Finite Element Method and Computational Structural Dynamics

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Lecture - 32
Mapped Elements - II

Ok friends. So, we have seen the promise that is that the parametric mapping concept holds in extending the concept of straight edge or straight phase elements to model any arbitrary distorted domains or any arbitrary shaped domains. And by using the interpolation model that we use for interpolation of the primary variables. So, there are various possibilities of parametric modelling.

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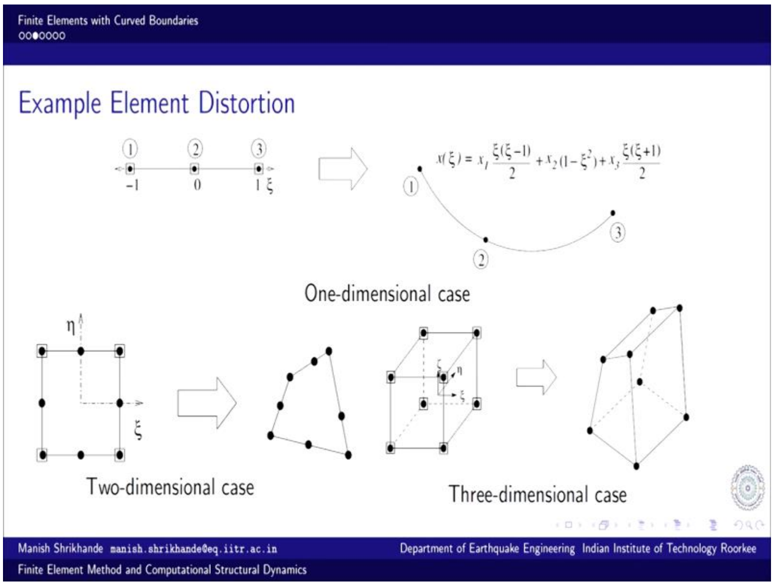

For example, just as in this case there are three nodes. I was referring to field circles they are used. These are the nodes used for approximation of the primary variable and small squares; these are the nodes which are used for the modelling of geometry. So, in this particular case both of them are identical. So, the geometry modelling is also based on three nodes and the variation of the primary variables within the element is also based on three nodes; interpolation between the values at these three nodes.

So, these are called iso-parametric variations. So, the modelling of the geometry as well as the or the polynomial approximation for the geometry is of the same degree as the polynomial approximation for the primary variable in element domain.

In this particular case the rectangle case, we used only four nodes for modelling the geometry four corner nodes, while the primary variable is modelled by using the interpolation between eight nodes. The primary values or the nodal variables at these eight number of nodes in the element defining the element eight noded serendipity element rectangle.

So, the degree of primary variable here is 2, but the degree of geometry approximation is only bi-linear. So, the because that is based on only these four nodes. So, this is what we call as sub parametric. So, the geometry variation is of lower degree than the variation of the primary variable. And by corollary, we also have super parametric variation where the geometry is interpolated at a higher degree of polynomial variation than the primary variable variation in the element domain.

Now, if you look carefully, then sub parametric variation is included within iso-parametric variation, because a higher degree approximation can always account for a lower degree variation, because we always retain the complete lower order terms in our interpolation function.

So, if interpolation is complete up to second degree, then the first-degree interpolation is also included. So, sub parametric is always is of course, a subset of iso-parametric variation. And this is again an iso-parametric variation for three-dimensional geometry. All eight nodes are used both for the variation of the primary variable as well as for the geometry.

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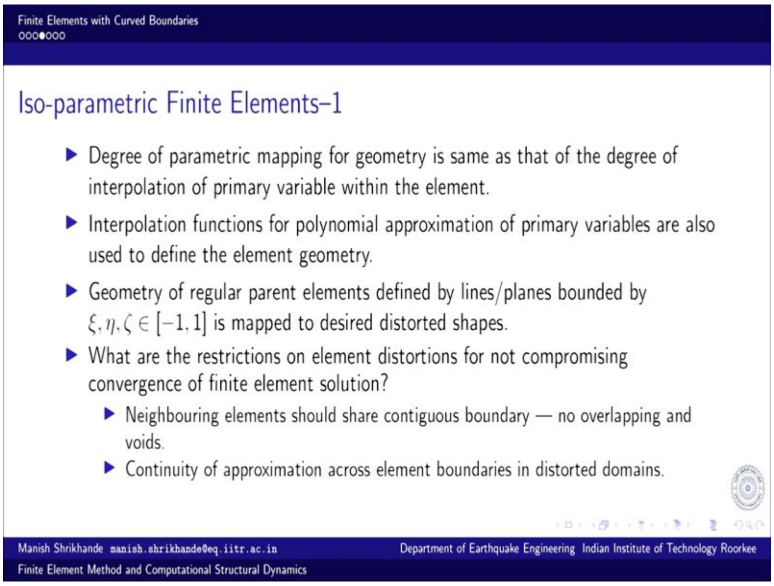

So, we use iso-parametric formulation in our subsequent discussion. We will not refer to sub parametric, because as I said earlier, it is a subset of iso-parametric formulation. Super parametric is generally not used in the context of finite element analysis that is not such a need for such a high degree variation of geometry.

So, for iso-parametric element formulation again the it is more, because of the ease that the degree of parametric mapping for geometry is same as the that of the degree of interpolation of primary variables within the element. And a direct fall out of this is we can use the same interpolation functions that were used for the approximation of the primary variable for the approximation of for the mapping of geometry.

So, distortion of geometry. So, interpolation functions for polynomial approximation of primary variables are also used to define the element geometry in exactly same form the structures remains the same $u = n_i u_i$ and $x = n_i x_i$. So, geometry of regular parent elements is defined by lines or planes bounded by those straight lines or areas defined by the coordinates local coordinates xi is equal to plus or minus 1.

So, this plus or minus 1 is the domain which puts a boundary on the elements. For example, in the case of one-dimensional element it is bounded between minus 1 to plus 1.

So, in case of a two-noded element this is of course, three node element, but it is possible to have a two-noded element, then in that case, two-node element; one node would be at minus 1 and another node would be second node would be at plus 1. For three-noded element, the two extreme nodes are at plus and minus 1 and mid side node is located at 0.

Similarly, for two-dimensional element, the boundaries coincide with the lines defined by ξ= + 1 or ξ= -1. η=+1 and η =-1. And similarly, here, the similar concepts hold. So, this plane is defined by η=+1, the bottom plane is defined by η=-1, this side is defined by ξ = 1, this plane this side is defined by ξ=-1 and these two phases are defined by η=-1 and + 1.

So, they together represent the finite domains and bounded by the natural coordinates ξ η bounded ranging between - 1 to + 1. So, that gives us a very standard format. So, that also facilitates standardized element library while writing the finite element code, because the local coordinates only vary between - 1 to + 1 or between 0 and 1, in case of triangle and tetrahedron elements.

So, geometry of regular parent elements is defined by lines or planes bounded by constant values of the local coordinates. And these regular parent elements are distorted or mapped to desired distorted shapes. So, what are the restrictions on the element distortions?

For not compromising convergence of finite element solution, we talked about the necessity of preserving the constant derivative condition, necessity of preserving the constant term in the approximation which will model the rigid body motion and constant strain states being retained. So, that was applicable. It was shown that for when we have the regular geometry and interpolation functions are chosen in certain way. Then, these properties can be satisfied and that is how the convergence of finite element solution can be guaranteed. Now, how this convergence of finite element solution can be guaranteed; under what conditions can it be guaranteed if at all, if we are transforming the element domains.

So, whatever was there earlier in the regular domain that may or may not hold in the transformed domain that depends on the mapping. So, what are the conditions under which this kind of distortions are permissible and they will not compromise on the convergence of finite element solution.

So, first thing that domain representation the discretization should be faithful or complete fidelity should be there in finite element mesh and the actual domain. So, that neighbouring nodes or neighbouring elements if they are adjacent I mean it is a continuous domain.

So, they should adjacent element should have contiguous boundaries it should not be like this or it should not be overlapping over each other like this right. So, no overlapping or no voids in between the adjacent elements neighboring elements, if the domain does not have a problem domain does not have that right. So, it is a continuum if it is a continuous domain in the distortion. After distortion, the distorted elements mapped elements should represent the continuous domain should represent that contiguity. So, all the adjacent elements, they should have contiguous boundaries.

Secondly, the continuity of approximation of primary variable across the element boundaries in distorted domains. So, in the regular domain, we define that if we choose the interpolation function in such a in a certain way, then the continuity of the primary variables would be satisfied.

So, C_0 continuity C_1 continuity depending on the problem that we discussed and that was established that if we do this, then the continuity required degree of continuity of primary variables can be ensured. So, that also needs to be satisfied in the case of distorted domains. So, primary variable in the one element and the adjacent element in the at the boundaries of these two adjacent elements the primary variable should take unique value. So, it should be continuous. So, piecewise continuity should be maintained desired degree of continuity. And then finally, completeness of the interpolated field in distorted domains.

So, completeness requirement; that is the rigid body motion, the constant term, the constant strain states so that at least in the limit, it should have the capacity to represent the constant state of strain in the limit as the element size coalesces to a point. So, that would be a constant strain at a point. So, that capacity has to be there. That capacity should not should never be compromised after distortion.

So, all these three requirements are essential before we can believe or before we can we start looking at the results of these mapped elements with any confidence. So, how do we do this. What are the under what conditions or how do we ensure these conditions these three basic requirements, because if any one of these is violated then, it is gone. The problem is not worth pursuing further.

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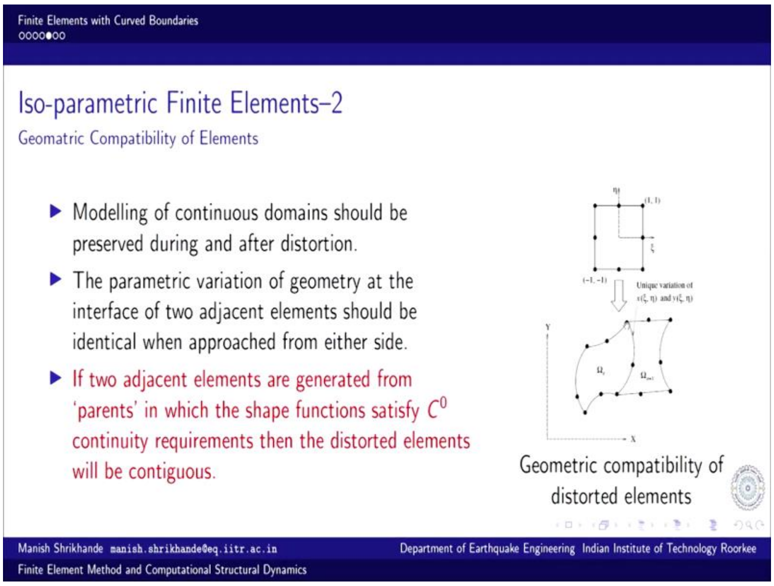

So, let us start with geometric compatibility of the elements. So, it so, happens that modelling of continuous domains should be preserved during and after distortion as I said. If the original domain is continuous, then after then the finite element mesh should also represent that continuity of the domains and individual element boundaries, they should be contiguous. There should not be any gaps and there should not be any overlap.

So, the parametric variation of geometry at the interface of two adjacent elements should be identical. So, if I have an element or two elements and they have been distorted. So, there is one element like this and another element like this. So, this parametric variation of this common boundary, it should be identical whether I approach it from this element side or whether I approach it from this element side right. So, if that happens then; obviously, if there is no chance. I mean both of them share the same nodes and the variation parametric variation is same. So, there is; obviously, they will be mapped onto same curve or same surface.

So, the degree of parametric variation has to be identical and that is guaranteed if the two adjacent elements are generated from the parent elements regular geometry elements, in which the shape functions satisfy C_0 continuity requirement.

C 0 continuity requirement ensures the continuity of the variable being interpolated. What is the primary variable of interpolation. So, that is ensured continuity across the inter element boundaries. So, in case of geometry, what we are interpolating is the geometric parameters the coordinates x coordinate how the x coordinate of each point along a line varies. So, the locus of a point and x and y coordinates x y and z coordinates for that matter. So, that would be satisfied, if the two elements that are adjacent, they have been derived from parent elements in which C^0 continuity is guaranteed. And of course, when we do that we also insist that is that goes without saying that the number of nodes and the parameters that are shared they are of course, identical.

There is no way continuity can be satisfied, when I use one model one element one side as quadratic and the other element as a linear. So, both the I mean quadratic and linear both of them individually they satisfy C^0 continuity, but they cannot they are not compatible with each other. So, if it is a quadratic mapping quadratic variation, the adjacent element has to be of quadratic mapping. So, that is a essential realization essential requirement of this mapping. So, again taking the example of eight node serendipity element.

So, element omega e is distorted into this shape and this particular common boundary between element e and e +1, because these three nodes are shared between the two elements and the variation is quadratic along these at this edge, because this is defined by quadratic variation in eta, because ξ value is corresponds to ξ=1 in this for this element.

And similarly, for this element, this edge corresponds to ξ=-1. So, this edge variation of this edge is second degree polynomial in η and second-degree polynomial is defined is uniquely defined by three distinct points.

So, if I have the three distinct points and geometries defined by coordinates of these three points, then the contiguity or the contiguity of the edge boundaries of these two elements is guaranteed, because each point coordinate of along this line as we approach this edge from element omega e is going to be same as the coordinates of each point of this on this line as we evaluate as we approach this curve from element e + 1.

And that is, because the interpolation functions for this particular element they satisfy the C^0 continuity. So, the variable being interpolated will have unique variation across the inter element boundaries. So, in this case the variation that is being approximated is the geometry how this locus of points along this line vary. So, the coordinates of the points. So, that is being approximated and that is unique variation that is defined for both the elements. And that is how that is why the elements will be contiguous. There will not be any voids in between or there will not be any overlap in between, because these parent element the interpolation functions satisfy the C^0 continuity requirements. So, that takes care of the geometric compatibility.

So, if they are derived from the same elements and then as long as they end up with the same variation with the same number of parameters to define that variation, then there is no question of any error on that count, as far as geometric compatibility is concerned.

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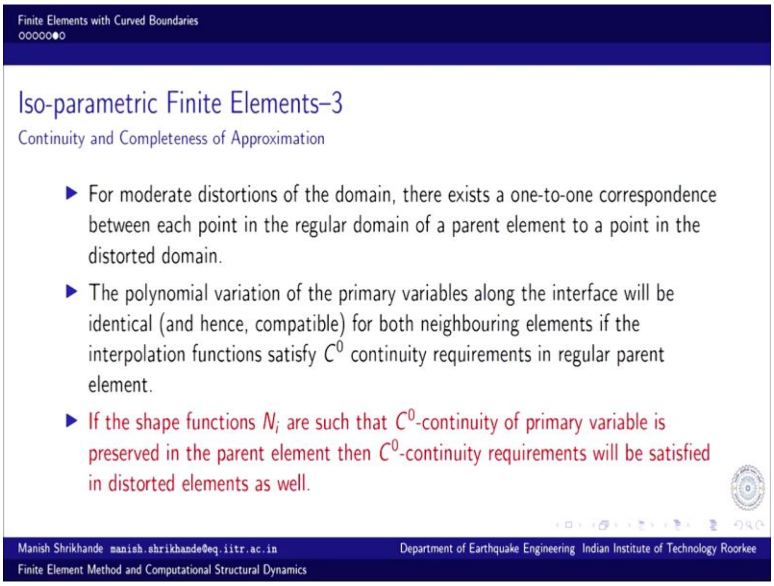

Now, we come to the continuity and completeness requirement for the approximation within the element. So, for moderate distortions of the domain, as I said there exists one-to-one correspondence between each point in the regular domain of the parent element to a point in the distorted domain.

Now, this happens only when the distortions are moderate. Extreme distortions can be when this thing is translated into is mapped on to I mean it is like right hand coordinate system mapped on to left hand coordinate system and so on. So, completely topsy turvy.

So, that will happen when you roll over the matter for example, a particular rectangle let us say this is a rectangle and I map it on to this geometry. So, this rectangular geometry is mapped on to half of the domain is mapped on to other half of the domain.

So, now, there is no one-to-one correspondence and this kind of mapping is what we call as severe distortion and this is not admissible in for finite element analysis. So, that one-to-one correspondence between the original parent domain and a distorted domain is essential assuming that this one-to-one correspondence exists.

Then, the primary variable polynomial variation of the primary variable along the interface will be identical and hence, compatible for both neighboring elements. If the interpolation function satisfy C^0 continuity in the regular element. So, just as the geometry instead of x, if you are interpolating primary variable u and v.

Again, we come up with the same argument that along the interface boundaries there will be a certain degree of variation and if as long as we have common nodes and common variables ensuring unique variation of that unique definition of the variation along that line continuity will be achieved. So, for example, in the case of eight noded rectangle, I mean if we have these three values u at this point u at this point and u at this point; the three values are sufficient to define a quadratic variation and that is what it will be if I evaluated at ξ = +1 or ξ=-1. If I come from this element.

So, three nodes they evaluate they are sufficient to give quadratic variation with respect to η. And similarly, for any of these nodes on this edge or at this edge, it will be eta is equal to 1 and it will be quadratic in ξ.

So, again three nodes are enough to define the quadratic variation with respect to ξ. So, that way the continuity of the primary variable in the distorted domain will be guaranteed, if the interpolation functions that are used to define the variation they are consistent with C^0 continuity requirement.

So, the conclusion is if the shape functions N i are such that C^0 continuity of primary variables is preserved in the parent element then C^0 continuity requirements will be satisfied in distorted domain as well; distorted elements as well. So, pretty straightforward, but it needs to be realized that it is not. So, obvious and it is not so trivial. I mean this will not be applicable or this will not hold in case of severe distortion.

So, this line may appear to be trivial, but this line actually this conclusion holds C^0 continuity in the is preserved in parent element, then C^0 continuity will be satisfied in the distorted elements as well; provided the distortions are moderate and one-to-one correspondence between the two domains exist right.

So, each point on the parent element is mapped on to some point in distorted element. So, there is one-to-one correspondence. So, as long as that distortion is of ensures that one-to-one mapping, this C^0 continuity of primary variables in the parent element will be carried over to the distorted elements as well.

Now, the completeness. As we saw the requirement is for minimum requirement for convergence is, it should have constant term and it should have a linear term. Constant term for modelling rigid body motion and linear terms for modelling constant strain states. So, this is the bare minimum for ensuring convergence of finite element analysis.

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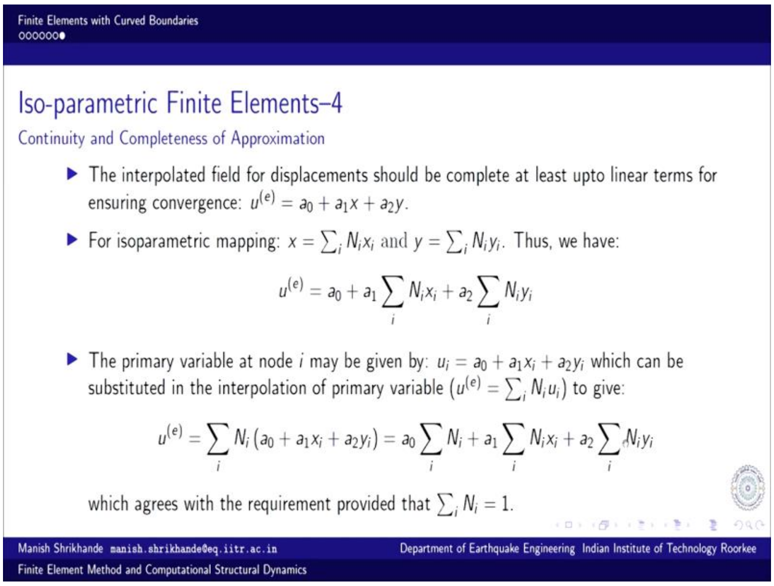

So, the displacement field for the interpolated field for the displacement primary variables should be complete at least up to linear term. So, again we take example for two-dimensions, just for ease of analysis; the arguments are exactly similar for three-dimensions it is just one more term.

So, for iso-parametric mapping, this x is defined by interpolation of geometry. So, N i x i and y is defined by variable y is varied according to N i Y i. So, we can substitute this expression for x and y in this approximation for u. So, u_e within the element that will be given as shown in the slide. Now, this when I evaluate at this is the variation of displacement across the entire element and this should hold at each node as well.

So, if I evaluate this primary variable at any node i; that will be given by just substituting for x_i and y_i the coordinates of the nodes. So, that is that is given by $u_i = a_0 + a_1 x_i + a_2 y_i$ right. So, this we can substitute in the approximation of primary variable.

So, because u is given by $N_i u_i$. So, I can substitute this value of u_i into that expression interpolated form and that gives us u variation of primary variable within the element is given by N i plus u i. So, $a_0 + a_1 x_i + a_2 y_i$ and that I can collect the terms individual terms.

The constant term that remains here, that constant term will be retained only if; if and only if, shape functions or the interpolation functions are partition of unity; that is this summation of interpolation function is equal to 1. Unless this condition is satisfied, this completeness requirement is not guaranteed. And fortunately, we have been insisting on this condition, because this constant I mean summation of interpolation functions being equal to unity that is of course, the requirement of admitting the constant term in approximation.

So, if the constant term approximation representation in approximation is satisfied in the parent element, then the completeness of the approximation would be satisfied in the distorted elements as well. So, that completes our discussion on the convergence. I mean under what conditions the finite element results would converge to true solution in the limit as the discretization is improved.

So, we looked at the various possibilities. And in general, if we pay attention to general rules of construction of interpolation functions good practices for construction of interpolation functions for parent element, then there is no problem or the all the requirements for correct finite element analysis would be satisfied in the distorted domains, as long as the distortions are not very severe. As long as the distortions are not severe, which lead to violation of one-to-one correspondence between the points in parent element and the points in distorted elements. We will continue our discussion regarding some other implementation issues we now, establish the theoretical basis why iso-parametric element should work in the finite element analysis taking the good from two different fields.

So, we retain the regular use of regular geometry for defining the interpolation. And we use parametric mapping to enhance our power of distortion and to model the curved geometries or for arbitrary domains modelling of arbitrary domains. In our next lecture, we will discuss how the domains the eventually the domain integrals have to be evaluated those integrals two-dimensional three-dimensional integrals; they have to be evaluated.

Now, the problem is our interpolation functions and the variation they are now defined in terms of local coordinates xi eta zeta whereas, the domain integrals integrand in those domain integrals and also, boundary integrals; they involve derivatives with respect to Cartesian coordinates x y z.

So, we solved one problem and we landed it landed into another problem. So, let us look at how to solve the new problem. So, let we will next lecture, we will study the what is the problem that we have landed into new problem. And then, we also look at how do we solve that problem.

Thank you.