do not.

And modeling is based on certain kinematic constraints. Just to make life simpler, because of this small dimension small along thickness direction, it is possible to make certain simplifying assumptions and that allows us to develop simpler theories which will allow us to model less using smaller number of variables and the problem can be solved in a more reasonable or small number of variables, and the number domains can be represented using smaller number of elements.

So, that is the primary reason for using the plates and theories for plates and shells, so that certain kinematic assumptions can be imposed on the displacement field, and that leads to reduction in the number of variables.

So, since thickness is very small in comparison with the other two-dimension, the out of plane deformations are assumed to be invariant throughout the thickness. So, that is the first assumption. So, the normal to the surface, the deformations, they are assumed to be same at each lamina. If we consider divide this entire thickness into as several laminates, then each lamina has the same deformation, so that continuity, I mean there is no void or no gaps in between during the process of deformation. So, if it was a homogeneous solid plate and during after the deformation it remains homogeneous solid plate and that can only happen when the normal to surface deformation is same across the thickness.

So, continuity of the medium is maintained. And this allows development of a working model for the strain field with reference to the out of plane deformation of the mid-surface. So, now, if thickness is, I mean is the deformation is invariant with respect to thickness, then it is very convenient frame of reference to align the origin or the mid-surface plane that is at Z coordinate to be 0, that to be the reference plane. And the deformations are measured with respect to this particular reference plane at Z = 0.

So, this invariants of the deformation through thickness, effectively throws out the third dimension and the entire problem can be modeled in a two-dimensional reference frame as we discussed at Z = 0, the reference plane in, and that would be lying in the X-Y plane.

And the material filaments, normal to the mid-surface. So, if this is mid-surface, and the filam, the normal planes to planes normal to this mid-surface, we assume that these planes remain normal and straight after the deformation as well. During the deformation, before deformation, during deformation, and after deformation.

So, this assumption is analogous to our very familiar assumption of Euler-Bernoulli beam flexure, that is plane sections, normal to neutral axis remain plane and normal after the deformation. So, these mid filaments, normal to the mid-surface. They extend up to h by 2 or - h by 2. So, half of thickness above and below the mid-surface.

And the bounding surface at these two at these extremes, half of thickness about the mid surf mid-surface, and half of thickness below mid surface they are referred to as top surface and bottom surface, respectively.

So, in case of a plate, I mean what we call as a plate the thickness is small, but not too small in comparison to other dimensions.

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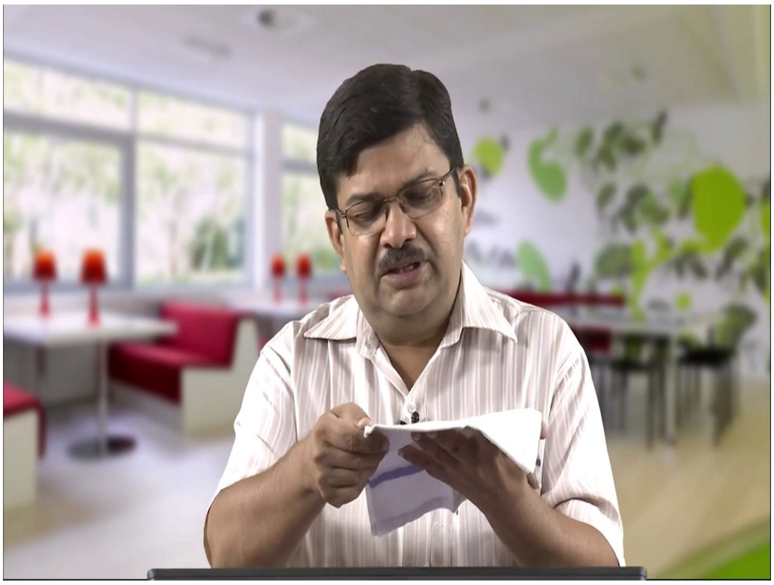

I mean for example, I would probably it might I mean this let us say handkerchief, right. So, in this case two-dimensions, they are very large compared to the thickness dimension, right. So, but this does not qualify to be called as a plate. This is not a plate, because it has no flexural restraint, it has no flexure flexural capacity. So, this is what we call as a fabric, right. So, this is a fabric. This is not a plate.

So, that is an important distinction that one needs to realize, and one needs to appreciate that the thickness of a plate is small, but not too small. Because as we all know from the our study of flexural bending of beams depth of the beam plays an important role in its flexural rigidity the Young's, the moment of inertia, I varies proportional to third power of the depth. And that is what derives the flexural rigidity for the beams and same thing happens for the plates.

So, thickness is important parameter. So, flexural rigidity is contributed by thickness and also by the Young's modulus of elasticity. No doubt. But, if thickness is too small, then it will behave as a more as a fabric and not as a plate. The predominant mode of deformation will not be flexural in that case. And plate by definition is a flexural member.

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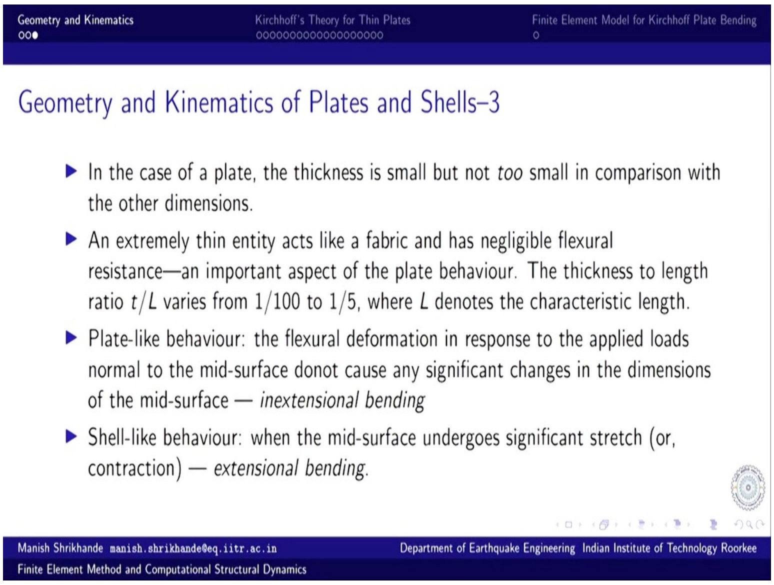

So, an extremely thin entity acts like a fabric, and has negligible flexural resistance which is an important aspect of the plate behaviour. So, as a rough guideline, so thickness to length ratio t by L varies from 1 over 100 to 1 over 5, where L represents the general characteristic length of a plate element.

So, a plate like behaviour, what is it to call something as a plate like behaviour? So, essentially the flexural deformations in response to the applied loads. Loads which are coming which are being applied normal to the mid-surface, and they do not cause any significant changes in the dimension of the mid-surface. Mid-surface itself remains invariant.

Similar thing, I mean exactly parallel to I mean you can always draw parallel with Euler-Bernoulli hypothesis. Neutral axis is one which does not change length, during deformation, bending and after bending. So, this is what is happening in plates also. So, replace neutral axis of being with mid-surface of the plate. The mid-surface does not change in its dimension.

So, this is what we call as in extensional bending. So, that is essence of plate like behaviour. So, it is a flexural deformation which is in response to the loads applied normal to the surface, but the mid-surface geometry, mid-surface dimensions, they do not change appreciably. So, though those can be ignored. So, this is called in extensional bending, and this is what is critical what is known as plate like behaviour.

So, it is inextensibility, inextensional only at the mid-surface, above mid-surface and below mid-surface, there are changes happening, right. Just as in the case of beam, neutral axis is what is does not change the length. Above neutral axis there is compression, below neutral axis there is tension in case of sagging moment being applied or concave curvature, the beam taking concave curvature.

So, if this is plate like behaviour, what is it to say a shell like behaviour? So, a shell like behaviour is where the mid-surface undergoes significant stretch or contraction and that is called extensional bending. I mean quite naturally because as we said shell predominant behaviour is membrane action, in plane action, the normal force is converted into axial forces axial thrust in this member surface I mean the shell geometry or arch members.

So, that will act axial forces they will of course, cause axial deformation. So, extensional deformations would be there. So, a shell like behaviour is one where the bending action is facilitated by extension or contraction of the mid-surface. So, that is what we call as shell like behaviour.

So, now, we can have plates, let us start with the plates. We can have plates which are which can be thin plates or thick plates. For example, I mean this as we said the thickness can range from 100s of representative dimension; 1, dimension representative length, or 2, one-fifth of the representative length. So, there is a wide range of thickness.

So, if to the other x, of one extreme we have very thin plates, and at the other extreme we have thickness, which is slightly I mean one-fifth of length is not too much. It is not even one order lower, I mean about there. So, again the it is similar to the difference between slender beams and thick beams or deep beams. So, where the depth beam, depth is very high very large depth.

So, we have what distinguishes deep beams from slender beams. That is in slender beams the Euler-Bernoulli hypothesis governs. That means, the shear deformation, shear strains are 0, there is no shear distortion of the section. Whereas, in the deep beams the difference is the shear deformations are no longer negligible it may still be predominantly flexural, but shear deformations do account for significant deformation, transfer displacement is contributed by shear deformations as well.

So, in that situation shear deformations cannot be ignored completely. And the same thing happens in case of thin plates and thick plates. In thin plates, shear deformations are negligible, and those are ignored and in case of thick plates the shear deformations are accounted for.

So, we start with theory of thin plates which is more popularly known as in popular literature. It is referred to as Kirchhoff's theory for thin plates. It is the same Kirchhoff which give us Kirchhoff's laws in electrical engineering circuit, network analysis. So, the first is of course, assumption similar to plane section remain plane before and after bending. So, mid surface material normal's to mid-surface, before deformation remains straight and normal to mid-surface after deformation.

And transverse shear is assumed to be negligible. So, again that is a condition which is approximated by thin plates under small deformation. So, when small this transverse shear is negligible, then that allows, I mean if we consider the transverse shear to be vanishing, then it allows us to have a special constraint on the deformation.

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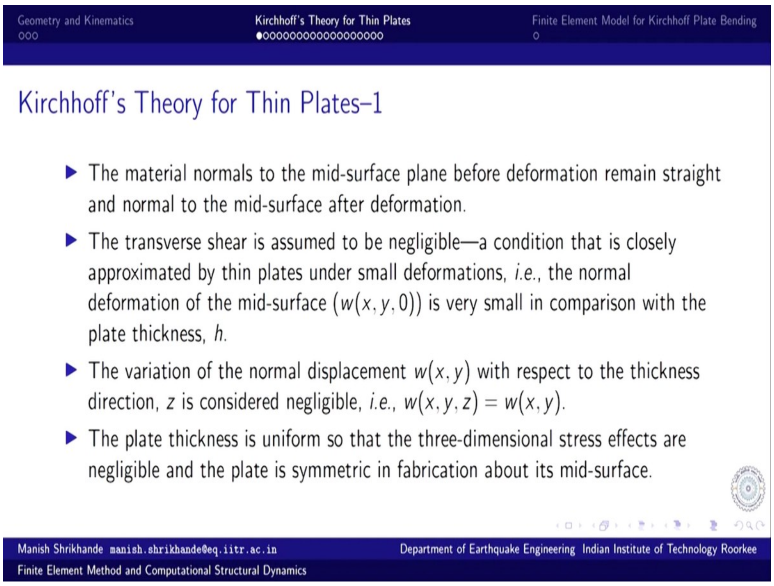

And of course, the variation of normal displacement with respect to thickness is 0. So, there is no variation. So, along the z direction, it the transverse displacement w along z displacement along z axis is invariant with respect to z. So, w as a function of x, y, and z can be represented as just two-dimensional variation w as a function of x and y.

And plate thickness is uniform, so that three-dimensional stress effects are negligible, and the plate is symmetric in fabrication about its mid-surface. So, that all the beautiful symmetric geometry, similar triangles, that we use for derivation of flexural formula, these are, those are all applicable.

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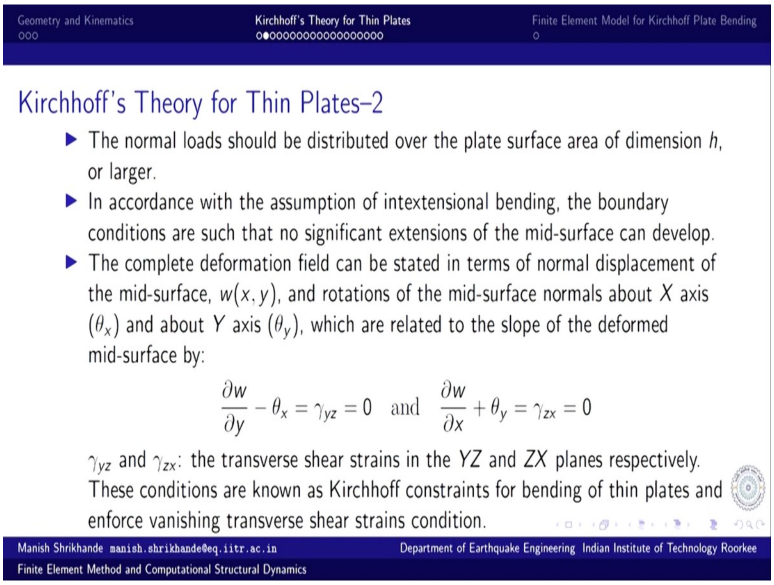

So, the normal loads should be distributed on the plate surface area of dimension h or larger. So, I mean again it is a the distributed load on the surface. It should not be just one point that will lead to very heavy stress concentration and difficult to distribute it because. And so, about the thickness extent of thickness, so the distributed normal loads, distributed on the surface, they should be distributed over an surface area of dimension let us say which is comparable to the thickness or larger than that.

So, in accordance with the assumption of in extensional bending, the boundary conditions are such that no significant extensions of the mid-surface can develop. We will discuss more on the boundary conditions for Kirchhoff plate theory subsequently.

So, now, the vanishing shear strain. So, there are no shear deformations. So, the complete deformation field can be stated in terms of normal displacement of the mid-surface that is given as a function of x and y. So, w as a function of x and y, and the rotation of the mid-surface normal’s about X and Y. So, theta x and theta y. And once we have theta x theta y multiplied by lever arm, so that gives us the deformation.

So, the point is when we define the rotations, section rotations, they give us the distortion plane distortion of the cross section and that is what the shear strain is. So, we define. Since, the we the basic assumption is shear strains are negligible or shear deformations are negligible, we compute the shear strain and impose it to 0 by the definition.

So, γ_{yz} and γ_{zx} , these are the shear strains, transverse shear strains in YZ and ZX planes, respectively. And the shear strains are given by, $\frac{\partial w}{\partial y} - \theta_x$, so that is the rotation of mid-surface normal about X axis and $\frac{\partial w}{\partial x} + \theta_y = \gamma_{zx}$ and that is also equal to 0. So, these are the constraints, shear constraints that we impose on the deformation field. And based on this we develop the geometry.

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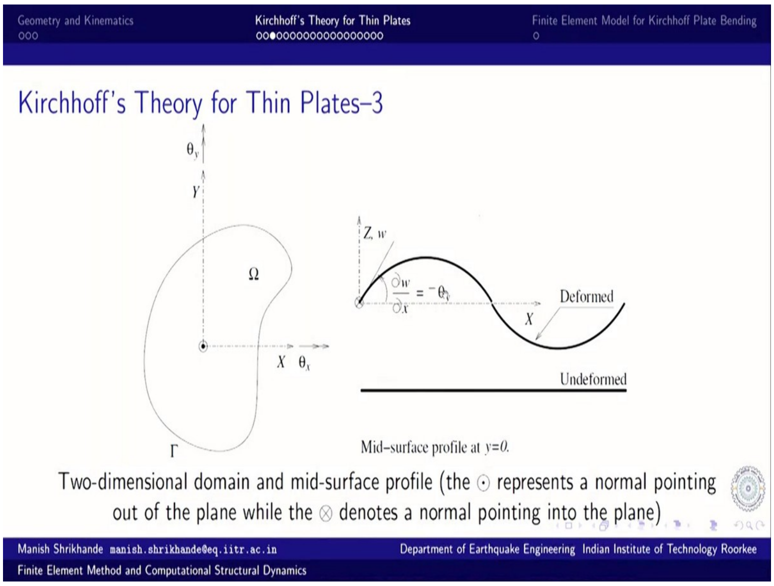

So, this is what the deformation field is. So, theta x that is the rotation. So, this double arrow notation is when the positive sense is when the right-hand thumb points to the direction of arrow, then the direction of curl of fingers is the sense of rotation. So, this is theta x rotation, and this is theta y rotation, rotation about Y axis, if this is X plane X axis and this one is Y axis.

So, if you look at it, so theta x will lead to deformation. So, this represents the deformation of the $\frac{\partial w}{\partial x}$, and the relationship, why the deformation strain, deformation in ZX plane would be like this. So, $\frac{\partial w}{\partial x}$ = negative of θ_y ; rotation about Y axis. And similarly, we can have normal to this axis, so along Y. And we can have similar geometry which will define the constraint shear constraint for YZ plane.

So, how these deformations allow us to develop the governing differential equations of motion starting from moment curvature relationships, and then relating it to strains, and then two stresses, and from there we develop the equilibrium equation, we will discuss in our next lecture.

Thank you.