Finite Element Method and Computational Structural Dynamics
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Lecture - 35
Mapped Elements - V

Hello friends. So, we have seen the effect of distortion on the reduction in the rate of convergence or adversely a impacting the rate of convergence by virtue of inadequacy in complete quadratic field or higher order complete polynomial which are otherwise present in the undistorted geometry the same degree of completeness may not be carried over in the distorted domains.

But at least the constant strain states are maintained and the first degree completeness is always assured. So, the convergence is of course, assured, but the rate of convergence might be little slower than undistorted or parent regular geometry elements in the case of distortion element distortion.

Now this parametric mapping and all as I said this has enhanced the applicability and appeal of finite element method and it has led to several problems and several interesting applications and some of the problems which were difficult to address earlier they became quite feasible and quite easy to handle within the framework of these isoparametric formulation or in general distorted elements.

So, we now look at some of the problems that crop up in finite element analysis and what do we do about those, I mean seemingly all the rigorous formulations I mean finite element formulation all the requirements are rigid rigorously or religiously adhere to, but even then the derived finite elements may not behave well may not perform according to expectations you will realize I mean you will appreciate that in a while.

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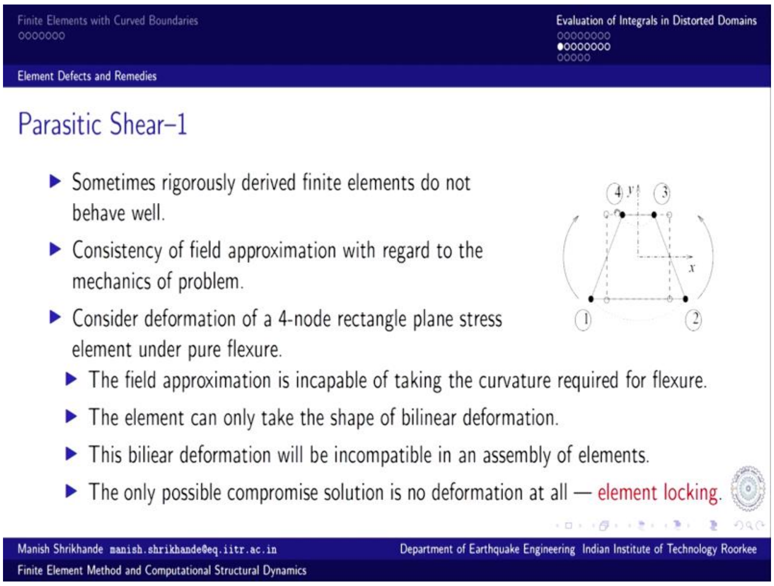

And the reason for that is the consistency of field approximation. We talked about the requirement of approximation field the what are the requirements of finite element approximation or polynomial interpolation and in addition to compatibility and continuity requirement, we also had a specific mention of consistency of field approximation and this aspect is related to the consistency requirement and we will explore this in a little greater detail now in this lecture.

So, let us consider the deformation of a 4 node rectangle plane stress element under plane stress conditions under pure flexure. So, it is only trying to bend the bracket kind of a system so just pure flexure, like this. So, this is a 4 node rectangle and it is subjected to pure bending on the two sides.

Now when I say 4 noded rectangle; obviously, they geometry the deformation that is modelled for both u and v it is it includes a constant term let us say a_0 then a linear term a_1 x linear term in y so, a_2 y and then fourth term would be a higher order term that would be x y so, a_3 x y.

So, idea is I mean it is bilinear it is linear in x multiplied by linear in y. So, the displacement approximation is inherently I mean it is incomplete it is incapable of representing what we would expect under the pure flexure I mean the curvature. So, this curvature the displacement assumed displacement field is incapable of modelling because vertical displacement there is no way this kind of curvature can be modelled using the deformation pattern that is available.

And similarly, if I apply the moment along y axis considering the y axis as the principle axis then the same kind of a problem will happen in with respect to deformation along x. So, this field approximation that we have for both displacement along x and displacement along y the field approximation is essentially incapable of taking the curved shape or the curvature that is required for modelling pure flexure.

So, what do we do, the problem in I mean since it is it cannot take the curved shape. So, the it can it will try to do the element will try to do whatever is the best possible. So, the nodes will move there will be compression on the top edge and extension on the bottom edge that is all it can model, but the edges would keep on moving along the same line right.

So, there is a compression on top edge and there is a extension on the bottom edge that much can be accommodated that much can be modelled by this element. So, compression on top fibers and extension on the bottom fibers and in between there is a linear variation. So, that is all accommodated, but rest of the things I mean this curvature curved shape deformation is not possible.

Now what happens is I mean this is fine as long as this is only one element, what happens if I have a series of elements modelling the same thing and there is a this assembly of elements is subjected to pure flexure. Then there is one element which is a bending like this, next element will also have to bend like this. So, there is a discontinuity here there is no problem of incompatibility because if I have.

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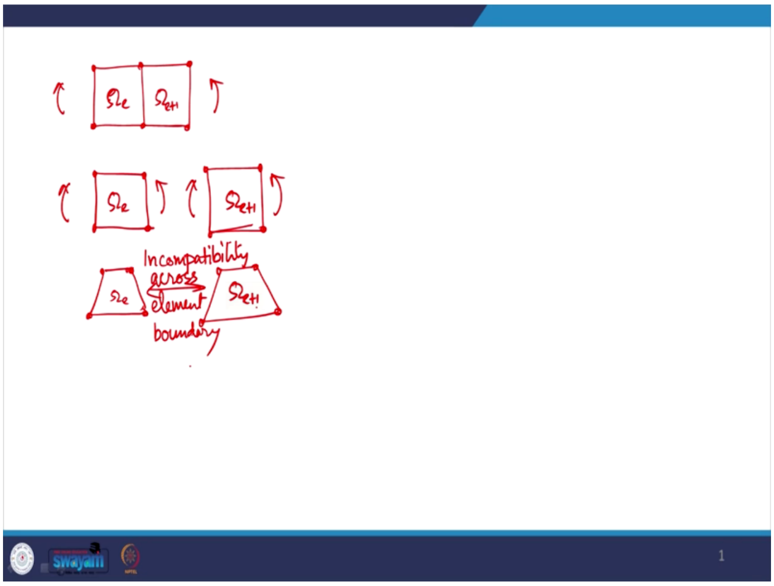

For example, if I take this two element assembly right. So, the free body diagram would be each element right. So, this element will now try to move like this and this element will also try to move like this in response to this pure flexure and there is no way we can achieve compatibility between these 2 nodes if this deformation pattern is to be entertained is to be adhered to incompatibility across element boundary right.

Now obviously incompatibility cannot be I mean we cannot have I mean medium which is continuous medium we cannot suddenly in response to this low flexure it cannot suddenly develop voids. So, what is the best possible way to do I mean compatibility is a very strong requirement. So, if it has to be compatible then the only possibility is that the element just does not deform at all right.

So, all the body entire element remains as it is and that is what we call as element has locked or this is called element locking and now this is a serious problem because it can be erroneous very misleading no matter what amount of load I apply or what magnitude of flexure I apply the predicted deformations would be zero deformation in the element and; obviously, zero stresses, zero strains and all subsequent design interpretation would be all wrong because the basic result is wrong.

Now, what has happened we have not really violated any of the requirements for finite element analysis or development of the weak form and construction of appropriate approximation and perfectly honoring all the requirements that can be expected that were put forth in the solution of the weak form of weighted residual statement and still rigorously derived element would still fail in this particular very simple test of pure flexure the element just does not work.

Now, why this happens? So, let us analyze this problem in greater detail and that will suggest I mean once you analyze this problem that will suggest the solution. So, to analyze the problem I mean the problem is we look at the displacement field as we know 4 noded rectangle, we will have 4 terms of polynomial terms and that will be constant term linear term in x, linear term in y, and a cross term x y.

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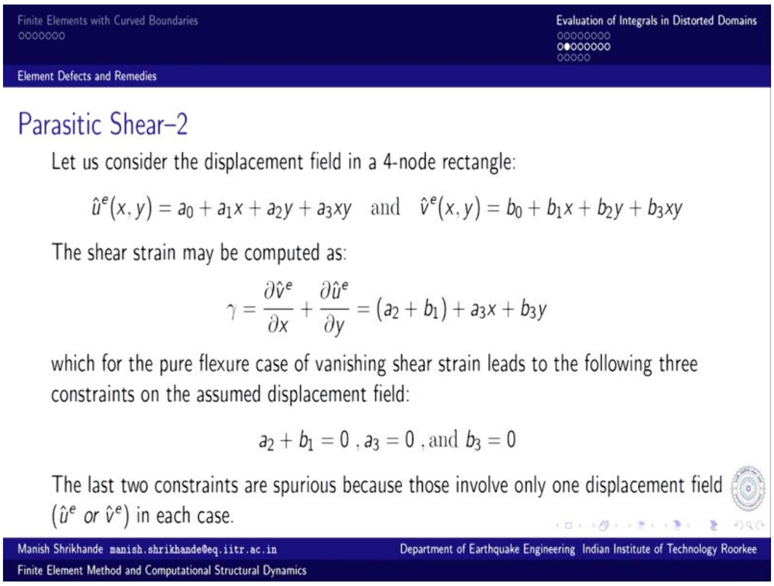

So, $a_0 + a_1 x + a_2 y + a_3 x y$ that is let us say that is for displacement along x direction u component and similarly for displacement along y direction v component is given by other I mean the coefficients may be different so, $b_0 + b_1 x + b_2 y + b_3 x y$.

Let us look at the shear strain. Shear strain can be evaluated as $a_2 + b_1$. So, if we will look at this constant term and then this term will produce linear terms so, $a_3 x + b_3 y$.

And if under the pure flexure shear strains have to vanish, there are no shear strains in case of pure flexure, no shear deformation, no shear strains. So, in that case we apply this vanishing shear strain condition in the displacement field or the strain field shear strain what do we get.

So, all of these terms are independent x variation is independent from y variation. So, all of these terms should vanish independently. So, $a_2 + b_1$ the constant term should vanish, then this term should vanish, the only way it can be guaranteed is when a_3 vanishes and the only way this term can vanish is when the coefficient itself vanishes b_3 vanishes.

What is wrong with that? The problem is shear strain is supposed to be contributed by both deformation fields it is a coupling, it is a coupling term I mean it involves contribution both from u as well as v. So, distortions are essentially a rate of change of u with respect to y and rate of change of v with respect to x.

Now, so any shear strain term should involve combination of both u and v fields and these last two terms they contain representation of only one component either at u component or v component. For example, this a_3 is equal to 0 imposes condition or constraint only on the u component only on the displacement field along x, b_3 is equal to 0 imposes constraint only on the displacement component along v right and that is inconsistent.

When we are talking about shear strain we cannot talk about individual or one displacement component only at a time, it has always to be a combination of I mean constraint should always involve both fields of deformation along x as well as along y and because our displacement model is such that it allows for independent constraints in individual displacement components and that is leading to this problem and this is what inconsistency of finite element approximation is.

And this enhances this emphasizes the importance of consistency of approximation, if finite element approximation may be continuous compatible and required degree of smoothness as well as geometric isotropy everything is satisfied, but if they are not consistent with the basic mechanics of the problem all the results may be useless rendered completely useless as we see in this problem.

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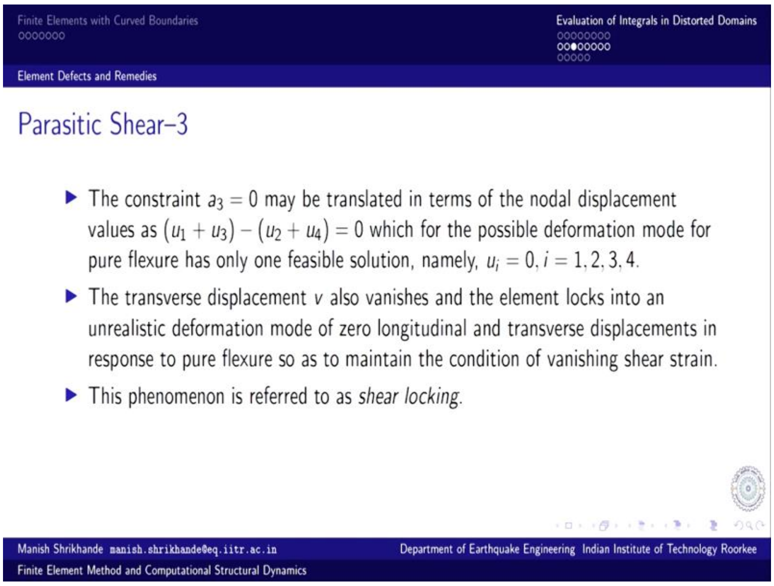

And the constraint a_3 is equal to 0. It may be translated in terms of a nodal displacement I can we can all take that this exercise is easily done. If we can model this using interpolation expression u_e within the element e is equal to $u_i + u_i n_i$ same summation. So, all these coefficients of this polynomial approximation can be related to the primary variable value at the nodes nodal variables.

So, a_3 is essentially it will evaluate to this combination of primary variables. So, that is $u_1 + u_3 - u_2 + u_4$ that is equal to that will be equal to the coefficient of x y term or a_3 and that is required to be 0 and when that is required to be 0 what is u_1 u_3 and what are u_2 and u_4. So, you can look at this. So, u_1 is this, u_3 is this. So, $u_1 + u_3$ and $- u_2 + u_4$ that has to be 0.

So, there will be corresponding I mean whatever by be the stretch of one node 1 that will be offset by corresponding compression at node 3, similarly whatever is this stretch at node 2 that will be offset by corresponding compression shrink at the node 4 and when you add them $u_1 + u_3$ is 0, $u_2 + u_4$ is 0, 0 - 0 is 0. So, that is what leads to this constraint 0 a_0 a_3 is equal to 0. So, essentially all the deformation would be 0.

So, transverse displacement also vanishes by the same argument I mean b_3 is equal to 0. So, it will be similar kind of argument that will happen so, $v_1 + v_3$ _ $v_2 + v_4$ b_4.

So, this phenomena is referred to as shear locking because it is the shear strain that is a causing problem, the problem is in representing vanishing shear strain conditions under pure flexure. If the problem is not of pure flexure then probably this may not appear this may not be noticed, but if the problem is of pure flexure then this will hit very strongly.

Higher order elements I mean if I took if I take 8 node serendipity elements or 9 node lagrangian elements in place of this 4 node rectangle they do not lock in such a drastic fashion, but they are still adversely affected in convergence. So, while higher order terms are always preferred for convergence higher convergence, but we may not get the expected rate of convergence Because of this shear locking problem and very fine mesh may be required.

So, how do we solve this problem, what are these possibilities to address this problem? So, we identified the problem the first step towards solution of the problem is to identify what is causing the problem. So, we identified the problem that the problem is in representation of the shear strain should vanish and somehow our displacement field is such that we are not able to ascertain that correctly the because of the inconsistency of the displacement field.

So, there are several possibilities to rectify this problem some of them were discovered accidentally and were more of a adhoc nature, I mean somehow while playing some tricks actually appear to give better quality results then rigorous formulation and those were propagated and became popular.

So, reduced integration is one such trick. So, domain integral as we noticed domain integral has to be exactly evaluated so that is what is the order of quadrature rule which will exactly evaluate the domain integral that is called full quadrature. So, we may intentionally under evaluate this domain integral.

Since the problem is essentially if we look at it from other end other point of view I mean displacements being 0 can also be interpreted as stiffness being infinite, for an infinite stiffness the deformation would be 0. So, it is a problem of stiffness calculation. So, a stiffness is being estimated very high.

So, how do we estimate this stiffness that is by sampling the integrand b^T d b at different gauss points and accumulating the weighted sum. So, we can try to accumulate less number of terms. So, instead of using the higher order quadrature rule we may use a little lower degree quadrature rule.

And it so happens that if we do that then the results improve dramatically there is a dramatic improvement in the behavior of this term. And we will see why that happens, but that is one of the tricks that has been suggested. So, reduced integration and also another cousin selective integration is also there.

Then next trick is inclusion of incompatible modes I mean so far we derive the element assuming or enforcing all kinds of requirements compatibility, continuity everything right. So, that inter element I mean piece wise continuity is always maintained all through, but that has led to this problem of inconsistency.

So, in order to bring consistency in the field approximation we need to add extra terms, we need to add extra terms for example, that curvature modelling. So, in order to model that curvature we need to add extra terms now when we add those extra terms in one element; obviously, these additions are going to be independent over each of the elements in the neighborhood.

And accordingly the displacement field including these additional modifications a element level modifications is not going to be compatible, because these node less parameters which will define these because there are only 4 nodes. So, primary variables are only those field variables defined to the nodes. So, those are the primary variables.

So, any additional parameter which will add this additional flexibility in the deformation pattern which will bring in consistency will cause in compatibility in the deformation mode in different elements adjacent elements. So, those are incompatible modes. So, we solve one problem by violating one of the requirements another problem and we create another problem elsewhere.

So, these additional terms as I explained lead to incompatibility of primary variables across the element boundaries. So, what is the acceptance criteria I mean; obviously, this cannot be allowed indiscriminately. So, the acceptance criteria is the only those incompatibility incompatible modes are used which do not contribute to the domain integrals under the constant state of stress.

So, we always adhere to that constant state of stress as a convergence criteria in the limit as the element becomes very small it is a constant stress condition and those conditions should be preserved.

So, as long as that incompatible mode that we are adding does not affect or does not have any contribution to the domain integral under the constant state of stress then the convergence of the finite element solution will occur.

Please note that I am only referring to here the convergence of finite element solution, earlier. If we adhere to all the basic tenets of finite element formulation the convergence that we would achieve is going to be monotonic convergence, by violating those principles cardinal principles we will still achieve convergence, but we are sacrificing monotonic convergence and what we will now achieve is a non-monotonic convergence; that means, the error will fluctuate from positive to negative positive to negative so it will.

Now why these so, monotonic convergence is so important and non-monotonic convergence is considered to be inferior to monotonic convergence is because monotonic convergence if I have a series of sequence of finite element analysis I can get an asymptotic value of that a study that trend of monotonic convergence and when I can project it asymptotically and that asymptotic value will actually be the true solution converged true state of solution that is and that is not possible for a non-monotonic convergence or not easily available for non-monotonic convergence condition..

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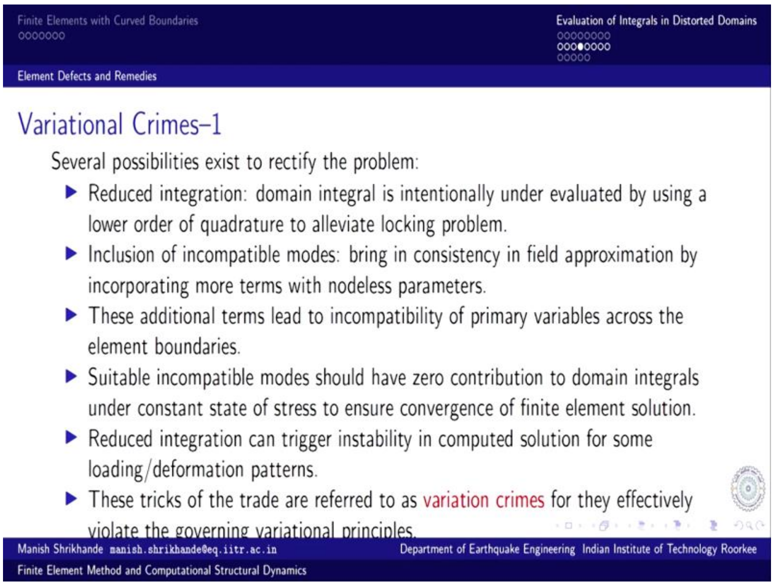

Another issue is reduced integration can trigger instability in computed solution for some loading or deformation patterns. So, the element may perform very well under some conditions, but it may suddenly get unstable or the solution may go completely haywire for certain loading conditions or certain deformation patterns that are imposed on the structure or the model.

And we will study more of these all these tricks of the trade, these tricks of the trade are referred to as variational crimes. So, variational crimes this phrase was coined by Gilbert Strang a Professor in MIT who analyze these I mean basic issues from the point of view of more rigorous mathematical analysis of finite element analysis and then he coin these terms because these improvement in behavior come at the cost of violating the basic governing variational principles.

And we are under evaluating the domain integral so, essentially we are not computing the strain energy contributions correctly and then we are adding more displacements; that means, we are not really adhering to the deformation pattern that is governed by minimum potential energy configuration. So, we are adding something more.

So, either way it is what we are all these tricks they seem to violate the basic governing variational principles of the problem and still the these tricks seem to work and these are referred to as variational crimes and essentially to alert the user that sometimes the crime pays, but one has to be extremely careful while committing these crimes and be very objective in interpreting the results.

We will discuss more on these variational crimes and a kind of trouble that they may land us in and how to assess whether these crimes are paying off or not in our next lecture.

Thank you.