Finite Element Method and Computational Structural Dynamics

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Lecture - 36
Variational Crimes

Hello friends. So, last lecture we discussed about several possibilities in which erroneous behavior or not up to the mark when the element behavior is not up to the mark can be rectified by using some of the tricks of the trade. And those tricks of the trade are collectively referred to as Variational Crimes, because seemingly the approach that we take violates the governing variational principles of the problem in one way or the other.

But there are certain reason certain ways in which the results can be improved and we will study why those tricks actually work in some cases. And how do we choose what is an acceptable violation of variational principle and what is an variational crime which is acceptable for analysis and convergence of the finite element result. Although in this particular case when variational crimes are being used are being committed.

Then the convergence is no longer monotonic no longer guarantee to be monotonic, it will be a non-monotonic convergence. But nevertheless it is possible to converge to the correct solution in the limit. So, reduced integration, inclusion of incompatible modes and that will bring in consistency in the field approximation that explains the consistency requirement in approximation field.

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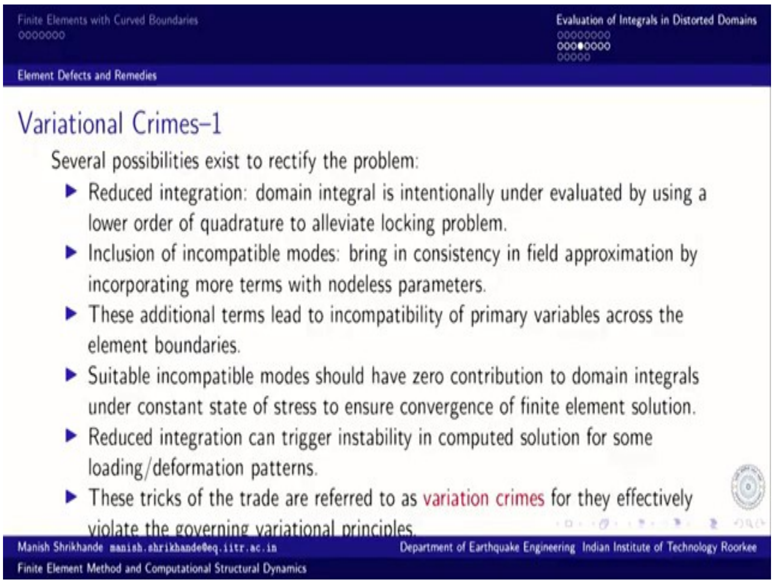

So, all of these I mean whether incompatible modes or reduced integration they all have some side effects undesirable side effects and these have to be under maintained under control for them not to pollute the computed finite element solution. So, let us start let us discuss these variational crimes particularly incompatible modes and reduced integration in little greater detail. And then we will see how it works and why it works.

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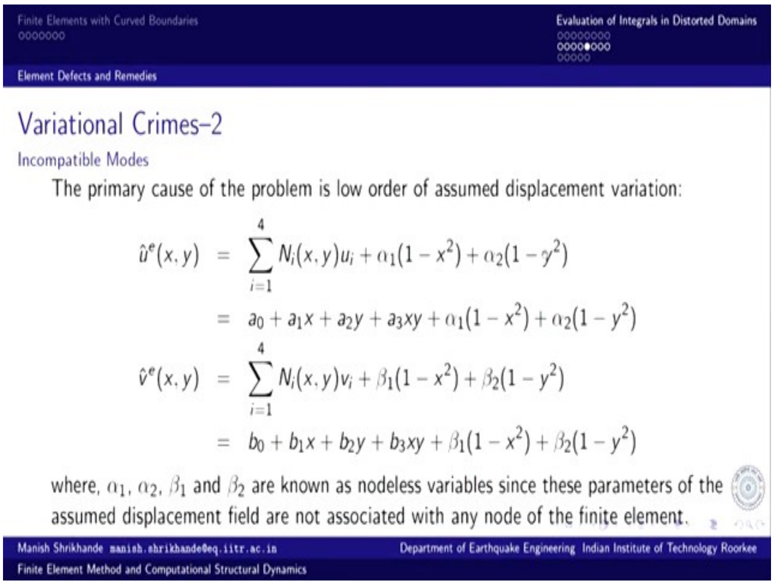

So, the primary cause as we discussed in the modeling of pure flexure by 4 noded rectangle element is the inability to take the curved path and that curvature curved geometry is possible, if we admit this second degree polynomial 1 - x square and 1 - y square in the approximation. So, the reason for the inclusion of this is these effects I mean are 0 on the boundaries wherever those are required to be 0.

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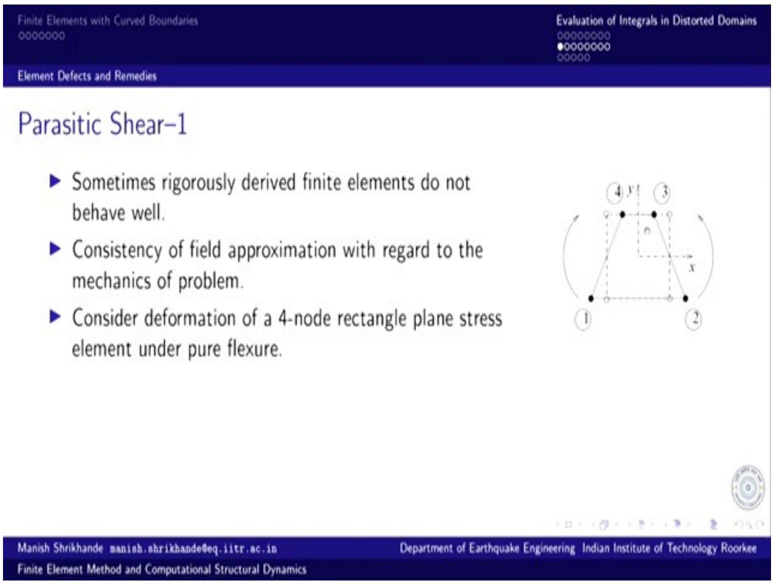

You may look at in this case in this particular case; so, at eta = + or - 1 this has to this has to take the quadratic geometry and at this edge the displacement has to be 0 or on this particular thing the displacements have to be are I mean with respect to x on this edge there is no variation. So, the variation has to vanish in this case on this geometry and similarly for moment applied in this direction it will be the other way round.

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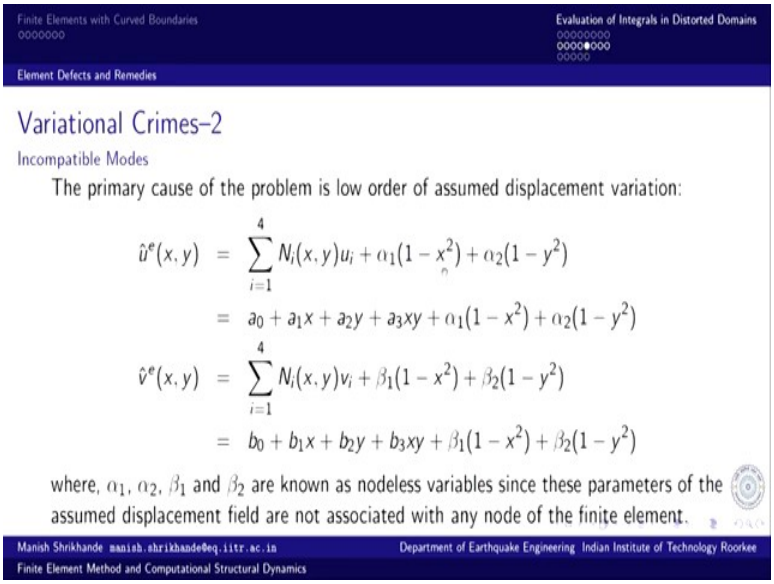

So, basic idea is we need to include these terms quadratic terms. So, that the complete displacement approximation becomes complete in quadratic. So, we already had one term from the quadratic second degree polynomial x y term, now we also have one x square term and y square term and these are included in such a way that their contribution on the boundaries is 0. Wherever we require it to be 0, so they can vanish at the boundaries if they need to be vanish, vanishing.

So, similarly we have a similar modification in case of displacement component v. So, these coefficients of these additional terms, so α_1, α_2, β_1, β_2 these are not associated with any node of the finite element and these are called nodeless variables and these govern the waiting of participation of these additional modes into the displacement field within the element.

And how do they work and if at all they work, why do they work? First of all why do they work is easy to understand because now by inclusion of this they allow these elements allow us to model the curved geometries and the element displacement field will admit a curved shape.

Compatibility; because, compatibility across the element is only defined in terms of the nodal variables, so that is as far as the primary variables of the node are concerned those are maintained those are maintained cum compare as you had shared node. So, the variation between the primary variable as far as the variation of primary variables along the nodes defined at the nodes is concerned that is maintained, but the effect of these additional nodeless variables they will add to incompatibilities between adjacent elements.

So, why these incompatibilities not affect the solution the convergence properties?

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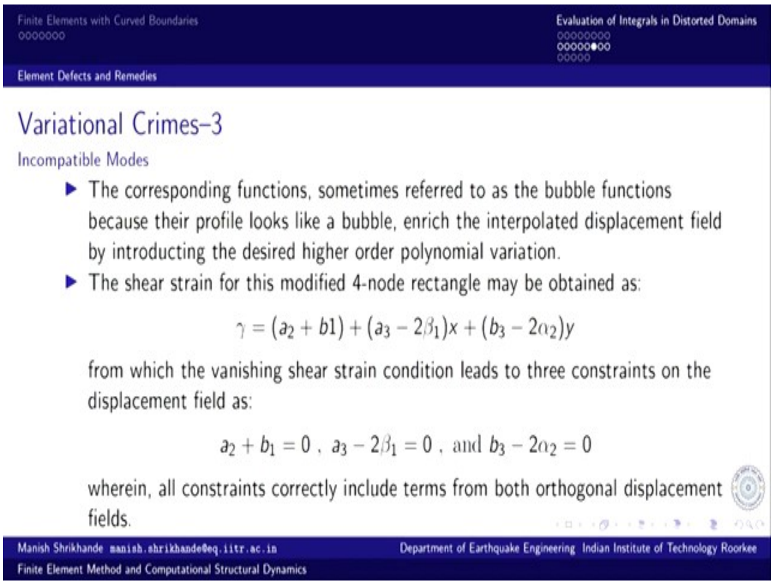

So, these corresponding functions in compatible modes these are sometimes referred to as bubble functions because their profile looks like a bubble (1 - x^2), so it is like quadratic and vanishing at x = 1 and x = - 1. So, it is looks like a bubble kind of a thing and 1 - y square is bubble like this. So, they enrich the interpolated displacement field by introducing the higher order polynomial variation in the displacement.

Now shear strain; if we compute shear strain, then this shear strain can be evaluated and once the varnishing shear strain condition is imposed then we again see that the consistency is now achieved all the constraints involved both degrees of freedom of both displacement components.

So, u displacement along x as well as displacement along y is incorporated in all the three constraints that are arising out of the vanishing shear strain condition. And therefore, the strain field is consistent the displacement is assume displacement pattern will lead to correct strain field in the element domain.

But an unfortunate side effect of this modification is that the modified displacement field is now incompatible along the inter element boundaries, because these nodeless variables are specific to an element. So, for adjacent elements these nodeless variables are going to be different.

So, accordingly the deformation corresponding to these additional functions additional variation quadratic variation that extent or amplitude of those quadratic variation is going to be different from element to element. And therefore there is going to be incompatibility across the inter element boundaries. So, it is because of this incompatibility that the higher order polynomial terms 1 - x^2 and 1 - y^2 that we saw just little while back associated with the nodeless variables are termed as incompatible modes.

Now, what under what conditions these incompatible modes are acceptable? So, as we have been saying loss of compatibility is a serious breach of the basic requirement of the convergence of finite element solution, all through we have been insisting on piece wise continuity of the finite element solution. But in this case we are willingly violating that piece wise continuity of the finite element solution continuity across the element boundaries.

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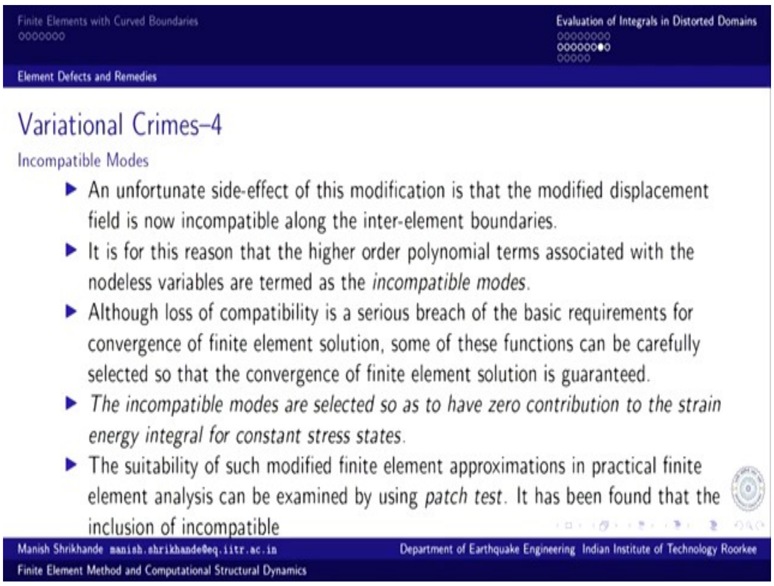

So it obviously, cannot be done arbitrarily. It only some kind of incompatibility may be acceptable. So, what is that condition of acceptability or acceptance criteria for these incompatible modes to be included in the analysis? It so happens that the incompatible modes are selected, so as to have zero contribution to the strain energy integral for constant strain states.

So, when we compute the strain energy so obviously strain energy is going to be quadratic function of the displacement field. So, sigma multiplied by the strain and integrated over the volume and stress is of course proportional to strain. So, essentially it is the quadratic wave function of the strain.

So, that strain field as we saw that will contribute that will have contributions from primary variable and the interpolation between the nodes nodal values of primary variable as well as some contribution coming from these nodeless variables additional incompatible modes.

So, under the specific condition that under the when the applied stress traction is constant and the element as a whole has a constant stress state then in that condition in that situation the contribution I mean this strain energy contributed by these additional terms that has to vanish. If we can find some of these functions such that under the constant and when the element strain element stress is constant that is D times B and multiplied by u.

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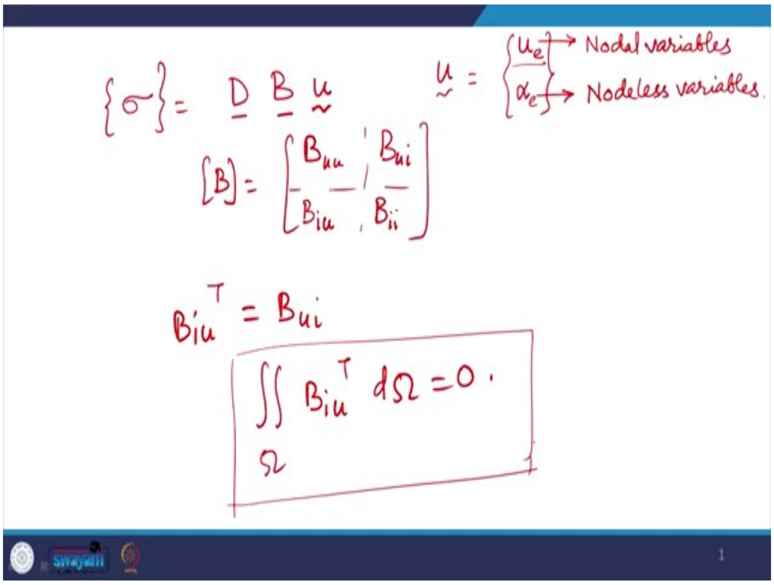

So, stress = D B multiplied by nodal displacement. So, nodal displacement so u in this case = the nodal variables as well as the nodeless variables. So, these are nodal variables and these are nodeless variables and obviously the strain displacement matrix will have some elements which are corresponding to which will operate on these nodal variables and there will be some elements which will operate on which will be contributed by the nodeless variables.

So, the total strain that we have will be B transpose u. So, that is strain displacement matrix multiplied by the constitutive relation matrix. So, this is the stress D times constitutive relation matrix multiplied by strain will provide as the stress field. So, when this stress is constant in the element in that case we need to look at these we can partition this B matrix as so B_ui and B_iu.

So, this will be transpose of this so B_ii is the term that corresponds to nodeless variables. Now what we need is the contribution that comes from the stress that comes from the interdisciplinary these incompatible modes and this B_iu is going to be is same as B_ui.

So, then what we need is essentially this can be defined this can be derived those interpolation incompatible modes are to be used which under which will satisfy this particular criteria. It can be derived proved that once the incompatible modes are such that the integral of this strain displacement matrix over the domain is 0, corresponding to the inter terms corresponding to relating the direct strains or the with respect to the nodeless variables.

Then under the constant stress condition then the solution that we compute will converge to the true solution. So, that is the basis of this crucial very important statement the incompatible modes are selected so as to have zero contribution to the strain energy integral for the constant stress states. Now, this result can be proved and you can look at the formal proof in the text book.

So, how do we ascertain that the convergence will actually occur? Suitability of such modified finite element approximations; so, whatever the conventional finite element formulation meeting all the requirements of the weak form of weighted residual statement, continuity and smoothness requirement.

And if we modify that then whether that modification will work in analysis or whether that is admissible this minimum I mean vanishing strain energy contribution is of course one way of doing that. But a more formal way is or more popular way of doing that is by using patch test.

So, patch test is a technique which is used for I mean several things now, but it was essentially derived or developed to examine the suitability of these inclusion of incompatible modes in the finite element solution. So, we will examine this patch test in our subsequent discussions and how do we do this patch test and what are the issues involved in patch test we will discuss in our next lecture. Now, we come to the next trick of the trade so to say the reduced integration.

So, as we can see that the problem of locking is essentially the displacement response is 0, irrespective of whatever the amount of loading whatever the magnitude of loading we impose. So, the we can keep on increasing the magnitude of amplitude of the moment and still the element would still not deform and the results predicted would be 0 or very very negligible displacements.

So, another interpretation of that is the stiffness is infinite, displacement is 0 in response to a load only because the stiffness is very high. So, that is a simple mechanical explanation. So, if this problem is low displacement because of very high stiffness, then that problem can possibly be resolved by somehow reducing the stiffness computed stiffness. Since, it is anyway computed by using numerical integration quadrature accumulating the contribution of domain integral from several sampling points.

So, maybe we can if we try to do that using a lower order integration situation would resolve and that is what was done. Because, the correct full quadrature requirement for integration of the stiffness terms for 4 node rectangle is 2 by 2 quadrature because of the presence of quadratic terms x square and y square terms in the integrand.

So, two term two point quadrature is required for exactly integration and two point quadrature is arrived at the sampling points are + or - 1 by root 3 if the domain is from - 1 to + 1. So, the 1 degree lower quadrature would be one point quadrature and one point quadrature sampling point is the centroid.

So, centroid is the that is the midpoint of the 4 node rectangle. So, one of the earliest now tricks to circumvent this problem of shear locking based on the observation, that the computed shear strain in a 4 node rectangle subjected to pure flexure vanishes at the centroid of the element. Let us again look at the problem.

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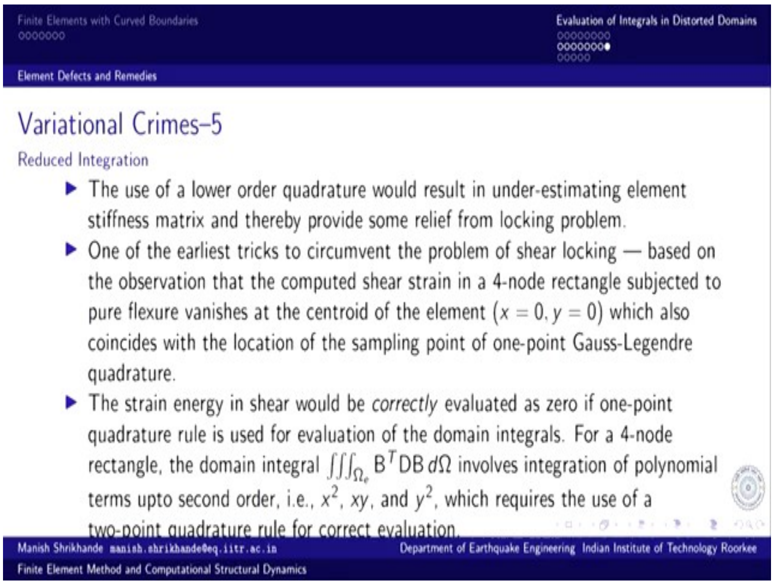

So, at this point centroid of the element the there the shear strain vanishes and the Gauss quadrature one point Gauss quadrature point sampling point coincides with this point centroid. So, if we evaluate the quadrature at this point shear strain would be 0, everywhere else there is distortion shear distortion but at this point this shear strain would be evaluated to be 0.

So, that was the basic idea and it coincides with the location of sampling point of one point Gauss Legendre quadrature and that was evaluated in the sense a strain energy in shear would be correctly evaluated as 0, if one point quadrature is used in the domain integral in the evaluation of domain integral.

For the 4 node rectangle the domain integral $B^T D B d\Omega$ involves integration of the polynomial terms up to second order that is x^2, xy and y^2 which require two point quadrature rule for the correct evaluation. Now, there are I mean there is no problem with two point quadrature, if we are evaluating the direct strain terms.

The product of $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ direct strains are not a problem, the problem is only when we evaluate the shear strain terms. So, what is done is we make use of 2 by 2 quadrature for evaluation of the bending terms flexural terms and one point quadrature for evaluating the shear strain term integrand involving shear strains.

So, when we do that so we are using two different orders of quadrature rule, one for bending and one for shear. So, this is what we call as selective integration selective quadrature or selective integration. So, we use a different orders of quadrature rules for flexural for the direct strains, we involve we use the full two point quadrature and for shear strains we use reduced quadrature that is one point quadrature and the results seemingly are very good so to say.

I mean the element behaves this with this kind of trick the element behaves magically as if all I mean this was the best thing to have happened. But obviously, this is wrong and care needs to be taken. And this is only because the coincidence of sampling point of lower one point Gauss-Legendre quadrature coinciding with the point where the shear strain also vanishes in the problem of pure flexure.

So, that is a coincident a very happy coincidence and it kind of solves the problem and this was the first approach towards these locking problems until the problem of locking was understood what is causing the problem and then a systematic solution for locking problem was derived by using the consistency requirement in the interpolation for the element for the primary variable within the element.

And before that this reduced integration and selective integration were promoted and they were used very exhaustively. Of course, it was not guaranteed whether it will work correctly or not and that is why the patch test was developed to ascertain whether these tricks can in the limit produce reproduce the results that we so religiously adhere to that is the constant displacement condition and the constant strain state or constant stress state.

If those requirements can be reproduced by in the elements which involve use of these tricks ad-hoc measure so to say. Then the element use of elements will not adversely affect the analysis and the results would converge albeit mono non monotonically, but still it is a useful analysis useful result. Now, as I said the reduced integration can often lead to instabilities.

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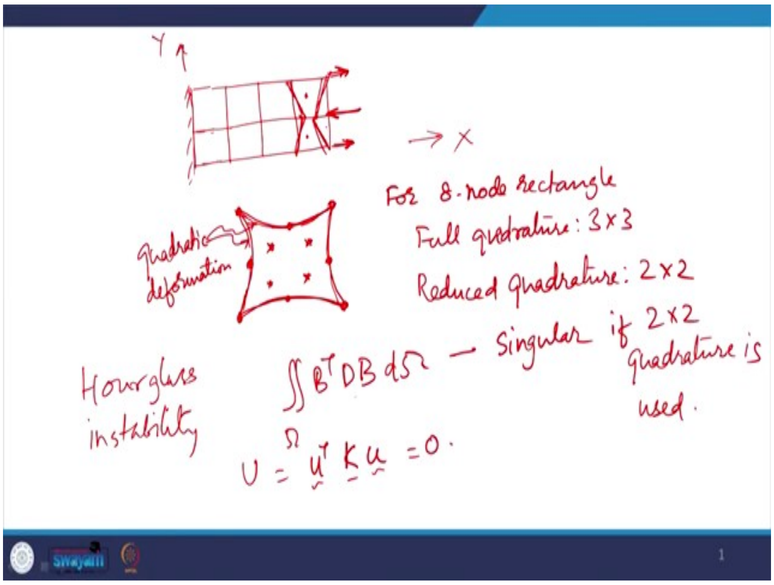

For example, if the loading condition is such let us say if we have a patch of elements and that will also explain why it is important to use selective integration instead of reduced integration as a whole. So, let us say this is constrained these nodes are constrained and if the loading conditions are such that there is a load in - x direction so this is positive x and positive y direction. So, if this is the middle level it is under negative x direction and top and bottom level it is positive x direction.

So, these elements would try to model I mean kind of achieve this kind of behavior, if these strains I mean this loading combination will encourage this kind of mode of deformation and this mode of deformation would be is inbuilt into the formulation because of these reduced one point quadrature at the middle point.

And similarly if I use Gauss quadrature I mean 8 node quadrilateral which has been distorted I mean the, if the deformation shape is kind of parabolic variation and if this element has been evaluated reduced integration 2 by 2 reduce integration. So, for 8 node rectangular quadrilateral element the full quadrature for 8 node rectangle full quadrature rule is 3 by 3 and if we use reduce quadrature one order lower, so that is 2 by 2.

Now, it is so happens that if this is quadratic and this is also quadratic deformation on both sides, then at these two 2 by 2 quadrature level this strains vanish that is a numerical value that is a numerical its again coincidence in unfortunate coincidence in this case.

For this particular mode of deformation quadratic when all the sides are moving deformed in quadratic fashion then the strains at this point these which coincide with the sampling points of 2 by 2 quadrature they are 0 and once that is evaluated to be 0 then the contribution one it comes to evaluation of stiffness matrix B transpose D B d omega and this is evaluated using the contribution accumulating from this.

So, they will all be evaluated to be this stiffness matrix that we compute is going to be singular if 2 by 2 quadrature is used right. I mean singular; that means, the strain energy computation would be 0 if it is a multi associated with these quadratic deformation pattern.

So, it is of course the matrix is not to be decoupled from this deformation pattern. So, strain energy arising because of this u transpose k u will be 0, if u is corresponding to this deformed deformation pattern and k is evaluated using these 2 by 2 quadrature rules.

So, this is obviously wrong I mean this is not a 0 strain energy mode. So, this is what we call as hourglass instability, if you see this deformation is like looks like an hourglass figure and this is also like an hourglass figure. So, if the deformation pattern is such that hourglass instability is triggered this has to be triggered it will not I mean it will not cause problem in any arbitrary load or the deformation case.

But if the deformation mode is such that it is close to this or closely resembles this hourglass figures, then we are looking at a problem potential problem and this has to be guarded against. One has to be aware of these situations otherwise the results could be difficult to interpret to say the least.

So, these are the issues with reduced integration. So, incompatible modes obviously they cause incompatibility in the displacement field across the element boundaries, so and reduced integration is also not without its own share of problems, it may trigger instability in the solution if the loading condition or the impose deformation pattern is such that which will trigger the hourglass instability mode. And one has to be cautious about these issues and closely guard against any such problems and evaluate the answers interpret the computed solution keeping these possibilities in mind.

In our next lecture we will discuss the patch test which is an extremely important tool for analysis of suitability of these tricks of the trade variational crimes in finite element analysis. And also it is the new found used for patch test is also in the error analysis posteriori error analysis. So, we will next lecture we will discuss the patch test.

Thank you.