Finite Element Method and Computational Structural Dynamics
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Lecture - 34
Mapped Elements - IV

Hello friends. So, in our last lecture, we have seen how the domain integrals are in terms of Cartesian coordinates and Cartesian derivatives with respect to Cartesian coordinates are transformed into local coordinates and integral is evaluated in terms of local coordinates in the range of minus 1 to plus 1 and the use of gauss quadrature for very efficiently computing those integrals.

So, the entire process as I have been maintaining from since beginning is very well suited for automatic computation and even the evaluation of integrals can be processed as a sequential evaluation and accumulation of weighted sum of integrand at a different sampling points.

And the position of sampling points and the corresponding weights is already available for different orders of inter quadrature rule and appropriate sampling locations and coefficients can be picked up from the library. So, that leaves us with the only remaining term in the weak form of the weighted residual statement that is the evaluation of the boundary term.

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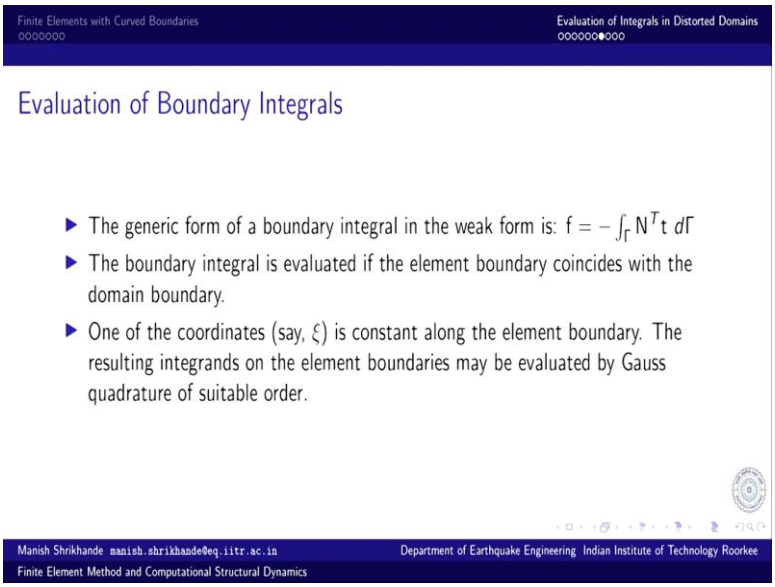

Boundary term is evaluated in terms of as a these are surface tractions and t is this vector of surface tractions on the boundaries and the boundary integral is only evaluated, if the element boundary coincides with the domain boundary because all other in the interior of the domain element boundaries, they would be cancelled out. To these integrals, boundary integrals would be cancelled out by the term coming from the corresponding contribution coming from the adjacent element. Because these boundary integrals are always to be evaluated in while move traversing the boundary in counter clockwise direction.

So, one of the coordinates, let us say ξ is constant along the element boundary because that is how the elements are defined. Elements are defined for local coordinates ranging from minus 1 to plus 1. So, element boundary is always going to be some surface or some line some curve, where one of the coordinate local coordinates is a constant and that is what defines the boundary, element boundary.

So, the resulting integrands on the element boundaries may be evaluated by Gauss quadrature of suitable order because once we evaluate that constant value, then remaining terms on the boundary for the remaining variables can be evaluated by Gauss quadrature.

For example, for three-dimensional domain, the boundary will be defined by a plane and it two-dimensional Gauss quadrature will be suit of suitable order may be used for evaluating those integrals of this type. And for two-dimensional problems, the boundary will be defined by a curve or one-dimensional only one-dimensional will be variable and then, one-dimensional quadrature can be used for evaluation of these boundary terms and with this evaluation of these boundary term, boundary integral and the evaluation of the domain integral as discussed earlier, the element level equilibrium equations are completely defined.And then, these element level contributions can be assembled into global system of equations and then, the computation can proceed to next element for establishing the element level equilibrium for equilibrium equations for the next element, which can again then be put into proper place holders in the global system of equations and once, this operation is completed over all the elements in the mesh covering the whole problem domain, then the essential boundary conditions are on the primary variable are imposed; whatever those boundary conditions may be and then, the solution

the unknown variables are solved for and rest of the process is exactly same; the post processing etcetera.

So, this is primarily the basic idea of how regular elements are not necessarily a restriction. They give us great opportunity or great facilitation for constructing polynomial approximation over regular geometry; but by using isoparametric formulation, we extend their application to model curved or complex boundaries complex domains. And the complete problem may be modelled by using smaller number of elements while minimizing the discretization error, domain discretization error. So, while we discuss element distortion as in the positive frame, positive sense by extending the capacity capabilities of finite element analysis it; obviously, nothing comes for free.

So, this facility of modelling for curve domains comes at a cost and that cost is reduced degree of accuracy in approximation of primary variable or field approximation, whatever the basic approximation is constructed for the primary variable and that affects the rate of convergence. Of course, convergence is guaranteed as long as the distortion is moderate and those requirement for ensuring convergence is met. Those requirements are there too, then the convergence will of course happen. But the results may not converge as rapidly as expected.

For example, if we are using second order quadratic elements or bi-quadratic element, 9-node Lagrangian or 8-node serendipity element and the if the elements are distorted in any curved shape, then in the distorted domain the degree of complete polynomial is only up to first degree. The second degree variation, which was complete in the undistorted domain unfortunately is no longer guaranteed in the case of distorted domain, if it is a curved distortion. For bi-linear distortion, then 9-node Lagrangian includes complete description; 9-node Lagrangian element is better. Then 8-node serendipity elements 8-node serendipity elements do not have complete polynomial approximation in distorted geometry for second degree approximation. The polynomial is complete only up to the first degree.

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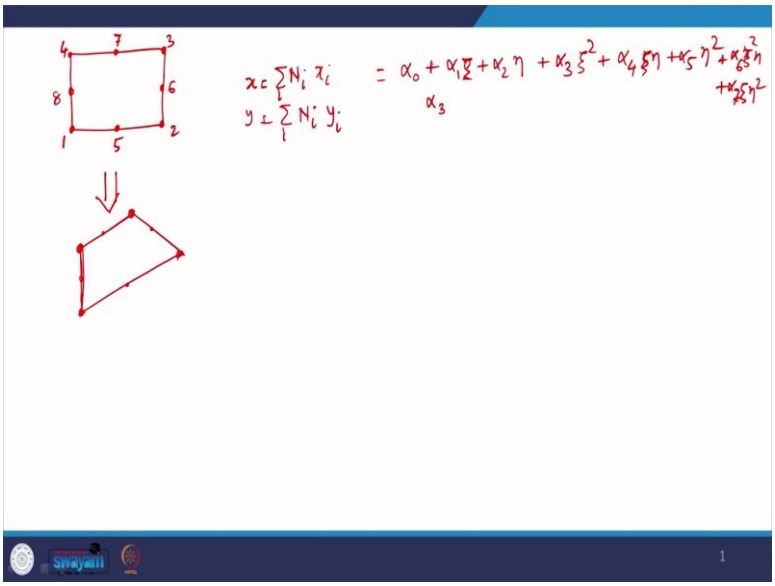

So, this is a standard, I mean very interesting result and proof is very simple. So, to say for example, let me just explain that to you. So, we have 8-node serendipity element; 1, 2, 3, 4, 5, 6, 7 , 8. So, x is defined in terms of $N_i x_i$ and the terms that are there and similarly, for y summation over i and the terms that are there x i can write it as α_0. So, these are the quadratic in $\xi \eta$ in terms of quadratic variation, in terms of this and so.

So, this is for if x is mapped using these 8 number of nodes, so isoparametric mapping. But let us say if this element is distorted into a bi-linear distortion. So, we are essentially using only 4 nodes; mid side nodes, we can also use. But the value of this mid side node coordinates would be linear interpolation between the coordinates of these corner nodes. So, effectively, it is same as using these 4 corner nodes. So, if this is the case, then all that we need to do here is the approximation will be including $\xi \eta$ term because it s a bilinear distortion. So, 4 nodes, so there will be 4 terms; so, the 4th term will be $\xi \eta$term.

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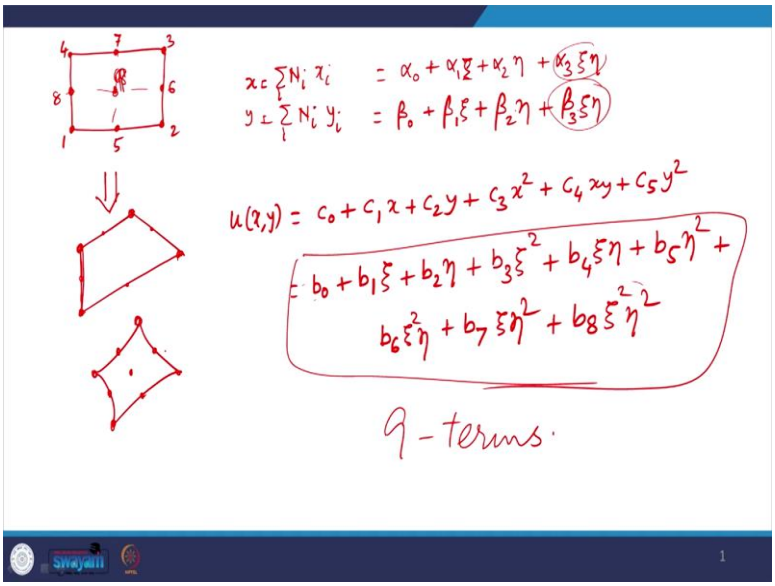

And similarly, for y. And if the displacement has to be second degree complete because if this polynomial variation u, if this has to be second degree complete u as a function of x and y. So, what I can do is this is the bilinear distortion of x and y. So, I can substitute this expansion bilinear expansion of x and y in this expression and then, try to collect terms what are the terms that I get right. So, what are the terms that we have? so, quadratic terms we will have. So, the terms would be shown in the slide. So, there would be these terms, There are total of 9 terms here; 1, 2, 3, 4, 5, 6, 7, 8, 9.

So, total of 9 terms. These 9 terms, 9 independent terms cannot be defined by 8-nodes; interpolation of 8 variables, interpolation of between 8 values. And therefore, this quadratic variation is not complete because there is no way to incorporate this 9th term in the 8-node approximation and therefore, the behaviour of this 9-node Lagrangian element is often superior under this kind of bilinear distortion. Of course, both the elements suffer I mean are incomplete, if the distortions were of this kind, quadratic distortion. If the element was distorted into this shape, then it really does not matter whether I use 8-node serendipity or 9-node Lagrangian element; both of them fail to meet the completeness requirement for second degree polynomial and they are accurate only up to first degree polynomial.

So, that is the first fall out of distortion. So, degree of completeness suffers is adversely takes an adverse hit and therefore, the accuracy is what we gain in terms of domain

discretization and larger element size is somehow to some extent compensated by this loss of accuracy in the distorted domain.

Nevertheless, the mapping, the power of parametric mapping to generate elements of any arbitrary shape is a great asset and the isoparametric formulation has led to tremendous growth of finite element method for modelling complex geometries. But one should be aware of the possible sources of errors and the reduced convergence because of distortion. So, large distortion will have greater effect on the rates of convergence. But element distortions allow for mesh gradation. For example, I can how do I change? I mean if I have steep strain gradients in certain location for example, at the re-entrant corners. So, strains very quickly very rapidly, stresses are varying very rapidly.

So, there in the region of that stress concentration, I need to use elements which are very fine; how do I change to a coarser mesh or larger element, if the strain field or the stress field is no longer varying very sharply, the gradients are very smooth. So, I can afford to use larger element.

For example, constraint strain triangle; if the strain is constant, not changing significantly. Then, I can afford to use a large constraint strain in that range or a large rectangle in that particular zone.

So, how do I ensure this transition between small elements in the zone of stress concentration and large elements in the zone of smoother gradients? So, this is facilitated by the mapping, the element distortion because I can define the elements in such a way that for example, just to show case for example, if I have two elements here.

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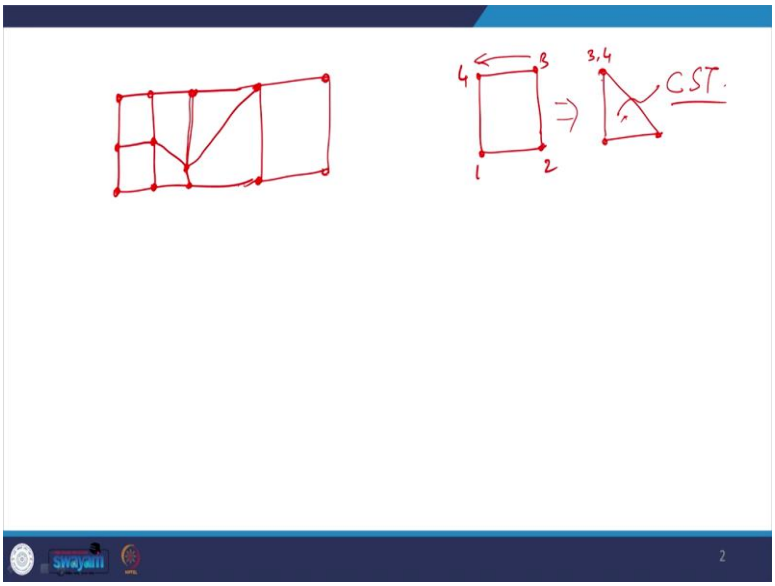

Now, of course, I need for continuity, I need that there should be a linear variation for any element that comes here. So, I can define this something similar and then, this can be taken as. So, this kind of element distortions, they facilitate the transition between finer mesh to large mesh and it is possible to define I mean of course, this is not a very good mesh because this is kind of two skewed element dimensions. So, but it is possible to make a better mesh with more reasonably sized element size sides; but the basic idea is by virtue of distorting the elements, it is possible to go from small elements to large elements gradually.

So, that is what is known as the mesh gradation and that allows smooth transition from fine mesh to coarse mesh and that is a great benefit in finite element modelling while maintaining perfect balance between the accuracy and the efficiency of solution. The numerical stability of computations is strongly influenced by the extent of distortions. If I distort it too much, then as I said the Jacobian gets very strongly influenced and it may be near singular and if the Jacobian is near singular, then the inverse of Jacobian will be very erroneous and the gradient calculation, the derivative calculation will all be questionable and that means.

So, the evaluation of the domain integral itself is in doubt; the accuracy of that domain integral is in doubt and the reliability of the computations is doubtful. So, there are guidelines about the distortions and how the distortion should take place. The operating

keyword is distortions should be moderate and that means, if there are 3-nodes in the side; for example, 8-node serendipity or 9-node Lagrangian, the position of the mid side node is best position is in the middle third of the edge. It should not deviate to beyond the middle third region of the element side. So, as long as that node remains within the middle third of that, the element computations are going to be the distortions are going to be acceptable. But these distortions can be used effectively for modelling singularity, as I said it can become singular.

The Jacobian can become singular, if in case of when the one to one correspondence is lost. But for some problems we do need singularity modelling, for example, the stress concentration at the tip of the crack. So, the stress is actually infinite. So, as you as we approach the crack tip, then the strain gradients increase exponentially.

So, this is what we in linear fracture mechanics, we call this square root singularity at the crack tip, if I if we truncate the strain field at the first term, after the first term of the series. So, it is a square root singularity; it is a proportional to inversely proportional to the square root from the distance from the crack tip, the strain field. So, this crack tip square root singularity can be very easily modelled by locating that mid side node in the quadratic elements of bi-quadratic element to the quarter point.

Just now I said the best location is within the middle third. If I venture outside this, then I approach singularity. Sometimes, I do intend to model singularity and then, if I take that node towards one-fourth position on the both sides for both the edges then that node corner node will have the strain field will have square root singularity. You can work it out and work out these strain field and that you will be in for a very amazing result. And that helps in this simple mapping helps in finite element modelling. I mean the infinity can never be modelled, if I use I mean the stress any finite element, finite size, the stress computed will always remain finite. So, by this simple mathematical parametric mapping, we can represent the infinity or singularity of the stress concentration at the crack tip and that leads to again very efficient finite element analysis of the entire problem with very small number of elements.

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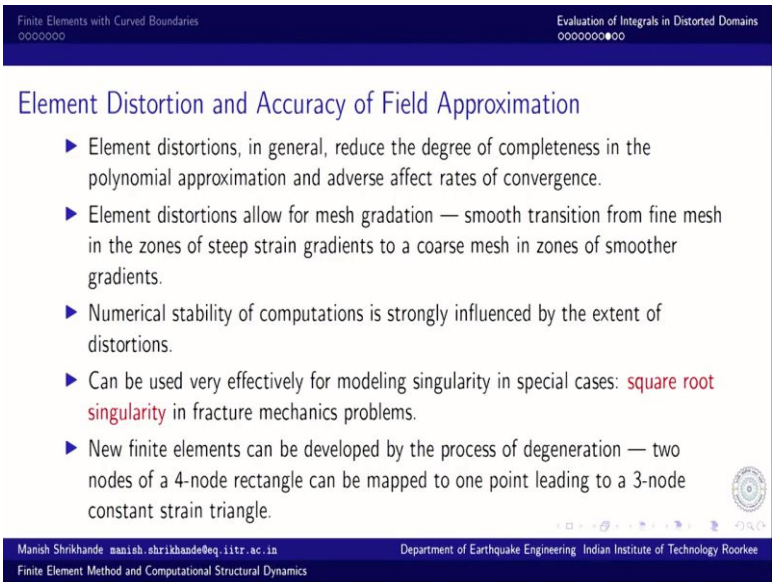

Further. new finite elements can be developed by the process of degeneration. For example, two nodes of a 4-node rectangle can be mapped on to for example, these 4-nodes. So, I can map this node on to this node. So, that the element becomes like this or it can go to this way whichever. So, if I do this, then this becomes a 3-node triangle and if I formulate this, if we go through the entire process of this mapping and assign the same coordinates in this physical domain for node 3 and node 4 and then, compute the strain field we will find that this particular triangle, 3-node triangle that we have generated will be sim the strain field will be constant.

So, we can generate a 3-node triangle, 3-node constant strain triangle from collapsing 1-node, 1 corner node of 4-node rectangle on to another corner node to make a 3-node triangle. Of course, at this particular point one to one mapping has been lost and the Jacobian will be singular at this point.

But for a constant strain triangle, we do not need to evaluate Jacobian at this point; it is constant everywhere. So, we can simply evaluate it at the centroid of the element and get all the computations done and we will be safely away from the point of singularity that is here. So, if we try to do any computation around this point around this location; obviously, that is questionable. But fortunately, we can operate in the return in the neighbourhood of centroid or at the centroid because all the strain fields are constants.

So, it really does not matter. I do not we do not really need to evaluate it at that point of singularity and things are under control.

So, new finite elements can be developed and this process is called the processor degeneration and this process of degeneration is very popular for developing the elements for analysis of plates and shells. Plates and shells are a difficult problem for analysis using finite elements and not without reason, they are very special structures, special structural forms and with behaviour governed by geometry as well as by the mechanics. And therefore, special considerations are required .

So, more on that when we discuss plate and shells. So, this distortion of course, helps us a lot, but there are certain element defects and sometimes a very rigorously formulated elements, finite element fail to give us adequate result. So, when does that happen? It is often the consistency of field approximation with respect to mechanics of the problem that is violated. We will consider this process and the element defects and the consistency of approximation in our next lecture.

Thank you.