Finite Element Method and Computational Structural Dynamics
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Lecture - 28
Finite Elements of C^0 Continuity in 2-D and 3-D-X

Hello friends. So, we were discussing about the rectangular family of a elements. And, we discussed about how the interpolation functions, they have to satisfy the partition of unity concept to be able to represent the rigid body modes.

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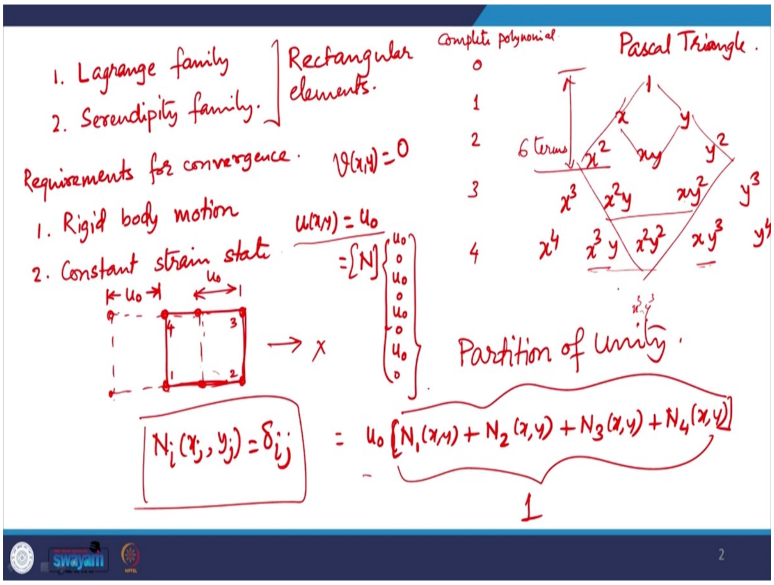

So, some of that is essential and this requirement is holds equally well for the triangular elements as well. So, all the finite elements for solid mechanics that are used for solid mechanics problems, they need to satisfy and also for other constant value of primary variable.

If, that requirement is to be included in the approximation, then it is essential that the interpolation functions have to satisfy they have to add up to unity at every point. So, this has to be satisfied uniformly at all points of within the domain, within the element. So, at all points of x y all possibilities of x y within the element domain, these interpolation functions they should add up to unity right.

And, this is; obviously, true at the nodes, because at nodes only one of them will be unity all other interpolation functions will vanish at node. For example, at node 1 N1 = unity N2, N3, N4, they will all be 0 at node 1, at node 2 N2 will be unity, but N1, N3 and N4 they will all vanish at node 2. Similarly for node 3 and 4. But this requirement is not just to be satisfied at the nodes, but at every point in the domain, in the interior of the domain. And, that is crucial because this interpolation condition is easy to satisfy, that is $N_i(x_j, y_j) = \delta_{ij}$. So, this is not that difficult to satisfy.

So, interpolation condition can be derived and it can be managed in several ways, but we need to look at that set of interpolation functions, which will in addition to satisfying this interpolation condition, they also satisfy the requirement of those being partition of unity, those divide the unity amongst themselves right. So, now we go to the next point of discussion the particular ways in which Lagrange interpolation and the serendipity interpolation functions are derived.

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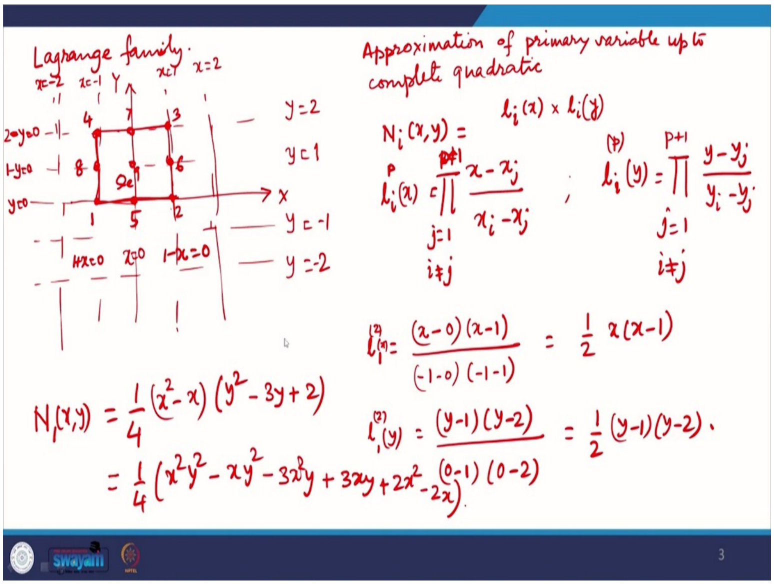

So, Lagrange interpolation and Lagrange family is simply defined as if I have the coordinate frame. So, there can be grid lines for constant value of x and y and I define the nodes at these grid points.

So, for example, for 4 nodded triangle or 4 nodded rectangle, we defined the intersection between these 4 grid lines right. So, this = $x = 1$, let us say this = $x = 2$, this is $x = -1$,

$x=-2$. And, similarly there will be $y=2$, $y=1$, $y=-1$, $y=-2$. So, these are the grid lines.

So, 4 nodded rectangle is defined by any 4 points, which are defined by the grid lines not necessarily same as mentioned above. They can be any 4 points which are located at the grid points of the Cartesian coordinate system, or coordinate system more precisely.

Now, I need to define variation of primary variable that is complete up to first degree so that it contains the constant term, which will model the rigid body motion, and it also contains the linear terms, which will model the constant strain conditions. But, if I wish to and the error in approximation is of the order of second degree polynomial, approximation of displacements is second degree.

Although, the second degree term is present in the approximation in the form of x and y, product x y term cross product term, but the second degree other 2 terms of the second degree polynomial x square and y square, they are not included in the approximation. So, they do not contribute to the general quadratic variation. So, this element 4 nodded element is not capable of representing complete quadratic.

So, I can define high r order elements by let us say quadratic term if I need to include 4 quadratic I mean the polynomial approximation, approximation of primary variable up to complete quadratic. So, up to complete quadratic; that means, I need to have 2 terms in each. So, second degree in x and second degree in y.

So, what is done is for the interpolation function? So, $N_i(x,y)=l_i(x).l_i(y)$.

So, Lagrange's interpolation of pth degree will have these terms ranging from 1 to p + 1, and 1 term less when i is not equal to j right. So, that it becomes for example, for second degree polynomial it will have two terms right and that is what it will have. And, similarly for l_i for y pth degree polynomial this is

$$\prod_{j=1;\,j\neq i}^{p+1} \frac{y-y_j}{y_i-y_j}$$

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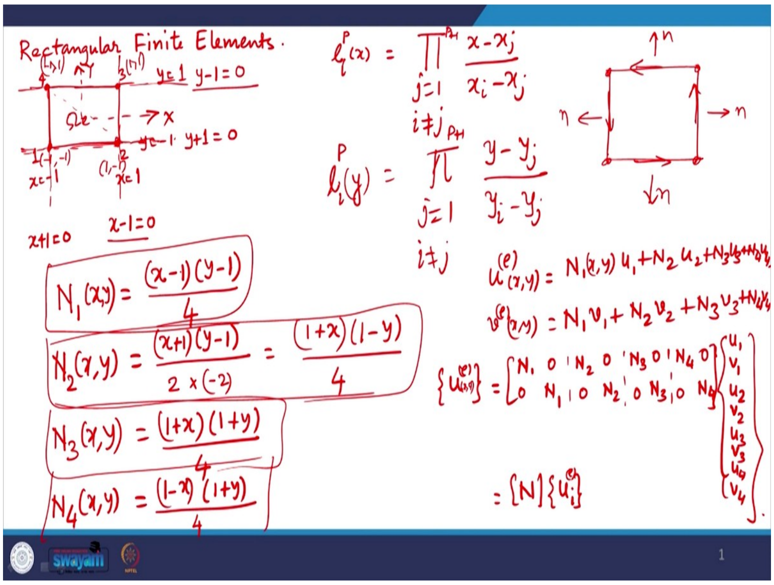

So, these are the two terms and if you look at the previous derivation, this is all first degree Lagrangian interpolation in x multiplied by first degree Lagrangian interpolation in y right. So, this should be actually up to $p+1$ yeah. Only then there will be by removing one term for which i is not equal to j there will be p number of terms in this product.

So, so, this is first degree polynomial in x multiplied by first degree polynomial in y and together they give us the required terms. And, the what are the terms here it would be instructive. So, there is a term x multiplied by y. So, that is cross term then 1 multiplied by x, then 1 multiplied by y and then 1 multiplied by 1. So, those are the terms in the in the Pascal triangle.

So, 1 multiplied by 1 that is the constant term x multiplied by 1. So, that is called linear term y multiplied by 1, that is linear term and then x multiplied by y that becomes x y term. And, similarly if I multiply by the quadratic terms quadratic interpolation model so, I will have for example, I need to take the elements what are the elements that will be required right?

So, what are the elements that we will be required? So, we will need to have 3 nodes in each side. So, I can have these possible nodes and this becomes the interpolation this becomes the element right.

So, there can be actually sorry it is possible to have interpolation. So, this is the node, this is the element that we will have Ω_e nodes being 1, 2, 3, 4, 5, 6, 7, 8 and 9 so, a total of 9 nodes. So, as you can see the nodes are all placed along the intersection of the grid lines constant values of coordinate lines. And, each of these constant values of coordinate lines are defined by one of these equations those; these are called the 0s of those lines or the equations of those lines $x - x_j = 0$ of $y - y_j = 0$.

So, we just multiply by the 0s and then normalize the numerator evaluate the numerator at the corresponding node, substitute the values of the coordinates of the corresponding node and then divide the numerator by this value. So, that it becomes normalized to unit value. So, this way we can have in this case the for in interpolation function for node 1, we can have $y - y_1$. So, that is in this case it is simply or this is only $x = 0$, because y axis is are oriented along this.

So, this is $x=0$ line and this is $1-x=0$ line and this is $1 + x = 0$ line, and this one is $y = 0$, and this is $1 - y = 0$, and this is $2 - y = 0$. So, that is the equation of those lines and that is what we need to substitute here. So, these are what those lines are $y - y_j$, $y - y_j$, and product evaluated at i^{th} node divide by that and that becomes normalized.

So, in this particular case it will have. So $l_1 = x - 0$. So, to say this is the coordinate 0, and multiplied by $x - 1$ divided by this is -1 multiplied by (-1) (-1). So, this numerator evaluated at $x = 1$ at the node 1. So, at node 1, x coordinate is (-1). So, that is what it becomes here.

So, this becomes (1/2) $(x - 1)$. And, similarly for l_2 so, this is quadratic and second degree polynomial for y in terms of y for node 1 will require this equation and this equation. So, $1 - y$ multiplied by $y - 2$, and then the value of the y coordinate of node 1 is 0. So, it becomes 0 - 1 multiplied by 0 - 0. So, that = again (1/2) $(y - 1)(y -2)$.

So, this is what it is and the product the N_1, then becomes (1/4) $(x^2 - x_i)$ multiplied by $(y^2 - 3)(y + 2)$. So, I take the product of these two terms and then these two terms they need to be multiplied. So, what is the product, what are the terms that we are looking at? So, there will be a constant term in general.

So, there would be some approximation I will just say 1 by 4 x square y square, then there will be x y square and then - 3 x square y + 3 x y + 2 x square - 2 x. So, these are the terms in shape function for node 1. And, similarly there would be other terms current coming from the shape functions of node 2, node 3, node 4, node 5, node 6, node 7, node 8 and node 9.

So, in general there will be 9 terms in the approximation, total polynomial approximation will have 9 terms and what are those 9 terms going to be? So, let us calculate again going from the Pascal triangle. So, 1 so, up to this 1, 2, 3, 4, 5, 6. So, these are all there, then beyond this we need 3 more terms to complete the total of 9, because there are 9 nodes, 9 nodes that come on the grid lines.

So, these grid lines they intersect total at 9 points. So, there have to be 9 nodes in the approximation and that will require 9 number of polynomial terms. So, what would be those 9 polynomial terms? So, up to this quadratic complete it is 6 terms right. So, these are 6 terms. In addition to these 6 terms, we need to include 3 more terms and those three more terms. Obviously, have to be located symmetrically in the Pascal triangle to maintain geometric isotropy.

So, those three terms would be x^2y, xy^2 and x^2y^2. So, these three terms, they are spurious terms, they do not contribute to the convergence requirement rather they pollute the approximation. The approximation is only complete up to second degree polynomial. So, these higher degree polynomial terms, they are only placeholders just to ensure unique approximation unique solution.

So, that we have 1 to 1 mapping between the number of nodes and the terms of the polynomial right. So, these are the three interpolation polynomials and as we see for 4 nodded rectangle, we had one spurious term, that is extra term beyond the required or intended degree of completeness. In case of quadratic completeness target to when we target to have complete quadratic variation, then we have three spurious terms, three extra terms from higher order.

Now, they are also not of same degree. So, there are two from the cubic variation and one from the quadratic variation, fourth degree variation. And, similarly if we try to go further up for the cubic complete variation, there would be still much further number of much larger number of spurious terms, in the polynomial approximation that need to be included. And, that large number of higher degree polynomials that is actually not good for the quality of approximation.

We have seen this in our interpolation while our discussion of interpolation, approximation, in approximation of runges function, higher degree polynomial terms they lead to very wild oscillations between the nodes of interposition. So, these higher degree interpolation terms, which are not can contributing to the convergence requirement or the accuracy of the finite element solution, they are they should be minimized.

Because, they are trying to they will corrupt the polynomial approximation, quality of approximation. And, it is for this reason that serendipity elements were developed to cut down on these extra terms that figure in the Lagrange interpolation. So, Lagrange interpolation model is very easy to develop.

It is just product of Lagrangian interpolation in x direction multiply it with the desired like Lagrangian interpolation in y direction. And, that is all and for the 3 dimensions, it would be another Lagrangian interpolation in z direction and all is well.

So, complete interpolation model is defined, but that would involve a lot number of spurious higher order terms, which are not really contributing to the problem, contributing to the accuracy of the solution. And, further they also another issue is they also require more number of nodes and more number of nodes; that means, it increases the problem size, because those primary variables are part of the solution. And, in this case there is one internal node here, interior node here, 9th node.

So, for quadratic approximation there is one interior node and this interior node does not contribute anything towards the inter element compatibility, inter element continuity. So, inter element continuity of the solution approximation, which will ensure piece wise continuous variation of the approximation from one element to the next element.

So, this interior node does not contribute at all. And, these number of interior nodes will keep on increasing more and more. For second degree complete, we have one interior node, if we go for third degree complete approximation, we will have 4 interior nodes. There will be total of 16 terms, 16 nodes right 4 on each side and then 4 in the interior node. So, there will be problem.

So, that not only increases the problem size and it also adds extraneous higher order terms, which will distort the approximation, polynomial approximation. And, with this objective to cut down on the to eliminate these interior to eliminate these interior nodes from the approximation.

And, at the same time that will also have the added benefit of reducing the number of spurious terms, these serendipity elements were developed. And, those serendipity elements are developed in a very straightforward logical way, I would say the if not straightforward.

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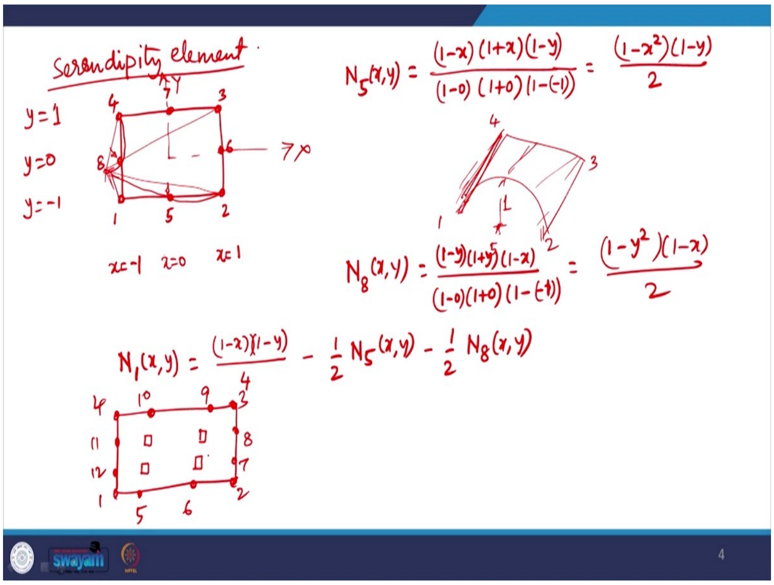

So, what are those techniques let us discuss by using an example. So, for serendipity element the quadratic complete, first degree first element is same in Lagrangian and serendipity element 4 nodded rectangle.

So, that is the simplest rectangular geometry we can have. So, there is no difference between the serendipity element and the Lagrangian element as far as 4 nodded rectangle is concerned. But for second degree complete; we aim for an element of this type. The problem was an interior node in the Lagrangian inter element. So, this 9th node that was in the interior of the domain in case of Lagrangian interpolation, we try we will eliminate, we eliminate that, because the inter element continuity is defined only by the nodes on the edges of the element.

So, let us number these nodes 1, 2, 3, 4, 4 corner nodes and then the mid side nodes 5, 6, 7, 8. Now, again the equations would be given by x = 1, and this is x = -1, and this is x=0. And, similarly this one is y = -1, y = 0, and y = +1. So, what is done here is? It is easier to visualize the interpolation functions for mid side nodes 5, 6, 7 and 8. In the sense that, these interpolation functions they have to be 0 on, let us say N5. N5 has to vanish on these 3 lines and then it has to take unit value at node 5 right. So, N5 has to vanish along this line, this line, and this line. So, a very simple way to do that is multiply the 0s of this line this line and this line and that is in some way proportional to the interpolation function for node 5. And, then I evaluate it at the coordinates of the node 5 and normalize it to unit length unit value.

So, that way it becomes very simple let me just say N5(x, y) is simply given by 1 - x and 1 + x and this is 1 - y right. And, then I divide it what are the coordinates of node 5, y = 0 no, y = - 1 and x = 0. So, this is 1 - 0, 1 + 0 and y = 1-(-1) . So, this is simply (1 - x)^2, 1 - y divided by 2.

So, this is N5 interpolation function for node 5 and if we look at the variation it looks like it will have quadratic variation. So, this is one and it vanishes to 0 at the edge 1 and 4, and 2 and 3, and 3 4. So, this is node 5. So, this is N5 N5 has quadratic variation and then linear variation in with respect to y.

So, it linearly vanishes to 0 at the other with respect to y, but with respect to x it has a quadratic variation. Similarly, for N6 I can write or it would be instructive to or more useful to take N7 or N8 right 5 and 8, because, that will help us in defining the interpolation function for node 1.

So, for N8 interpolation function for node 8 I can multiply the equation of line 3 4 and equation of line 1 2 and equation of line 2 3. So, that would ensure that N8 satisfies the interpolation condition of vanishing on all other nodes except node 8. So, that would be 1 - y, 1 + y, and this is 1 - x, and that has to be substituted for node 8 coordinates.

So, coordinate of node 8 is y = 0 and x = - 1. So, 1 - 0, 1 + 0 and 1 - (-1) so, again that = (1 - y)^2, 1 - x y 2. So, it will be similar kind of quadratic variation along y, and then it linearly vanishing to 0, with respect to x along edge 2 3. And, similarly we can define the interpolation functions for 6 and 7, by suitable product of 0s of the 3 lines.

Now, what is done is we can define N1(x, y) the interpolation function for node 1 is x - 1, y - 1, by 4 for the 4 nodded rectangle. So, we make use of that. So, N(x, y) = 1 - x and 1 - y by 4. So, when what is this function, (1 - x) (1 - y) will have variation of 1 at node 1 and it will linearly vary to 0 at nodes 2 and 3 right.

And, this will have because node 5 is at midpoint. So, this will have a value of half at node 5. And, similarly this will have a value of I mean this by linear interpolation for 4 node rectangle, this will have a value of half at node 8. So, what I need to do is I can pull this down such. So, us to have 0 value at node 5 and node 8 suitably. Now, how can I do that I know that N5 evaluates to unity at node 5. And, it has no effect on other nodes right, at all other nodes N5 is 0.

So, all that I need to do is subtract half of N5. So, this will reduce, this will pull down this by linear by half at node 5 and it will not have any effect on rest of the nodes. And, similarly I can pull down this at node 8. So, that this interpolation function for node 1 is given by bilinear variation, that was derived for 4 nodded rectangle and then modifications local modifications by the mid side nodes, appropriate mid side node, so, for N5 and N8.

Similarly for N2, it can be derived from bilinear shape function for 4 node rectangle and then suitable corrections incorporated for node 5 and node 6. For node 3, it will be similarly done by using suitable corrections for node 6 and node 7. And, for node 4, it will involve corrections for node 7 and node 8 and that way we will have the complete interpolation functions.

And, we will see if we try to examine, it will we will find that they all satisfy the partition of unity concept as well as satisfying the basic requirement of the interpolation criteria, that is N i evaluated at x j y j should be equal to delta i j the kronecker delta.

So, we can derive this for higher order terms and for third degree complete, we will have only fewer number of elements and for quadratic element as we can see there are 8 number of node. So, there will be 8 terms in the polynomial and that would involve only these two terms.

So, this extra term that was there in the Lagrangian interpolation that is eliminated, that is not required. So, we can cut down on the higher degree polynomial, higher degree terms. And, similarly we will have 10, 12 node rectangle for the cubic variation. So, I can have at one-third points. So, total 12 number of nodes.

So, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 total 12 number of nodes and together with this 12 number of nodes, we can derive the interpolation functions for which are complete up to third degree polynomial. Contrast these with the Lagrangian interpolation which would require nodes at these 4 points as well 4 interior nodes. And, those 4 interior nodes would require 4 extra terms in the polynomial and for 12 node interpolation there would be this would involve 6 + 4.

So, these are 10 terms and from 10 terms we will only require 1 extra term here, 1 extra term here. So, only 2 extra terms higher order terms in case of serendipity element, but for the Lagrangian element there would be much more I mean 4 there would be 4 extra terms. So, there it will involve all these 3 terms and then 1 extra term here. So, that would be x cube y cube term.

So, total of 4 extra terms. In the polynomial approximation and that is a major problem, but increasingly as we go above higher order interpolation, it becomes increasingly more and more difficult to find appropriate interpolation functions for serendipity elements. Fortunately we do not require to go for very higher order interpolation we generally do not go beyond quadratic cubic is about the maximum that we need for a finite element approximation and for that purpose serendipity elements are a very good choice.

But, certain application areas the Lagrangian and Interpol interpolation is also useful and particularly for row lower order for example, quadratic element the is not much to choose between Lagrangian and inter serendipity. It is only a matter of 1 extra node not much of a difference, but for cubic element of course, serendipity element and Lagrangian element they make lot of difference in the number of variables extra variables to be dealt with.

So, we will consider I mean we stop our discussion of rectangular elements here. In our next lecture we will consider practical solution of solving a stress analysis problem by using finite elements. A complete step by step process that will explain the entire process of finite element analysis.

Thank you.