Finite Element Method and Computational Structural Dynamics
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Lecture - 31
Mapped Element - I

Hello friends. So, we have seen how a stress analysis problem is typically modeled using Finite Elements. We look into the aspects of the problem, what kind of framework or what kind of conditions and boundary conditions are there, loading conditions and boundary conditions and then, we take a call whether any simplifying assumptions can be made and a simplified model can be developed. For example, we looked for the suitability of plane stress conditions and thereafter, we model the entire 3 D problem as a two-dimensional idealization, two-dimensional simplification as a plane stress problem.

And subsequently, the entire problem domain was divided by using finite elements; I mean two a elements we used, 4-noded rectangle and one 3-noded triangle, constraint strength triangle and subsequently, the formulation element level equilibrium and evaluation of the boundary integrals etcetera, the these are all very standard processes. And eventually, we develop element level equilibrium equations for each element successively and those element level equilibrium equations are assembled to form a global system of equations for representing the entire domain and then, the essential boundary conditions of the problem are imposed. And subsequently, the equations are solved for unknowns of the problem, the primary variables. Those are the nodal displacements in x along x and y for that bracket problem. And subsequently, once we have these primary variables, we can go back to element level equations and evaluate the strains and stresses within each element and then, those stresses are again combined and we can draw the visualization. I mean because the table of numbers it is generates finite element analysis generates voluminous data and it is not very practical to look at the tabular form of result. So, very often, it is presented the results are presented in a graphical form in the form of contours. So, displacement contours or stress contours different types of stress contours, I mean we can have normal sigma x x sigma y y tau x y along the with respect to the chosen frame of reference or we can have a principal stresses or other stress criteria, one mistress or some similar equivalent transformation of stress tensor. And appropriately, the results can be interpreted for further analysis and design.

So, that in all defines the and also explains or shows or demonstrates the power of finite element method, how simple the entire problem becomes although the problem geometry that we discussed was very simple. But the process remains the same, even if we deal with a complicated geometry or more complex geometry.

The, it is completely process is automatic and the only human intervention is required is in preparation of the input data. Subsequent analysis is can be interested entirely to the computer. Now, this brings us to an important problem. I mean, we used; I mean if you have been paying attention, you would have noticed that the elements that we use finite elements, they are all regular geometry that is the essence of finite element. Because the complex geometry the approximation is difficult to develop over the entire domain; but it is easier to develop approximation, if the geometry is regular.

So, that is why we develop these finite elements that we will have regular geometry. For example, rectangle or triangle, very well known geometric figures, where in the interpolation the approximations can be developed very easily and those can be used for modeling of the domains.

Now, this is fine as long as the domain is straight; the edges are I mean the element boundaries straight element boundaries. For example, state edges in case of two-dimensional finite elements or plane areas, plane surfaces in case of three-dimensional domains. If those are aligned with the domain boundaries, domain boundaries are also straight lines or plain phases.

But if those are curved. So, curved lines or curved surfaces in case of three-dimensional domains, then we have a problem in representing this domain. Because a curve cannot be represented by a straight edge or a straight face or plane face. So, we can develop that, we can reduce the discretization error. So, to say, so by reducing the size of the element, so very very small elements, I mean any arbitrary degree of curve can be represented to sufficient accuracy by a straight line, provided we look at a small enough segment of that curve.

So, by extending the same logic, we can actually divide the use of a very small element size and we can reduce the error of approximation in the curved boundaries by using regular geometry, geometrical shapes or regular finite elements right. So, large number of small size elements, but when we say I mean it is easier said than done. When we say large size of large number of small size elements, then I am forcibly I am forced to use large number of nodes; I mean large number of elements and large number of nodes.

So, that leads to increase in the problem size. So, while they may not be required for from the accuracy point of view, the such are a fine mesh, may not be required from the accuracy point of view. But because of the accuracy, I mean the finite element approximation point of view, but to reduce the error in discretization of the domain because if for a curved surface, if I model this curved surface by the straight line, then this portion of the domain is left un modeled.

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So, this is discretization error and no this cannot be compensated by any clever interpolation within the element. This part of the domain is unrepresented and that remains a problem. So, this can be reduced, if I can model this by small size elements of smaller size. So, a curvature can be modeled by straight lines using smaller and smaller segments. So, the problem, but that will require large number of elements and large number of nodes. So, the problem size increases due to increased number of elements and nodes.

So, this can be reduced, if I can somehow develop finite elements which can have curved boundaries. And yet, I retain the advantage of developing approximation over regular geometry. Because finite element has the biggest advantage from that very point from that very fact that a regular geometry, well-known geometry, standard geometrical shapes, I define we can define the interpolation and they can be combined to represent the whole domain. So, that advantage has to be retained; at the same time, there we need to explore possibility of somehow modeling the curved geometry using reasonably sized elements.

Because reducing the element size just to model the curvature or the domain representation is an overkill. We do not really need, I mean because if I use a small element near the element near the domain boundary, then I am forced to use the same size of element all through the domain. So, whether there is a stress gradient or strain gradient or not, where the accuracy of solution is important and mesh refinement may be required; but I will be required to use the same size of elements all through.

So, that would be an overkill and may be not justified. So, that is why it is very important to look for possibilities to x model the curved boundaries, curved domains by using suitable. Modification of the finite element formulation that we have seen and look for possibilities, if we can use it for modeling the curved domains.

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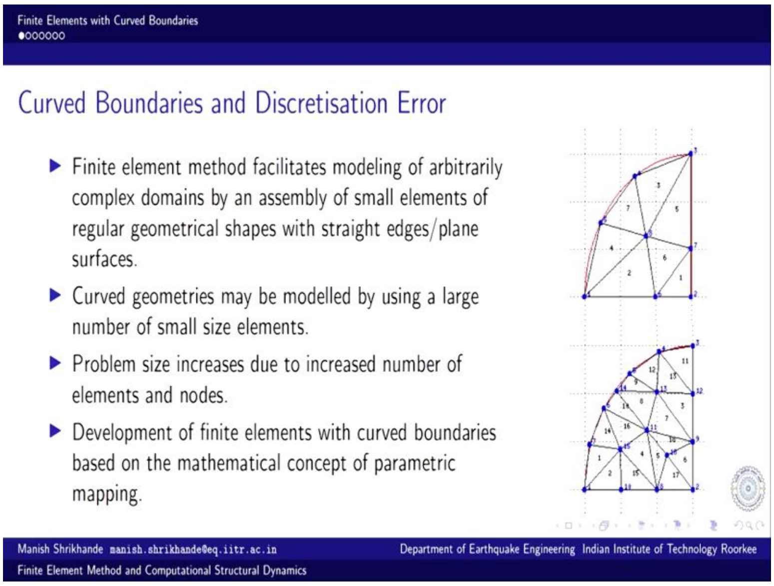

So, just to explain the concept of discretization error and the need for modeling curved boundaries, this one quadrant of a circle; I mean it can be plane stress model or whatever two-dimensional geometry, so one quadrant. And in this case, it is modeled by using 7 finite elements, triangular constant strain triangular constant strain triangle and you can see the discretization error here.

So, the original domain is this circle arc of this circle and that this part is left unrepresented and similarly for this. Now, this can be reduced, this discretization error can be reduced, if I use smaller size elements. For example, in this case same domain is discretized using about 18 number of finite elements of again 3-node triangles.

But you can see that the error in the modeling of the domain itself. The gap between the original problem domain and the boundary of the finite element mesh is much smaller, much reduced in this refined mesh. And this can be further refined if I use further smaller elements. But this is only to explain the concept or the issue that we are discussing here.

This error, this representation error in representation of the domain is a permanent error. This cannot be compensated in any way. No matter what I do in this interior of the element domain, whatever has been left out remains out of the consideration all through the analysis. And that is why it is important to explore.

If it is possible to look for mapping this or somehow modeling this edge of the element to coincide with the problem boundary right. So, this curved boundary, if I can somehow make this element boundary to coincide with the problem domain, domain boundary, the error in discretization of the domain would be greatly reduced and brought under control. Now, to this effect, we make use of a very powerful concept of mathematical analysis or mathematics coordinate geometry. So, all of us have probably some a studied sometime or the other equations of various curves coordinate geometry.

So, and at that time, there are different ways in which the equations can be defined. For example, the parametric form of ellipse; until I discovered or I found its application in this finite element analysis, I always used to wonder what do I do with this parametric form of equation. Normal coordinate geometry is so appealing, so much more convenient and so much more easy to visualize; parametric form is not that easy to visualize.

So, why do we bother about parametric form, but then this parametric mapping, I hope I will be able to explain to you the beauty of this parametric mapping and how powerful tool it becomes to enhance the applicability of finite element method and then, open up new vistas. We will see how; I mean this parametric mapping or the mapped elements as they are called, development of this mapped element, theory of mapped elements has led to enormous increase in scope of application of the finite element method in engineering analysis. So, for example, I was referring to parametric form of the ellipse.

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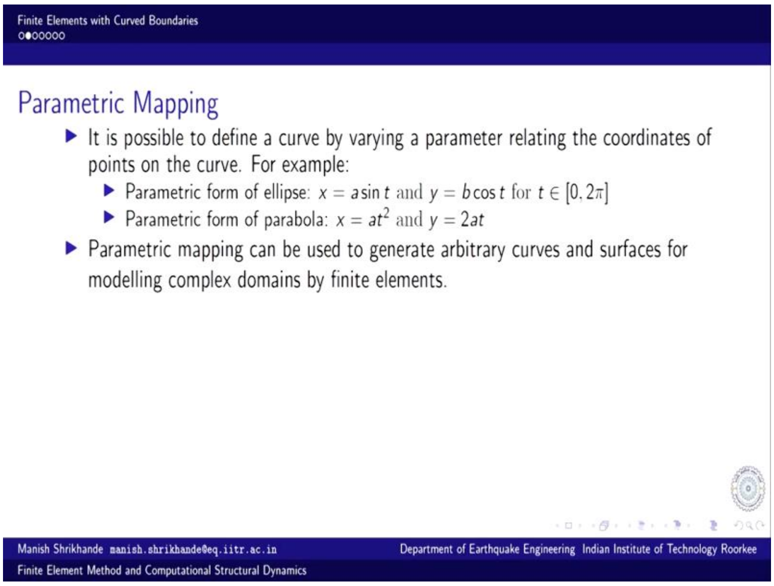

So, the coordinates x and y can be referred to as x= a sin t and y = b cos t; t is a parameter varying from 0 to 2π. I mean that is of course, the principal range, t can vary from anything. It will, because it is a argument of trigonometric function. It is it gets mapped to the range to 0 to 2 π because trigonometric functions are periodic with period 2 π. So, t can be anything, but it will eventually roll back onto this range and a and b are the semi major and semi minor axes. And this is what I mean we can always convert it to that familiar form of equation $x^2 / a^2 + y^2 / b^2 = 1$ that is the equation of ellipse. So, that can be very easily be achieved by in this by using this form.

But important point to note here is although there are two coordinates, but knowing the semi major and semi minor axes, I just need to vary one parameter t and the entire ellipse can be traced out. And this actually forms a basis of actual construction of ellipse on ground or on a field. So, elliptical figure, how to draw an electrical figure; so, this is one of the ways in which we can actually trace out an ellipse on field.

Similarly, we can have parametric form of parabola. So, x can be defined as a t^2 and y can be defined as 2 a t. Now, if we can substitute it, I mean we can confirm it by comparing it with this standard equation of parabola $y^2 = 4$ a x and you will find that this is what it leads to. Now, again t can be anything and we it will nicely trace out the parabola. So, t can be from minus infinity to plus infinity and it will generate parabolic equation.

So, this gives us power. So, point is we can generate arbitrary curves arbitrary functions; I mean geometries by suitable variation of a parameter. We just need to define appropriate parametric variation and any kind of curves can be developed using those parametric variation. So, this parametric mapping concept can be used to generate arbitrary curves and surfaces curves for two-dimensional domains and surfaces for three-dimensional domains for modeling complex domains by finite elements.

So, if the domain is very complex and boundaries are very curved, not straight edge regular geometries, then no issues; we can model those curve geometries very easily, very closely without having to reduce the element size to a very small element impractical impractically small size and still reduce the discretization error significantly.

So, how do we do this? Essential idea is we transform the geometric shapes the parent finite elements which are as we all as we know these are regular geometries; rectangle, triangle, tetrahedron or cuboid or prism in three-dimensions to a desired shape where whatever shape we want. So, we start with a regular geometry, regular finite element that we have a discussed in our earlier lectures.

But those regular geometries, regular elements, they are distorted to take an appropriate shape; desired shape. So, this process is called element distortion and this is achieved by the use of what we call as local or natural or curvilinear coordinate system and using these coordinates for parametric transformation, definition of the desired geometry. How do we do this?

We do this exactly in the same the way the way we define the interpolation of primary variables. So, if you recall the primary variable within the element is defined as let us say u is the primary variable. So, variation of primary variable within the element domain is defined as σ n_i u_i. So, n_1 u_1 + n_2 u_2 + n_3 u_3 so on.

So, interpolation function for a node multiplied by the primary variable at that node. So, this summation extending over all nodes of the element that gives us the approximation of the primary variable over the entire element. Now, we do use the similar kind of interpolation; but this is to define the geometry. So, we have a set of standard elements in an element library. So, those are regular elements that we have discussed in our earlier lectures. So, rectangle elements, triangular elements or even tetrahedron or cuboid or prism in case of three-dimensional finite elements with well-defined interpolation functions as we discussed in our previous lectures how to develop the interpolation functions for them. Now, those elements those regular elements straight edge or straight phase elements they can be I mean they of course, enclose area or volume defined by the coordinates x and y. So, these can be mapped. So, this variation of the geometry the x, y coordinate Cartesian coordinates, they can be suitably mapped.

So, how the elements can we can change the domain or what kind of domain is represented by an element that is considered to be in the similar way to be govern in a similar way as we do for primary variable. So, for example, the variation of x coordinate within the element is defined as an interpolation model. So, just as u is equal to $u_i n_i$, we define x is equal to $x_i n_i$ summation over i. So, coordinates, we just define the coordinates, where do I want to place node 1 or ith node of the element parent element. So, parent element let us say it is 4-noded rectangle. So, 4-noded rectangle defined by i j k l, 4-nodes. So, where do I need to where do I want to place ith node? Where do I want to place jth node? Where do I want to place kth node? And where do I want to place lth node?

So, once I do that, I already have the coordinates of those points defined and that is what defines the variation of geometry $x = x_i n_i$; $y= y_i n_i$; $z = z_i n_i$. So, straightforward concept. Just to crystallize those ideas, let us look at these mapping what we are trying to do and that will explain the concept.

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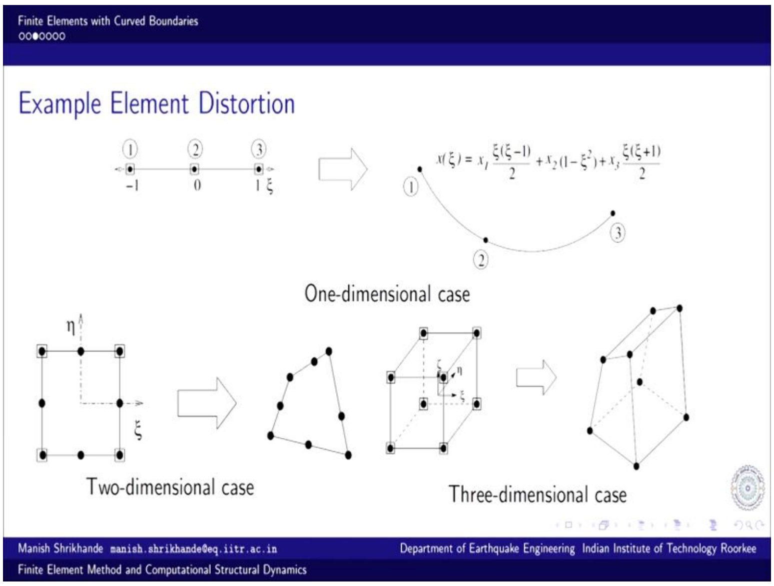

So, this is what we have parent element 3-noded parent element, straight one-dimensional, ranging from I mean this x_i one-dimension x_i local coordinate ranging from -1 to + 1. So, node one placed at -1, xi is equal to -1 and node 3 placed at x_i is equal to + 1 and node 2 is the midpoint that is at x_i is equal to 0.

Now, if I have to define a second degree curvature, second degree curved line, I can very easily define this by using this kind of interpolation. Now, interpolation function if you will if you try to work out using these coordinates, so these are the interpolation for node 1, this is the interpolation function for node 2 and this is the interpolation function for node 3. So, all that we need to do is find the coordinates, I mean any circular arc is uniquely defined by three distinct points. So, all that we need to do is find the coordinates of these three points and interpolate it using this, interpolation formula. So, x as a function of x_i is defined by this curve right. I just have to define what is the coordinate of x 1, what is the coordinate of x_2 and what is the coordinate of x_3.

So, entire this parent element one-three, straight element one-three gets mapped on to this quadratic, I mean second degree curve. So, I can distort this, I can stretch it, I can compress it, I can distort it in any shape; I can rotate it as well whatever just I need to specify appropriate values of the coordinates, where these nodes are likely to be placed or they should be placed; what is the destination of this distorted geometry.

So, this is one-dimensional case. In two-dimension, we have two coordinates. So this 8-noded rectangle can be I mean in this case, we only used bilinear distortion. So, we just defined coordinates of corner nodes that are indicated by these squares small squares. So, these are the small squares, these are the nodes which are used for defining or defining the geometry. Field circles are the nodes which are used for defining the primary variables.

So, there are two kinds of interpolation as we said, as we saw earlier; one is the variation of the primary variable with respect to within the interior of the element domain and second is the geometry how the geometry, how this parent element is going to be mapped in actual problem domain.

So, the geometry in this particular example, in this particular case is mapped using this small square nodes indicated by small square. So, in this case 4 corner nodes. So, 4 corner nodes the interpolation will be bilinear. So, 4-noded rectangle. So, those 4-noded rectangle that can be so all these straight edges will be distorted using straight lines.

So, this rectangle, regular rectangle can be mapped into an arbitrary trapezoid right. So, different edges can be mapped a along different ways. So, formula, I mean it is the same interpolation formula $x = x_i n_i$; y is equal to y i n i summation over i. So, all the nodes defining the geometry. And similarly, now the for three-dimensional, we can this cube 8-noded cube can be distorted into a in this case first term of a pyramid or something.

So, any arbitrary geometry, just we need to define a unique coordinates of the points one-to-one correspondence, this node goes here. So, what are the coordinates of this point? This node goes here. So, what are the coordinates of this destination point and so on. This node goes here, what are the coordinates of this target point and so on.

So, each node has a one-to-one correspondence. So, each node of the parent element or for that matter, every point in the parent domain has one-to-one correspondence with a point in the distorted domain that is essential. We will come to that in our subsequent discussions. So, this is this mapping is very important and this is how we can generate arbitrary shapes and so, the whole idea is we still use regular geometry.

So, the interpolation is still done in the case of in this regular geometry because that is the biggest advantage. Finite element method has to offer the possibility of using a very comfortable regular geometry and we develop our approximation within these regular geometry parent elements and then, transform that map that onto a distorted a geometry and that would does distorted geometry or the mapping transformation into an arbitrary shape will be according to the problem domain. Because once I have the problem domain, I obviously, we all obviously know the coordinates, where these nodes have to be placed. Now, the only point that needs to be mentioned here is one-to-one correspondence. Every point in this domain, element domain, parent element domain, these are called parent elements, undistorted elements are referred to as parent elements and these are called distorted elements. So, every point in the interior of the domain of parent elements has to have a one-to-one correspondence with a point corresponding point in the distorted domain.

Now, this can be violated. For example, if I try to map this point here and this point here. So, that would actually go like this. So, this point go and gets mapped like this. So, there would be an overlap right. Similarly, I can possibly try to map this such that this edge folds back on to the element itself.

So, that would lose one-to-one correspondence and that will be indicated there are mathematical indications of that happening. So, that would be caught on the in the tracks during the process of analysis, if ever that kind of error happens and the analysis will not progress any further. Any finite element analysis could would make this check. And because the results are going to be useless, there is no point in moving further with this kind of wrong representation of the domain.

So, this is how the mapping is done. We will discuss more little some theoretical aspects because of the convergence because we developed finite elements in regular geometry using all sorts of arguments, what are the requirements of convergence and all and during the and we are all throwing it away and distorting the shape to whatever we want. So, what happens to the convergence requirement? Does the finite element solution converge at all and does it converge to the exact result? We will see that in our next lecture.

Thank you.